# EXAMPLES OF HOPF ALGEBRAS OF DIMENSION $2^m$.

YEVGENIA KASHINA

Abstract. In this paper we construct two families of semisimple Hopf algebras of dimension $2^{n+1}$, $n \geq 3$. They are all constructed as Radford's biproducts. For these examples and their duals we compute their grouplike elements, centers, character algebras and Grothendieck rings. Comparing these facts we are able to show that depending on the dimension, representatives of one of the families are selfdual.

## 1. Introduction.

In this paper we construct two families of semisimple Hopf algebras of dimension $2^{n+1}$ for $n \geq 3$ over an algebraically closed field of characteristic 0. They are all constructed as biproducts $A = R \star H$, as in Radford [R], where $H = k\mathbb{Z}_2$ and $R$ is a Hopf algebra in the Yetter-Drinfeld category ${}^H_H YD$. This construction is similar to the one in [FMS, 5.11] and [R, Section 4]. The definitions of the Yetter-Drinfeld category and biproducts are given in Section 2.

In Section 3 we construct two families, $R_n^+$ and $R_n^-$, of semisimple Hopf algebras in the Yetter-Drinfeld category ${}^{k\mathbb{Z}_2}_{k\mathbb{Z}_2}YD$ of dimension $2^n$, $n \geq 3$. In Section 4 we construct two families, $A_n^+$ and $A_n^-$, of semisimple Hopf algebras of dimension $2^{n+1}$ as biproducts of our constructions in Section 3.

The study of these families appears to be very interesting, since their behavior depends on the dimension. For example, $A_n^+$ and $A_n^-$ exchange their roles with respect to groups of grouplikes in the case of $n = 3$ and $n \geq 4$. As will be shown, the same "exchange" happens with respect to self-duality. Moreover, it follows from the results of [K1], that $A_3^+ \cong \left(A_3^+\right)^*$ and $\left(A_3^-\right)^*$ are the unique Hopf algebras of dimension 16 with "exceptional" Grothendieck ring structures. That is $\mathbf{G}\left(A_3^+\right) = \mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbf{G}\left(A_3^-\right) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and 2-dimensional representations are not selfdual.

In Section 5 we describe the structure of the Hopf algebras dual to $A_n^\pm$. In Section 6 we describe character algebras of $A_n^\pm$ and $(A_n^\pm)^*$ and compute their central orthogonal idempotents. We show:

**Proposition 6.1.** The character algebras of $A_n^\pm$ and $(A_n^\pm)^*$ are commutative of dimension $5 \cdot 2^{n-2}$. Thus, since a character algebra of a semisimple Hopf algebra is semisimple, all these algebras are isomorphic to $k^{5 \cdot 2^{n-2}}$.

In Section 7 we prove the following result:

**Theorem 7.1.** The Hopf algebra $A_3^+$ of dimension 16 is selfdual.

---

Research at MSRI is supported in part by NSF grant DMS-9701755.

To construct the isomorphism between $A_3^+$ and $\left(A_3^+\right)^*$ we compare the corresponding central orthogonal idempotents of $A_3^+$ and $\left(A_3^+\right)^*$, $C\left(A_3^+\right)$ and $C\left(\left(A_3^+\right)^*\right)$. In Section 8 we describe the Grothendieck rings of $A_n^\pm$ and $(A_n^\pm)^*$. The computation of 2-dimensional representations of $(A_n^\pm)^*$ uses the isomorphism between $A_3^+$ and $\left(A_3^+\right)^*$. Combining these descriptions with the results in [K2] we prove the following theorem:

**Theorem 8.3.** The Hopf algebras $A_n^-$ of dimension $2^{n+1}$ for $n \geq 4$ are selfdual.

## 2. The Yetter-Drinfeld category and biproducts.

Let $H$ be a finite-dimensional semisimple Hopf algebra. The Yetter-Drinfeld category ${}_H^HYD$, introduced in [Y], consists of vector spaces $V$ over a field $k$ such that $V$ is both a left $H$-module and a left $H$-comodule (with coaction $\rho$) and it satisfies the compatibility condition

$$\sum h_1 v_{-1} \otimes h_2 \cdot v_0 = \sum \left(h_1 \cdot v\right)_{-1} h_2 \otimes \left(h_1 \cdot v\right)_0 \tag{2.1}$$

for all $h \in H$, $v \in V$ and

$$\rho\left(v\right) = \sum v_{-1} \otimes v_0$$

${}_H^HYD$ is a braided monoidal category with the braiding $\tau : V \otimes W \to W \otimes V$ given by

$$\tau\left(v \otimes w\right) = \sum v_{-1} \cdot w \otimes v_0$$

If $A$ and $B$ are algebras in the category ${}_H^HYD$ (that is $m_A$, $m_B$, $u_A$ and $u_B$ are all category morphisms), then $A \otimes B$ is also an algebra in this category via the multiplication $m_{A\otimes B} = \left(m_A \otimes m_B\right) \circ \left(id_A \otimes \tau \otimes id_B\right)$.

If $C$ and $D$ are coalgebras in the category ${}_H^HYD$ (that is $\Delta_C$, $\Delta_D$, $\varepsilon_C$ and $\varepsilon_D$ are all category morphisms), then $C \otimes D$ is also a coalgebra in this category via the comultiplication $\Delta_{C\otimes D} = \left(id_C \otimes \tau \otimes id_D\right) \circ \left(\Delta_C \otimes \Delta_D\right)$.

**Definition 2.1.** $R$ is a Hopf algebra in ${}_H^HYD$ if it is both an algebra and a coalgebra in this category, the bialgebra structure maps are all category morphisms, comultiplication and counit are multiplicative in the category and it has an antipode $S_R$, which is a convolution inverse to the identity and is a morphism in the category.

The condition for $\Delta_R$ to be multiplicative in ${}_H^HYD$, using $\tau$ on $R \otimes R$ is

$$\Delta_R \circ m_R = \left(m_R \otimes m_R\right) \circ \tau \circ \left(\Delta_R \otimes \Delta_R\right)$$

or if we denote $\Delta_R\left(r\right) = \sum r^1 \otimes r^2$,

$$\Delta_R\left(rs\right) = \sum r^1 \left(\left(r^2\right)_{-1} \cdot s^1\right) \otimes \left(r^2\right)_0 s^2 \tag{2.2}$$

The antipode $S_R$ is an anti-algebra morphism in the category ${}_H^HYD$, that is

$$\begin{aligned} S_R \circ m &= \left(m \otimes m\right) \circ \left(id \otimes \tau \otimes id\right) \circ \left(S_R \otimes S_R\right) \qquad \text{or} \\ S_R\left(rs\right) &= \sum \left(r_{-1} \cdot S_R\left(s\right)\right) S_R\left(r_0\right) \end{aligned} \tag{2.3}$$

**Definition 2.2.** Let $R$ be a Hopf algebra in ${}_H^HYD$. The *biproduct* $A = R\star H$, defined in [R, Theorem 1], is an ordinary Hopf algebra with the following unit, multiplication, counit, comultiplication and antipode:

$$1_A = 1_R \star 1_H$$

$$(r\star x)(s\star y) = \sum r(x_1\cdot s)\star x_2 y \tag{2.4}$$

$$\varepsilon_A(r\star x) = \varepsilon_R(r)\,\varepsilon_H(x) \tag{2.5}$$

$$\Delta_A(r\star x) = \sum \left(r^1\star\left(r^2\right)_{-1}x_1\right)\otimes\left(r^2\right)_0\star x_2 \tag{2.6}$$

$$S_A(r\star x) = \sum\left(1\star S_H(r_{-1}x)\right)\left(S_R(r_0)\star 1\right) \tag{2.7}$$

where $r, s\in R$, $x, y\in H$.

For more details on the Yetter-Drinfeld category and biproducts see [M, §10.6] and [R].

Let $H = kG$, where $G$ is an abelian group with a bicharacter $\langle\ |\ \rangle : G\times G\to k^*$. The category of $kG$-comodules ${}^{kG}\mathcal{M}$, which coincides with the category of $G$-graded modules, is contained in ${}_H^HYD$, since any $V\in {}^{kG}\mathcal{M}$ is also a left $kG$-module via

$$g\cdot v_h = \langle g\mid h\rangle\, v_h \tag{2.8}$$

for all $g, h\in G$ and $v_h\in V_h$ , the $h$-component of $V$, and the compatibility condition (2.1) is satisfied. The braiding $\tau : V\otimes W\to W\otimes V$ is given by

$$\tau(v\otimes w) = \langle g\mid h\rangle\, w\otimes v \tag{2.9}$$

for all homogeneous elements $v\in V_g$ and $w\in W_h$ . When $k$ contains $\omega$, a primitive $m^{\text{th}}$root of 1, and $G = \mathbb{Z}_m = k\langle g\rangle$ we can define the bicharacter $\langle\ |\ \rangle : \mathbb{Z}_m\times\mathbb{Z}_m\to k^*$ by

$$\left\langle g^i\mid h^j\right\rangle = \omega^{ij} \tag{2.10}$$

$R$ is a $G$-graded Hopf algebra if it is both a $G$-graded algebra and $G$-graded coalgebra, $\Delta_R$ is multiplicative in this category (using $\tau$, defined by the formula 2.9) and the antipode $S_R$ is a $G$-graded map. The $G$-graded Hopf algebra is a Hopf algebra in ${}_H^HYD$.

