A COMBINATORIAL APPROACH TO QUANTIFICATION OF LIE ALGEBRAS

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ABSTRACT. We propose a notion of a quantum universal enveloping algebra for an arbitrary Lie algebra defined by generators and relations which is based on the quantum Lie operation concept. This enveloping algebra has a PBW basis that admits the Kashiwara crystalization. We describe all skew primitive elements of the quantum universal enveloping algebra for the classical nilpotent algebras of the infinite series defined by the Serre relations and prove that the set of PBW-generators for each of these enveloping algebras coincides with the Lalonde–Ram basis of the ground Lie algebra with a skew commutator in place of the Lie operation. The similar statement is valid for Hall–Shirshov basis of any Lie algebra defined by one relation, but it is not so in general case.

1. *Introduction*

Quantum universal enveloping algebras appeared in the famous papers by Drinfeld [**14**] and Jimbo [**17**]. Since then a great deal of articles and number of monographs were devoted to their investigation. All of these researches are mainly concerned with a particular quantification of Lie algebras of the classical series. This is accounted for first by the fact that these Lie algebras have applications and visual interpretations in physical speculations, and then by the fact that a general, and commonly accepted as standard, notion of a quantum universal enveloping algebra is not elaborated yet (see a detailed discussion in [**1**, **31**]).

In the present paper we propose a combinatorial solution of this problem by means of the quantum (Lie) operation concept [**21, 23, 24**]. In line with the main idea of our approach, the skew primitive elements must play the same role in quantum enveloping algebras as the primitive elements do in the classical case. By the Friedrichs criteria [**12**, **15**, **30**, **32**, **33**], the primitive elements form the ground Lie algebra in the classical case. For this reason we consider the space spanned by the skew primitive elements and equipped with the quantum operations as a quantum analogue of a Lie algebra.

In the second section we adduce the main notions and consider some examples. These examples, in particular, show that the Drinfeld–Jimbo enveloping algebra as well as its modifications are quantum enveloping algebras in our sense.

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In the third section with the help of the Heyneman–Radford theorem we introduce a notion of a *combinatorial rank* of a Hopf algebra generated by skew primitive semi-invariants. Then we define the quantum enveloping algebra of an *arbitrary rank* that slightly generalises the definitions given in the preceding section.

The basis construction problem for the quantum enveloping algebras is considered in the fourth section. We indicate two main methods for the construction of *PBW-generators*. One of them modifies the Hall–Shirshov basis construction process by means of replacing the Lie operation with a skew commutator. The set of the PBW-generators defined in this way, the values of *hard super-letters*, plays the same role as the basis of the ground Lie algebra does in the PBW theorem. At first glance it would seem reasonable to consider the $\mathbf{k}[G]$-module generated by the values of hard super-letters as a quantum Lie algebra. However, this extremely important module falls far short of being uniquely defined. It essentially depends on the ordering of the main generators, their degrees, and it is almost never antipode stable. Also we have to note the following important fact. Our definition of the hard super-letter is not constructive and, of course, it cannot be constructive in general. The basis construction problem includs the word problem for Lie algebras defined by generators and relations, while the latter one has no general algorithmic solution (see [**4**, **7**]).

The second method is connected with the Kashiwara crystallisation idea [**19**, **20**] (see also a development in [**11**, **25**]). M. Kashiwara has considered the main parameter q of the Drinfeld–Jimbo enveloping algebra as a temperature of some physical medium. When the temperature tend to zero, the medium crystallises. The PBW-generators must crystallise as well. In our case under this process no one limit quantum enveloping algebra appears since the existence conditions normally include equalities of the form $\prod p_{ij} = 1$ (see [**23**]). Nevertheless if we equate all quantification parameters to zero, the hard super-letters would form a new set of PBW-generators for the given quantum universal enveloping algebra. To put this another way, the PBW-basis defined by the super-letters admits the Kashiwara crystallisation.

In the fifth section we bring a way to construct a Groebner–Shirshov relations system for a quantum enveloping algebra. This system is related to the main skew primitive generators, and, according to the Diamond Lemma (see [**3**, **5**, **37**]), it determines the crystal basis. The usefulness of the Groebner–Shirshov systems depends upon the fact that such a system not only defines a basis of an associative algebra, but it also provides a simple diminishing algorithm for expansion of elements on this basis (see, for example [**2**]).

In the sixth section we adapt a well known method of triangular splitting to the *quantification with constants*. The original method appeared in studies of simple finite dimensional Lie algebras. Then it has been extended into the field of quantum algebra in a lot of publications (see, for example [**8**, **29**, **38**]). By means of this method the investigation of the Drinfeld–Jimbo enveloping algebra amounts to a consideration of its positive and negative homogeneous components, *quantum Borel sub-algebras*.

In the seventh section we consider more thoroughly the quantum universal enveloping algebras of nilpotent algebras of the series A_n, B_n, C_n, D_n defined by the

Serre relations. We adduce first lists of all hard super-letters in the explicit form, then Groebner–Shirshov relations systems, and next spaces $L(U_P(\mathfrak{g}))$ spanned by the skew primitive elements (i.e. the Lie algebra quantifications $\mathfrak{g}_P$ proper). In all cases the lists of hard super-letters (but the hard super-letters themselves) turn out to be independent of the quantification parameters. This means that the PBW-generators result from the Hall–Shirshov basis of the ground Lie algebra by replacing the Lie operation with the skew commutator. The same is valid for the Groebner–Shirshov relations systems. Note that the Hall–Shirshov bases, under the name *standard Lyndon bases*, for the classical Lie series were constructed by P. Lalonde and A. Ram [**26**], while the Groebner–Shirshov systems of Lie relations were found by L.A. Bokut' and A.A. Klein [**6**].

Furthermore, in all cases $\mathfrak{g}_P$ as a quantum Lie algebra (in our sense) proves to be very simple in structure. Either it is a coloured Lie super-algebra (provided that the main parameter p_{11} equals 1), or values of all non-unary quantum operations equal zero on $\mathfrak{g}_P$. In particular, if $\mathrm{char}(\mathbf{k}) = 0$ and $p_{11}^t \neq 1$ then the partial quantum operations may be defined on $\mathfrak{g}_P$, but all of them have zero values. Thus, in this case we have a reason to consider $U_P(\mathfrak{g})$ as an algebra of 'commutative' quantum polynomials, since the universal enveloping algebra of a Lie algebra with zero bracket is the algebra of ordinary commutative polynomials. From this standpoint the Drinfeld–Jimbo enveloping algebra is a 'quantum' Weyl algebra of (skew) differential operators. Immediately afterwards a number of interesting questions appears. What is the structure of other algebras of 'commutative' quantum polynomials? Under what conditions are the quantum universal enveloping algebras of homogeneous components of other Kac–Moody algebras defined by the Gabber–Kac relations [**16**] the algebras of 'commutative' quantum polynomials? When do the PBW-generators result from a basis of the ground Lie algebra by means of replacing the Lie operation with the skew commutator? These and other questions we briefly discuss in the last section.

It is as well to bear in mind that the combinatorial approach is not free from flaws: the quantum universal enveloping algebra essentially depends on a combinatorial representation of the ground Lie algebra, i.e. a close connection with the abstract category of Lie algebras is lost.

2. *Quantum enveloping algebras*

Recall that a variable x is called a quantum variable if an element g_x of a fixed Abelian group G and a character $\chi^x \in G^*$ are associated with it. A noncommutative polynomial in quantum variables is called a *quantum operation* if all of its values in all Hopf algebras are skew primitive provided that every variable x has a value $x = a$ such that

$$\Delta(a) = a \otimes 1 + g_x \otimes a, \quad g^{-1}ag = \chi^x(g)a, \quad g \in G. \tag{1}$$

Let $x_1, \ldots, x_n$ be a set of quantum variables. For each word u in $x_1, \ldots, x_n$ we denote by g_u an element of G that appears from u by replacing of all x_i with g_{x_i}. In the same way we denote by χ^u a character that appears from u by replacing of all x_i with χ^{x_i}. Thus on the free algebra $\mathbf{k}\langle x_1, \dots, x_n\rangle$ a grading by the group $G\times G^*$ is defined. For each pair of homogeneous elements u, v we fix the denotations $p_{uv} = \chi^u(g_v) = p(u, v)$.

The quantum operation can be defined equivalently as a $G\times 1$-homogeneous polynomial that has only primitive values in all *braided bigraded Hopf algebras* provided that all quantum variables have primitive homogeneous values $g_a = g_x$, $\chi^a = \chi^x$ (see [**21**, Sect. 1–4]).

Recall that a *constitution* of a word u is a sequence of non-negative integers $(m_1, m_2, \dots, m_n)$ such that u is of degree m_1 in x_1, $\deg_1(u) = m_1$; of degree m_2 in x_2, $\deg_2(u) = m_2$; and so on. Since the group G is abelian, all constitution homogeneous polynomials are homogeneous with respect to the grading. Let us define a bilinear skew commutator on the set of graded homogeneous noncommutative polynomials by the formula

$$[u, v] = uv - p_{uv}vu. \tag{2}$$

These brackets satisfy the following Jacobi and skew differential identities:

$$[[u, v], w] = [u, [v, w]] + p_{wv}^{-1}[[u, w], v] + (p_{vw} - p_{wv}^{-1})[u, w]\cdot v; \tag{3}$$

$$[[u, v], w] = [u, [v, w]] + p_{vw}[[u, w], v] + p_{uv}(p_{vw}p_{wv} - 1)v\cdot [u, w]; \tag{4}$$

$$[u, v\cdot w] = [u, v]\cdot w + p_{uv}v\cdot [u, w]; \quad [u\cdot v, w] = p_{vw}[u, w]\cdot v + u\cdot [v, w], \tag{5}$$

where by the dot we denote the usual multiplication. It is easy to see that the following conditional *restricted identities* are valid as well

$$[u, v^n] = [\dots[[u, v], v]\dots, v]; \quad [v^n, u] = [v, [\dots [v, u]\dots]], \tag{6}$$

provided that p_{vv} is a primitive t-th root of unit, and $n = t$ or $n = tl^k$ in the case of characteristic $l > 0$.

Suppose that a Lie algebra $\mathfrak{g}$ is defined by the generators $x_1, \dots, x_n$ and the relations $f_i = 0$. Let us convert the generators into quantum variables. For this associate to them elements of $G\times G^*$ in arbitrary way. Let $P = ||p_{ij}||$, $p_{ij} = \chi^{x_i}(g_{x_j})$ be the *quantification matrix*.

Definition 2.1. A *braided quantum enveloping algebra* is a braided bigraded Hopf algebra $U_P^b(\mathfrak{g})$ defined by the variables $x_1, \dots, x_n$ and the relations $f_i = 0$, where the Lie operation is replaced with (2), provided that in this way f_i are converted into the quantum operations f_i^*. The coproduct and the commutation relations in the tensor product are defined by

$$\Delta(x_i) = x_i\underline{\otimes}1 + 1\underline{\otimes}x_i, \tag{7}$$

$$(x_i\underline{\otimes}x_j)\cdot(x_k\underline{\otimes}x_m) = (\chi^{x_k}(g_{x_j}))^{-1}x_ix_k\underline{\otimes}x_jx_m. \tag{8}$$

Definition 2.2. A *simple quantification* or a *quantum universal enveloping algebra* of $\mathfrak{g}$ is an algebra $U_P(\mathfrak{g})$ that is isomorphic to the skew group algebra

$$U_P(\mathfrak{g}) = U_P^b(\mathfrak{g}) * G, \tag{9}$$

where the group action and the coproduct are defined by

$$g^{-1}x_ig = \chi^{x_i}(g)x_i,\ \Delta(x_i) = x_i \otimes 1 + g_{x_i} \otimes x_i,\ \Delta(g) = g \otimes g. \tag{10}$$

Definition 2.3. A *quantification with constants* is a simple quantification where additionally some generators x_i associated to the trivial character are replaced with the constants $\alpha_i(1 - g_{x_i})$.

The formulae (10) and (7) correctly define the coproduct since by definition of the quantum operation $\Delta(f_i^*) = f_i^* \otimes 1 + g \otimes f_i^*$ in the case of ordinary Hopf algebras and $\Delta(f_i^*) = f_i^* \underline{\otimes} 1 + 1 \underline{\otimes} f_i^*$ in the braided case.

We have to note that the defined quantifications essentially depend on the combinatorial representation of the Lie algebra. For example, an additional relation $[x_1, x_1] = 0$ does not change the Lie algebra. At the same time if $\chi^{x_1}(g_1) = -1$ then this relation admits the quantification and yields a nontrivial relation for the quantum enveloping algebra, $2x_1^2 = 0$.

Example 1. Suppose that the Lie algebra is defined by a system of constitution homogeneous relations. If the characters χ^i are such that $p_{ij}p_{ji} = 1$ for all i, j then the skew commutator itself is a quantum operation. Therefore on replacing the Lie operation all relations become quantum operations as well. This means that the braided enveloping algebra is the universal enveloping algebra $U(\mathfrak{g}^{col})$ of the coloured Lie super-algebra which is defined by the same relations as the given Lie algebra is. The simple quantification appears as the Radford biproduct $U(\mathfrak{g}^{col}) \star \mathbf{k}[G]$ or, equivalently, as the universal G-enveloping algebra of the coloured Lie super-algebra $\mathfrak{g}^{col}$ (see [**35**] or [**21**, Example 1.9]).

Example 2. Suppose that the Lie algebra $\mathfrak{g}$ is defined by the generators $x_1, \dots, x_n$ and the system of nil relations

$$x_j(adx_i)^{n_{ij}} = 0, \quad 1 \leq i \neq j \leq n. \tag{11}$$

Usually instead of the matrix of degrees (without the main diagonal) $||n_{ij}||$ the matrix $A = ||a_{ij}||$, $a_{ij} = 1 - n_{ij}$ is considered. The Coxeter graph $\Gamma(A)$ is associated to every such a matrix. This graph has the vertices $1, \dots, n$, where the vertex i is connected by $a_{ij}a_{ji}$ edges with the vertex j.

