THE LMO-INVARIANT OF 3-MANIFOLDS OF RANK ONE AND THE ALEXANDER POLYNOMIAL

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Abstract. We prove that the LMO-invariant of a 3-manifold of rank one is determined by the Alexander polynomial of the manifold, and conversely, that the Alexander polynomial is determined by the LMO-invariant. Furthermore, we show that the Alexander polynomial of a null-homologous knot in a rational homology 3-sphere can be obtained by composing the weight system of the Alexander polynomial with the Århus invariant of knots.

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Introduction

In analogy with the theory of Vassiliev invariants of links, different notions of finite type invariants of 3-manifolds have been introduced. For integral homology spheres these different notions coincide with the original definition of Ohtsuki ([Oht]). The LMO-invariant Z^{LMO} assembles all $\mathbb{Q}$-valued finite type invariants of integral homology spheres in a formal series and is therefore called a universal finite type invariant ([LMO], [Le1]). For connected closed manifolds M the following is known about Z^{LMO}:

For M with ...	Z^{LMO} is determined by and determines ...
rank $H_1(M) = 0$	all $\mathbb{Q}$-valued invariants of Goussarov and Habiro ([Habi])
$H_1(M) = \mathbb{Z}$	the Alexander polynomial (Theorem 1 of [GaH])
rank $H_1(M) \geq 2$	the Casson-Walker-Lescop invariant ([HaB], [Hab], [Les])

In this article we fill in the missing puzzle piece for the interpretation of the LMO-invariant of manifolds of rank ≥ 1 in terms of classical invariants. We prove the following generalization of Theorem 1 of [GaH].

Theorem 1. *Let M be a closed oriented 3-manifold of rank 1. Then the LMO-invariant $Z^{LMO}(M)$ is determined by the Alexander polynomial $\nabla(M)$, and conversely, $\nabla(M)$ is determined by $Z^{LMO}(M)$.*

In the proof of Theorem 1 of [GaH] it was used that the Alexander polynomial ∇ of links L in S^3 can be obtained from the universal Vassiliev invariant Z of links in S^3 via a map W_∇ as follows:

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$$\frac{h}{e^{h/2}-e^{-h/2}}\nabla(L)_{|\ t^{1/2}=e^{h/2}} = W_\nabla \circ Z(L). \tag{1}$$

We generalize Equation (1) by replacing Z by the Århus invariant $\AA$ ([BGRT1]) of knots in a rational homology sphere:

Theorem 2. *Let K be a null-homologous knot in a rational homology 3-sphere. Then*

$$\frac{h}{e^{h/2}-e^{-h/2}}\nabla(K)_{|\ t^{1/2}=e^{h/2}} = W_\nabla \circ \AA(K).$$

Theorem 2 is an important ingredient in the proof of Theorem 1. Theorems 1 and 2 will be proven in Section 4. In Sections 1–3 we prepare these proofs by recalling definitions and properties of the Alexander polynomial of links and manifolds, of unitrivalent diagrams and the map W_∇, and of the universal finite type invariants Z, Z^{LMO} and $\AA$.

1. The Alexander polynomial

In this section we make preliminary definitions and recall some facts about the Alexander polynomial ∇ from [Les].

All manifolds and submanifolds in this paper are oriented. Let M be a rational homology 3-sphere (meaning that M is a connected closed manifold of dimension 3 with $H_1(M,\mathbb{Q})=0$). Let K be a knot in M. Choose a tubular neighborhood T of K. A meridian of K is a simple closed curve m on the boundary ∂T of T that is null-homologous in T. The curve m is oriented by the right-hand rule. There exists a unique isomorphism $i_K : H_1(M\setminus K,\mathbb{Q}) \longrightarrow \mathbb{Q}$ that sends a meridian of K to 1. As a $\mathbb{Q}$-linear map i_K is uniquely determined by the property that for any oriented surface $\Sigma\subset M$ with $\partial\Sigma\cap K=\emptyset$ the value $i_K(\partial\Sigma)$ is the intersection number of K with Σ. For disjoint knots K_1, K_2 the linking number $\mathrm{lk}(K_1,K_2)$ is defined as $i_{K_1}(K_2)$. The linking number $\mathrm{lk}(.,.)$ is symmetric.

Denote the number of components of a link L by $|L|$. A framed link L is a link with a simple closed curve μ_i on the boundary ∂T_i of a tubular neighborhood T_i of each component K_i ($i=1,\dots,|L|$). Inside of T_i, μ_i is homologous to q_iK_i for some $q_i\in\mathbb{Z}$. The *linking matrix* (l_{ij}) of L is defined by $l_{ij}=\mathrm{lk}(K_i,\mu_j)/q_j$. The link L has integral framing if all q_i are 1. The values $l_{ii}\in\mathbb{Q}$ are called framing of K_i. We denote by M_L the manifold obtained by surgery on $L\subset M$.

Let $L\subset M$ be a null-homologous link. Then there exists an oriented connected surface Σ embedded in M such that $\partial\Sigma=L$. Any surface with this property is called a Seifert surface of L. Let $\Sigma^\pm=\Sigma^+\cup\Sigma^-$ be a tubular neighborhood of Σ such that $\Sigma=\Sigma^+\cap\Sigma^-$ and Σ^+ lies on the positive side of Σ. The Seifert form of $\Sigma\subset M$ is the $\mathbb{Z}$-bilinear form $s: H_1(\Sigma)\times H_1(\Sigma)\longrightarrow\mathbb{Q}$ defined by sending homology classes a, b to $\mathrm{lk}(A^-,B^+)$ where A^- is a knot in Σ^- representing a and B^+ is a knot in Σ^+ representing b. In this section a matrix of s with respect to an arbitrary basis of $H_1(\Sigma)$ is called a Seifert matrix of Σ (later we will choose a particular basis of $H_1(\Sigma)$). Define the bilinear form s^* by $s^*(a,b)=s(b,a)$. Then $s-s^*$ is the intersection form of Σ. Denote the transpose of a matrix V by V^*.