## 3. Construction of Hopf algebras of dimension $2^n$ in ${}_{k\mathbb{Z}_2}^{k\mathbb{Z}_2}YD$.

In the rest of the paper $k$ will be an algebraically closed field of $\operatorname{char} k = 0$, $H = k\mathbb{Z}_2 = k\langle g\rangle$, $n\geq 3$. In this section we construct two families, $R^-$ and $R^+$, of Hopf algebras in ${}_H^HYD$. Let's define two $k$-algebras $R_n^-$ and $R_n^+$ of dimension $2^n$:

$$R_n^{\pm} = k\left\langle u_0, u_1\mid (u_0)^{2^{n-1}+1} = u_0, (u_1)^{2^{n-1}+1} = \pm u_1, u_0u_1 = u_1u_0 = 0\right\rangle \tag{3.1}$$

Then $R_n^{\pm}$ is a $\mathbb{Z}_2$-graded module with $\left(R_n^{\pm}\right)_1 = k\left\langle u_0^j, u_1^{2s}\right\rangle_{j=1,\ldots,2^{n-1};s=1,\ldots,2^{n-2}}$, $\left(R_n^{\pm}\right)_g = k\left\langle u_1^{2s-1}\right\rangle_{s=1,\ldots,2^{n-2}}$. Thus $R_n^{\pm}\in {}_H^HYD$ via the following action (see formulas (2.8) and (2.10)) and coaction: 1

$$\begin{aligned} 1\cdot r &= r\\ g\cdot r_1 &= r_1\\ g\cdot r_g &= -r_g\\ \rho(r_1) &= 1\otimes r_1\\ \rho(r_g) &= g\otimes r_g \end{aligned}$$

where $r\in R_n^{\pm}$, $r_1\in (R_n^{\pm})_1$, $r_g\in (R_n^{\pm})_g$.

Define the comultiplication and the counit in $R_n^{\pm}$ by

$$\begin{aligned} \Delta_{R_n^{\pm}}(u_0) &= u_0\otimes u_0+u_1\otimes u_1, \quad \varepsilon_{R_n^{\pm}}(u_0)=1\\ \Delta_{R_n^{\pm}}(u_1) &= u_0\otimes u_1+u_1\otimes u_0, \quad \varepsilon_{R_n^{\pm}}(u_1)=0 \end{aligned}$$

and extend to $\Delta_{R_n^{\pm}}\left(u_0^j\right)$ and $\Delta_{R_n^{\pm}}\left(u_1^j\right)$ via the formula (2.2). Then

$$\begin{aligned} \Delta_{R_n^{\pm}}\left(u_0^j\right) &= u_0^j\otimes u_0^j+(-1)^{\frac{j(j-1)}{2}}u_1^j\otimes u_1^j\\ \Delta_{R_n^{\pm}}\left(u_1^j\right) &= u_0^j\otimes u_1^j+u_1^j\otimes u_0^j \end{aligned}$$

Define the antipode in $R_n^{\pm}$ by

$$\begin{aligned} S_{R_n^{\pm}}(u_0) &= u_0^{2^{n-1}-1}\\ S_{R_n^{\pm}}(u_1) &= \pm u_1^{2^{n-1}-1} \end{aligned}$$

and extend it to $S_{R_n^{\pm}}\left(u_0^j\right)$ and $S_{R_n^{\pm}}\left(u_1^j\right)$ via the formula (2.3). Then

$$\begin{aligned} S_{R_n^{\pm}}\left(u_0^j\right) &= u_0^{2^{n-1}-j} & j=1,\ldots,2^{n-1}-1\\ S_{R_n^{\pm}}\left(u_1^j\right) &= \pm(\pm 1)^{\frac{j(j-1)}{2}}u_1^{2^{n-1}-j} & j=1,\ldots,2^{n-1}-1\\ S_{R_n^{\pm}}\left(u_0^{2^{n-1}}\right) &= u_0^{2^{n-1}}\\ S_{R_n^{\pm}}\left(u_1^{2^{n-1}}\right) &= u_1^{2^{n-1}} \end{aligned}$$

**Lemma 3.1.** *$R_n^{\pm}$ with the structures defined above is a semisimple $\mathbb{Z}_2$-graded Hopf algebra.*

*Proof.* The comultiplication $\Delta_{R_n^{\pm}}$ is well defined since

$$\begin{aligned} \Delta_{R_n^{\pm}}\left(u_0^{2^{n-1}+1}\right) &= u_0^{2^{n-1}+1}\otimes u_0^{2^{n-1}+1}+u_1^{2^{n-1}+1}\otimes u_1^{2^{n-1}+1}\\ &= u_0\otimes u_0+(\pm u_1)\otimes(\pm u_1)=\Delta_{R_n^{\pm}}(u_0)\\ \Delta_{R_n^{\pm}}\left(u_1^{2^{n-1}+1}\right) &= u_0^{2^{n-1}+1}\otimes u_1^{2^{n-1}+1}+u_1^{2^{n-1}+1}\otimes u_0^{2^{n-1}+1}\\ &= \pm u_0\otimes u_1\pm u_1\otimes u_0=\Delta_{R_n^{\pm}}(\pm u_1) \end{aligned}$$

The antipode $S_{R_n^{\pm}}$ is well defined since

$$\begin{aligned} \left(S_{R_n^{\pm}}*id\right)(u_0) &= S_{R_n^{\pm}}(u_0)\,u_0+S_{R_n^{\pm}}(u_1)\,u_1\\ &= u_0^{2^{n-1}}\pm u_1^{2^{n-1}}=1_{R_n^-}=\varepsilon(u_0)\,1_{R_n^-}\\ \left(S_{R_n^{\pm}}*id\right)(u_1) &= S_{R_n^{\pm}}(u_0)\,u_1+S_{R_n^{\pm}}(u_1)\,u_0=0=\varepsilon(u_1)\,1_{R_n^-} \end{aligned}$$

$$
\begin{aligned}
S_{R_n^{\pm}}\left(u_0^{2^{n-1}+1}\right) &= u_0^{2^{n-1}} u_0^{2^{n-1}-1} = u_0^{2^{n-1}-1} = S_{R_n^-}(u_0) \\
S_{R_n^{\pm}}\left(u_1^{2^{n-1}+1}\right) &= \left(g \cdot u_1^{2^{n-1}}\right)\left(\pm u_1^{2^{n-1}-1}\right) = \pm u_1^{2^{n-1}+2^{n-1}-1} = \pm u_1^{2^{n-1}+1} u_1^{2^{n-1}-2} \\
&= u_1 u_1^{2^{n-1}-2} = u_1^{2^{n-1}-1} = S_{R_n^{\pm}}(-u_1)
\end{aligned}
$$

$R_n^{\pm}$ has a unit $1_{R_n^-} = (u_0)^{2^{n-1}} \pm (u_1)^{2^{n-1}}$, it is easy to check that all the structure maps are $\mathbb{Z}_2$-graded and comultiplication is multiplicative in ${}_H^HYD$, since $\Delta_{R_n^{\pm}}$ was defined using the formula (2.2). Thus $R_n^{\pm}$ is a Hopf algebra in ${}_H^HYD$.

$R_n^{\pm}$ is semisimple, since $t = \sum_{j=1}^{2^{n-1}} u_0^j$ is an integral in $R_n^{\pm}$, with $\varepsilon_{R_n^-}(t) = 2^{n-1} \neq 0$. $\square$

## 4. Construction of Hopf algebras of dimension $2^{n+1}$.

In this section we construct two families of Hopf algebras $A^-$ and $A^+$, using Hopf algebras $R^-$ and $R^+$ in ${}_H^HYD$ described in the previous section.

Define $A_n^{\pm} = R_n^{\pm} \star H$, Radford's biproduct. Then by [R, Theorem 1], $A_n^{\pm}$ is an ordinary Hopf algebra of dimension $2^{n+1}$. Moreover, $A_n^{\pm}$ is semisimple since both $R_n^{\pm}$ and $H$ are semisimple. $A_n^{\pm}$ is spanned by the elements $\left\{u_i^k \star g^s\right\}_{i=0,1,k=1,\ldots,2^{n-1},s=0,1}$. We will sometimes omit $\star$ in this notation.