If $a_{ij} = 0$ then the relation $x_j \mathrm{ad} x_i = 0$ is in the list (11), and the relation $x_i(\mathrm{ad}x_j)^{n_{ji}} = 0$ is a consequence of it. The skew commutator $[x_j, x_i]$ is a quantum operation if and only if $p_{ij}p_{ji} = 1$. Under this condition we have $[x_i, x_j] = -p_{ij}[x_j, x_i]$. Therefore both in the given Lie algebra and in its quantification one may replace the relation $x_i(\mathrm{ad}x_j)^{n_{ji}} = 0$ with $x_i \mathrm{ad} x_j = 0$. In other words, without loss of generality, we may suppose that $a_{ij} = 0 \leftrightarrow a_{ji} = 0$. By the Gabber-Kac theorem [**16**] we get that the algebra $\mathfrak{g}$ is the positive homogeneous component $\mathfrak{g}_1^+$ of a Kac-Moody algebra $\mathfrak{g}_1$.

Theorem 6.1 [**21**] describes the conditions for a homogeneous polynomial in two variables which is linear in one of them to be a quantum operation. From this theorem we have the following corollary.

COROLLARY 2.4. *If n_{ij} is a simple number or unit and in the former case p_{ii} is not a primitive n_{ij}-th root of unit, then the relation* (11) *admits a quantification if and only if $p_{ij}p_{ji} = p_{ii}^{a_{ij}}$.*

Theorem 6.1 [**21**] provides no essential restrictions on the non-diagonal parameters p_{ij} : if the matrix P correctly defines a quantification of (11) then for every set $Z = \{z_{ij} | z_{ij} z_{ji} = z_{ii} = 1\}$ the following matrix does as well:

$$P_Z = \{p_{ij} z_{ij} | p_{ij} \in P, z_{ij} \in Z\}. \tag{12}$$

EXAMPLE 3. Let G be freely generated by $g_1, \dots g_n$ and A be a generalised Cartan matrix symmetrised by $d_1, \dots, d_n$, while the characters are defined by $p_{ij} = q^{-d_i a_{ij}}$. In this case the simple quantification is the positive component of the Drinfeld–Jimbo enveloping algebra together with the group-like elements, $U_P(\mathfrak{g}) = U_q^+(\mathfrak{g}) * G$. By means of an arbitrary deformation (12) one may define a 'colouring' of $U_q^+(\mathfrak{g}) * G$.

The braided enveloping algebra equals $U_q^+(\mathfrak{g})$ where the coproduct and braiding are defined by (7) and (8) with the coefficient $q^{d_k a_{kj}}$. The formula (12) correctly defines its 'colouring' as well.

EXAMPLE 4. If in the example above we complete the set of quantum variables by the new ones $x_1^-, \dots, x_n^-; z_1, \dots, z_n$ such that

$$\chi^{x^-} = (\chi^x)^{-1}, \quad g_{x^-} = g_x, \quad \chi^z = \mathrm{id}, \quad g_{z_i} = g_i^2, \tag{13}$$

then by [**21**, Theorem 6.1] the Gabber–Kac relations (2), (3), and $[e_i, f_j] = \delta_{ij} h_i$ (see [**16**, Theorem 2]) under the identification $e_i = x_i$, $f_i = x_i^-$, $h_i = z_i$ admit the quantification with constants $z_i = \varepsilon_i (1 - g_i^2)$. (Unformally we may consider the obtained quantification as one of the Kac–Moody algebra identifying g_i with q^{h_i}, where the rest of the Kac–Moody algebra relations, $[h_i, e_j] = a_{ij} e_i$, $[h_i, f_j] = -a_{ij} f_j$, is quantified to the G-action: $g_j^{-1} x_i^{\pm} g_j = q^{\mp d_{ij} a_{ij}} x_i^{\pm}$.) This quantification coincides with the Drinfeld–Jimbo one under a suitable choice of x_i, x_i^-, and ε_i depending up the particular definition of $U_q(\mathfrak{g})$:

[**28**]	$x_i = E_i,\ g_i = K_i,\ x_i^- = F_i K_i,\ p_{ij} = v^{-d_i a_{ij}},\ \varepsilon_i = (v^{-d_i} - v^{d_i})^{-1};$
[**29**]	$x_i = E_i,\ g_i = \tilde{K}_i,\ x_i^- = F_i \tilde{K}_i,\ p_{i\mu} = v^{-\langle \mu, i' \rangle},\ \varepsilon_i = (v_i^{-1} - v_i)^{-1};$
[**19**]Δ_+	$x_i = e_i,\ g_i = t_i,\ x_i^- = t_i f_i,\ p_{ij} = q_j^{-\langle h_j, \alpha_i \rangle},\ \varepsilon_i = (q_i - q_i^3)^{-1};$
[**19**]Δ_-	$x_i = f_i,\ g_i = t_i,\ x_i^- = e_i t_i,\ p_{ij} = q_j^{\langle h_j, \alpha_i \rangle},\ \varepsilon_i = (q_i^{-1} - q_i)^{-1};$
[**34**]	$x_i = E_i K_i,\ g_i = K_i^2,\ x_i^- = F_i K_i,\ p_{ij} = q^{-2 d_i a_{ij}},\ \varepsilon_i = (1 - q^{4 d_i})^{-1}.$

By (13) the brackets $[x_i, x_j^-]$ are quantum operations only if $p_{ij} = p_{ji}$. So in this case the 'colourings' (12) may be only black-white, $z_{ij} = \pm 1$.

In the perfect analogy the Kang quantification [**18**] of the generalised Kac-Moody algebras [**9**] is a quantification in our sense as well.

3. *Combinatorial rank*

By the above definitions the quantum enveloping algebras (with or without constants) are character Hopf algebras (see [**21**, Definition 1.2]). In this section by means of a combinatorial rank notion we identify the quantum enveloping algebras in the class of character Hopf algebras.

Let H be a character Hopf algebra generated by $a_1, \dots, a_n$:

$$\Delta(a_i) = a_i \otimes 1 + g_{a_i} \otimes a_i, \qquad g^{-1} a_i g = \chi^{a_i}(g) a_i, \quad g \in G. \tag{14}$$

Let us associate a quantum variable x_i with the parameters $(\chi^{a_i},\, g_{a_i})$ to a_i. Denote by $G\langle X\rangle$ the free enveloping algebra defined by the quantum variables $x_1 \dots, x_n$. (see [**21**, Sec. 3] under denotation $H\langle X\rangle$).

The map $x_i \to a_i$ has an extension to a homomorphism of Hopf algebras $\varphi : G\langle X\rangle \to H$. Denote by I the kernel of this homomorphism. If $I \neq 0$ then by the Heyneman–Radford theorem (see, for example [**34**, pages 65–71]), the Hopf ideal I has a non-zero skew primitive element. Let I_1 be an ideal generated by all skew primitive elements of I. Clearly I_1 is a Hopf ideal as well. Now consider the Hopf ideal I/I_1 of the quotient Hopf algebra $G\langle X\rangle/I_1$. This ideal also has non-zero skew primitive elements (provided $I_1 \neq I$). Denote by I_2/I_1 the ideal generated by all skew primitive elements of I/I_1, where I_2 is its preimage with respect to the projection $G\langle X\rangle \to G\langle X\rangle/I_1$. Continuing the process we will find a strictly increasing, finite or infinite, chain of Hopf ideals of $G\langle X\rangle$:

$$0 = I_0 \subset I_1 \subset I_2 \subset \ldots \subset I_n \subset \ldots . \tag{15}$$

Definition 3.1. The length of (15) is called a *combinatorial rank* of H.

By definition, the combinatorial rank of any quantum enveloping algebra (with constants) equals one. In the case of zero characteristic the inverse statement is valid as well.

Theorem 3.2. *Each character Hopf algebra of the combinatorial rank 1 over a field of zero characteristic is isomorphic to a quantum enveloping algebra with constants of a Lie algebra.*

Proof. By definition, I is generated by skew primitive elements. These elements as noncommutative polynomials are the quantum operations. Consider one of them, say f. Let us decompose f into a sum of homogeneous components $f = \sum f_i$. All positive components belongs to $\mathbf{k}\langle X\rangle$ and they are the quantum operations themselves, while the constant component has the form $\alpha(1-g), g \in G$ (see [**21**, Sec. 3 and Prop. 3.3]). If $\alpha \neq 0$ then we introduce a new quantum variable z_f with the parameters (id, g). Each f_i has a representation through the skew commutator. Indeed, by [**21**, Theorem 7.5] the complete linearization f_i^{lin} of f_i has the required representation. By the identification of variables in a suitable way in f_i^{lin} we get the required representation for f_i multiplied by a natural number, $m_i f_i = f_i^{[]}$.

Now consider a Lie algebra $\mathfrak{g}$ defined by the generators x_i, z_f and the relations $\sum m_i^{-1} f_i^{[]} + z_f = 0$, with the Lie multiplication in place of the skew commutator. It is clear that H is the quantification with constants of $\mathfrak{g}$.

In the same way one may introduce the notion of the combinatorial rank for the braided bigraded Hopf algebras. In this case all braided quantum enveloping algebras are of rank 1, and all braided bigraded algebras of rank 1 are the braided quantification of some Lie algebras.

Now we are ready to define a quantification of *arbitrary rank*. For this in the definitions of the above section it is necessary to change the requirement that all f_i^* are quantum operations with the following condition.

The set F splits in a union $F=\cup_{j=1}^n F_j$ such that F_1^* consists of quantum operations; the set F_2^* consists of skew primitive elements of $G\langle X||F_1^*\rangle$; the set F_3^* consists of skew primitive elements of $G\langle X||F_1^*, F_2^*\rangle$, and so on.

The quantum enveloping algebras of an arbitrary rank are character Hopf algebras also. But it is not clear if any character Hopf algebra is a quantification of some rank of a suitable Lie algebra. It is so if the Hopf algebra is homogeneous and the ground field has a zero characteristic (to appear). Also it is not clear if there exist character Hopf algebras, or braided bigraded Hopf algebras, of infinite combinatorial rank; while it is easy to see that $\cup I_n = I$. Also it is possible to show that F_1 always contains all relations of a minimal constitution in F. For example, each of (11) is of a minimal constitution in (11). Therefore the quantification of arbitrary rank with the identification $g_i=\exp(h_i)$ of any (generalized) Kac–Moody algebra $\mathfrak{g}$, or its nilpotent component $\mathfrak{g}^+$, is always a quantification in the sense of the above section.

4. *PBW-generators and crystallisation*

The next result yields a PBW basis for the quantum enveloping algebras.

THEOREM 4.1. *Every character Hopf algebra H has a linearly ordered set of constitution homogeneous elements $U=\{u_i \mid i\in I\}$ such that the set of all products $gu_1^{n_1}u_2^{n_2}\cdots u_m^{n_m}$, where $g\in G$, $u_1<u_2<\ldots<u_m$, $0\leq n_i<h(i)$ forms a basis of H. Here if $p_{ii}\stackrel{df}{=}p_{u_iu_i}$ is not a root of unity then $h(i)=\infty$; if $p_{ii}=1$ then either $h(i)=\infty$ or $h(i)=l$ is the characteristic of the ground field; if p_{ii} is a primitive t-th root of unity, $t\neq 1$, then $h(i)=t$.*

The set U is referred to as a set of *PBW-generators* of H. This theorem easily follows from [**22**, Theorem 2]. Let us recall necessary notions.

Let $a_1,\ldots,a_n$ be a set of skew primitive generators of H, and let x_i be the associated quantum variables. Consider the lexicographical ordering of all words in $x_1>x_2>\ldots>x_n$. A non-empty word u is called *standard* if $vw>wv$ for each decomposition $u=vw$ with non-empty v,w. The following properties are well known (see, for example [**10**, **13**, **27**, **36**, **37**]).

1s. A word u is standard if and only if it is greater than each of its ends.

2s. Every standard word starts with a maximal letter that it has.

3s. Each word c has a unique representation $c=u_1^{n_1}u_2^{n_2}\cdots u_k^{n_k}$, where $u_1<u_2<\cdots<u_k$ are standard words (the Lyndon theorem).

4s. If u,v are different standard words and u^n contains v^k as a sub-word, $u^n=cv^kd$, then u itself contains v^k as a sub-word, $u=bv^ke$.

The set of *standard nonassociative* words is defined as the smallest set SL that contains all variables x_i and satisfies the following properties.

1) If $[u]=[[v][w]]\in SL$ then $[v],[w]\in SL$, and $v>w$ are standard.

2) If $[u]=[[[v_1][v_2]][w]]\in SL$ then $v_2\leq w$.

The following statements are valid as well.

5s. Every standard word has the only alignment of brackets such that the appeared nonassociative word is standard (the Shirshov theorem [**36**]).

6s. The factors v, w of the nonassociative decomposition $[u] = [[v][w]]$ are the standard words such that $u = vw$ and v has the minimal length ([**37**]).

DEFINITION 4.2. A *super-letter* is a polynomial that equals a nonassociative standard word where the brackets mean (2). A *super-word* is a word in super-letters. By 5s every standard word u defines a super-letter $[u]$.

Let D be a linearly ordered Abelian additive group. Suppose that some positive D-degrees $d_1, \ldots, d_n \in D$ are associated to $x_1, \ldots, x_n$. We define the degree of a word to be equal to $m_1 d_1 + \ldots + m_n d_n$ where $(m_1, \ldots, m_n)$ is the constitution of the word. The order and the degree on the super-letters are defined in the following way: $[u] > [v] \iff u > v$; $\mathrm{D}([u]) = \mathrm{D}(u)$.