Proposition 3. *Let L be a null-homologous link in a rational homology sphere M. Choose a Seifert surface Σ of L. Let V be a Seifert matrix of Σ. Then*

$$\nabla(L) = \det(t^{1/2}V - t^{-1/2}V^*) \in (t^{1/2} - t^{-1/2})^{|L|-1}\mathbb{Q}[(t^{1/2} - t^{-1/2})^2] \subset \mathbb{Q}[t^{\pm 1/2}]$$

is an invariant of the pair $L \subset M$ up to homeomorphism; in particular it is an isotopy invariant of L.

Proposition 3 can be proven by using sign-determined Reidemeister torsion (see Proposition 2.3.13 of [Les], [Tur]).

Up to sign the invariant $\nabla(L)$ can be described as follows. Let N be a connected 3-manifold and let $\varphi : H_1(N) \longrightarrow \mathcal{Z} = \mathbb{Z}$ be a homomorphism. Let $\widetilde{N}$ be the connected cover of N corresponding to $\mathrm{Ker}(\varphi)$. Then $H_1(\widetilde{N})$ is a module over the group ring $\mathbb{Z}[\mathcal{Z}] \cong \mathbb{Z}[t^{\pm 1}]$. Let $J \subset \mathbb{Z}[t^{\pm 1}]$ be the order ideal of $H_1(\widetilde{N})$. Let $\Delta_\varphi(N)$ be a generator of the smallest principal ideal containing J. Then $\Delta_\varphi(N)$ is unique up to multiplication by $\pm t^i$. For a link in a rational homology sphere M we denote $\Delta_\varphi(M \setminus L)$ by $\Delta(L)$, where $\varphi : H_1(M \setminus L) \longrightarrow \mathbb{Z}$ is given by the sum of the linking numbers with the components of L. The following lemma (see Proposition 2.3.13 of [Les]) relates $\nabla(L)$ and $\Delta(L)$.

Lemma 4. *Let L be a null-homologous link in a rational homology sphere M. Then there exists a unique $i \in \mathbb{Z}$ such that $t^{i/2}\Delta(L)$ is invariant under the replacement of $t^{1/2}$ by $-t^{-1/2}$. For some $\epsilon \in \{\pm 1\}$ we have $\epsilon t^{i/2}\Delta(L) = |H_1(M)|\nabla(L)$.*

Now consider a connected closed 3-manifold N of rank 1. Denote the quotient of $H_1(N)$ by its torsion subgroup $\mathrm{Tor}(H_1(N))$ by $H_1^{\#}(N)$. Choose an isomorphism $\psi : H_1^{\#}(N) \longrightarrow \mathbb{Z}$. Denote the composition of the canonical projection $H_1(N) \longrightarrow H_1^{\#}(N)$ with ψ by $\bar{\psi}$. The following two lemmas (see [Les], Section 5.1) allow to compare $\Delta_{\bar{\psi}}(N)$ with a knot invariant.

Lemma 5. *Every connected closed 3-manifold N of rank 1 can be obtained by 0-framed surgery on a null-homologous knot K in a rational homology sphere M. We then have $\mathrm{Tor}(H_1(N)) \cong H_1(M)$.*

Lemma 6. *Let K be a null-homologous 0-framed knot in a rational homology 3-sphere M. Then $\Delta(K)$ is equal to $\Delta_{\bar{\psi}}(M_K)$ up to multiplication by $\pm t^i$.*

We see by Lemmas 4, 5 and 6 that there exists $j \in \mathbb{Z}$ such that $t^{j/2}\Delta_{\bar{\psi}}(N)$ is invariant under the replacement of $t^{1/2}$ by $-t^{-1/2}$. Furthermore, we can choose $\epsilon \in \{\pm 1\}$ such that $\epsilon\Delta_{\bar{\psi}}(N)_{|t=1} = |H_1(\mathrm{Tor}(N))| > 0$. Denote $(\epsilon t^{j/2}/|H_1(\mathrm{Tor}(N))|)\Delta_{\bar{\psi}}(N)$ by $\nabla(N)$. The definition of $\nabla(N)$ does not depend on the choice of the isomorphism ψ because $\nabla(N) \in \mathbb{Q}[(t^{1/2} - t^{-1/2})^2]$. The invariant ∇ satisfies

$$\nabla(K) = \nabla(M_K). \tag{2}$$

for all null-homologous 0-framed knots K in a rational homology sphere M.

2. Unitrivalent diagrams and W_∇

In this section we briefly recall facts about unitrivalent diagrams and use them to state properties of the Vassiliev invariants in the Alexander polynomial ∇.

Let Γ be a compact oriented 1-manifold whose boundary $\partial\Gamma$ is partitioned into two ordered sets called upper and lower boundary. Let X be a set. A *unitrivalent diagram* with *skeleton* Γ is a graph D with distinguished subgraph Γ such that all vertices of D are either univalent or trivalent. Trivalent vertices not lying on Γ are called *internal* and are oriented by a cyclic order of the incident edges. Univalent vertices are also called *legs*. Each leg of a unitrivalent diagram is labeled by an element of X. We allow connected components in D that do not intersect Γ whenever these components contain at least one trivalent vertex. Recall the definition of a $\mathbb{Q}$-vector space $\mathcal{A}(\Gamma, X)$ generated by unitrivalent diagrams modulo relations called (STU), (IHX), and (AS) ([BN1]). When Γ is equipped with additional information (for example: dots on circle-components of Γ, a set Y in bijection with circle-components of Γ, a distinguished subset of the components of Γ,...), we require in the definition of $\mathcal{A}(\Gamma, X)$ that homeomorphisms between unitrivalent diagrams also preserve this additional data. The space $\mathcal{A}(\Gamma, X)$ is graded by half of the number of vertices of unitrivalent diagrams. Denote $\mathcal{A}(\Gamma, \emptyset)$ by $\mathcal{A}(\Gamma)$.