**Theorem 4.1.** *The semisimple Hopf algebras $A_n^{\pm}$ have the following properties:*

1. The unit in $A_n^{\pm}$ is given by the formula
$$1_{A_n^{\pm}} = 1_{R_n^{\pm}} \star 1_H = (u_0)^{2^{n-1}} \star 1_H \pm (u_1)^{2^{n-1}} \star 1_H$$
and the multiplication is given by formula (2.4). Since $g^s \cdot u_j^l = (-1)^{jls} u_j^l$ we get:
$$\left(u_i^k \star g^s\right)\left(u_j^l \star g^t\right) = (-1)^{jls}\left(u_i^k u_j^l\right) \star g^{s+t} \tag{4.1}$$
2. The counit and the comultiplication in $A_n^{\pm}$ are given by the formulas (2.5) and (2.6). Then
$$
\begin{aligned}
\Delta_{A_n^{\pm}}(u_0 \star 1) &= (u_0 \star 1) \otimes (u_0 \star 1) + (u_1 \star g) \otimes (u_1 \star 1) \\
\Delta_{A_n^{\pm}}(u_1 \star 1) &= (u_0 \star g) \otimes (u_1 \star 1) + (u_1 \star 1) \otimes (u_0 \star 1) \\
\Delta_{A_n^{\pm}}(1 \star g) &= (1 \star g) \otimes (1 \star g)
\end{aligned} \tag{4.2}
$$
$$
\begin{aligned}
\Delta_{A_n^{\pm}}\left(u_0^j \star g^k\right) &= \left(u_0^j \star g^k\right) \otimes \left(u_0^j \star g^k\right) + (-1)^{\frac{j(j-1)}{2}}\left(u_1^j \star g^{k+j}\right) \otimes \left(u_1^j \star g^k\right) \\
\Delta_{A_n^{\pm}}\left(u_1^j \star g^k\right) &= \left(u_0^j \star g^{k+j}\right) \otimes \left(u_1^j \star g^k\right) + \left(u_1^j \star g^k\right) \otimes \left(u_0^j \star g^k\right)
\end{aligned}
$$
3. The antipode $S_{A_n^{\pm}}$ is defined by the formula (2.7).
4. $A_n^{\pm}$ are not commutative, since
$$
\begin{aligned}
(u_1 \star g)(u_1 \star 1) &= -u_1^2 \star g \\
(u_1 \star 1)(u_1 \star g) &= u_1^2 \star g
\end{aligned}
$$
5. $A_n^{\pm}$ are not cocommutative (see formula (4.2)).

6. The center of $A_n^{\pm}$ is $Z(A_n^{\pm}) = k\langle u_0 \star 1, u_0 \star g, u_1^2 \star 1\rangle$ of dimension $2^n + 2^{n-2} = 5 \cdot 2^{n-2}$. Here we list the central orthogonal idempotents of $A_n^{\pm}$:

| $A_n^-$ | $A_n^+$ |
|---|---|
| $\frac{1}{2^n}\sum_{j=1}^{2^{n-1}}\sum_{k=0}^{1}\omega^{jl}u_0^j g^k$ | $\frac{1}{2^n}\sum_{j=1}^{2^{n-1}}\sum_{k=0}^{1}\omega^{jl}u_0^j \star g^k$ |
| $\frac{1}{2^{n-2}}\sum_{s=1}^{2^{n-2}}\omega^{(2m-1)s}u_1^{2s}$ | $\frac{1}{2^{n-2}}\sum_{s=1}^{2^{n-2}}\omega^{2ms}u_1^{2s}$ |
| $\frac{1}{2^n}\sum_{j=1}^{2^{n-1}}\sum_{k=0}^{1}(-1)^k\omega^{jl}u_0^j g^k$ | $\frac{1}{2^n}\sum_{j=1}^{2^{n-1}}\sum_{k=0}^{1}(-1)^k\omega^{jl}u_0^j g^k$ |

where $\omega$ is a primitive $2^{n-1}$-root of unity, $l = 1, ..., 2^{n-1}$, $m = 1, ..., 2^{n-2}$.

7. $A_n^- \ncong A_n^+$ since they have different groups of grouplikes:
   (a) When $n = 3$, i.e. $\dim A_n^{\pm} = 16$,
   $\mathbf{G}(A_3^-) = \langle u_0^4 + u_1^4\rangle \times \langle u_0^2 + iu_1^2\rangle \times \langle g\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$,
   $\mathbf{G}(A_3^+) = \langle u_0^2 + iu_1^2\rangle \times \langle g\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$.
   (b) When $n > 3$,
   $\mathbf{G}(A_n^-) = \langle u_0^2 + iu_1^2\rangle \times \langle g\rangle \cong \mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_2$,
   $\mathbf{G}(A_n^+) = \langle u_0^2 + iu_1^2\rangle \times \left\langle u_0^{2^{n-1}} - u_1^{2^{n-1}}\right\rangle \times \langle g\rangle \cong \mathbb{Z}_{2^{n-2}} \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

## 5. The structure of the duals $(A_n^{\pm})^*$.

5.1. **The algebra structure of $(A_n^{\pm})^*$.** Both $(A_n^-)^*$ and $(A_n^+)^*$ have the same algebra structure. They are spanned by elements $p_{i,j,k}$, where $i, k \in \{0, 1\}$ and $j = 1, ..., 2^{n-1}$, such that

$$\left\langle p_{i,j,k}, u_q^s \star g^t\right\rangle = \delta_{i,q}\delta_{j,s}\delta_{k,t} \tag{5.1}$$

The unit in $(A_n^{\pm})^*$ is

$$1_{(A_n^{\pm})^*} = \varepsilon_{A_n^{\pm}} = \sum_{j=1}^{2^{n-1}}\sum_{k=0}^{1} p_{0,j,k}$$

Let's determine the multiplication rules in $(A_n^{\pm})^*$:

1. $p_{i,j,k} * p_{q,s,t} = 0$, if $j \neq s$
2. (a) $\left\langle p_{0,j,k} * p_{0,j,t}, u_1^l g^m\right\rangle = \left\langle p_{0,j,k}, u_0^l \star g^{m+l}\right\rangle\left\langle p_{0,j,t}, u_1^l \star g^m\right\rangle + \left\langle p_{0,j,k}, u_1^l \star g^m\right\rangle\left\langle p_{0,j,t}, u_0^l \star g^m\right\rangle = 0$
   (b) $\left\langle p_{0,j,k} * p_{0,j,t}, u_0^l g^m\right\rangle = \left\langle p_{0,j,k}, u_0^l \star g^m\right\rangle\left\langle p_{0,j,t}, u_0^l \star g^m\right\rangle + (-1)^{\frac{j(j-1)}{2}}\left\langle p_{0,j,k}, u_1^l \star g^{m+l}\right\rangle\left\langle p_{0,j,t}, u_1^l \star g^m\right\rangle = \delta_{j,l}\delta_{k,m}\delta_{t,m}$
3. (a) $\left\langle p_{1,j,k} * p_{0,j,t}, u_0^l g^m\right\rangle = \left\langle p_{1,j,k}, u_0^l \star g^m\right\rangle\left\langle p_{0,j,t}, u_0^l \star g^m\right\rangle + (-1)^{\frac{j(j-1)}{2}}\left\langle p_{1,j,k}, u_1^l \star g^{m+l}\right\rangle\left\langle p_{0,j,t}, u_1^l \star g^m\right\rangle = 0$
   (b) $\left\langle p_{1,j,k} * p_{0,j,t}, u_1^l g^m\right\rangle = \left\langle p_{1,j,k}, u_0^l \star g^{m+l}\right\rangle\left\langle p_{0,j,t}, u_1^l \star g^m\right\rangle + \left\langle p_{1,j,k}, u_1^l \star g^m\right\rangle\left\langle p_{0,j,t}, u_0^l \star g^m\right\rangle = \delta_{j,l}\delta_{k,m}\delta_{t,m}$