DEFINITION 4.3. A super-letter $[u]$ is called *hard in H* provided that its value in H is not a linear combination of values of super-words of the same degree in less than $[u]$ super-letters and G-super-words of a lesser degree.

DEFINITION 4.4. We say that a *height* of a super-letter $[u]$ of degree d equals $h = h([u])$ if h is the smallest number such that: first p_{uu} is a primitive t-th root of unity and either $h = t$ or $h = tl^r$, where $l = \mathrm{char}(\mathbf{k})$; and then the value in H of $[u]^h$ is a linear combination of super-words of degree hd in less than $[u]$ super-letters and G-super-words of a lesser degree. If there exists no such number then the height equals infinity.

Clearly, if the algebra H is D-homogeneous then one may omit the underlined parts of the above definitions.

THEOREM 4.5. ([**22**, Theorem 2]). *The set of all values in H of all G-super-words W in the hard super-letters $[u_i]$,*

$$W = g[u_1]^{n_1}[u_2]^{n_2} \cdots [u_m]^{n_m}, \tag{16}$$

where $g \in G$, $u_1 < u_2 < \ldots < u_m$, $n_i < h([u_i])$ is a basis of H.

In order to find the set of PBW-generators it is necessary first to include in U the values of all hard super-letters, then for each hard super-letter $[u]$ of a finite height, $h([u]) = tl^k$, to add the values of $[u]^t, [u]^{tl}, \ldots [u]^{tl^{(k-1)}}$, and next for each hard super-letter of infinite height such that p_{uu} is a primitive t-th root of unity to add the value of $[u]^t$.

Obviously the set of PBW-generators plays the same role as the basis of the Lie algebra in the PBW theorem does. Nevertheless the $\mathbf{k}[G]$-bimodule generated by the PBW-generators is not uniquely defined. It depends on the ordering of the main generators, the D-degree, and under the action of antipode it transforms to a different bimodule of PBW-generators $\mathbf{k}[G]S(U)$.

Another way to construct PBW-generators is connected with the M.Kashiwara crystallisation idea [**19**, **20**]. M.Kashiwara considered the main parameter of the Drinfeld–Jimbo enveloping algebra as the temperature of some physical medium.

When the temperature tends to zero the medium crystallises. By this means the 'crystal' bases must appear. If we replace p_{ij} with zero then $[u,v]$ turns into uv, while $[u]$ turns into u.

LEMMA 4.6. (Bases Crystallisation). *Under the above crystallisation the set of PBW-generators constructed in Theorem 4.5 turns into another set of PBW-generators.*

Proof. See [**22**, Corollary 1].

LEMMA 4.7. (Super-letters Crystallisation). *A super-letter $[u]$ is hard in H if and only if the value of u is not a linear combination of lesser words of the same degree and G-words of a lesser degree.*

Proof. See [**22**, Corollary 2].

LEMMA 4.8. *Let B be a set of the super-letters containing $x_1, \dots, x_n$. If each pair $[u],[v] \in B$, $u > v$ satisfies one of the following conditions*
1) *$[[u][v]]$ is not a standard nonassociative word;*
2) *the super-letter $[[u][v]]$ is not hard in H;*
3) *$[[u][v]] \in B$,*

then the set B includes all hard in H super-letters.

Proof. Let $[w]$ be a hard super-letter of minimal degree such that $[w] \notin B$. Then $[w] = [[u][v]], u > v$ where $[u]$, $[v]$ are hard super-letters. Indeed, if $[u]$ is not hard then by Lemma 4.7 we have $u = \sum \alpha_i u_i + S$, where $u_i < u$ and $D(u_i) = D(u)$, $D(S) < D(u)$. We have $uv = \sum \alpha_i u_i v + Sv$, where $u_i v < uv$. Therefore by Lemma 4.7, the super-letter $[w] = [uv]$ can not be hard in H. Contradiction. Similarly, if $[v]$ is not hard then $v = \sum \alpha_i v_i + S$, $v_i < v$, $D(v_i) = D(v)$, $D(S) < D(v)$. Therefore $uv = \sum \alpha_i uv_i + uS$, $uv_i < uv$, and again $[w]$ can not be hard.

Thus, according to the choice of $[w]$, we get $[u],[v] \in B$. Since this pair satisfies neither condition 1) nor 2), the condition 3), $[uv] \in B$, holds. □

LEMMA 4.9. *If $\mathbf{T} \in H$ is a skew primitive element then*

$$\mathbf{T} = [u]^h + \sum \alpha_i W_i + \sum \beta_j g_j W_j'$$

where $[u]$ is a hard super-letter, W_i are basis super-words in super-letters less than $[u]$, $D(W_i) = hD([u])$, $D(W_j') < hD([u])$. Here if p_{uu} is not a root of unity then $h = 1$; if p_{uu} is a primitive t-th root of unity then $h = 1$, or $h = t$, or $h = tl^k$, where l is the characteristic.

Proof. Consider an expansion of $\mathbf{T}$ in terms of the basis (16)

$$\mathbf{T} = \alpha g U + \sum_{i=1}^{k} \gamma_i g_i W_i + W', \quad \alpha \neq 0, \tag{17}$$

where gU, g_iW_i are different basis elements of maximal degree, and U is one of the biggest words among U, W_i with respect to the lexicographic ordering of words in the super-letters. On basis expansion of tensors, the element $\Delta(\mathbf{T}) - \mathbf{T} \otimes 1 - g_t \otimes \mathbf{T}$ has only one tensor of the form $gU \otimes \dots$ and this tensor equals $gU \otimes \alpha(g-1)$. Therefore $g = 1$ and one may apply [**22**, Lemma 13]. □

5. *Groebner–Shirshov relations systems*

Let $x_1, \dots, x_n$ be variables that have positive degrees $d_1, \dots, d_n \in D$. Recall that a *Hall ordering* of words in $x_1, \dots, x_n$ is an order when the words are compared firstly by the degree and then words of the same degree are compared by means of the lexicographic ordering . Consider a set of relations

$$w_i = f_i, \quad i \in I, \tag{18}$$

where w_i is a word and f_i is a linear combination of Hall lesser words. The system (18) is said to be *closed under compositions* or a *Groebner–Shirshov relations system* if first none of w_i contains w_j, $i \neq j \in I$ as a sub-word, and then for each pair of words w_k, w_j such that some non-empty terminal of w_k coincides with an onset of w_j, that is $w_k = w_k'v$, $w_j = vw_j'$, the difference (a *composition*) $f_k w_j' - w_k' f_j$ can be reduced to zero *in the free algebra* by means of a sequence of one sided substitutions $w_i \to f_i$, $i \in I$.

LEMMA 5.1. (Diamond Lemma [**3**, **5**, **37**]). *If the system* (18) *is closed under compositions then the words that have none of w_i as sub-words form a basis of the algebra H defined by* (18).

If none of the words w_i has sub-words w_j, $j \neq i$, then the converse statement is valid as well. Indeed, any composition by means of substitutions $w_i \to f_i$ can be reduced to a linear combination of words that have no sub-words w_i. Since $f_i w_j' - w_i' f_j = (f_i - w_i)w_j' - w_i'(f_j - w_j)$, this linear combination equals zero in H. Therefore all the coefficients have to be zero.

Since Bases Crystallisation Lemma provides the basis that consists of words, the above note gives a way to construct the Groebner–Shirshov relations system for any quantum enveloping algebra.

Let H be a character Hopf algebra generated by skew primitive semi-invariants $a_1, \dots, a_n$ (or a braided bigraded Hopf algebra generated by graiding homogeneous primitive elements $a_1, \dots, a_n$) and let $x_1, \dots, x_n$ be the related quantum variables. A non-hard in H super-letter $[w]$ is referred to as a *minimal* one if first w has no proper standard sub-words that define non-hard super-letters, and then w has no sub-words u^h, where $[u]$ is a hard super-letter of the height h.

By the Super-letters Crystallisation Lemma, for every minimal non-hard in H super-letter $[w]$ we may write a relation in H

$$w = \sum \alpha_i w_i + \sum \beta_j g_j w_j, \tag{19}$$

where w_j, $w_i < w$ in the Hall sense, $D(w_i) = D(w)$, $D(w_j) < D(w)$. In the same way if $[u]$ is a hard in H super-letter of a finite height h then

$$u^h = \sum \alpha_i u_i + \sum \beta_j g_j u_j, \tag{20}$$

where u_j, $u_i < u^h$ in the Hall sense, $D(u_i) = hD(u)$, $D(u_j) < hD(u)$. The relations (14) and the group operation provide the relations

$$x_i g = \chi^{x_i}(g) g x_i, \quad g_1 g_2 = g_3. \tag{21}$$

Theorem 5.2. *The set of relations* (19), (20), *and* (21) *forms a Groebner–Shirshov system that defines H. The basis determined by this system in Diamond Lemma coincides with the crystal basis.*

Proof. The property 4s implies that none of the left hand sides of (19), (20), (21) contains another one as a sub-word. Therefore by the Bases Crystallisation Lemma it is sufficient to show that the set of all words c determined in the Diamond Lemma coincides with the crystal basis. By 3s we have $c = u_1^{n_1} u_2^{n_2} \cdots u_k^{n_k}$, where $u_1 < \ldots < u_k$ is a sequence of standard words. Every word u_i define a hard super-letter $[u_i]$ since in the opposite case u_i, and therefore c, contains a sub-word w that defines a minimal non-hard super-letter $[w]$. In the same way n_i does not exceed the height of $[u_i]$. □

Lemma 5.3. *In terms of Lemma* 4.8 *the set of all super-letters* $[[u][v]]$ *that satisfy the condition* 2) *contains all minimal non-hard super-letters, but non-hard generators* x_i.

Proof. If $[w]$ is a minimal non-hard super-letter then $[w] = [[u][v]]$, where $[u]$, $[v]$ are hard super-letters. By Lemma 4.8 we have $[u], [v] \in B$, while $[[u][v]]$ neither satisfies 1) nor 3). □

6. *Quantification with constants*

By means of the Diamond Lemma in some instances the investigation of a quantification with constants can be reduced to one of a simple quantification.

Let $H_1 = \langle x_1, \ldots, x_k || F_1\rangle$ be a character Hopf algebra defined by the quantum variables $x_1, \ldots, x_k$ and the grading homogeneous relations $\{f = 0 : f \in F_1\}$, while $H_2 = \langle x_{k+1}, \ldots, x_n || F_2\rangle$ is a character Hopf algebra defined by the quantum variables $x_{k+1}, \ldots, x_n$ and the grading homogeneous relations $\{h = 0 : h \in F_2\}$. Consider the algebra $H = \langle x_1, \ldots, x_n || F_1, F_2, F_3\rangle$, where F_3 is the following system of relations with constants

$$[x_i, x_j] = \alpha_{ij}(1 - g_i g_j), \quad 1 \leq i \leq k < j \leq n. \tag{22}$$

If the conditions below are met then the character Hopf algebra structure on H is uniquely determined:

$$p_{ij}p_{ji} = 1,\ 1 \leq i \leq k < j \leq n; \quad \chi^{x_i}\chi^{x_j} \neq 1 \Longrightarrow \alpha_{ij} = 0. \tag{23}$$

Indeed, in this case the difference w_{ij} between the left and right hand sides of (22) is a skew primitive semi-invariant of the free enveloping algebra $G\langle x_1, \ldots, x_n\rangle$. Consider the ideals of relations $I_1 = \mathrm{id}(F_1)$ and $I_2 = \mathrm{id}(F_2)$ of H_1 and H_2 respectively. They are, in the present context, Hopf ideals of $G\langle x_1, \ldots, x_k\rangle$ and $G\langle x_{k+1}, \ldots, x_n\rangle$, respectively. Therefore $V = I_1 + I_2 + \sum \mathbf{k} w_{ij}$ is an antipode stable coideal of $G\langle X\rangle$. Consequently the ideal generated by V is a Hopf ideal. It remains to note that this ideal is generated in $G\langle X\rangle$ by w_{ij} and F_1, F_2.

Lemma 6.1. *Every hard in H super-letter belongs to either H_1 or H_2, and it is hard in the related algebra.*

Proof. If a standard word contains at least one of the letters x_i, $i \leq k$ then it has to start with one of them (see s2). If this word contains a letter x_j, $j > k$ then it has a sub word of the form x_ix_j, $i \le k < j$. Therefore by Lemma 4.7 and relations (22) this word defines a non-hard super-letter. $\square$

The converse statement is not universally true. In order to formulate the necessary and sufficient conditions let us define partial skew derivatives:

$$\begin{aligned} &(x_j)'_i = (x_i)'_j = \alpha_{ij}(1 - g_ig_j), \quad i \le k < j; \\ &(v \cdot w)'_i = (v)'_i \cdot w + p(x_i, v)v \cdot (w)'_i, \ i \le k, \ v, w \in G\langle x_{k+1}, \dots , x_n\rangle; \\ &(u \cdot v)'_j = p(v, x_j)(u)'_j \cdot v + u \cdot (v)'_j, \ j > k, \ u, v \in G\langle x_1, \dots , x_k\rangle. \end{aligned} \tag{24}$$

Lemma 6.2. *All hard in H_1 or H_2 super-letters are hard in H if and only if $(h)'_i = 0$ in H_2 for all $i \le k$, $h \in F_2$ and $(f)'_j = 0$ in H_1 for all $j > k$, $f \in F_1$. If these conditions are met then*

$$H \cong H_2 \otimes_{\mathbf{k}[G]} H_1 \tag{25}$$

as $\mathbf{k}[G]$-bimodules and the space generated by the skew primitive elements of H equals the sum of these spaces for H_1 and H_2.