The invariants of ℓ-component links in S^3 that are coefficients of $z^i = (t^{1/2} - t^{-1/2})^i$ in $\nabla(L)$ induce linear forms $W_i : \mathcal{A}_i^\ell \longrightarrow \mathbb{Q}$ on the degree-i part $\mathcal{A}_i^\ell$ of $\mathcal{A}^\ell := \mathcal{A}(S^{1 \sqcup \ell})$ (see Section 3 of [BNG]) . For a in the completion of $\mathcal{A}^\ell$ by the degree, we define

$$W_\nabla(a) = \sum_i W_i(a) h^i \in \mathbb{Q}[[h]].$$

It will follow from Theorem 2 and can also be seen directly that the Alexander polynomial of links in a rational homology sphere induces the same map W_∇ (see skein relation 2.3.16 of [Les], or Exercise 3.10 of [BNG]). The map W_∇ and its extensions to $\mathcal{A}(\Gamma, X)$ obtained from representations of the Lie superalgebra gl$(1|1)$ have the following property (see Proposition 7.1 of [Vai], consider the degree-1 part of $\mathcal{A}(\emptyset)$ seperately).

Lemma 7. *Let $D \in \mathcal{A}(\Gamma, X)$ be a unitrivalent diagram. Assume that D has an internal vertex u such that all edges incident to u are connected to internal vertices. Then we have $W_\nabla(D) = 0$.*

Let $I_x \cong I := [0, 1]$ $(x \in X)$. Denote the disjoint union by $\sqcup$. For every partition of $\partial(\Gamma \sqcup \bigsqcup_{x\in X} I_x)$ into two ordered sets called upper and lower boundary there exists an isomorphism

$$\chi_X : \mathcal{A}(\Gamma, X \sqcup Y) \longrightarrow \mathcal{A}\left(\Gamma \sqcup \bigsqcup_{x\in X} I_x, Y\right) \tag{3}$$

given by the average over all permutations of putting x-labeled univalent vertices of a diagram on the corresponding skeleton component $I_x \cong I$ of $\Gamma \sqcup \bigsqcup_{x\in X} I_x$. The inverse of χ_X will be denoted by σ_X and the set X will not be specified when it is clear from the context. Obviously, there exists an isomorphism of $\mathcal{A}(\Gamma \sqcup \bigsqcup_{x\in X} I_x, Y)$

with a space $\mathcal{A}(\Gamma\sqcup\bigsqcup_{x\in X}S_x^{1*},Y)$, where the circles S_x^{1*} have a dot and are in bijection with the set X. Similarly, we have a surjective map from $\mathcal{A}(\bigsqcup_{x\in X}I_x\sqcup\bigsqcup_{y\in Y}I_y,Z)$ to $\mathcal{A}(\bigsqcup_{x\in X}I_x\sqcup\bigsqcup_{y\in Y}S_y^1,Z)$ given by closing the intervals I_y to form the circles S_y^1. Denote the composition of χ_Y with this surjective map by $\bar\chi_Y$.
An important special case is $\mathcal{A}(S^1)\cong\mathcal{A}(S^{1*},\emptyset)\cong\mathcal{A}(I)=:\mathcal{A}$ (see [BN1]). The space $\mathcal{A}$ is a commutative algebra with multiplication $\#$ induced by the connected sum of the skeletons S^1 of diagrams (resp. by the concatenation of skeletons I of diagrams). More generally, the connected sum of S^1 with any distinguished skeleton component C of a unitrivalent diagram turns $\mathcal{A}(\Gamma\cup C,X)$ into an $\mathcal{A}$-module. Let $\bar{\mathcal{A}}$ be the quotient of $\mathcal{A}$ by the ideal generated by the degree-1 part of $\mathcal{A}$ and let $\pi:\mathcal{A}\longrightarrow\bar{\mathcal{A}}$ be the canonical projection. There exists a unique inclusion of algebras $i:\bar{\mathcal{A}}\longrightarrow\mathcal{A}$ with the property that $i(D)=D$ for all diagrams D such that $D\setminus S^1$ is connected and D contains an internal vertex ([BN1], Equation (5), Exercise 3.16). The map $P_{\mathrm{defr}}=i\circ\pi:\mathcal{A}\longrightarrow\mathcal{A}$ is called *deframing projection.*
The disjoint union of unitrivalent diagrams turns $\mathcal{A}(\emptyset,X)$ into a commutative algebra and $\mathcal{A}(\Gamma,X)$ into an $\mathcal{A}(\emptyset,X)$-module. Important examples of diagrams in $\mathcal{A}(\emptyset,X)$ are so-called *struts* ${}^{i}\frown{}^{j}$ with labels $i,j\in X$, and so-called *wheels* $\omega_n=$ having n internal vertices lying on a circle and n univalent vertices with the same label ($n=4$ in this example). Let $\mathcal{A}(\emptyset,X)_{\mathrm{strut}}\subset\mathcal{A}(\emptyset,X)$ be the subalgebra generated by struts and $\mathcal{A}(\emptyset,X)_{\mathrm{wh}}$ be the subalgebra generated by wheels. It is known that $\mathcal{A}(\emptyset,X)_{\mathrm{strut}}$ is a polynomial algebra in the $n(n+1)/2$ different struts ($n=|X|$) and $\mathcal{A}(\emptyset,X)_{\mathrm{wh}}$ is a polynomial algebra in wheels with an even number of univalent vertices. There exist unique projections from $\mathcal{A}(\Gamma,X)$ to $\mathcal{A}(\emptyset,X)_{\mathrm{strut}}$ (resp. $\mathcal{A}(\emptyset,X)_{\mathrm{wh}}$) that send all diagrams to 0 that have a connected component that is not a strut (resp. a wheel). Define $P_{\mathrm{strut}}:\mathcal{A}(\Gamma\sqcup\bigsqcup_{x\in X}I_x,\emptyset)\longrightarrow\mathcal{A}(\Gamma,X)$ as the composition of σ with the projection to $\mathcal{A}(\emptyset,X)_{\mathrm{strut}}\subset\mathcal{A}(\Gamma,X)$. The map P_{strut} descends to $\mathcal{A}(\Gamma\sqcup\bigsqcup_{x\in X}S_x^1,\emptyset)$ where the circle-components S_x^1 are in bijection with X. Define $P_{\mathrm{wh}}:\mathcal{A}\longrightarrow\mathcal{A}(\emptyset,\{x\})$ as the composition of $\sigma\circ P_{\mathrm{defr}}$ with the projection to $\mathcal{A}(\emptyset,\{x\})_{\mathrm{wh}}$. We have $P_{\mathrm{wh}}(a\#b)=P_{\mathrm{wh}}(a)\sqcup P_{\mathrm{wh}}(b)$ for all $a,b\in\mathcal{A}$. The map P_{wh} is related to W_∇ as follows (see [Vai], [Kri]).