4. (a) $\left\langle p_{0,j,k} * p_{1,j,t}, u_0^l g^m \right\rangle = \left\langle p_{0,j,k}, u_0^l \star g^m \right\rangle \left\langle p_{1,j,t}, u_0^l \star g^m \right\rangle +$
$(-1)^{\frac{j(j-1)}{2}} \left\langle p_{0,j,k}, u_1^l \star g^{m+l} \right\rangle \left\langle p_{1,j,t}, u_1^l \star g^m \right\rangle = 0$
   (b) $\left\langle p_{0,j,k} * p_{1,j,t}, u_1^l g^m \right\rangle = \left\langle p_{0,j,k}, u_0^l \star g^{m+l} \right\rangle \left\langle p_{1,j,t}, u_1^l \star g^m \right\rangle +$
$\left\langle p_{0,j,k}, u_1^l \star g^m \right\rangle \left\langle p_{1,j,t}, u_0^l \star g^m \right\rangle = \delta_{j,l}\delta_{k,m+l}\delta_{t,m}$
5. (a) $\left\langle p_{1,j,k} * p_{1,j,t}, u_0^l g^m \right\rangle = \left\langle p_{1,j,k}, u_0^l \star g^m \right\rangle \left\langle p_{1,j,t}, u_0^l \star g^m \right\rangle +$
$(-1)^{\frac{j(j-1)}{2}} \left\langle p_{1,j,k}, u_1^l \star g^{m+l} \right\rangle \left\langle p_{1,j,t}, u_1^l \star g^m \right\rangle = (-1)^{\frac{j(j-1)}{2}} \delta_{j,l}\delta_{k,m+l}\delta_{t,m}$
   (b) $\left\langle p_{1,j,k} * p_{1,j,t}, u_1^l g^m \right\rangle = \left\langle p_{1,j,k}, u_0^l \star g^{m+l} \right\rangle \left\langle p_{1,j,t}, u_1^l \star g^m \right\rangle +$
$\left\langle p_{1,j,k}, u_1^l \star g^m \right\rangle \left\langle p_{1,j,t}, u_0^l \star g^m \right\rangle = 0$

The above computations imply the following multiplication rules:

$$p_{0,j,k} * p_{0,j,k} = p_{0,j,k} \tag{5.2}$$
$$p_{1,j,k} * p_{0,j,k} = p_{1,j,k} \tag{5.3}$$
$$p_{0,j,k} * p_{1,j,k} = p_{1,j,k}, \text{ if } j \equiv 0 \,(\mathrm{mod} 2) \tag{5.4}$$
$$p_{0,j,k+1} * p_{1,j,k} = p_{1,j,k}, \text{ if } j \equiv 1 \,(\mathrm{mod} 2) \tag{5.5}$$
$$p_{1,j,k} * p_{1,j,k} = (-1)^{\frac{j}{2}} p_{0,j,k}, \text{ if } j \equiv 0 \,(\mathrm{mod} 2) \tag{5.6}$$
$$p_{1,j,k+1} * p_{1,j,k} = (-1)^{\frac{j-1}{2}} p_{0,j,k}, \text{ if } j \equiv 1 \,(\mathrm{mod} 2) \tag{5.7}$$
$$p_{i,j,k} * p_{q,s,t} = 0, \text{ otherwise} \tag{5.8}$$

### 5.2. **The centers of $(A_n^{\pm})^*$.**

**Proposition 5.1.** $(A_n^{\pm})^*$ *have the centers of dimension* $5 \cdot 2^{n-2}$ *and the following central orthogonal idempotents:*

$$\left\{ \begin{array}{l} \frac{1}{2}\left(p_{0,2s,0} \pm i^s p_{1,2s,0}\right) \\ p_{0,2s-1,0} + p_{0,2s-1,1} \\ \frac{1}{2}\left(p_{0,2s,1} \pm i^s p_{1,2s,1}\right) \end{array} \right\}, \qquad \textit{where } s = 1, ..., 2^{n-2}.$$

*Proof.* If $\sum \alpha_{i,j,k} p_{i,j,k} \in Z\left((A_n^{\pm})^*\right)$, then

$$\begin{aligned}
\left(\sum \alpha_{i,j,k} p_{i,j,k}\right) p_{0,s,t} &= \alpha_{0,s,t} p_{0,s,t} + \alpha_{1,s,t} p_{1,s,t} \\
p_{0,s,t} \left(\sum \alpha_{i,j,k} p_{i,j,k}\right) &= \alpha_{0,s,t} p_{0,s,t} + \alpha_{1,s,t-s} p_{1,s,t-s} \\
\left(\sum \alpha_{i,j,k} p_{i,j,k}\right) p_{1,s,t} &= \alpha_{0,s,t+s} p_{1,s,t} + (-1)^{\frac{s(s-1)}{2}} \alpha_{1,s,t+s} p_{0,s,t} \\
p_{1,s,t} \left(\sum \alpha_{i,j,k} p_{i,j,k}\right) &= \alpha_{0,s,t} p_{1,s,t} + (-1)^{\frac{s(s-1)}{2}} \alpha_{1,s,t-s} p_{0,s,t-s}
\end{aligned}$$

Thus

$$\alpha_{1,j,k} = 0, \text{ when } j \equiv 1 \,(\mathrm{mod} 2)$$
$$\alpha_{0,j,0} = \alpha_{0,j,1}, \text{ when } j \equiv 1 \,(\mathrm{mod} 2)$$

Therefore $Z\left((A_n^{\pm})^*\right) = k\left\{p_{0,2s,0}, p_{0,2s,1}, p_{1,2s,0}, p_{1,2s,1}, p_{0,2s-1,0} + p_{0,2s-1,1}\right\}$,
$s = 1, ..., 2^{n-2}$ and $\dim Z\left((A_n^{\pm})^*\right) = 2^{n-2} + 2^{n-2} + 2^{n-2} + 2^{n-2} + 2^{n-2} = 5 \cdot 2^{n-2}$. $\square$

5.3. **The coalgebra structures of $(A_n^-)^*$ and $(A_n^+)^*$.** In $(A_n^-)^*$ the counit and comultiplication are given by the formulas:

$$\varepsilon_{\left(A_n^-\right)^*}(p_{i,j,k}) = \langle p_{i,j,k}, 1_A\rangle = \delta_{i,0}\delta_{2^{n-1},0}\delta_{k,0} - \delta_{i,1}\delta_{2^{n-1},0}\delta_{k,0}$$

$$\begin{aligned}\Delta_{\left(A_n^-\right)^*}(p_{0,j,k}) &= \sum_{s+t\equiv j(2^{n-1}),l+m\equiv k(2)} p_{0,s,l}\otimes p_{0,t,m}\\ \Delta_{\left(A_n^-\right)^*}(p_{1,j,k}) &= \sum_{s+t\equiv j(2^{n-1}),l+m\equiv k(2)} (-1)^{\delta_{s+t-j,2^{n-1}}+lt} p_{1,s,l}\otimes p_{1,t,m}\end{aligned}$$

In $(A_n^+)^*$ the counit and comultiplication are given by the formulas:

$$\varepsilon_{\left(A_n^+\right)^*}(p_{i,j,k}) = \left\langle p_{i,j,k}, 1_{A_n^+}\right\rangle = \delta_{i,0}\delta_{2^{n-1},0}\delta_{k,0} + \delta_{i,1}\delta_{2^{n-1},0}\delta_{k,0}$$

$$\begin{aligned}\Delta_{\left(A_n^+\right)^*}(p_{0,j,k}) &= \sum_{s+t\equiv j(2^{n-1}),l+m\equiv k(2)} p_{0,s,l}\otimes p_{0,t,m}\\ \Delta_{\left(A_n^+\right)^*}(p_{1,j,k}) &= \sum_{s+t\equiv j(2^{n-1}),l+m\equiv k(2)} (-1)^{lt} p_{1,s,l}\otimes p_{1,t,m} \qquad (5.9)\end{aligned}$$

The following result could be verified by straightforward computations:

**Proposition 5.2.** *The group of grouplike elements of* $(A_n^\pm)^*$ *is* $\mathbf{G}\left((A_n^\pm)^*\right) = \left\langle \sum_{j=1}^{2^{n-1}}\sum_{k=0}^{1}(-1)^k p_{0,j,k}\right\rangle \times \left\langle \sum_{j=1}^{2^{n-1}}\sum_{k=0}^{1}\omega^j p_{0,j,k}\right\rangle \cong \mathbb{Z}_2\times\mathbb{Z}_{2^{n-1}}$, *where* $\omega$ *is a primitive* $2^{n-1}$*-root of unity.*

## 6. The character algebras of $A_n^\pm$ and $(A_n^\pm)^*$.

In this section we will describe the character algebras of $A_n^\pm$ and $(A_n^\pm)^*$. In particular, the following proposition will follow from this description:

**Proposition 6.1.** *The character algebras of* $A_n^\pm$ *and* $(A_n^\pm)^*$ *are commutative of dimension* $5\cdot 2^{n-2}$. *Thus, since a character algebra of a semisimple Hopf algebra is semisimple, all these algebras are isomorphic to* $k^{5\cdot 2^{n-2}}$.

**Definition 6.2.** The *character algebra* $C(A)$ of a Hopf algebra $A$ is a subalgebra of $A^*$ which is generated by the characters of representations of $A$. Equivalently, $C(A) = \{f\in A^* \mid \langle f, xy\rangle = \langle f, yx\rangle \text{ for any } x, y\in A\}$.