Proof. By (5) and (24) the following equalities are valid in H :

$$0 = [x_i, h] = (h)'_i; \quad 0 = [f, x_j] = (f)'_j, \ i \le k < j. \tag{26}$$

If all hard in H_1 or H_2 super-letters are hard in H then H_1, H_2 are sub-algebras of H. So (26) proves the necessity of the lemma conditions.

Conversely. Let us consider an algebra R defined by the generators $g \in G$, $x_1, \dots , x_n$ and the relations (21), (22). Evidently this system is closed under the compositions. Therefore by Diamond Lemma the set of words gvw forms a basis of R where $g \in G$; v is a word in x_j, $j > k$; and w is a word in x_i, $i \le k$. In other words R as a bimodule over $\mathbf{k}[G]$ has a decomposition

$$R = G\langle x_{k+1}, \dots , x_n\rangle \otimes_{\mathbf{k}[G]} G\langle x_1, \dots , x_k\rangle. \tag{27}$$

Let us show that the two sided ideal of R generated by F_2 coincides with the right ideal $I_2R = I_2 \otimes_{\mathbf{k}[G]} G\langle x_1, \dots , x_k\rangle$. It will suffice to show that I_2R admits left multiplication by x_i, $i \le k$. If v is a word in $x_{k+1}, \dots , x_k$, $h \in F_2$, $r \in R$ then $x_ivhr = [x_i, vh]r + p(x_i, vh)vhx_ir$. The second term belongs to I_2R, while the first one can be rewritten by (5): $[x_i, v]h + p(x_i, v)v[x_i, h]$. Both of these addends belong to I_2R since $[x_i, v] = (v)'_i \in G\langle x_{k+1}, \dots , x_n\rangle$ and $[x_i, h] = (h)'_i \in I_2$.

Furthermore, consider a quotient algebra $R_1 = R/I_2R$:

$$R_1 = (G\langle x_{k+1}, \dots , x_n\rangle \otimes_{\mathbf{k}[G]} G\langle x_1, \dots , x_k\rangle)/(I_2 \otimes_{\mathbf{k}[G]} G\langle x_1, \dots , x_k\rangle) =$$

$$H_2 \otimes_{\mathbf{k}[G]} G\langle x_1, \dots , x_k\rangle,$$

where the equality means the natural isomorphism of $\mathbf{k}[G]$-bimodules.

Along similar lines, the left ideal $R_1I_1 = H_2 \otimes_{\mathbf{k}[G]} I_1$ of this quotient algebra coincides with the two sided ideal generated by F_1. Therefore

$$H = R_1/R_1I_1 = H_2 \otimes_{\mathbf{k}[G]} G\langle x_1, \dots , x_k\rangle/H_2 \otimes_{\mathbf{k}[G]} I_1 = H_2 \otimes_{\mathbf{k}[G]} H_1.$$

Thus the monotonous restricted G-words in hard in H_1 or H_2 super-letters form a basis of H. This, in particular, proves the first statement.

Now let $T=\sum \alpha_t g_t V_t W_t$ be the basis decomposition of a skew primitive element, $g_t\in G$, $V_t\in H_2$, $W_t\in H_1$, $\alpha_t\neq 0$. We have to show that for each t one of the super-words V_t or W_t is empty. Suppose that it is not so. Among the addends with non-empty V_t, W_t we choose the largest one in the Hall sense, say $g_sV_sW_s$. Under the basis decomposition of $\Delta(T)-T\otimes 1-g(T)\otimes T$ the term $\alpha_s g_s g(V_s)W_s\otimes g_sV_s$ appears and cannot be cancelled with other. Indeed, since the coproduct is homogeneous (see [**22**, Lemma 9]) and since under the basis decomposition the super-words are decreased (see [**22**, Lemma 7]) the product $\alpha_s(g_s\otimes g_s)\Delta(V_s)\Delta(W_s)$ has the only term of the above type. By the same reasons $\alpha_t(g_t\otimes g_t)\Delta(V_t)\Delta(W_t)$ has a term of the above type only if $V_t\geq V_s$ and $W_t\geq W_s$ with respect to the Hall ordering of the set of all super-words. However, by the choise of s, we have $D(V_sW_s)\geq D(V_tW_t)$. Hence $D(V_t)=D(V_s)$ and $D(W_t)=D(W_s)$. In particular V_t is not a proper onset of V_s. Therefore $V_t=V_s$ since otherwise the inequality $V_t>V_s$ yields a contradiction $V_tW_t>V_sW_s$. The inequality $W_t>W_s$ get the same contradiction. Therefore $V_t=V_s$ and $W_t=W_s$, in which case $g_tg(V_t)W_t\otimes g_tV_t=g_sg(V_s)W_s\otimes g_sV_s$. Thus $g_t=g_s$ and $t=s$. □

7. *Quantification of the classical series*

In this section we apply the above general results to the infinite series A_n, B_n, C_n, D_n of nilpotent Lie algebras defined by the Serre relations (11). Let $\mathfrak{g}$ be any such Lie algebra.

Lemma 7.1. *If a standard word u has no sub words of the type*

$$x_i^s x_j x_i^m,\ \text{where } s+m=1-a_{ij} \tag{28}$$

then $[u]$ is a hard in $U_P(\mathfrak{g})$ super-letter.

Proof. Let R be defined by the generators $x_1,\dots,x_n$ and the relations

$$x_i^s x_j x_i^m=0,\quad \text{where}\quad s+m=1-a_{ij}. \tag{29}$$

Clearly (29) implies (11) with the skew commutator in place of the Lie operation. Therefore R is a homomorphic image of $U_P(\mathfrak{g})$. The system (29) is closed under compositions since a composition of monomial relations always has the form $0=0$.

Let u have no sub-words (28). If $[u]$ is not hard then, by the Super-letters Crystallisation Lemma, u is a linear combination of lesser words in $U_P(\mathfrak{g})$. Therefore u is a linear combination of lesser words in R as well. This contradicts the fact that u belongs to the Groebner–Shirshov basis of R, since every word either belongs to this basis or equals zero in R. □

Theorem $\mathbf{A_n}$. *Suppose that $\mathfrak{g}$ is of the type A_n, and $p_{ii}\neq -1$. Denote by B the set of the super-letters given below:*

$$[u_{km}]\stackrel{df}{=}[x_kx_{k+1}\dots x_m],\quad 1\leq k\leq m\leq n. \tag{30}$$

The following statements are valid.

1. *The values of $[u_{km}]$ in $U_P(\mathfrak{g})$ form a PBW-generators set.*

2. *Each of the super-letters* (30) *has infinite height in* $U_P(\mathfrak{g})$.
3. *The values of all non-hard in* $U_P(\mathfrak{g})$ *super-letters equal zero.*
4. *The following relations with* (21) *form the Groebner–Shirshov relations system that determines the crystal basis of* $U_P(\mathfrak{g})$:

$$\begin{aligned} &[u_0] \stackrel{df}{=} [x_k x_m] = 0, && 1 \leq k < m-1 < n; \\ &[u_1] \stackrel{df}{=} [x_k x_{k+1} \dots x_m x_{k+1}] = 0, && 1 \leq k < m \leq n; \\ &[u_2] \stackrel{df}{=} [x_k x_{k+1} \dots x_m x_k x_{k+1} \dots x_{m+1}] = 0, && 1 \leq k \leq m < n. \end{aligned} \tag{31}$$

5. *If* $p_{11} \neq 1$ *then the generators* x_i, *the constants* $1-g$, $g \in G$, *and, in the case that* p_{11} *is a primitive t-th root of 1, the elements* $x_i^t, x_i^{tl^k}$ *form a basis of* $\mathfrak{g}_P = L(U_P(\mathfrak{g}))$. *Here* l *is the characteristic of the ground field.*
6. *If* $p_{11} = 1$ *then the elements* (30) *and, in the case* $l > 0$, *their* l^k*-th powers, together with* $1-g$, $g \in G$ *form a basis of* $\mathfrak{g}_P$.

By Corollary 2.4 the relations (11) with a Cartan matrix A of type A_n admit a quantification if and only if

$$p_{ii} = p_{11}, \quad p_{ii+1}p_{i+1i} = p_{11}^{-1}; \quad p_{ij}p_{ji} = 1, \ i-j > 1. \tag{32}$$

In this case the quantified relations (11) take up the form

$$x_i x_{i+1}^2 = p_{ii+1}(1+p_{i+1i+1})x_{i+1}x_i x_{i+1} - p_{ii+1}^2 p_{i+1i+1} x_{i+1}^2 x_i, \tag{33}$$

$$x_i^2 x_{i+1} = p_{ii+1}(1+p_{ii})x_i x_{i+1} x_i - p_{ii+1}^2 p_{ii} x_{i+1} x_i^2, \tag{34}$$

$$x_i x_j = p_{ij} x_j x_i, \qquad i-j > 1. \tag{35}$$

Let us introduce a congruence $u \equiv_k v$ on $G\langle X\rangle$. This congruence means that the value of $u-v$ in $U_P^b(\mathfrak{g})$ belongs to the subspace generated by values of all words with the initial letters $x_i, i \geq k$. Clearly, this congruence admits right multiplication by arbitrary polynomials as well as left multiplication by the independent of x_{k-1} ones (see (35)). For example, by (33) and (34) we have

$$x_i x_{i+1}^2 \equiv_{i+1} 0; \quad x_i x_{i+1} x_i \equiv_{i+1} \alpha x_i^2 x_{i+1}, \quad \alpha \neq 0. \tag{36}$$

LEMMA 7.2. *If* $y = x_i$, $m+1 \neq i > k$ *or* $y = x_i^2$, $m+1 = i > k$ *then*

$$u_{km} y \equiv_{k+1} 0. \tag{37}$$

Proof. Let $y = x_{m+1}^2$, $m+1 > k$. By (36) and (35) we have that $u_{km}y = u_{k\,m-1}\underline{x_m x_{m+1}^2} \equiv_{m+1} 0$. If $y = x_i$ and $m+1 \neq i > k$ then we get $u_{km}y = \alpha u_{k\,i-1}\underline{x_i x_{i+1} x_i}$ $u_{i+2m} \equiv_{i+1} \beta \underline{u_{k\,i-1} x_i^2} u_{i+1m} \equiv_{k+1} 0$ by the above case. □

LEMMA 7.3. *The brackets in* $[u_{km}]$ *are left-ordered,* $[u_{km}] = [x_k[u_{k+1m}]]$.

Proof. The statement immediately follows from the properties 6s and 2s. □

LEMMA 7.4. *If a nonassociative word* $[[u_{km}][u_{rs}]]$ *is standard then* $k = m \leq r$; *or* $r = k+1$, $m \geq s$; *or* $r = k$, $m < s$.

Proof. By definition, $u_{km} > u_{rs}$ if and only if either $k < r$; or $k = r$, $m < s$. If $k = m$ then $u_{km} = x_k$ and $m \leq r$. If $k \neq m$ then $[u_{km}] = [x_k[u_{k+1m}]]$. Therefore $u_{k+1m} \leq u_{rs}$, i.e. either $k+1 > r$; or $k+1 = r$ and $m \geq s$. The former case contradicts $k < r$ while the latter one does $k = r$. Thus only the possibilities set in the lemma remain. □

LEMMA 7.5. *If $[w] = [[u_{km}][u_{rs}]]$, $n \geq 1$ is a standard nonassociative word then the constitution of $[w]^h$ does not equal the constitution of any super-word in less than $[w]$ super-letters from B.*

Proof. The inequalities at the last column of the following tableaux are valid for all $[u] \in B$ that are less than the super-letters located on the same row, where as above $\deg_i(u)$ means the degree of u in x_i.

$$\begin{array}{lll} [x_k u_{k+1s}] & & \deg_k(u) \leq \deg_{s+1}(u); \\ [x_k u_{rs}], & k \leq r \neq k+1 & \deg_k(u) \leq \deg_{k+1}(u); \\ [u_{km} u_{k+1s}], & m \geq s & \deg_k(u) \leq \deg_{m+1}(u); \\ [u_{km} u_{ks}], & m < s & \deg_k(u) \leq \deg_{m+1}(u). \end{array} \tag{38}$$

If all super-letters of a super-word U satisfy one of these inequalities then U does as well. Clearly, no one of the super-letters in the first column satisfies the degree inequality on the same row. Finally, by Lemma 7.4 the first column contains all standard nonassociative words of the type $[[u_{km}][u_{rs}]]$. □

LEMMA 7.6. *If $p_{11} \neq 1$ then the values of $[u_{km}]^h$, $k < m$, $h \geq 1$ are not skew primitive, in particular they are non-zero.*

Proof. The sub-algebra generated by $x_2, \ldots x_n$ is defined by the Cartan matrix of the type A_{n-1}. This allows us to use induction on n. If $n = 1$ then the lemma is correct in the sense that $[u_{km}]^h = x_1^h \neq 0$.

Let $n > 1$. If $k > 1$ then we may use the inductive supposition directly. Consider the decomposition $\Delta([u_{1m}]) = \sum u^{(1)} \otimes u^{(2)}$. Since

$$[u_{1m}] = x_1[u_{2m}] - p(x_1, u_{2m})[u_{2m}]x_1, \tag{39}$$

we have

$$\begin{aligned} \Delta([u_{1m}]) = (x_1 \otimes 1 + g_1 \otimes x_1)\Delta([u_{2m}]) - \\ p(x_1, u_{2m})\Delta([u_{2m}])(x_1 \otimes 1 + g_1 \otimes x_1). \end{aligned} \tag{40}$$

Therefore the sum of all tensors $u^{(1)} \otimes u^{(2)}$ with $\deg_1(u^{(2)}) = 1$, $\deg_k(u^{(2)}) = 0$, $k > 1$ has the form $\varepsilon g_1[u_{2m}] \otimes x_1$, where $\varepsilon = 1 - p(x_1, u_{2m})p(u_{2m}, x_1)$ since $[u_{2m}]g_1 = p(u_{2m}, x_1)g_1[u_{2m}]$. By (32) we have $p_{ij}p_{ji} = 1$ for $i - 1 > j$. Therefore $\varepsilon = 1 - p_{12}p_{21} = 1 - p_{11}^{-1} \neq 0$.