Lemma 8. *For $D\in\mathcal{A}$ the value $W_\nabla(D)$ depends only on $P_{\mathrm{wh}}(D)$ and is determined by*

$$W_\nabla(D_1\#D_2)=W_\nabla(D_1)W_\nabla(D_2)\quad\textit{and}\quad W_\nabla(\bar\chi(\omega_{2n}))=-2h^{2n}.$$

Lemma 8 was used in proofs of the Melvin-Morton-Rozansky conjecture ([BNG]).

3. Universal finite type invariants

Recall from Section 3 of [LM2] that a non-associative framed tangle (or q-tangle) T is a usual tangle with integral framing, except that source(T) and target(T) are equipped with parentheses on the sequences of $\pm$-symbols associated with the lower and upper boundary points of T. We denote by Z the universal Vassiliev invariant of non-associative framed tangles (see [LM2]). Denote the underlying 1-manifold of a tangle T (together with the partition of ∂T into two ordered sets and possibly

together with a decoration of T such as dots, distinguished components, ...) by $\Gamma(T)$. Then the values $Z(T)$ lie in the completion of $\mathcal{A}(\Gamma(T))$ by the degree.
Let $\nu = Z(O)$ be the invariant of the trivial knot with 0-framing. Let $T = L' \cup T''$ be a diagram of a framed non-associative tangle where the components of the sublink L' of T are in bijection with a set X' and each component of L' has a dot on its circle. Define $\check{Z}(T)$ as the connected sum of $Z(T)$ with $\nu^{\otimes |L'|}$ along the components of $\Gamma(L')$. Cut the chord diagrams in $\check{Z}(T)$ at the dots and apply the isomorphism $\sigma_{X'}$. The result lies in the completion of $\mathcal{A}(\Gamma(T''), X')$ and is called $\check{Z}^{\sigma}(T)$. The value $\check{Z}^{\sigma}(T)$ is not invariant under isotopies of the tangle represented by the diagram T. For tangles T with dotted circles L' invariants $Z_0^{LMO}(T)$ and $\mathring{A}_0(T)$ of isotopy (that are also invariant under second Kirby moves along L') are obtained from $C = \check{Z}^{\sigma}(T)$ as follows (see [LMO], [Le2], [BGRT2]).
Definition of Z_0^{LMO}: The degree-n part of $Z_0^{LMO}(T) :=< C >$ is obtained from the degree $n+|L'|n$ part of C by forgetting the diagrams in C that do not have exactly $2n$ legs of each color $x \in X'$, by summing over all the $((2n-1)!!)^{|L'|} = ((2n)!/2^n n!)^{|L'|}$ possible ways of gluing pairs of legs of diagrams in C with the same label and by replacing circles that do not belong to $\Gamma(T'')$ by $-2n$.
Definition of $\mathring{A}_0$: $\mathring{A}_0$ is only defined when the linking matrix (l_{ij}) of L' is invertible (or equivalently, when $S^3_{L'}$ is a rational homology sphere). Write C in the form

$$C = P \sqcup \exp\left(\frac{1}{2}\sum_{i,j\in X'} l_{ij}\; {}^{i}\frown^{j}\right)$$

where P contains no struts. Let (l^{ij}) be the inverse matrix of (l_{ij}). Then

$$\mathring{A}_0(T) :=< P, \exp\left(-\frac{1}{2}\sum_{i,j\in X'} l^{ij}\; {}_{\partial i}\smile_{\partial j}\right) >,$$

where $< D_1, D_2 >$ is 0 if for some i the number of i-labeled legs of D_1 is not equal to the number of ∂i-labeled legs of D_2, and is given by the sum of all ways of gluing all legs with i-labels to legs with ∂i-labels in the remaining case.
Let $L \subset M$ be a link in a 3-manifold. Represent $L \subset M$ by a diagram $L' \cup L''$ of a link in S^3, such that $S^3_{L'} \cong M$ and the image of L'' in $S^3_{L'}$ is mapped to L by this homeomorphism. Put a dot on each component of L'. Two invariants Z^{LMO} and $\mathring{A}$ of homeomorphisms of the pair (M, L) are obtained from $Z_0^{LMO}(L' \cup L'')$ and $\mathring{A}_0(L' \cup L'')$ by normalization (making it invariant under the first Kirby move) as follows:

$$Z^{LMO}(L) = Z_0^{LMO}(U_+)^{-\sigma_+} Z_0^{LMO}(U_-)^{-\sigma_-} Z_0^{LMO}(L' \cup L''), \tag{4}$$

$$\mathring{A}(L) = \mathring{A}_0(U_+)^{-\sigma_+} \mathring{A}_0(U_-)^{-\sigma_-} \mathring{A}_0(L' \cup L''), \tag{5}$$

where $U_\pm$ is the trivial knot with a dot and framing ± 1 and σ_+ (resp. σ_-) is the number of positive (resp. negative) eigenvalues of the linking matrix (l_{ij}) of L'. The invariants of the empty link $Z^{LMO}(\emptyset)$ and $\mathring{A}(\emptyset)$ are also denoted by $Z^{LMO}(M)$ and $\mathring{A}(M)$, respectively. The series $\mathring{A}_0(U_\pm)$ have degree-0 term 1. Therefore Lemma 7 implies

$$W_\nabla \circ \mathring{A}_0(L' \cup L'') = W_\nabla \circ \mathring{A}(L). \tag{6}$$

We will make use of the following result of [BGRT3] (Equation (7) follows from Proposition 1.2 of [BGRT3] in the same way as Theorem 1 of [BGRT3]):

$$\mathring{A}(L) = |H_1(M)|^{-deg} Z^{LMO}(L), \tag{7}$$

where $|H_1(M)|^{-deg}$ denotes the operation of multiplying diagrams of degree m by $|H_1(M)|^{-m}$.