6.1. **The character algebra of $A_n^\pm$.** $p\in C(A_n^\pm)$ if $\langle p, ab\rangle = \langle p, ba\rangle$ for any $a$, $b\in A_n^\pm$. Write $p = \sum \alpha_{i,j,k}p_{i,j,k}$. By the formula (4.1) we have the following multiplication rules in $A_n^\pm$:

$$\begin{aligned}u_0^m g^q u_r^n g^t &= u_r^n g^t u_0^m g^q \qquad \text{for } r\in\{0,1\}\\ u_1^m g^q u_1^n g^t &= (-1)^{qn} u_1^m u_1^n g^{q+t}\\ u_1^n g^t u_1^m g^q &= (-1)^{mt} u_1^m u_1^n g^{q+t}\end{aligned}$$

Then $p = \sum \alpha_{i,j,k} p_{i,j,k} \in C(A_n^{\pm})$ if and only if

$$(-1)^{qn} \sum \alpha_{i,j,k} \left\langle p_{i,j,k}, u_1^{m+n} g^{q+t} \right\rangle = (-1)^{mt} \sum \alpha_{i,j,k} \left\langle p_{i,j,k}, u_1^{m+n} g^{q+t} \right\rangle$$

and we get $C(A_n^{\pm}) = k\{p_{0,j,k}, p_{1,2s,0}\}$, where $j = 1, ..., 2^{n-1}$, $s = 1, ..., 2^{n-2}$, $k = 0, 1$. Then $C(A_n^{\pm})$ is commutative and $C(A_n^{\pm}) \cong k^{5 \cdot 2^{n-2}}$.

The orthogonal idempotents of $C(A_n^{\pm})$ are

$$\left\{ \begin{array}{l} \frac{1}{2}(p_{0,2s,0} \pm i^s p_{1,2s,0}) \\ p_{0,2s-1,k} \\ p_{0,2s,1} \end{array} \right\}$$

where $s = 1, ..., 2^{n-2}$, $k = 0, 1$.

Note that the orthogonal idempotents of $C(A_n^-)$ and $C(A_n^+)$ are the same since the character algebras are the same and $(A_n^-)^*$ and $(A_n^+)^*$ have the same algebra structure.

6.2. **The character algebra of** $(A_n^{\pm})^*$**.** $a \in C\left((A_n^{\pm})^*\right)$ if $\langle \varphi\psi, a \rangle = \langle \psi\varphi, a \rangle$ for any $\varphi$, $\psi \in (A_n^{\pm})^*$. Write $a = \sum \alpha_{i,j,k} u_i^j g^k$. Then by formulas (5.2) - (5.8):

$$\begin{array}{rcl} \langle p_{1,2s-1,k}, a \rangle & = & \langle p_{1,2s-1,k} p_{0,2s-1,k}, a \rangle = \langle p_{0,2s-1,k} p_{1,2s-1,k}, a \rangle = 0 \\ (-1)^{s-1} \langle p_{0,2s-1,0}, a \rangle & = & \langle p_{1,2s-1,1} p_{1,2s-1,0}, a \rangle = \langle p_{1,2s-1,0} p_{1,2s-1,1}, a \rangle \\ & = & (-1)^{s-1} \langle p_{0,2s-1,1}, a \rangle \end{array}$$

Thus $\alpha_{1,2s-1,0} = \alpha_{1,2s-1,1} = 0$, $\alpha_{0,2s-1,0} = \alpha_{0,2s-1,1}$ and $C\left((A_n^{\pm})^*\right) = k\left\{u_i^{2s} g^k, u_0^{2s-1} + u_0^{2s-1} g\right\}$, where $s = 1, ..., 2^{n-2}$, $i = 0, 1$. Then $C\left((A_n^{\pm})^*\right)$ is commutative and $C\left((A_n^{\pm})^*\right) \cong k^{5 \cdot 2^{n-2}}$.

The orthogonal idempotents of $C\left((A_n^{\pm})^*\right)$ are

| $C\left((A_n^-)^*\right)$ | $C\left((A_n^+)^*\right)$ |
|---|---|
| $\frac{1}{2^n} \sum_{j=1}^{2^{n-1}} \sum_{k=0}^{1} \omega^{jl} u_0^j \star g^k$ | $\frac{1}{2^n} \sum_{j=1}^{2^{n-1}} \sum_{k=0}^{1} \omega^{jl} u_0^j \star g^k$ |
| $\frac{1}{2^{n-1}} \sum_{s=1}^{2^{n-2}} \sum_{k=0}^{1} (-1)^{rk} \omega^{(2m-1)s} u_1^{2s} \star g^k$ | $\frac{1}{2^{n-1}} \sum_{s=1}^{2^{n-2}} \sum_{k=0}^{1} (-1)^{rk} \omega^{2ms} u_1^{2s} \star g^k$ |
| $\frac{1}{2^{n-1}} \sum_{s=1}^{2^{n-2}} \sum_{k=0}^{1} (-1)^{k} \omega^{2sm} u_0^{2s} \star g^k$ | $\frac{1}{2^{n-1}} \sum_{s=1}^{2^{n-2}} \sum_{k=0}^{1} (-1)^{k} \omega^{2sm} u_0^{2s} \star g^k$ |

where $\omega$ is a primitive $2^{n-1}$-root of unity, $l = 1, ..., 2^{n-1}$, $m = 1, ..., 2^{n-2}$, $r = 0, 1$.

## 7. The Hopf algebras $A_3^+$ and $\left(A_3^+\right)^*$ of dimension 16.

We now specialize to the case $n = 3$. Let's write $A_3^+ = A^+$. In this section we prove the following result:

**Theorem 7.1.** *The Hopf algebra $A_3^+$ of dimension 16 is selfdual.*

7.1. **Central orthogonal idempotents of $C(A^+)$, $(A^+)^*$, $C\left((A^+)^*\right)$ and $A^+$.** We would like to determine the correspondence between orthogonal idempotents of $C(A^+)$ and $C\left((A^+)^*\right)$ and between central orthogonal idempotents of $(A^+)^*$ and $A^+$. We will use this correspondence to construct an isomorphism between $A^+$ and $(A^+)^*$. All the properties of orthogonal idempotents here can be easily checked by straightforward computations.

The orthogonal idempotents of $C\left((A^+)^*\right)$ are
$f_1 = \frac{1}{8}\left(u_0^4 + u_0^4 g + u_0^2 + u_0^2 g + u_0 + u_0 g + u_0^3 + u_0^3 g\right)$
$f_2 = \frac{1}{8}\left(u_0^4 + u_0^4 g + u_0^2 + u_0^2 g - u_0 - u_0 g - u_0^3 - u_0^3 g\right)$
$f_3 = \frac{1}{8}\left(u_0^4 + u_0^4 g - u_0^2 - u_0^2 g + iu_0 + iu_0 g - iu_0^3 - iu_0^3 g\right)$
$f_4 = \frac{1}{8}\left(u_0^4 + u_0^4 g - u_0^2 - u_0^2 g - iu_0 - iu_0 g + iu_0^3 + iu_0^3 g\right)$
$f_5 = \frac{1}{4}\left(u_1^4 + u_1^4 g + u_1^2 + u_1^2 g\right)$
$f_6 = \frac{1}{4}\left(u_1^4 - u_1^4 g + u_1^2 - u_1^2 g\right)$
$f_7 = \frac{1}{4}\left(u_1^4 - u_1^4 g - u_1^2 + u_1^2 g\right)$
$f_8 = \frac{1}{4}\left(u_1^4 + u_1^4 g - u_1^2 - u_1^2 g\right)$
$f_9 = \frac{1}{4}\left(u_0^4 - u_0^4 g + u_0^2 - u_0^2 g\right)$
$f_{10} = \frac{1}{4}\left(u_0^4 - u_0^4 g - u_0^2 + u_0^2 g\right)$