This implies that in the decomposition $\Delta([u_{1m}]^h) = \sum v^{(1)} \otimes v^{(2)}$ the sum of all tensors $v^{(1)} \otimes v^{(2)}$ with $\deg_1(v^{(2)}) = h$, $\deg_k(v^{(2)}) = 0$, $k > 1$ equals $\varepsilon^h[u_{2m}]^h \otimes x_1^h$. Thus $[u_{1m}]^h$ is not skew primitive in $U_P(\mathfrak{g})$. □

Proof of Theorem $\mathbf{A_n}$. Let us show firstly that B satisfies the conditions of Lemma 4.8. By the Super-letter Crystallisation Lemma $[w] = [[u_{km}][u_{rs}]]$ is non-hard if the value of $u_{km}u_{rs}$ is a linear combination of lesser words. For $k = m$, $r = k+1$ we have

$[w] = [u_{ks}] \in B$. If $k = m$, $r > k+1$ then the word $x_k u_{rs}$ can be diminished by (34) or (35). If $k \neq m$ then by Lemma 7.4 the word $u_{km}u_{rs}$ has a sub-word of the type u_1 or u_2. Thus we need show only that the values in $U_P(\mathfrak{g})$ of u_1 and u_2 are linear combinations of lesser words.

The word u_1 has such a representation by Lemma 7.2. Consider the word u_2. Let us show by downward induction on k that

$$u_{km}u_{k\,m+1} \equiv_{k+1} \gamma u_{k\,m+1}u_{km}, \quad \gamma \neq 0. \tag{41}$$

If $k = m$ then one may use (34) with $i = k$. Let $k < m$. Let us transpose the second letter x_k of u_2 as far to the left as possible by (35). We get

$$u_2 = \alpha \underline{x_k x_{k+1} x_k} x_{k+2} \cdots x_m x_{k+1} \cdots x_{m+1}, \quad \alpha \neq 0.$$

By (34) we have

$$u_2 \equiv_{k+1} \beta x_k^2 (x_{k+1}x_{k+2} \cdots x_m x_{k+1} \cdots x_{m+1}), \quad \beta \neq 0.$$

Let us apply the inductive supposition to the word in the parentheses. Since x_i, $i > k+1$ commutes with x_k^2 according to the formulae (35), we get

$$u_2 \equiv_{k+1} \gamma \underline{x_k^2 x_{k+1}} x_{k+2} \cdots x_{m+1} x_{k+1} \cdots x_m.$$

Now it remains to replace the underlined sub-word according to (34) and then to transpose the second letter x_k to its former position by (35).

Note that for the diminishing of u_1, u_2 we did not use, and we could not use, the relation $[x_{n-1}x_n^2] = 0$ since $\deg_n(u_1) \leq 1$, $\deg_n(u_2) \leq 1$.

Thus B satisfies the conditions of Lemma 4.8. Since none of $[u_{km}]$ has sub-words (28), Lemmas 7.1 and 4.8 show that the first statement is correct.

If $[u_{km}]$ has a finite height h then the value of the polynomial $[u_{km}]^h$ in $U_P(\mathfrak{g})$ is a linear combination of words in hard super-letters that are less than $[u_{km}]$. However by Lemma 7.5 this linear combination is trivial, $[u_{km}]^h = 0$, since the defining relations are homogeneous. By Lemma 7.6 the second statement is correct for $p_{11} \neq 1$.

Similarly consider the skew primitive elements. Since both the defining relations and the coproduct are homogeneous, all the homogeneous components of a skew primitive element are skew primitive itself. Therefore it remains to describe all skew primitive elements homogeneous in each x_i. Let T be such an element. By Lemma 4.9 we have

$$T = [u]^h + \sum \alpha_i W_i,$$

where $[u]$ is a hard super-letter, $u = u_{km}$, and W_i are super-words in less than $[u]$ super-letters from B. By the homogeneity all W_i have the same constitution as $[u_{km}]^h$ does. However by Lemma 7.5 there exist no such super-words. This means that the only case possible is $T = [u_{km}]^h$. Thus, by Lemma 7.6 the fifth statement is valid as well.

If $p_{11} = 1$ then $p_{ij}p_{ji} = p_{ii} = 1$ for all i, j. So we are under the conditions of Example 1, that is $U_P^b(\mathfrak{g})$ is the universal enveloping algebra of the colour Lie algebra $\mathfrak{g}^{col}$. Further, $[u_{km}] \in \mathfrak{g}^{col}$ and $[u_{km}]$ are linearly independent in $\mathfrak{g}^{col}$ since they are hard super-letters and no one of them can be a linear combination of the lesser ones. Let us complete B to a homogeneous basis B' of $\mathfrak{g}^{col}$. Then by the PBW theorem for the colour Lie algebras the products $b_1^{n_1}\cdots b_k^{n_k}$, $b_1<\ldots<b_k$ form a basis of $U(\mathfrak{g}^{col})=U_P^b(\mathfrak{g})$. However, the monotonous restricted words in B form a basis of $U_P^b(\mathfrak{g})$ also. Thus $B'=B$ and all hard super-letters have the infinite height.

In particular, we get that the second statement is valid in complete extent. Moreover, if $p_{11}=1$ then $p(u_{km},u_{km})=1$, thus for $l=0$ all homogeneous skew primitive elements became exhausted by $[u_{km}]$, while for $l>0$ the powers $[u_{km}]^{l^k}$ are added to them (of course, here $l\neq 2$ since $-1\neq p_{ii}=1$).

So we have proved all statements, but the third and fourth ones. These statements will follow Theorem 5.2 and Lemma 5.3 if we prove that all non-hard super-letters $[[u_{km}][u_{rs}]]$ equal zero in $U_P(\mathfrak{g})$. By the homogeneous definition, $[[u_{km}][u_{rs}]]$ is a linear combination of super-words in lesser hard super-letters. However, by Lemma 7.5, there exist no such super-words of the same constitution. Therefore, by the homogeneity, the above linear combination equals zero. □

Theorem $\mathbf{B_n}$. *Let $\mathfrak{g}$ be of the type B_n, and $p_{ii}\neq -1$, $1\leq i<n$, $p_{nn}^{[3]}\neq 0$. Denote by B the set of the super-letters given below:*

$$\begin{array}{ll}[u_{km}]\stackrel{df}{=}[x_kx_{k+1}\ldots x_m], & 1\leq k\leq m\leq n;\\ [w_{km}]\stackrel{df}{=}[x_kx_{k+1}\ldots x_n\cdot x_n\ldots x_m], & 1\leq k<m\leq n.\end{array}\tag{42}$$

The following statements are valid.

1. *The values of* (42) *in $U_P(\mathfrak{g})$ form the PBW-generators set.*
2. *Every super-letter $[u]\in B$ has infinite height in $U_P(\mathfrak{g})$.*
3. *The relations* (21) *with the following ones form a Groebner–Shirshov system that determines the crystal basis of $U_P(\mathfrak{g})$.*

$$\begin{array}{ll}[u_0]\stackrel{df}{=}[x_kx_m]=0, & 1\leq k<m-1<n;\\ [u_1]\stackrel{df}{=}[u_{km}x_{k+1}]=0, & 1\leq k<m\leq n,\ k\neq n-1;\\ [u_2]\stackrel{df}{=}[u_{km}u_{k\,m+1}]=0, & 1\leq k\leq m<n;\\ [u_3]\stackrel{df}{=}[w_{km}x_{k+1}]=0, & 1\leq k<m\leq n,\ k\neq m-2;\\ [u_4]\stackrel{df}{=}[w_{kk+1}x_{k+2}]=0, & 1\leq k<n-1;\\ [u_5]\stackrel{df}{=}[w_{km}w_{k\,m-1}]=0, & 1\leq k<m-1\leq n-1;\\ [u_6]\stackrel{df}{=}[u_{kn}^2x_n]=0, & 1\leq k<n.\end{array}\tag{43}$$

4. *If $p_{11}\neq 1$ then the generators x_i and their powers $x_i^t, x_i^{tl^k}$, such that p_{ii} is a primitive t-th root of 1, together with the constants $1-g$, $g\in G$ form a basis of $\mathfrak{g}_P=L(U_P(\mathfrak{g}))$. Here l is the characteristic of the ground field.*
5. *If $p_{nn}=p_{11}=1$ then the elements* (42) *and, for $l>0$, their l^k-th powers, together with $1-g$, $g\in G$ form a basis of $\mathfrak{g}_P$. If $p_{nn}=-p_{11}=-1$ then $[u_{kn}]^2$, $[u_{kn}]^{2l^k}$ are added to them.*

Recall that in the case B_n the algebra $U_P^b(\mathfrak{g})$ is defined by (33), (34), (35) where in (33) the last relation, $i=n-1$, is replaced with

$$x_{n-1}x_n^3=p_{n-1n}p_{nn}^{[3]}x_nx_{n-1}x_n^2-p_{n-1n}^2p_{nn}p_{nn}^{[3]}x_n^2x_{n-1}x_n+p_{n-1n}^3p_{nn}^3x_n^3x_{n-1}.\tag{44}$$

By Corollary 2.4 we get the existence conditions

$$p_{ii} = p_{11},\ p_{ii+1}p_{i+1i} = p_{11}^{-1} = p_{nn}^{-2},\ 1 \le i \le n-1;\ p_{ij}p_{ji} = 1,\ i-j > 1. \tag{45}$$

The relations (33) and (44) show that

$$x_i x_{i+1}^2 \equiv_{i+1} 0, \quad i < n-1; \qquad x_{n-1}x_n^3 \equiv_n 0, \tag{46}$$

while the relations (34) imply

$$x_i x_{i+1} x_i \equiv_{i+1} \alpha x_i^2 x_{i+1}, \quad \alpha \ne 0. \tag{47}$$

By means of these relations and (35), (44) we have

$$x_{n-2}x_{n-1}\underline{x_n^2 x_{n-1} x_n} \equiv_{n-1} 0. \tag{48}$$

Lemma 7.7. *The brackets in* $[w_{km}]$ *are set by the recurrence formulae:*

$$\begin{array}{ll} [w_{km}] = [x_k[w_{k+1m}]], & \text{if } 1 \le k < m-1 < n; \\ [w_{kk+1}] = [[w_{kk+2}]x_{k+1}], & \text{if } 1 \le k < n. \end{array} \tag{49}$$

Here by the definition $w_{k\,n+1} = u_{kn}$.

Proof. It is enough to use the property 6s and then 1s and 2s. □

Lemma 7.8. *The nonassociative word* $[[w_{km}][w_{rs}]]$ *is standard only in the following two cases:* 1) $s \ge m > k+1 = r$; 2) $s < m,\ r = k$.

Proof. If $[[w_{km}][w_{rs}]]$ is standard then $w_{km} > w_{rs}$ and by (49) either $w_{k+1} \le w_{rs}$, or $m = k+1$ and $x_{k+1} \le w_{rs}$. The inequality $w_{km} > w_{rs}$ is correct only in two cases: $k < r$ or $k = r, m > s$. We get four possibilities: 1) $k < r,\ k < m-1,\ w_{k+1m} \le w_{rs}$; 2) $k < r,\ m = k+1,\ x_{k+1} \le w_{rs}$; 3) $k = r,\ m > s,\ k < m-1,\ w_{k+1m} \le w_{rs}$; 4) $k = r,\ m > s,\ m = k+1,\ x_{k+1} \le w_{rs}$. Only the first and third ones are consistent since in the second case $x_{k+1} \le w_{rs}$ implies $k+1 > r$, while in the fourth case $r < s$ and $k = r < s < m = k+1$. If now we decode $w_{k+1m} \le w_{rs}$ in the first and third cases, we get the two possibilities mentioned in the lemma. □

Lemma 7.9. *The nonassociative word* $[[u_{km}][w_{rs}]]$ *is standard only in the following two cases:* 1) $k = r$; 2) $k = m < r$.

Proof. The inequality $u_{km} > w_{rs}$ means $k \le r$. Since $[u_{km}] = [x_k[u_{k+1m}]]$, for $k \ne m$ we get $u_{k+1m} \le w_{rs}$, so $k+1 > r$ and $k = r$. If $k = m \ne r$ then $x_m > w_{rs}$ and $m < r$. □

Lemma 7.10. *The nonassociative word* $[[w_{km}][u_{rs}]]$ *is standard only in the following two cases:* 1) $r = k+1 < m$; 2) $r = k+1 = m = s$.

Proof. The inequality $w_{km} > u_{rs}$ implies $r > k$. If $k < m-1$ then by the first formula (49) we have $w_{k+1m} \le u_{rs}$ that is equivalent to $k+1 \ge r$. Therefore $r = k+1 < m$. If $k = m-1$ then by the second formula (49) we get $x_{k+1} \le u_{rs}$, i.e. either $k+1 > r$ or $k+1 = r = s$. The former case contradicts $r > k$ while the latter one is mentioned in the lemma. □

Lemma 7.11. *If $[u],[v]\in B$ then one of the statements below is correct.*
1) $[[u][v]]$ *is not a standard nonassociative word;*
2) uv *contains a sub-word of one of the types* $u_0,u_1,u_2,u_3,u_4,u_5,u_6$;
3) $[[u][v]]\in B$.