Let us recall some notation used in Lemma 9 below. Let T be a non-associative framed tangle T with a distinguished subset $\tilde{T}$ of its components. Denote by $d(T)$ the non-associative framed tangle given by replacing each component in $\tilde{T}$ by two copies that are parallel with respect to the framing. The symbols $a \in \{+,-\}$ in source(T) (resp. target(T)) that belong to $\tilde{T}$ are replace by $(a\ a)$ in source$(d(T))$ (resp. target$(d(T))$). Define $s(T)$ by reversing the orientation of each component in $\tilde{T}$. Define $\epsilon(T)$ by deleting $\tilde{T}$. Now let D be a unitrivalent diagram D with a distinguished subset $\tilde{\Gamma}$ of its skeleton components. Define $d(D)$ by replacing each skeleton component in $\tilde{\Gamma}$ by two copies, and by summing over all ways of lifting vertices of D that lie on $\tilde{\Gamma}$ to the new skeleton. Define $s(D)$ by reversing the orientation of the components in $\tilde{\Gamma}$ and by multiplying with $\prod_{C\in\tilde{\Gamma}}(-1)^{n_C}$ where n_C is the number of vertices lying on the skeleton component C of D. If $n_C > 0$ for some component C of $\tilde{\Gamma}$, then define $\epsilon(D) = 0$. Define $\epsilon(D)$ by deleting the components in $\tilde{\Gamma}$ in the remaining case. The composition $T_1 \circ T_2$ of non-associative tangles T_1, T_2 with source(T_1) = target(T_2) is defined by placing T_1 on the top of T_2. For diagrams D_i in $\mathcal{A}(\Gamma(T_i))$ a composition $D_1 \circ D_2$ is defined similarly. In the following lemma we state generalizations of well-known properties of Z.

Lemma 9. *Let T, T_1, T_2 be non-associative tangles with dotted circles.*
(1) Assume that some of the components of T without dots are distinguished. Then we have[1]

$$d(\mathring{A}_0(T)) = \mathring{A}_0(d(T)) \quad , \quad s(\mathring{A}_0(T)) = \mathring{A}_0(s(T)) \quad , \quad \epsilon(\mathring{A}_0(T)) = \mathring{A}_0(\epsilon(T)).$$

(2) Assume that source(T_1) = target(T_2). *Then*

$$\mathring{A}_0(T_1 \circ T_2) = \mathring{A}_0(T_1) \circ \mathring{A}_0(T_2).$$

(3) We have

$$\mathring{A}_0(T) = \bar{\chi}_Y((\exp(P)),$$

where P is a series of connected diagrams in $\mathcal{A}(\emptyset, Y)$ and Y is a set in bijection with the components of T without dots.

[1] As in [LM3] we must assume for the first property of $\mathring{A}_0$ that an even associator is used in the definition of Z. This causes no restrictions in Theorems 1 and 2 because for links L, the invariants $\mathring{A}(L)$ and $Z^{LMO}(L)$ do not depend on the choice of an associator.

The proof of Lemma 9 is straightforward. Statements similar to Lemma 9 hold for $\mathring{A}(L)$.

4. Proofs of Theorems 1 and 2

Recall from Equation (1) that for links L in S^3 we have $c\nabla(L)_{|\,t^{1/2}=e^{h/2}} = W_\nabla \circ Z(L)$ with $c = h/(e^{h/2} - e^{-h/2})$. Equation (1) is proven in [LM1] and [BNG] by showing that $W_\nabla \circ Z$ satisfies a skein relation and $W_\nabla \circ Z(O) = c$. With the methods of this proof one can show directly that $W_\nabla \circ \mathring{A}$ satisfies the same skein relation for links in a rational homology sphere and $W_\nabla \circ \mathring{A}(O) = c$, but this does not imply Theorem 2. In this section we present a proof of Theorem 2 based on Equation (1). Then we prove Theorem 1 by using Theorem 2.
Let $L = L' \cup L''$ be a framed link in a rational homology sphere M. Denote the components of L' (resp. L'') by K_x with $x \in X'$ (resp. $x \in X''$) and their framings by μ_x. For $x, y \in X' \cup X''$ let $l_{xy} = \mathrm{lk}(\mu_x, K_y)$ be linking numbers in M, let the submatrix corresponding to L' be invertible and denote its inverse by $(l^{xy})_{x,y\in X'}$. In the following lemma we recall how the linking numbers transform under surgery.

Lemma 10. *For $i, j \in X''$ the linking numbers $\tilde{l}_{ij} = \mathrm{lk}(\mu_i, K_j)$ of $L'' \subset M_{L'}$ are given by*

$$\tilde{l}_{ij} = l_{ij} - \sum_{x,y\in X'} l_{ix} l^{xy} l_{yj}.$$

Proof. Denote the meridians of the components of L by m_x. In $H_1(M \setminus (L' \cup L''), \mathbb{Q})$ the framings μ_y can uniquely be expressed as $\mu_y = \sum_{j\in X'\cup X''} l_{yj} m_j$. This implies for $x \in X'$ that

$$\sum_{y\in X'} l^{xy}\mu_y = m_x + \sum_{y\in X', j\in X''} l^{xy} l_{yj} m_j.$$

In $H_1(M_{L'} \setminus L'', \mathbb{Q}) = H_1(M \setminus (L' \cup L''), \mathbb{Q})/(\mu_x)_{x\in X'}$ we obtain the following unique expression of μ_i ($i \in X''$) in terms of the meridians m_j ($j \in X''$) of $L'' \subset M_{L'}$:

$$\mu_i = \sum_{j\in X'\cup X''} l_{ij} m_j = \sum_{j\in X''} l_{ij} m_j - \sum_{x\in X', y\in X', j\in X''} l_{ix} l^{xy} l_{yj} m_j.$$

This implies the lemma. □

The following lemma tells us that the linking numbers of a link $L \subset M$ can be recovered from $P_{\mathrm{strut}}(\mathring{A}(L))$.