The central orthogonal idempotents of $A^+$ are
$F_1 = \frac{1}{8}\left(u_0^4 + u_0^4 g + u_0^2 + u_0^2 g + u_0 + u_0 g + u_0^3 + u_0^3 g\right)$
$F_2 = \frac{1}{8}\left(u_0^4 + u_0^4 g + u_0^2 + u_0^2 g - u_0 - u_0 g - u_0^3 - u_0^3 g\right)$
$F_3 = \frac{1}{8}\left(u_0^4 + u_0^4 g - u_0^2 - u_0^2 g + iu_0 + iu_0 g - iu_0^3 - iu_0^3 g\right)$
$F_4 = \frac{1}{8}\left(u_0^4 + u_0^4 g - u_0^2 - u_0^2 g - iu_0 - iu_0 g + iu_0^3 + iu_0^3 g\right)$
$F_5 = \frac{1}{2}\left(u_1^4 + u_1^2\right)$
$F_6 = \frac{1}{2}\left(u_1^4 - u_1^2\right)$
$F_7 = \frac{1}{8}\left(u_0^4 - u_0^4 g + u_0^2 - u_0^2 g + u_0 - u_0 g + u_0^3 - u_0^3 g\right)$
$F_8 = \frac{1}{8}\left(u_0^4 - u_0^4 g - u_0^2 + u_0^2 g + iu_0 - iu_0 g - iu_0^3 + iu_0^3 g\right)$
$F_9 = \frac{1}{8}\left(u_0^4 - u_0^4 g + u_0^2 - u_0^2 g - u_0 + u_0 g - u_0^3 + u_0^3 g\right)$
$F_{10} = \frac{1}{8}\left(u_0^4 - u_0^4 g - u_0^2 + u_0^2 g - iu_0 + iu_0 g + iu_0^3 - iu_0^3 g\right)$

We have the following relations between the orthogonal idempotents of $C\left((A^+)^*\right)$ and the central orthogonal idempotents of $A^+$:
$F_1 = f_1$, $F_2 = f_2$, $F_3 = f_3$, $F_4 = f_4$,
$F_5 = f_5 + f_7$, $F_6 = f_6 + f_8$,
$F_7 + F_9 = f_9$, $F_8 + F_{10} = f_{10}$.

Moreover, $C\left((A^+)^*\right) \cap Z(A^+) = k\left\{u_0^2, u_0^2 g, u_0^4, u_0^4 g, u_0 + u_0 g, u_0^3 + u_0^3 g, u_1^2, u_1^4\right\}$ and $\dim\left(C(A^+) \cap Z\left((A^+)^*\right)\right) = 8$.

The orthogonal idempotents of $C(A^+)$ are
$e_1 = \frac{1}{2}\left(p_{0,4,0} + p_{1,4,0}\right)$ $e_5 = p_{0,1,0}$ $e_9 = p_{0,4,1}$
$e_2 = \frac{1}{2}\left(p_{0,4,0} - p_{1,4,0}\right)$ $e_6 = p_{0,3,0}$ $e_{10} = p_{0,2,1}$
$e_3 = \frac{1}{2}\left(p_{0,2,0} + ip_{1,2,0}\right)$ $e_7 = p_{0,1,1}$
$e_4 = \frac{1}{2}\left(p_{0,2,0} - ip_{1,2,0}\right)$ $e_8 = p_{0,3,1}$

The central orthogonal idempotents of $(A^+)^*$ are

$$\begin{array}{lll} E_1=\frac{1}{2}(p_{0,4,0}+p_{1,4,0}) & E_5=p_{0,1,0}+p_{0,1,1} & E_7=\frac{1}{2}(p_{0,4,1}+p_{1,4,1}) \\ E_2=\frac{1}{2}(p_{0,4,0}-p_{1,4,0}) & E_6=p_{0,3,0}+p_{0,3,1} & E_8=\frac{1}{2}(p_{0,2,1}+ip_{1,2,1}) \\ E_3=\frac{1}{2}(p_{0,2,0}+ip_{1,2,0}) & & E_9=\frac{1}{2}(p_{0,4,1}-p_{1,4,1}) \\ E_4=\frac{1}{2}(p_{0,2,0}-ip_{1,2,0}) & & E_{10}=\frac{1}{2}(p_{0,2,1}-ip_{1,2,1}) \end{array}$$

We have the following relations between the orthogonal idempotents of $C(A^+)$ and the central orthogonal idempotents of $(A^+)^*$:
$E_1=e_1$, $E_2=e_2$, $E_3=e_3$, $E_4=e_4$,
$E_5=e_5+e_7$, $E_6=e_6+e_8$,
$E_7+E_9=e_9$, $E_8+E_{10}=e_{10}$.

Moreover, $C(A^+)\cap Z\left((A^+)^*\right)=k\{p_{0,2,0},p_{0,4,0},p_{0,2,1},p_{0,4,1},p_{1,2,0},p_{1,4,0},$ $p_{0,1,0}+p_{0,1,1},p_{0,3,0}+p_{0,3,1}\}$ and $\dim\left(C(A^+)\cap Z\left((A^+)^*\right)\right)=8$.

7.2. **Proof of Theorem 7.1.** If there exists a Hopf algebra isomorphism $\varphi$ between $A^+$ and $(A^+)^*$ then central orthogonal idempotents of $A^+$ should be sent to central orthogonal idempotents of $(A^+)^*$ and orthogonal idempotents of $C\left((A^+)^*\right)$ should be sent to orthogonal idempotents of $C(A^+)$. We can express $u_0$ and $u_0g\in A^+$ as a linear combination of the central orthogonal idempotents $F_i$:

$$\begin{aligned} u_0 &= F_1-F_2-iF_3+iF_4+F_7-iF_8-F_9+iF_{10} \\ u_0g &= F_1-F_2-iF_3+iF_4-F_7+iF_8+F_9-iF_{10} \end{aligned}$$

Let's define $\varphi(F_i)=E_i$. Then

$$\begin{aligned} \varphi(u_0) &= p_{1,4,0}+p_{1,2,0}+p_{1,4,1}+p_{1,2,1} \\ \varphi(u_0g) &= p_{1,4,0}+p_{1,2,0}-p_{1,4,1}-p_{1,2,1} \end{aligned} \tag{7.1}$$

The map $\varphi$ should be a coalgebra map, that is

$$(\varphi\otimes\varphi)\circ\Delta_{A^+}=\Delta_{(A^+)^*}\circ\varphi$$

Thus, using formulas (4.2) and (5.9), we get only two possibilities for $\varphi(u_1)$ and $\varphi(u_1g)$:
$\varphi(u_1)=p_{1,1,0}+p_{1,3,0}+p_{1,1,1}+p_{1,3,1}$ and $\varphi(u_1g)=p_{1,1,0}+p_{1,3,0}-p_{1,1,1}-p_{1,3,1}$ or $\varphi(u_1)=-p_{1,1,0}-p_{1,3,0}-p_{1,1,1}-p_{1,3,1}$ and $\varphi(u_1g)=-p_{1,1,0}-p_{1,3,0}+p_{1,1,1}+p_{1,3,1}$. Let's define

$$\varphi(u_1) = p_{1,1,0}+p_{1,3,0}+p_{1,1,1}+p_{1,3,1} \tag{7.2}$$

$$\varphi(u_1g) = p_{1,1,0}+p_{1,3,0}-p_{1,1,1}-p_{1,3,1} \tag{7.3}$$

$g$ is a noncentral grouplike of order 2 in $A^+$, so it should be sent to a noncentral grouplike of order 2 in $(A^+)^*$. Thus there are only two possibilities for $\varphi(g)$ and only one of them is compatible with formulas (7.2) and (7.3), namely

$$\varphi(g)=p_{0,1,0}+p_{0,2,0}+p_{0,3,0}+p_{0,4,0}-p_{0,1,1}-p_{0,2,1}-p_{0,3,1}-p_{0,4,1} \tag{7.4}$$

It can be easily checked, that the map $\varphi:A_3^+\to\left(A_3^+\right)^*$, defined by formulas (7.1), (7.2) and (7.4) and extended multiplicatively, is a Hopf algebra isomorphism. For example

$$\varphi\left(u_0^5\right) = \left(p_{1,4,0}+p_{1,2,0}+p_{1,4,1}+p_{1,2,1}\right)^5$$

$$
\begin{aligned}
&= (p_{1,4,0}+p_{1,2,0}+p_{1,4,1}+p_{1,2,1})^4 (p_{1,4,0}+p_{1,2,0}+p_{1,4,1}+p_{1,2,1}) \\
&= (p_{0,4,0}-p_{0,2,0}+p_{0,4,1}-p_{0,2,1})^2 (p_{1,4,0}+p_{1,2,0}+p_{1,4,1}+p_{1,2,1}) \\
&= (p_{0,4,0}+p_{0,2,0}+p_{0,4,1}+p_{0,2,1}) (p_{1,4,0}+p_{1,2,0}+p_{1,4,1}+p_{1,2,1}) \\
&= (p_{1,4,0}+p_{1,2,0}+p_{1,4,1}+p_{1,2,1}) = \varphi(u_0) \\
\varphi(u_1^5) &= (p_{1,1,0}+p_{1,3,0}+p_{1,1,1}+p_{1,3,1})^5 \\
&= (p_{1,1,0}+p_{1,3,0}+p_{1,1,1}+p_{1,3,1})^4 (p_{1,1,0}+p_{1,3,0}+p_{1,1,1}+p_{1,3,1}) \\
&= (p_{0,1,0}-p_{0,3,0}+p_{0,1,1}-p_{0,3,1})^2 (p_{1,1,0}+p_{1,3,0}+p_{1,1,1}+p_{1,3,1}) \\
&= (p_{0,1,0}+p_{0,3,0}+p_{0,1,1}+p_{0,3,1}) (p_{1,1,0}+p_{1,3,0}+p_{1,1,1}+p_{1,3,1}) \\
&= (p_{1,1,0}+p_{1,3,0}+p_{1,1,1}+p_{1,3,1}) = \varphi(u_1)
\end{aligned}
$$

## 8. The Grothendieck rings of $A_n^{\pm}$ and $(A_n^{\pm})^*$.

In this section we completely describe the Grothendieck rings of $A_n^{\pm}$ and $(A_n^{\pm})^*$. First we will need the following definition, which was introduced in [S].