Proof. The proof results from Lemmas 7.4, 7.8, 7.9, 7.10. □

Lemma 7.12. *If a super-word W equals one of the super-letters $[u_1]$–$[u_6]$ or $[u_{km}]^h$, $[w_{km}]^h$, $h\geq 1$ then its constitution does not equal the constitution of any super-word in less than W super-letters from B.*

Proof. The proof is akin to Lemma 7.5 with the following tableaux:

$$\begin{array}{ll} [u_{km}],\ [u_{km}x_{k+1}],\ [u_{km}u_{k\,m+1}] & \deg_k(u)\leq \deg_{m+1}(u); \\ [w_{km}],\ [w_{km}x_{k+1}],\ [w_{km}w_{k\,m-1}] & 2\deg_k(u)\leq \deg_{m-1}(u); \\ [w_{kk+1}x_{k+2}] & \deg_k(u)=0; \\ [u_{kn}^2x_n] & \deg_k(u)\leq \deg_n(u). \end{array} \tag{50}$$

□

Lemma 7.13. *If $y=x_i,\ m-1\neq i>k$ or $y=x_i^2,\ m-1=i>k$ then*

$$w_{km}y\equiv_{k+1}0. \tag{51}$$

Proof. If $i<m-1$ then by means of (35) it is possible to permute y to the left beyond x_n^2 and use Lemma 7.2 with $m'=n-1$. If $y=x_i^2,\ m-1=i>k$ then by the above case, $i<m-1$, we get

$$w_{km}y=w_{k\,m+1}\underline{x_mx_{m-1}^2}=\underline{w_{k\,m+1}x_{m-1}}(\alpha x_mx_{m-1}+\beta x_{m-1}x_m)\equiv_{k+1}0, \tag{52}$$

where for $m=n$ by definition $w_{k\,n+1}=u_{kn}$, and $u_{kn}x_{n-1}\equiv_{n-1}0$.

If $y=x_i, i=m>k$ then for $m=n$ one may use the second equality (46). For $m<n$ we have $w_{km}y=w_{k\,m+1}y_1$ where $y_1=x_m^2$. Therefore for $k<n-1$ we may use (52) with $m+1$ in place of m. For $k=n-1$ we have $w_{km}x_n=x_{n-1}x_n^3\equiv_n 0$.

Finally, if $y=x_i,\ i>m>k$ then by (35) we have $w_{km}y=\alpha w_{ki+1}\underline{x_ix_{i-1}x_i}\cdot v$. For $i=n$ one may use (48), while for $i<n$, changing the underlined word according to (33), we may use the above considered cases: $m'-1=i'$, where $m'=i+1$, $i'=i$; and $i'<m'-1$, where $m'=i+1$, $i'=i-1$. □

Another interesting relation appears if we multiply (44) by x_{n-1} from the left and subtract (34) with $i=n-1$ multiplied from the right by x_n^2 :

$$x_{n-1}x_nx_{n-1}x_n^2\equiv_n \alpha x_{n-1}x_n^2x_{n-1}x_n, \tag{53}$$

in which case $\alpha=p_{n-1n}p_{nn}^{[3]}\neq 0$.

Lemma 7.14. *For $k<s<m\leq n$ the following relation is valid.*

$$w_{km}w_{ks}\equiv_{k+1}\varepsilon w_{ks}w_{km},\quad \varepsilon\neq 0. \tag{54}$$

Proof. Let us use downward induction on k. For this we first transpose the second letter x_k of $w_{km}w_{ks}$ as far to the left as possible by means of (35), and then change the onset $x_kx_{k+1}x_k$ according to (47). We get

$$w_{km}w_{ks} \equiv_{k+1} \alpha x_k^2(w_{k+1m}w_{k+1s}), \quad \alpha \neq 0. \tag{55}$$

For $k+1 < s$ we apply the inductive supposition to the word in the parentheses and then by (47) and (35) transpose x_k to its former position.

The case $k+1 = s$, the basis of the induction on k, we prove by downward induction on s.

Let $k+1 = s = n-1$. Then $m = n$. Let us show firstly that

$$\underline{x_{n-1}x_n^2x_{n-1}x_n}x_nx_{n-1} \equiv_n \alpha x_{n-1}x_n^2x_{n-1}^2x_n^2 + \beta x_{n-1}x_nx_{n-1}^2x_n^3, \quad \alpha \neq 0. \tag{56}$$

For this in the left hand side we transpose the first letter x_n by means of (53) to the penultimate position, and then replace the ending $x_n^3x_{n-1}$ by (44). We get a linear combination of three words. One of them equals the second word of (56), while two other have the following forms.

$$x_{n-1}x_n\underline{x_{n-1}x_nx_{n-1}}x_n^2, \ \underline{x_{n-1}x_nx_{n-1}x_n^2}x_{n-1}x_n.$$

The former word by (34) transforms into the form (56). The latter one, after the application of (53) and the replacing of $x_{n-1}x_nx_{n-1}$ by (34), will have an additional term $\underline{x_{n-1}x_n^3}x_{n-1}^2x_n$ to which it is possible to apply (46). The direct calculation of the coefficients shows that $\alpha = p_{n-1n}p_{nn} \neq 0$.

Now let us multiply (56) by x_{n-2}^2 from the left and use (34) with $i = n-2$. We get that $w_{n-2\,n}w_{n-2\,n-1}$ with respect to $\equiv_{n-1}$ equals

$$\gamma x_{n-2}x_{n-1}x_n^2\underline{x_{n-2}x_{n-1}^2}x_n^2 + \delta x_{n-2}x_{n-1}x_n\underline{x_{n-2}x_{n-1}^2}x_n^3, \ \gamma \neq 0. \tag{57}$$

Let us apply (46) and then (47) and (46) to the second word. We get that this word with respect to $\equiv_{n-1}$ equals zero. The first word after application of (34) takes up the form

$$\varepsilon w_{n-2\,n-1}w_{n-2\,n} + \varepsilon' \underline{w_{n-2\,n}x_{n-1}^2}x_{n-2}x_n^2, \quad \varepsilon \neq 0.$$

Thus, by Lemma 7.13, the basis of the induction on s is proved.

Let us carry out the inductive step. Let $k+1 = s < n-1$. If $m > s+1 = k+2$ then by the inductive supposition on s we may write

$$w_{km}w_{ks} = (w_{km}w_{kk+2})x_{k+1} \equiv_{k+1} \alpha w_{kk+2}w_{km}x_{k+1} =$$

$$\beta w_{kk+2}\underline{x_kx_{k+1}x_{k+2}x_{k+1}}w_{k+3\,m}. \tag{58}$$

Taking into account (51) we may neglect the words starting with x_{k+1}^2, x_{k+2} while transforming the underlined part:

$$x_k\underline{x_{k+1}x_{k+2}x_{k+1}} \equiv \gamma \underline{x_kx_{k+1}^2}x_{k+2} \equiv \delta x_{k+1}x_kx_{k+1}x_{k+2}. \tag{59}$$

In this way (58) is transformed into (54).

If $m = s+1 = k+2 < n$ then the relation (55) takes up the form

$$w_{km}w_{ks} \equiv_{k+1} \alpha x_k^2(w_{k+1k+2}w_{k+1k+3})x_{k+2}x_{k+1}.$$

Let us apply the inductive supposition with $k' = k+1$, $s' = k+2$, $m' = k+3$ to the word in the parentheses. We get

$$w_{km}w_{ks} \equiv_{k+1} \alpha\varepsilon^{-1}x_k^2 w_{k+1k+3}w_{k+1k+3}\underline{x_{k+2}^2 x_{k+1}},$$

or after an evident replacement

$$w_{km}w_{ks} \equiv_{k+1} \gamma x_k^2 w_{k+1k+3}w_{k+1k+2}\cdot x_{k+1}x_{k+2} + \delta x_k^2 w_{k+1k+3}^2 x_{k+1}x_{k+2}^2.$$

In both terms we may transpose one letter x_k to its former position by means of (47) and (35). We get

$$w_{km}w_{ks} \equiv_{k+1} \gamma' \underline{w_{kk+3}w_{kk+1}}x_{k+2} + \delta' w_{kk+3}^2 x_{k+1}x_{k+2}^2. \tag{60}$$

It is possible to apply (54) with $m' = k+3$, $s' = k+1$ to the first term since the case $m > s+1$ is completely considered. Therefore it is enough to show that the second term equals zero with respect to $\equiv_{k+1}$. When we transpose the third letter x_{k+1} as far to the left as possible we get the word

$$w_{kk+3}\underline{x_k x_{k+1}x_{k+2}x_{k+1}}w_{k+3\,k+3}x_{k+2}^2. \tag{61}$$

Taking into account (51) we may neglect the words starting with x_{k+1} while transforming the underlined part:

$$x_k\underline{x_{k+1}x_{k+2}x_{k+1}} \equiv x_{k+2}\underline{x_k x_{k+1}^2} \equiv x_{k+2}x_{k+1}x_k x_{k+1}. \tag{62}$$

Therefore the word (61) equals $w_{kk+1}\underline{w_{kk+3}x_{k+2}^2}$ with respect to $\equiv_{k+1}$ and it remains only to apply Lemma 7.13 twice. □

Lemma 7.15. *The set B satisfies the conditions of Lemma* 4.8.

Proof. By Lemmas 7.11 and 4.7 it is sufficient to show that in $U_P^b(\mathfrak{g})$ all words of the form $u_0, \dots, u_6$ are linear combinations of lesser ones. The words u_0 are diminished by (35). The words u_1, u_2 have been presented in this way, without using $[x_{n-1}x_n^2] = 0$, in the proof of the above theorem. The relation (51) shows that $u_3 \equiv_{k+1} 0$, $u_4 \equiv_{k+1} 0$. Lemma 7.14 with $s = m-1$ yields the necessary representation for u_5.

Let us prove by downward induction on k that

$$u_6 \stackrel{df}{=} u_{kn}^2 x_n \equiv_{k+1} \varepsilon u_{kn}x_n u_{kn}, \quad \varepsilon \neq 0.$$

For $k = n-1$ this equality takes up the form (53). Let $k < n-1$. Let us transpose the second letter x_k of $u_{kn}^2 x_n$ as far to the left as possible by means of (35) and then apply (33). We get

$$u_{kn}^2 x_n \equiv_{k+1} \alpha x_k^2 (u_{k+1n}^2 x_n), \quad \alpha \neq 0.$$

We may apply the inductive supposition to the term in the parentheses and then by (33), (35) transpose one of x_k's to its former position. □

Lemma 7.16. *If $p_{11} \neq 1$ then the values of polynomials $[v]^h$, where $[v] \in B$, $v \neq x_i$ $h \geq 1$ are not skew primitive, in particular, they are non-zero.*

Proof. Note that for $n>2$ the sub-algebra generated by $x_2,\dots x_n$ is defined by the Cartan matrix of the type B_{n-1}. This allows us to carry out the induction on n with additional supposition that the statements 1 and 2 of Theorem B_n are valid for lesser values of n. It is convenient formally consider the sub-algebras $\langle x_i\rangle$ as algebras of the type B_1. In this case for $n=1$ the lemma and the statements 1 and 2 are correct in the evident way. If v starts with $x_k\neq x_1$ then we may directly use the inductive supposition. If $v=u_{1m}$, one may literally repeat the arguments of Lemma 7.6 starting at the formula (39). Let $v=w_{1m}$. If $m>2$ then by Lemma 7.7 we have $w_{1m}=[x_1[w_{2m}]]$. This provides a possibility to repeat the same arguments of Lemma 7.6 with w in place of u.

Consider the last case $v=w_{12}$. By Lemma 7.7 we have

$$[w_{12}]=[w_{13}]x_2-p(w_{13},x_2)x_2[w_{13}], \tag{63}$$

$$[w_{13}]=x_1[w_{23}]-p(x_1,w_{23})[w_{23}]x_1. \tag{64}$$

Applying the coproduct first to (64) then to (63) we may find the sum Σ of all tensors $w^{(1)}\otimes w^{(2)}$ of $\Delta([w_{12}])$ with $\deg_1(w^{(2)})=1$, $\deg_k(w^{(2)})=0$, $k>1$ (in much the same way as (40)):

$$\begin{aligned}\Sigma=(\varepsilon g_1[w_{23}]\otimes x_1)(x_2\otimes 1)-p(w_{13},x_2)(x_2\otimes 1)(\varepsilon g_1[w_{23}]\otimes x_1)=\\ \varepsilon g_1([w_{23}]x_2-p(w_{13},x_2)p(x_2,x_1)x_2[w_{23}])\otimes x_1.\end{aligned} \tag{65}$$

For $n>2$, taking into account first the bicharacter property of p, then the equality $[x_2[w_{23}]]=x_2[w_{23}]-\,p(x_2,w_{23})[w_{23}]x_2$, and next the following relations $p_{ij}p_{ji}=1$, $i-j>1$; $p_{11}^{-1}=p_{12}p_{21}=p_{22}^{-1}=p_{23}p_{32}$, we may write

$$\Sigma=\varepsilon g_1(-p(w_{13},x_2)p_{21}[x_2w_{23}]+(1-p_{11}^{-1})[w_{23}]\cdot x_2)\otimes x_1. \tag{66}$$

Consider the left hand side of this tensor on applying the inductive supposition. Note that x_2w_{23} is a standard word and $[x_2w_{23}]$ equals $[x_2[w_{23}]]$. This super-letter is non-hard in $U_P(\mathfrak{g})$ since x_2w_{23} contains the sub-word $x_2^2x_3$. Thus $[x_2w_{23}]$ is a linear combination of monotonous non-decreasing super-words in lesser super-letters. Among these super-words there is no $[w_{23}]\cdot x_2$ since $x_2>x_2w_{23}$. On the other hand, $[w_{23}]\cdot x_2$ is a monotonous non-decreasing super-word and hence its value in $U_P(\mathfrak{g})$ is a basis element. Therefore for $n>2$ the left hand side W of Σ is non-zero.