Lemma 11. *Let L be a link with integral framing in a rational homology sphere M. Let the components of L be in bijection with a set X. Let $(\tilde{l}_{ij})_{i,j\in X}$ be the linking matrix of L. Then*

$$P_{\mathrm{strut}}(\mathring{A}(L)) = \exp\left(\frac{1}{2}\sum_{i,j\in X} \tilde{l}_{ij}\, {}^{i}\frown^{j}\right).$$

Proof. Choose a diagram of $L' \cup L'' \subset S^3$ such that $(S^3_{L'}, L'') \cong (M, L)$ and put dots on the components of L'. Let $(l_{xy})_{x,y \in X' \cup X''}$ be the linking matrix of $L' \cup L''$ and let $(l^{xy})_{x,y \in X'}$ be the inverse of the linking matrix of L'. Then for a series P (resp. $\tilde{P}$) of diagrams in $\mathcal{A}(\emptyset, X' \cup X'')$ (resp. in $\mathcal{A}(\emptyset, X''))$ that contains no struts and has degree-0-term 1, we have

$$
\begin{aligned}
\check{Z}^{\sigma}(L' \cup L'') &= \bar{\chi}_{X''}\left(P \sqcup \exp\left(\frac{1}{2}\sum_{x,y \in X' \cup X''} l_{xy}\, {}^{x}\frown{}^{y}\right)\right) \quad \text{and} \\
\mathring{A}_0(L' \cup L'') &= \bar{\chi}_{X''}\Bigg(< P \sqcup \exp\left(\frac{1}{2}\sum_{i,j \in X''} l_{ij}\, {}^{i}\frown{}^{j} + \sum_{i \in X'', x \in X'} l_{ix}\, {}^{i}\frown{}^{x}\right), \\
&\qquad \exp\left(-\frac{1}{2}\sum_{x,y \in X'} l^{xy}\, {}_{\partial x}\smile{}_{\partial y}\right) > \Bigg) \\
&= \bar{\chi}_{X''}\left(\tilde{P} \sqcup \exp\left(\frac{1}{2}\sum_{i,j \in X''} l_{ij}\, {}^{i}\frown{}^{j} - \frac{1}{2}\sum_{i,j \in X'', x,y \in X'} l_{ix} l^{xy} l_{yj}\, {}^{i}\frown{}^{j}\right)\right).
\end{aligned}
$$

Since $P_{\mathrm{strut}}(\mathring{A}(L)) = P_{\mathrm{strut}}(\mathring{A}_0(L' \cup L''))$, Lemma 11 follows from Lemma 10. □

For technical reasons we fix a representative of each homeomorphism-class of connected compact surfaces with boundary. We call this representative Σ a standard surface and equip it with a decomposition into a single vertex $v \cong I \times I$ (also called coupon) with bands $B_i \cong I \times I$ that are glued along $I \times \{0, 1\}$ to the upper boundary $I \times \{1\}$ of v. Call the part $I \times \{0\}$ of v its distinguished lower boundary. We orient the cores $I \times \{1/2\}$ of the bands B_i $(i = 1, \ldots, \operatorname{rank} H_1(\Sigma))$. An example is shown on the left side of Figure 1.

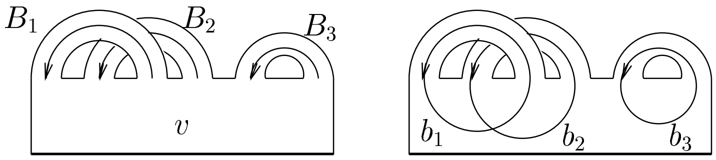

FIGURE 1. A standard surface Σ and a basis (b_i) of $H_1(\Sigma)$

We associate a basis of $H_1(\Sigma)$ to the ribbon graph decomposition of Σ as shown in Figure 1 by an example. The orientation of b_i is determined by the orientation of the core of the band B_i. An embedding of a standard surface into $R^2 \times I$ is an example of a ribbon graph in the sense of Section 8 of [KaT]. Ribbon graphs without vertices can canonically be identified with framed tangles. We will use this identification in the following.

From now on we use the term Seifert matrix of a Seifert surface $\Sigma \subset M$ always with respect to a basis of $H_1(\Sigma)$ obtained by identifying Σ with a standard surface in some freely chosen way. We use the same basis for a matrix of the intersection form of Σ.

Keylemma 12. *Let K be a knot in a rational homology sphere M bounding a Seifert surface Σ. Let V be a Seifert matrix of Σ. Then the power series $W_\nabla \circ \mathring{A}(K)$ depends only on V. The coefficient of h^i in this series is a polynomial in the entries of V.*

Let us prepare the proof of Keylemma 12. We will make some statements more generally for links instead of knots. Let $V = (v_{ij})$ be a Seifert matrix. Choose a null-homologous link L in a rational homology sphere M with Seifert surface Σ and Seifert matrix V. The homeomorphism type of Σ is determined by the similarity type of $V - V^*$. There exists a link with integral framing $\tilde{L} \subset M$ such that $M_{\tilde{L}} = S^3$. The link $\tilde{L}$ can be chosen to be disjoint from Σ because changing crossings between $L = \partial\Sigma$ and $\tilde{L}$ preserves the property that $M_{\tilde{L}} = S^3$. Therefore $\Sigma \subset M$ can be obtained from a surface $\Sigma'' \subset S^3$ by surgery along a link $L' \subset S^3 \setminus \Sigma''$. The identification of Σ with a standard surface induces an identification of Σ'' with a standard surface. In a diagram of $L' \cup \Sigma''$ the vertex v of Σ'' can be pulled downwards, such that the diagram $L' \cup \Sigma''$ is of the form $(L' \cup T_1'') \circ T_2''$ where T_2'' is a planar diagram of a neighborhood of v and the distinguished lower boundary of v is the lowest part of the diagram. Put dots on the components of L'. An example is shown in Figure 2.

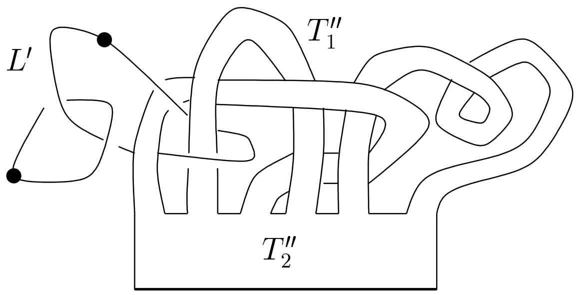

FIGURE 2. A diagram of $L' \cup \Sigma''$

The components of T_1'' are in bijection with the set $X'' = \{1, \dots, \operatorname{rank} H_1(\Sigma)\}$. Regard T_1'' as a non-associative framed tangle with parentheses of the form $((((\ldots).).).)$ on source(T_1''). Let $F = (f_{ij}) = V - V^*$ be the matrix of the intersection form of Σ and let $U = (u_{ij}) = 1/2(V + V^*) = V - 1/2F$.