**Definition 8.1.** Let $K_0(A)^+$ denote the abelian semigroup of all equivalence classes of representations of a semisimple Hopf algebra $A$ with the addition given by a direct sum. Then its enveloping group $K_0(A)$ has the structure of an ordered ring with involution $^*$ and is called the Grothendieck ring.

In [NR] the structure of $K_0(A)$ was described for comodules; it was then translated into the language of modules in [N]. The multiplication in this ring is defined as follows: let $[\pi_1]$ and $[\pi_2]$ denote the classes of representations equivalent to $\pi_1$ and $\pi_2$, then $[\pi_1] \bullet [\pi_2]$ is the class of the representation $(\pi_1 \otimes \pi_2) \circ \Delta$; the unit of this ring is the class $[\varepsilon]$ and $[\pi]^*$ is the equivalence class of the dual representation ${}^t(\pi \circ S)$ defined by $\langle {}^t(\pi \circ S(h))(f), v\rangle = \langle f, (\pi \circ S(h))(v)\rangle$. The equivalence classes of irreducible representations of $A$ form a basis of $K_0(A)$ and are called *basic elements*. If $[\pi_1], \ldots, [\pi_d]$ are the basic elements then $[\rho] = \sum_{i=1}^d \deg \pi_i [\pi_i]$ is called the *marked element*. For basic elements $x$ and $y$ we write

$$x \bullet y = \sum_{z \text{ - basic}} m(z, x \bullet y)\, z$$

where $m(z, x \bullet y)$ are non-negative integers. Then the following properties are true (see [NR] and [N]):

$$m(z, x \bullet y) = m(x^*, y \bullet z^*) \tag{8.1}$$

$$m(1, x \bullet y^*) = \delta_{x,y} \tag{8.2}$$

$$\sum m(z, x \bullet y) \deg(z) = \deg(x \bullet y) \tag{8.3}$$

For simplicity of notation we will write $\pi$ instead of $[\pi]$ for elements of $K_0(A)$. The 1-dimensional irreducible representations of $A$ are exactly the elements of $\mathbf{G}(H^*)$, or multiplicative characters of $A$.

**Lemma 8.2.** *The Hopf algebras $A_n^{\pm}$ and $(A_n^{\pm})^*$ have only 1-dimensional and 2-dimensional irreducible representations.*

*Proof.* Let $A$ be one of these Hopf algebras. Since $|\mathbf{G}(A^*)| = 2^n$ and $\dim Z(A) = 5 \cdot 2^{n-2}$, there are exactly $2^n$ irreducible representations of dimension 1 and $5 \cdot 2^{n-2} - 2^n = 2^{n-2}$ irreducible representations of dimension greater or equal then 2. Sum of the squares of these dimensions should be equal to $\dim A - |\mathbf{G}(A^*)| = 2^n$, thus all of them should be of dimension 2. $\square$

**Theorem 8.3.** *The Hopf algebras $A_n^-$ of dimension $2^{n+1}$ for $n \geq 4$ are selfdual.*

*Proof.* We will show in Propositions 8.4 and 8.8 that for $n \geq 4$ $A_n^-$ and $(A_n^-)^*$ have exactly the same structure of Grothendieck ring as an ordered ring with involution and marked element. In [K2] we show that there are exactly three nonisomorphic Hopf algebras of dimension $2^{n+1}$ with $\mathbf{G}(H) \cong \mathbf{G}((H)^*) \cong \mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_2$. Only one of them, $H_{c_1: \frac{1+i}{2}x + \frac{1-i}{2}x^{2^{n-2}+1}}$, has the Grothendieck ring with the same structure as $A_n^-$ and $(A_n^-)^*$. Thus $A_n^- \cong H_{c_1: \frac{1+i}{2}x + \frac{1-i}{2}x^{2^{n-2}+1}} \cong (A_n^-)^*$. $\square$

8.1. **The Grothendieck ring of $A_n^+$.** $\mathbf{G}\left((A_n^+)^*\right) = \langle \chi \rangle \times \langle \varphi \rangle \cong \mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_2$, where $\chi(u_0) = \omega$, $\chi(u_1) = 0$, $\chi(g) = 1$ and $\varphi(u_0) = 1$, $\varphi(u_1) = 0$, $\varphi(g) = -1$, where $\omega$ is a primitive $2^{n-1}$-th root of unity. There are $2^{n-2}$ degree 2 irreducible representations $\pi_m$ for $m = 0, \dots, 2^{n-2} - 1$, defined by

$$\pi_m(u_0) = 0 \quad \pi_m(u_1) = \omega^m \left(\sqrt{i}\right)^{1-\delta_{n,3}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \pi_m(g) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The product of two 2-dimensional representations $\pi = \pi_l \bullet \pi_m = (\pi_l \otimes \pi_m) \circ \Delta$ is defined by

$$\pi(u_0) = \omega^{l+m} i^{1-\delta_{n,3}} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \pi(u_1) = 0 \quad \pi(g) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and is equivalent to representation $\chi^{l+m} + \chi^{l+m+2^{n-2}} + \chi^{l+m}\varphi + \chi^{l+m+2^{n-2}}\varphi$. This together with formulas (8.1) and (8.2) completely determines the structure of $K_0(A_n^+)$ as an ordered ring with involution and marked element.

**Proposition 8.4.** $K_0(A_n^+) = \mathbb{Z}\langle \chi, \varphi, \pi_m \rangle_{m=0,\dots,2^{n-2}-1}$ *is a commutative ordered ring with involution, a marked element and following relations:*

$$\begin{array}{lll} \chi^* = \chi^{-1} & \chi^{2^{n-1}} = 1 & \chi \bullet \pi_m = \pi_{m+1 \bmod 2^{n-2}} \\ \varphi^* = \varphi & \varphi^2 = 1 & \varphi \bullet \pi_m = \pi_m \\ \pi_0^* = \pi_0 & \pi_m^* = \pi_{2^{n-2}-m}, & m = 1, \dots, 2^{n-2} - 1 \\ & \multicolumn{2}{l}{\pi_l \bullet \pi_m = \chi^{l+m} + \chi^{l+m+2^{n-2}} + \chi^{l+m}\varphi + \chi^{l+m+2^{n-2}}\varphi} \end{array}$$

8.2. **The Grothendieck ring of $A_n^-$.** $\mathbf{G}((A_n^-)^*) = \langle \chi \rangle \times \langle \varphi \rangle \cong \mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_2$, where $\chi(u_0) = \omega$, $\chi(u_1) = 0$, $\chi(g) = 1$ and $\varphi(u_0) = 1$, $\varphi(u_1) = 0$, $\varphi(g) = -1$, where $\omega$ is a primitive $2^{n-1}$-th root of unity. There are $2^{n-2}$ degree 2 irreducible representations $\pi_m$ for $m = 0, \dots, 2^{n-2} - 1$, defined by

$$\pi_m(u_0) = 0 \quad \pi_m(u_1) = \xi^{2m+1} \left(\sqrt{i}\right)^{1-\delta_{n,3}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \pi_m(g) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $\xi$ is a primitive $2^n$-th root of unity and $\xi^2 = \omega$.

The product of two 2-dimensional representations $\pi = \pi_l \bullet \pi_m = (\pi_l \otimes \pi_m) \circ \Delta$ is defined by

$$\pi(u_0) = \omega^{l+m+1} i^{1-\delta_{n,3}} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \pi(u_1) = 0 \quad \pi(g) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and is equivalent to representation $\chi^{l+m+1} + \chi^{l+m+2^{n-2}+1} + \chi^{l+m+1}\varphi + \chi^{l+m+2^{n-2}+1}\varphi$. This together with formulas (8.1) and (8.2) completely determines the structure of $K_0(A_n^-)$ as an ordered ring with involution and a marked element.