For $n=2$, by the definition $w_{23}=x_2$, $w_{13}=x_1x_2$, and the equality (65) takes up the form $\Sigma=\varepsilon g_1(1-p_{12}p_{22}p_{21})x_2^2\otimes x_1$. Since $1\neq p_{11}^{-1}=p_{12}p_{21}=p_{22}^{-2}$, we get $(1-p_{12}p_{22}p_{21})=1-p_{22}^{-1}\neq 0$. Therefore in this case $\Sigma\neq 0$ as well.

By [**22**, Corollary 10] the sub-algebra generated by $x_2,\dots,x_n$ has no zero divisors. In particular $W^h\neq 0$ and $\Sigma^h\neq 0$ in any case.

It remains to note that for $n>1$ the sum of all tensors $w^{(1)}\otimes w^{(2)}$ of $\Delta([w_{12}]^h)$ such that $\deg_1(w^{(2)})=h$, $\deg_k(w^{(2)})=0$, $k>1$ equals Σ^h, hence $[w_{12}]^h$ can not be skew-primitive. □

Proof of Theorem B_n. Since none of u_{km}, w_{km} contains sub-words (28), Lemmas 7.15, 7.1, 4.8 imply the first statement.

If $[v]\in B$ is of finite height then by Lemma 7.12 and the homogeneous version of Definition 4.4 we have $[v]^h=0$. For $p_{11}\neq 1$ this contradicts Lemma 7.16.

Along similar lines, by Lemma 4.9, every skew primitive homogeneous element has the form $[v]^h$. This, together with Lemma 7.16, proves the fourth statement and, for $p_{11} \neq 1$, the second one too.

If $p_{11} = 1$ then by (45) we have $p_{nn}^2 = 1$, $p_{ii} = 1$, $i < n$. Besides, $p_{ij}p_{ji} = 1$ for all i, j. This means that the skew commutator is a quantum operation. Hence all elements of B are skew primitive. In the case $p_{nn} = 1$ these elements span a colour Lie algebra, while in the case $p_{nn} = -1$ they span a colour Lie super-algebra. Now as in Theorem A_n, we may use the PBW-theorem for the colour Lie super-algebras.

The third statement will follow Theorem 5.2 and Lemmas 5.3, 7.11 if we prove that all super-letters (43) are zero in $U_P(\mathfrak{g})$. We have already proved that these super-letters are non-hard. Therefore it remains to use the homogeneous version of Definition 4.3 and Lemma 7.12. □

Theorem $\mathbf{C_n}$. *Suppose that $\mathfrak{g}$ is of the type C_n, and $p_{ii} \neq -1$, $1 \leq i \leq n$, $p^{[3]}_{n-1n-1} \neq 0$. Denote by B the set of the following super-letters:*

$$\begin{array}{llll} [u_{km}] & \stackrel{df}{=} & [x_k x_{k+1} \ldots x_m], & 1 \leq k \leq m \leq n; \\ [v_{km}] & \stackrel{df}{=} & [x_k x_{k+1} \ldots x_n \cdot x_{n-1} \ldots x_m], & 1 \leq k < m < n; \\ [v_k] & \stackrel{df}{=} & [u_{k\,n-1} u_{kn}], & 1 \leq k < n. \end{array} \tag{67}$$

The statements given below are valid.

1. *The values of the super-letters* (67) *in $U_P(\mathfrak{g})$ form the PBW-generators set.*
2. *Each of these super-letters has the infinite height in $U_P(\mathfrak{g})$.*
3. *The following relations with* (21) *form a Groebner–Shirshov system that determines the crystal basis of $U_P(\mathfrak{g})$.*

$$\begin{array}{llll} [u_0] & \stackrel{df}{=} & [x_k x_m] = 0, & 1 \leq k < m-1 < n; \\ [u_1] & \stackrel{df}{=} & [u_{km} x_{k+1}] = 0, & 1 \leq k < m \leq n,\ (k,m) \neq (n-2, n); \\ [u_2] & \stackrel{df}{=} & [u_{km} u_{km+1}] = 0, & 1 \leq k \leq m < n-1; \\ [w_3] & \stackrel{df}{=} & [v_{km} x_{k+1}] = 0, & 1 \leq k < m < n,\ k \neq m-2; \\ [w_4] & \stackrel{df}{=} & [v_{kk+1} x_{k+2}] = 0, & 1 \leq k < n-1; \\ [w_5] & \stackrel{df}{=} & [v_{km} v_{km-1}] = 0, & 1 \leq k < m-1 \leq n-1; \\ [w_6] & \stackrel{df}{=} & [u_{k\,n-1}^3 x_n] = 0, & 1 \leq k < n. \end{array} \tag{68}$$

4. *If $p_{11} \neq 1$ then the generators x_i and their powers $x_i^t, x_i^{tl^k}$, such that p_{ii} is a primitive t-th root of 1 together with the constants $1-g$, $g \in G$ form a basis of $\mathfrak{g}_P = L(U_P(\mathfrak{g}))$. Here l is the characteristic of the ground field.*
5. *If $p_{11} = 1$ then the elements* (67) *and in the case of prime characteristic l theirs l^k-th powers, together with the constants $1-g$, $g \in G$ form a basis of $\mathfrak{g}_P$.*

In the case C_n the algebra $U_P^b(\mathfrak{g})$ is defined by the same relations (33), (34), (35), where in (34) the last relation, $i = n-1$, is replaced with

$$\begin{aligned} x_{n-1}^3 x_n = {} & p_{n-1n} p^{[3]}_{n-1n-1} x_{n-1}^2 x_n x_{n-1} + \\ & -p_{n-1n}^2 p_{n-1n-1} p^{[3]}_{n-1n-1} x_{n-1} x_n x_{n-1}^2 + p_{n-1n}^3 p_{n-1n-1}^3 x_n x_{n-1}^3. \end{aligned} \tag{69}$$

By Corollary 2.4 we get the existence conditions

$$\begin{gathered} p_{ii}=p_{11},\ p_{i-1i}p_{ii-1}=p_{11}^{-1},\ 1<i<n, \\ p_{n-1n}p_{nn-1}=p_{nn}^{-1}=p_{n-1n-1}^{-2};\ p_{ij}p_{ji}=1,\ i-j>1. \end{gathered} \tag{70}$$

Therefore the following relations are correct

$$x_ix_{i+1}^2 \equiv_{i+1} 0, \qquad 1\le i<n; \tag{71}$$

$$x_ix_{i+1}x_i \equiv_{i+1} \alpha x_i^2x_{i+1}, \quad 1\le i<n-1,\ \alpha\neq 0; \tag{72}$$

$$x_{n-1}x_nx_{n-1}^2 \equiv_n \alpha x_{n-1}^3x_n+\beta x_{n-1}^2x_nx_{n-1}, \quad \alpha,\ \beta\neq 0. \tag{73}$$

The left multiplication by x_{n-2} of the last relation implies

$$x_{n-2}x_{n-1}x_nx_{n-1}^2 \equiv_{n-1} 0. \tag{74}$$

LEMMA 7.17. *The brackets in $[v_{km}], [v_k]$ are set according to the following recurrence formulae, where by the definition $v_{kn}=u_{kn}$.*

$$\begin{array}{lcll} [v_{km}] & = & [x_k[v_{k+1m}]], & \text{if } 1\le k<m-1<n-1; \\ [v_{kk+1}] & = & [[v_{kk+2}]x_{k+1}], & \text{if } 1\le k<n-1; \\ [v_k] & = & [[u_{k\,n-1}][u_{kn}]], & \text{if } 1\le k<n. \end{array} \tag{75}$$

Proof. It is enough to use the properties 6s, 1s and 2s. □

LEMMA 7.18. *If $[u],[v]\in B$ then one of the following statements is valid.*
1) *$[[u][v]]$ is not a standard nonassociative word;*
2) *uv contains a sub-word of one of the types $u_0,u_1,u_2,w_3,w_4,w_5,w_6$;*
3) *$[[u][v]]\in B$.*

Proof. The first two formulae (75) coincide with (49) up to replacement of v with w provided $k+1\neq n>m$. Obviously for $m<n$ the inequality $v_{km}>v_{rs}$ is equivalent to $w_{km}>w_{rs}$, while $v_{km}>u_{rs}$ is equivalent to $w_{km}>w_{rs}$. Hence Lemmas (7.8), (7.9), (7.10) are still valid under the replacement of w with v :

$$\begin{array}{lll} [[v_{km}][v_{rs}]] & \text{is standard} \Leftrightarrow & s\ge m>k+1=r\vee(s<m\&r=k); \\ [[u_{km}][v_{rs}]] & \text{is standard} \Leftrightarrow & k=r\vee k=m<r; \\ [[v_{km}][u_{rs}]] & \text{is standard} \Leftrightarrow & r=k+1<m\vee r=k+1=m=s. \end{array} \tag{76}$$

Further, $v_k>v_r$ if and only if $k<r$, and under this condition $[[v_k][v_r]]$ is not standard since $u_{kn}>u_{rn-1}u_{rn}$.

In a similar manner $v_k>u_{rm}$ is equivalent to $k<r$, while $v_k>v_{rm}$ is equivalent to $k\le r$. Therefore none of the words $[[v_k][u_{rm}]]$, $[[v_k][v_{rm}]]$ is standard since $u_{kn}>u_{rm}$ and $u_{kn}>v_{rm}$, respectively.

For the remaining two cases we have only two possibilities

$$\begin{array}{lll} [[u_{km}][v_r]] & \text{is standard} \Leftrightarrow & r=k\le m<n; \\ [[v_{km}][v_r]] & \text{is standard} \Leftrightarrow & r=k+1\&k<m-1. \end{array} \tag{77}$$

The treatment in turn of the eight possibilities (76), (77) proves the lemma. □

LEMMA 7.19. *If a super-word W equals one of the super-letters (68) or $[v]^h$, $[v]\in B$, $h\geq 1$, then its constitution does not equal the constitution of any word in less then W super-letters from B.*

Proof. The proof is akin to Lemma 7.5 with the following tableaux:

$$\begin{array}{ll} [u_{km}]^h,\ [u_{km}x_{k+1}],\ [u_{km}u_{km+1}] & \deg_k(u)\leq \deg_{m+1}(u); \\ [v_{km}]^h,\ [v_{km}x_{k+1}],\ [v_{km}v_{km-1}] & 2\deg_k(u)\leq \deg_{m-1}(u); \\ [v_{kk+1}x_{k+2}] & \deg_k(u)=0; \\ [v_k]^h & \deg_k(u)\leq \deg_n(u); \\ [u_{kn-1}^3x_n] & \deg_k(u)\leq 2\deg_n(u). \end{array} \tag{78}$$

□

LEMMA 7.20. *If $y=x_i,\ m-1\neq i>k$ or $y=x_i^2,\ m-1=i>k$ then*

$$v_{km}y\equiv_{k+1}0. \tag{79}$$

Proof. For $i<m-1$, we may transpose y by means of (35) to the left across x_n^2 and then use Lemma 7.2 with $m'=n-1$.

If $y=x_i^2,\ m-1=i>k$ then by the above case, $i<m-1$, we get

$$v_{km}y=v_{km+1}\underline{x_mx_{m-1}^2}=\underline{v_{km+1}x_{m-1}}(\alpha x_mx_{m-1}+\beta x_{m-1}x_m)\equiv_{k+1}0, \tag{80}$$

where by definition $v_{kn}=u_{kn}$ and $u_{kn}x_{n-2}\equiv_{n-2}0$, while $n-2=i>k$.

If $y=x_i$, $i=m>k$ then for $m=n-1$ we may use the inequality (74), while for $m<n-1$ we have $v_{km}y=v_{km+1}y_1$ where $y_1=x_m^2$. Hence we may use (80) replacing m by $m+1$.

If $y=x_i,\ i>m>k$ then by (35) we get $v_{km}y=\alpha v_{ki+1}\underline{x_ix_{i-1}x_i}\cdot w$. Changing the underlined by (33), we may apply the previously considered cases: $m'-1=i'$, where $m'=i+1$, $i'=i$; and $i'<m'-1$, where $m'=i+1$, $i'=i-1$. □

If we multiply (69) by x_n from the right and subtract (33) with $i=n-1$ multiplied from the left by x_{n-1}^2, then by means of $p_{n-1n-1}^{-2}=p_{nn-1}p_{n-1n}=p_{nn}^{-1}$ we get

$$x_{n-1}^2\underline{x_nx_{n-1}x_n}\equiv_n p_{n-1n}(p_{n-1n-1}^{[3]}x_{n-1}x_nx_{n-1}^2x_n-p_{n-1n-1}x_{n-1}^2x_n^2x_{n-1}). \tag{81}$$

Let us first multiply this relation by x_{n-2}^2 from the left and then apply (33) to the underlined sub-word. Taking into account the relation $x_{n-2}^2x_{n-1}^3\equiv_{n-1}0$, we get that the left hand side of the multiplied (81) equals $p_{n-1n}p_{nn}(1+p_{nn})^{-1}x_{n-2}^2x_{n-1}^2x_n^2x_{n-1}$ up to $\equiv_{n-1}$, i.e. it is proportional to the second term of the right hand side. As a result the relation below with $\alpha=p_{n-1n-1}^{-1}(1+p_{nn})\neq 0$ is correct.

$$x_{n-2}^2x_{n-1}^2x_n^2x_{n-1}\equiv_{n-1}\alpha x_{n-2}^2x_{n-1}x_nx_{n-1}^2x_n. \tag{82}$$

LEMMA 7.21. *If $k<s<m\leq n$ and as above $v_{kn}=u_{kn}$ then*

$$v_{km}v_{ks}\equiv_{k+1}\varepsilon v_{ks}v_{km},\quad \varepsilon\neq 0, \tag{83}$$

Proof. Let us use downward induction on k. For this we first transpose the second letter x_k of $v_{km}v_{ks}$ as far to the left as possible by means of (35), and then change the onset $x_kx_{k+1}x_k$ according to (72). We get

$$v_{km}v_{ks} \equiv_{k+1} \alpha x_k^2(v_{k+1m}v_{k+1s}), \quad \alpha \neq 0. \tag{84}$$

For $k+1 < s$ we may apply the inductive supposition to the word in the parentheses, and then transpose x_k to its former position by (72), (35).