Lemma 13. *With the notation from above we have*

$$P_{\mathrm{strut}}(\mathring{A}_0(L' \cup T_1'')) = \exp\left(\frac{1}{2}\sum_{i,j\in X''} u_{ij}\, {}^{i}\frown^{j}\right).$$

Proof. Let K_i^+ (resp. K_i^-) be a knot in the upper part $\Sigma^+ \subset M$ (resp. in the lower part $\Sigma^- \subset M$) of a tubular neighborhood of Σ representing the i-th basis element of $H_1(\Sigma)$. Let the knot $K_i^- \cong K_i^+$ have the framing $\operatorname{lk}(K_i^-, K_i^+) = v_{ii}$ induced by the surface Σ. First consider $i \neq j \in X''$. Define P_{ij} as the composition of P_{strut} with the projection to the part containing only powers of the strut ${}^{i}\frown^{j}$. Lemma 11 implies that $P_{ij}(\mathring{A}(K_i^- \cup K_j^+)) = \exp(v_{ij}\, {}^{i}\frown^{j})$. Represent $K_i^- \cup K_j^+ \subset M$ by a surgery diagram $(L' \cup S_1'') \circ S_2''$ where the tangle S_1'' consists of the i-th and j-th framed strands of T_1'' and S_2'' is a 0-framed tangle consisting of two intervals close to T_2''. See Figure 3 for an example (compare Figures 1 and 2). In this figure the dotted line separates S_2'' from $L' \cup S_1''$ and is not a part of the diagram.

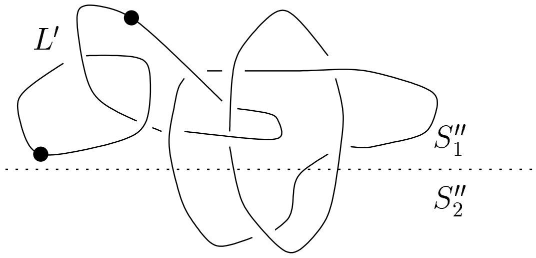

FIGURE 3. A surgery diagram $(L' \cup S_1'') \circ S_2''$ of $K_1^- \cup K_2^+$

We have $\mathring{A}_0(S_2'') = Z(S_2'')$ and the explicit description of Z (see [LM2]) implies

$$P_{\mathrm{strut}}(\mathring{A}_0(S_2'')) = \exp((1/2) f_{ij}\, {}^{i}\frown^{j}).$$

Observe the following property of P_{ij}:

$$P_{ij}\left(\mathring{A}_0(L' \cup S_1'') \circ \mathring{A}_0(S_2'')\right) = P_{ij}(\mathring{A}_0(L' \cup S_1'')) \sqcup P_{ij}(\mathring{A}_0(S_2'')).$$

The last two formulas and Part (2) of Lemma 9 imply

$$P_{ij}(\mathring{A}_0(L' \cup S_1'')) = \exp((v_{ij} - f_{ij}/2)\, {}^{i}\frown^{j}) = \exp(u_{ij}\, {}^{i}\frown^{j}).$$

Using Lemma 9 for ϵ we see that $P_{ij}(\mathring{A}(L' \cup T_1'')) = \exp(u_{ij}\, {}^{i}\frown^{j})$. For $i = j$ Lemma 11 implies $P_{ii}(\mathring{A}(K_i^{\pm})) = \exp((1/2) v_{ii}\, {}^{i}\frown^{i}) = \exp((1/2) u_{ii}\, {}^{i}\frown^{i})$. We apply Lemma 9 for ϵ as above and obtain $P_{ii}(\mathring{A}_0(L' \cup T_1'')) = \exp((1/2) u_{ii}\, {}^{i}\frown^{i})$. By Part (3) of Lemma 9 we have $P_{\mathrm{strut}}(\mathring{A}_0(L' \cup T_1'')) = \bigsqcup_{i \leq j} P_{ij}(\mathring{A}_0(L' \cup T_1''))$ which completes the proof. $\square$

Starting from V we made a lot of choices in the definition of T_1''. Since $\mathring{A}(L)$ is an invariant only the choice of $L \subset M$ can influence $W_\nabla \circ \mathring{A}(L)$. Now we are ready to show that for knots L the invariant $W_\nabla \circ \mathring{A}(L)$ depends only on V.

Proof of Keylemma 12. We use the notation from above. For suitable distinguished components of T_1'' and of $d(T_1'')$ the tangle $s(d(T_1''))$ coincides with the part of the framed oriented boundary of Σ'' that belongs to T_1''. Let T_3'' be the part of the framed oriented boundary of Σ'' that belongs to T_2''. We regard T_3'' as a non-associative tangle with $\mathrm{target}(T_3'') = \mathrm{source}(s(d(T_1'')))$. The invariant $Z(T_3'') = \mathring{A}_0(T_3'')$ depends only on rank $H_1(\Sigma)$. Since we know that the Seifert matrix V is chosen with respect to a basis induced by a standard surface, the definition of the map $\chi_{X''} : \mathcal{A}(\emptyset, X'') \longrightarrow \mathcal{A}(\Gamma(T_1''), \emptyset)$ depends only on V (see Equation (3)). We will show below that for knots L all terms in $\mathring{A}_0(L' \cup T_1'')$ that contain an internal vertex do not contribute to $W_\nabla(\mathring{A}(L))$. Equation (6), Lemma 9 and Lemma 13 then imply

$$\begin{aligned} W_\nabla(\mathring{A}(L)) &= W_\nabla(\mathring{A}_0(L' \cup \partial\Sigma'')) \\ &= W_\nabla(s(d(\mathring{A}_0(L' \cup T_1''))) \circ Z(T_3'')) \\ &= W_\nabla\left(s(d(\chi_{X''}(P_{\mathrm{strut}}(\mathring{A}_0(L' \cup T_1''))))) \circ Z(T_3'')\right) \\ &= W_\nabla\left(s \circ d \circ \chi_{X''}\left(\exp\left(\frac{1}{4}\sum_{i,j\in X''}(v_{ij}+v_{ji})\, {}^{i}\frown^{j}\right)\right) \circ Z(T_3'')\right). \end{aligned}$$