**Proposition 8.5.** *$K_0(A_n^-) = \mathbb{Z}\langle \chi, \varphi, \pi_m \rangle_{m=0,\dots,2^{n-2}-1}$ is a commutative ordered ring with involution, marked element and following relations:*

$$\begin{array}{ccc} \chi^* = \chi^{-1} & \chi^{2^{n-1}} = 1 & \chi \bullet \pi_m = \pi_{m+1 \bmod 2^{n-2}} \\ \varphi^* = \varphi & \varphi^2 = 1 & \varphi \bullet \pi_m = \pi_m \\ \pi_m^* = \pi_{2^{n-2}-m-1} & \multicolumn{2}{c}{\pi_l \bullet \pi_m = \chi^{l+m+1} + \chi^{l+m+2^{n-2}+1} + \chi^{l+m+1}\varphi + \chi^{l+m+2^{n-2}+1}\varphi} \end{array}$$

8.3. **The Grothendieck rings of $(A_n^{\pm})^*$.** There are $2^{n-2}$ degree 2 irreducible representations $\pi_m$ for $m = 0, \dots, 2^{n-2}-1$, defined by

$$\pi_m(p_{0,2m+1,0}) = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \tfrac{1}{2}X \qquad \pi_m(p_{1,2m+1,0}) = \tfrac{1}{2} i^{m^2} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

$$\pi_m(p_{0,2m+1,1}) = \tfrac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \tfrac{1}{2}Y \qquad \pi_m(p_{1,2m+1,1}) = \tfrac{1}{2} i^{m^2} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

$$\pi_m(p_{a,b,c}) = 0, \text{ otherwise}$$

The product of two 2-dimensional representations $\pi_{l,m} = \pi_l \bullet \pi_m = (\pi_l \otimes \pi_m) \circ \Delta$ is defined by

$$\pi_{l,m}(p_{0,2l+2m+2,0}) = \tfrac{1}{2}\begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} \quad \pi_{l,m}(p_{1,2l+2m+2,0}) = \frac{i^{m^2+l^2}}{2}(-1)^d \begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix}$$

$$\pi_{l,m}(p_{0,2l+2m+2,1}) = \tfrac{1}{2}\begin{pmatrix} Y & 0 \\ 0 & Y \end{pmatrix} \quad \pi_{l,m}(p_{1,2l+2m+2,1}) = \frac{i^{m^2+l^2}}{2}(-1)^d \begin{pmatrix} 0 & -Y \\ Y & 0 \end{pmatrix}$$

$$\pi_{l,m}(p_{a,b,c}) = 0, \text{ otherwise}$$

where in case of $A_n^+$ $d$ is always 0 and in case of $A_n^-$ $d = 0$ if $2l + 2m + 2 \le 2^{n-1}$ and $d = 1$ if $2l + 2m + 2 > 2^{n-1}$. The product $\pi_{l,m}$ is equivalent to the representation $\pi$ defined by

$$\pi(p_{0,2l+2m+2,0}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \pi(p_{1,2l+2m+2,0}) = i^{m^2+l^2} \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\pi(p_{0,2l+2m+2,1}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \pi(p_{1,2l+2m+2,1}) = i^{m^2+l^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

$$\pi(p_{a,b,c}) = 0, \text{ otherwise}$$

**Proposition 8.6.** *$K_0\left(A_3^+\right)^* = \mathbb{Z}\left\langle \chi = u_0^2 + iu_1^2, \varphi = g, \pi_0, \pi_1 \right\rangle$ is a commutative ordered ring with involution, marked element and following relations:*

$$\begin{array}{lll} \chi^* = \chi^{-1} & \chi^4 = 1 & \chi \bullet \pi_m = \pi_{m+1 \bmod 2} \\ \varphi^* = \varphi & \varphi^2 = 1 & \varphi \bullet \pi_m = \pi_m \\ \pi_0^* = \pi_1 & \multicolumn{2}{l}{\pi_l \bullet \pi_m = \chi^{l+m+1} + \chi^{l+m+3} + \chi^{l+m+1}\varphi + \chi^{l+m+3}\varphi} \end{array}$$

**Proposition 8.7.** *$K_0\left(A_3^-\right)^* = \mathbb{Z}\left\langle \chi = u_0^2 + iu_1^2, \varphi = g, \psi = u_0^4 + u_1^4, \pi_0, \pi_1 \right\rangle$ is a commutative ordered ring with involution, marked element and following relations:*

$$\begin{array}{lll} \chi^* = \chi & \chi^2 = 1 & \chi \bullet \pi_m = \pi_{m+1 \bmod 2} \\ \varphi^* = \varphi & \varphi^2 = 1 & \varphi \bullet \pi_m = \pi_m \\ \psi^* = \psi & \psi^2 = 1 & \psi \bullet \pi_m = \pi_m \\ \pi_0^* = \pi_1 & \multicolumn{2}{l}{\pi_l \bullet \pi_m = \chi^{l+m+1} + \chi^{l+m+1}\varphi + \chi^{l+m+1}\psi + \chi^{l+m+1}\varphi\psi} \end{array}$$

**Proposition 8.8.** *$K_0\left(A_n^-\right)^* = \mathbb{Z}\left\langle \chi = u_0^2 + iu_1^2, \varphi = g, \pi_m \right\rangle_{m=0,\dots,2^{n-2}-1}$ for $n \geq 4$ is a commutative ordered ring with involution, marked element and following relations:*

$$\begin{array}{rll} \chi^* = \chi^{-1} & \chi^{2^{n-1}} = 1 & \chi \bullet \pi_m = \pi_{m+1 \bmod 2^{n-2}} \\ \varphi^* = \varphi & \varphi^2 = 1 & \varphi \bullet \pi_m = \pi_m \\ \pi_m^* = \pi_{2^{n-2}-m-1} & \multicolumn{2}{l}{\pi_l \bullet \pi_m = \chi^{l+m+1} + \chi^{l+m+2^{n-2}+1} + \chi^{l+m+1}\varphi + \chi^{l+m+2^{n-2}+1}\varphi} \end{array}$$

**Proposition 8.9.** *$K_0\left(A_n^+\right)^* = \mathbb{Z}\left\langle \chi = u_0^2 + iu_1^2, \varphi = g, \psi = u_0^{2^{n-1}} - u_1^{2^{n-1}}, \pi_m \right\rangle_{m=0,\dots,2^{n-2}-1}$ for $n \geq 4$ is a commutative ordered ring with involution, marked element and following relations:*

$$\begin{array}{rll} \chi^* = \chi^{-1} & \chi^{2^{n-1}} = 1 & \chi \bullet \pi_m = \pi_{m+1 \bmod 2^{n-2}} \\ \varphi^* = \varphi & \varphi^2 = 1 & \varphi \bullet \pi_m = \pi_m \\ \psi^* = \psi & \psi^2 = 1 & \psi \bullet \pi_m = \pi_m \\ \pi_m^* = \pi_{2^{n-2}-m-1} & \multicolumn{2}{l}{\pi_l \bullet \pi_m = \chi^{l+m+1} + \chi^{l+m+1}\varphi + \chi^{l+m+1}\psi + \chi^{l+m+1}\varphi\psi} \end{array}$$


## ACKNOWLEDGMENTS

Part of this work is contained in the author's Ph.D. thesis at the University of Southern California. I am very grateful to my Ph.D. advisor Professor Susan Montgomery for numerous discussions and helpful comments.

The work on this paper was finished when the author was a postdoctoral fellow at MSRI. I would like to thank MSRI and the organizers of the Noncommutative Algebra program for their support and hospitality.


## References


[FMS] D. Fischman, S. Montgomery, H.-J. Schneider, Frobenius Extensions of Subalgebras of Hopf Algebras, *AMS Trans.* **349** (1997), no. 12, 4857-4895.

[K1] Y. Kashina, On the classification of semisimple Hopf algebras of dimension 16, submitted

[K2] Y. Kashina, Semisimple Hopf algebras of dimension $2^m$ with an abelian metacyclic group of grouplikes, preprint

[M] S. Montgomery, Hopf Algebras and Their Actions on Rings, CBMS Lectures vol. 82, AMS, Providence, RI, 1993.

[NR] W. Nichols, M. B. Richmond, The Grothendieck group of a Hopf algebra, *J. Pure and Appl. Algebra* **106** (1996), 297–306.

[N] D. Nikshych, On finite dimensional simple Hopf algebras, preprint

[R] D. Radford, The Structure of Hopf Algebras with a Projection, *J. Algebra* **92** (1985), 322-347.

[S] J.-P. Serre, Groups de Grothendieck des schemas en groups réductifs déployés, Publ. Math I.H.E.S., **34** (1968), 37–52.

[Y] D.N. Yetter, Quantum groups and representations of monoidal categories, *Math. Proc. Cambridge Phil. Soc.* **108** (1990), 261–290.



Yevgenia Kashina, Mathematical Sciences Research Institute, 1000 Centennial Drive, Berkeley, CA 94720-5070

*E-mail address*: `kashina@msri.org`