This will show that $W_\nabla(\mathring{A}(L))$ is determined by the Seifert matrix V. Obviously, the coefficients of h^i in $W_\nabla(\mathring{A}(L))$ are polynomials of degree $\leq i$ in the entries of V. This will prove the keylemma.
It remains to consider diagrams D in $\mathring{A}_0(L' \cup T_1'')$ with an internal vertex u. In $s(d(D))$ each of the edges incident to u is either connected to another internal vertex or appears twice, namely as the difference of the two ways of lifting it to the skeleton $\Gamma(s(d(T_1'')))$. We represent this difference by a box in Figure 4.

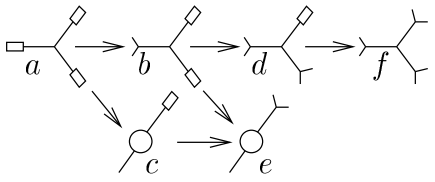

FIGURE 4. Replacing differences of univalent vertices by internal vertices

A neighborhood of the internal vertex u looks like in one of the possibilities (a)-(f) in Figure 4. When we push a lifted vertex in the box along the circle $\Gamma(\partial\Sigma'')$, then it will finally cancel with the second lifted vertex. By the (STU)-relation we can replace a box in Figure 4 by a sum of diagrams with an additional internal vertex. More precisely, a part of the diagram looking like in (a), (b), (c), or (d) in Figure 4 is replaced by a sum of diagrams where a neighborhood of u looks like in diagrams that can be reached by following a directed arrow in Figure 4. When we apply this procedure to all boxes, we will finally end up with possibilities (e) and (f). By Lemma 7 all diagrams that have a subdiagram as in (e) or (f) are sent to 0 by W_∇. □

Let us recall a fact about knots (and links) in S^3 (see Proposition 8.7 of [BuZ]).

Fact 14. *Let V be a $n \times n$-matrix over $\mathbb{Z}$ such that $V - V^*$ is a matrix of the intersection form of a surface. Then V is a Seifert matrix of a link in S^3.*

Since for all Seifert forms s the intersection form of Σ is equal to $s - s^*$, we see that Seifert forms of a fixed surface Σ are a subset of an affine space whose associated $\mathbb{Q}$-vector space are symmetric $\mathbb{Z}$-bilinear forms on $H_1(\Sigma)$ with values in $\mathbb{Q}$. By Fact 14 Seifert forms of Seifert surfaces Σ in S^3 are a lattice of full rank in this affine space.

Proof of Theorem 2. By Proposition 3 and Keylemma 12 the coefficients of h^i in the two power series $\frac{h}{e^{h/2}-e^{-h/2}}\nabla(K)_{|t^{1/2}=e^{h/2}}$ and $W_\nabla \circ \mathring{A}(K)$ only depend on a Seifert matrix V of a knot K and are polynomials p_i and q_i in the entries of V. By Equation (1) we have $p_i(V) = q_i(V)$ for all Seifert matrices of knots in S^3. Fact 14 implies that $p_i = q_i$ for all i. $\square$

The following lemma is a straightforward extension of a result of [GaH].

Lemma 15. *Let K be a 0-framed knot in a rational homology sphere M. Then any of the series $Z^{LMO}(M_K)$, $W_\nabla \circ Z^{LMO}(K)$, $W_\nabla \circ \mathring{A}(K)$ can be computed from any other of these series.*

Sketch of proof. The invariants Z^{LMO} and $\mathring{A}$ of $K \subset M$ differ only by normalization (see Equation (7)). Let $C = Z^{LMO}(K)$. Then we have $Z^{LMO}(M_K) = <\sigma(\nu\#C)>$ with $\nu = Z(O)$. The following four steps show that $W_\nabla(C)$ can be calculated from $Z^{LMO}(M_K)$ and vice versa. This will complete the proof.
1) $W_\nabla(C)$ depends only on the wheel-part $P_{\mathrm{wh}}(C)$ of C (Lemma 8).
2) $P_{\mathrm{wh}}(C)$ can be calculated from $W_\nabla(C)$ because $P_{\mathrm{wh}}(C) = \exp(P)$ where P is a formal series of connected wheels (see Part (3) of Lemma 9), $W_\nabla(C) = \exp(W_\nabla(P))$, and W_∇ is injective on connected wheels (Lemma 8).
3) $\sigma(\nu\#C)$ contains no struts because K is 0-framed (see Lemma 11 and Equation (7)). All remaining non-vanishing diagrams in $\mathcal{A}(\emptyset, \{x\})$ have at least as many internal vertices as univalent vertices. This implies that $Z^{LMO}(M_K)$ depends only on $P_{\mathrm{wh}}(\nu\#C) = P_{\mathrm{wh}}(\nu)\sqcup P_{\mathrm{wh}}(C)$.
4) The map $<\cdot>$ is injective on wheels (see [GaH], Lemma 3.1, use the sl_2-weight system on $\mathcal{A}(\emptyset)$ to see that $<\cdot>$ is injective on connected wheels). Therefore $P_{\mathrm{wh}}(\nu)\sqcup P_{\mathrm{wh}}(C)$ can be calculated from $Z^{LMO}(M_K) = <P_{\mathrm{wh}}(\nu)\sqcup P_{\mathrm{wh}}(C)>$. $P_{\mathrm{wh}}(\nu)$ is invertible. $\square$

Now we prove the main result of this paper.

Proof of Theorem 1. By Lemmas 5 and 15 and by Equation (2) it is sufficient to show that for a null-homotopic knot K in a rational homology sphere each of the invariants $\nabla(K)$ and $W_\nabla \circ \mathring{A}(K)$ can be computed from the other one. This statement follows from Theorem 2. $\square$

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