For $k+1 = s$ we will use downward induction on s.

Let $k+1 = s = n-1$. In this case $m = n$ and (84) becomes:

$$v_{n-2\,n}v_{n-2\,n-1} \equiv_{n-1} \beta x_{n-2}^2(x_{n-1}\underline{x_nx_{n-1}x_n}x_{n-1}).$$

Let us replace the underlined part according to (33). Since $x_{n-2}^2x_{n-1}x_n^2 \equiv_n 0$, we may continue by (82):

$$\equiv_{n-1} \beta_1x_{n-2}^2x_{n-1}^2x_n^2x_{n-1} \equiv_{n-1} \beta_2\underline{x_{n-2}^2x_{n-1}}x_nx_{n-1}^2x_n \equiv_{n-1}$$

$$\beta_3x_{n-2}x_{n-1}\underline{x_{n-2}x_n}x_{n-1}^2x_n \equiv_{n-1} \beta_4x_{n-2}x_{n-1}x_n\underline{x_{n-2}x_{n-1}^2}x_n.$$

With the help of (33) we get

$$= \varepsilon v_{n-2\,n-1}v_{n-2\,n} + \beta_5x_{n-2}\underline{x_{n-1}x_nx_{n-1}^2}x_{n-2}x_n, \quad \varepsilon \neq 0.$$

By (73) and (71) we see that the second term equals zero up to $\equiv_{n-1}$.

The inductive step on s coincides the inductive step on s in Lemma 7.14 up to replacing both the citations of Lemma 7.13 with the citations of Lemma 7.20 and w with v. □

LEMMA 7.22. *The set B satisfies the Lemma 4.8 conditions.*

Proof. According to the Super-letter Crystallisation Lemma and Lemma 7.11 it is sufficient to show that words of the form $u_0, u_1, u_2, w_3, w_4, w_5, w_6$ are linear combinations of lesser words in $U_P(\mathfrak{g})$. The words u_0 are diminished by (35). The words u_1, u_2 have been diminished in Theorem A_n since in the case C_n the words u_2 are independent of x_n, while u_1 depends on x_n only if $u_1 = x_{n-1}x_n^2$. The relation (79) shows that $w_3 \equiv_{k+1} 0$, $w_4 \equiv_{k+1} 0$. Lemma 7.21 with $s = m-1$ gives the required representation for u_5.

Consider the words w_6. For $k = n-1$ the relation (69) defines the required decomposition. Let $k < n-1$. Since $x_1, \ldots, x_{n-1}$ generate a sub-algebra of the type A_{n-1}, the crystal decomposition of $u_{kn-2}^3x_{n-1}$ has the form

$$u_{kn-2}^3x_{n-1} = \sum \alpha u_{m_1s_1}u_{m_2s_2}\cdots u_{m_ts_t}, \tag{85}$$

where $u_{m_1s_1} \leq u_{m_2s_2} \leq \ldots \leq u_{m_ts_t}$, that is $m_1 \geq m_2 \geq \ldots \geq m_t$ and $s_i \geq s_{i+1}$ if $m_i = m_{i+1}$. In particular, if $m_1 = k$ then $m_2 = \ldots = m_t = k$ and, due to the homogeneity, $t = 3$, $s_1 = n-1$, $s_2 = s_3 = n-2$. Therefore

$$u_{kn-2}^3x_{n-1} \equiv_{k+1} \varepsilon u_{kn-1}u_{kn-2}^2. \tag{86}$$

Along similar lines, the following relations are valid as well

$$u_{kn-2}^3x_{n-1}^2 \equiv_{k+1} \mu u_{kn-1}^2u_{kn-2}, \quad u_{k\,n-2}^2x_{n-1}^3 \equiv_{k+1} 0. \tag{87}$$

Now let us multiply (33) with $i = n-2$ by x_{n-1} from the right, and then add to the result the same relation multiplied by $p_{n-2\,n-1}(1+p_{n-1n-1})x_{n-1}$ from the left. We get the following relation with $\alpha = p_{n-2\,n-1}^2 p_{n-1n-1}^{[3]} \neq 0$.

$$x_{n-2}x_{n-1}^3 = \alpha x_{n-1}^2 x_{n-2}x_{n-1} + \beta x_{n-1}^3 x_{n-2}, \tag{88}$$

Further, we may write

$$u_{k\,n-1}^3 = \beta_1 u_{k\,n-2}u_{k\,n-3}\underline{x_{n-1}x_{n-2}x_{n-1}}u_{k\,n-1}, \quad \beta_1 \neq 0, \tag{89}$$

where for $k = n-2$ the term $u_{k\,n-3}$ is absent. Let us apply (33) with $i = n-2$ to the underlined word. Since $u_{k\,n-2}u_{k\,n-3}x_{n-1}^2 \equiv_{n-1} 0$, we have got

$$u_{k\,n-1}^3 \equiv_{n-1} \beta_2 u_{k\,n-2}^2 u_{k\,n-3}\underline{x_{n-1}^2 x_{n-2}x_{n-1}}. \tag{90}$$

Let us apply (88). Taking into account the second of (87) we get

$$u_{k\,n-1}^3 \equiv_{k+1} \beta_3 u_{k\,n-2}^3 x_{n-1}^3. \tag{91}$$

Let us multiply this relation from the right by x_n. By (69) we have

$$u_{k\,n-1}^3 x_n \equiv_{k+1} \alpha \underline{u_{k\,n-2}^3 x_{n-1}} x_n x_{n-1}^2 + \beta \underline{u_{k\,n-2}^3 x_{n-1}^2} x_n x_{n-1}. \tag{92}$$

By means of (86) and (87) we have got

$$u_{k\,n-1}^3 x_n \equiv_{k+1} \alpha_1 u_{k\,n-1}x_n u_{k\,n-2}^2 x_{n-1}^2 + \beta_1 u_{k\,n-1}^2 x_n u_{k\,n-2}x_{n-1},$$

and both of these words are less than $u_{k\,n-1}^3 x_n$. □

LEMMA 7.23. *If $p_{11} \neq 1$ then the values of $[v]^h$, where $[v] \in B$, $v \neq x_i$, $h \geq 1$ are not skew primitive. In particular they are non-zero.*

Proof. Note that for $n > 3$ the algebra generated by $x_2, \ldots x_n$ is a sub-algebra of the type C_{n-1}. Therefore we may use induction on n with additional supposition that the theorem statements 1 and 2 are valid for the lesser values of n. We will formally consider the sub-algebra generated by x_{n-1}, x_n as an algebra of the type C_2, and the sub-algebra generated by x_n as an algebra of type C_1. In this case for $n = 1$ the present lemma and the statements 1 and 2 are valid in obvious way.

If the first letter x_k of v is less than x_1 then we may use the inductive supposition directly. If $v = u_{1m}$ then one may literally repeat arguments of Lemma 7.6 starting at (39).

If $v = v_{1m}$ and $n > 3$ then we may repeat arguments of Lemma 7.16 starting at (63) up to replacing w with v. For $n = 3$ in these arguments the formula (66) assumes the form

$$\Sigma = \varepsilon g_1(-p(v_{13}, x_2)p_{21}[x_2^2 x_3] + (1 - p_{11}^{-1})[x_2 x_3] \cdot x_2) \otimes x_1. \tag{93}$$

Therefore the left component of the tensor Σ is a non-zero linear combination of the basis elements. For $n = 2$ the set B has no elements v_{1m} at all.

Consider the last case, $v = v_1 = [u_{1n-1}^2 x_n]$. Let S_k be the sum of all tensors of $\Delta([u_{kn}]) = \sum u^{(1)} \otimes u^{(2)}$ with $\deg_n(w^{(1)}) = 1$, $\deg_k(w^{(1)}) = 0$, $k < n$. Evidently

$S_n = x_n \otimes 1$. Let us show by downward induction on k that $S_k = (1-p_{11}^{-1})g(u_{kn-1})x_n \otimes [u_{kn-1}]$ at $k < n$. We have

$$\Delta([u_{kn}]) = \Delta(x_k)\Delta([u_{k+1n}]) - p(x_k, u_{k+1n})\Delta([u_{k+1n}])\Delta(x_k). \tag{94}$$

Consequently,

$$S_k = (g_k \otimes x_k)S_{k+1} - p(x_k, u_{k+1n})S_{k+1}(g_k \otimes x_k). \tag{95}$$

This implies the required formula since by (70) at $k < n-1$ we have

$$p(x_k, u_{k+1n})p(x_n, x_k) = p(x_k, u_{k+1n-1}),$$

while at $k = n-1$ we have $p(x_{n-1}, x_n)p(x_n, x_{n-1}) = p_{11}^{-1}$.

In a similar manner, consider the sum S of all tensors of $\Delta([u_{kn}^2 x_n]) = \sum w^{(1)} \otimes w^{(2)}$ with $\deg_n(w^{(1)}) = 1$, $\deg_i(w^{(1)}) = 0$, at $i < n$.

$$\Delta([[u_{1n-1}][u_{1n}]]) = \Delta([u_{1n-1}])\Delta([u_{1n}]) - p(u_{1n-1}, u_{1n})\Delta([u_{1n}])\Delta([u_{1n-1}]). \tag{96}$$

Since we now S_1, we may calculate S :

$$\begin{aligned} S = (g(u_{1n-1}) \otimes [u_{1n-1}])S_1 - p(u_{1n-1}, u_{1n})S_1(g(u_{1n-1}) \otimes [u_{1n-1}]) = \\ (1-p_{11}^{-1})g(u_{1n-1}^2)x_n \otimes (1 - p(u_{1n-1}, u_{1n})p(x_n, u_{1n-1}))[u_{1n-1}]^2. \end{aligned} \tag{97}$$

By (70), using the bicharacter property of p, we have

$$1 - p(u_{1n-1}, u_{1n})p(x_n, u_{1n-1}) = 1 - p(u_{1n-1}, u_{1n-1})p_{n-1n}p_{nn-1} =$$
$$1 - p_{n-1n-1}p_{n-1n-1}^{-2} = 1 - p_{11}^{-1} \neq 0.$$

Because of this, $S \neq 0$ and the sum of all tensors $w^{(1)} \otimes w^{(2)}$ with $\deg_n(w^{(1)}) = h$, $\deg_k(w^{(1)}) = 0$, $k < n$ of the basis decomposition of $\Delta([v_1]^h)$ equals $S^h \neq 0$. Therefore $[v_1]^h$ is not skew primitive. $\square$

Proof of Theorem C_n. For the first statement it will suffice to prove that all super-letters (67) are hard in $U_P(\mathfrak{g})$. Since none of u_{km}, v_{km} contains a sub-word (28), Lemma 7.1 implies that $[u_{km}]$, $[v_{km}]$ are hard.

If $[v_k]$ is not hard then, by the homogeneous version of Definition 4.3, its value is a polynomial in lesser hard super-letters. In line with Lemmas 7.22 and 4.8, all hard super-letters belong to B. Therefore, by Lemma 7.19, $[v_k] = 0$. Since $\deg_n(v_k) = 1$ and $\deg_{n-1}(v_k) = 2$, the equality $[v_k] = 0$ is valid in the algebra C' which is defined by all relations of $U_P(\mathfrak{g})$, but ones of degree greater than 1 in x_n and ones of degree greater than 2 in x_{n-1}, that is in the algebra defined by (33), (34) with $i < n-1$, and (35). These relations do not reverse the order of x_{n-1} and x_n in monomials since none of them has both x_{n-1} and x_n. This implies that the sum of all monomials of $[v_k] = [u_{kn-1}] \cdot [u_{kn}] - p(u_{kn-1}, u_{kn})[u_{kn}] \cdot [u_{kn-1}]$ in which x_n is prefixed to x_{n-1} equals zero in C', that is $[u_{kn}] \cdot [u_{kn-1}] = 0$. Especially, this equality is valid in $U_P(\mathfrak{g})$. Since, by Theorem 4.5, the super-word $[u_{kn}] \cdot [u_{kn-1}]$ is a basis element, the first statement is proved.

If $[v] \in B$ is of finite height then, by Lemma 7.19 and the homogeneous version of Definition 4.4, we have $[v]^h = 0$. For $p_{11} \neq 1$ this contradicts Lemma 7.23. In a similar manner, according to Lemma 4.9, every skew primitive homogeneous element has the form $[v]^h$. This, together with Lemma 7.23, proves the fourth statement and, for $p_{11}\neq 1$, the second one too. If $p_{11}=1$ then according to (70) we have $p_{ii}=p_{ij}p_{ji}=1$ at all i,j. In particular, the skew commutator is a quantum operation. Hence all elements of B are skew primitive. These elements span a colour Lie algebra. Now, as in Theorem A_n, we may use the coloured PBW theorem.

The third statement will follow from Theorem 5.2 and Lemmas 5.3, 7.18 provided we note that all super-letters (68) are zero in $U_P(\mathfrak{g})$. We have proved already that these super-letters are non-hard. So it remains to use first the homogeneous version of Definition 4.3 and then Lemma 7.26. □

Theorem $\mathbf{D_n}$. *Let $\mathfrak{g}$ be of the type D_n, and $p_{ii}\neq -1$, $1\leq i\leq n$. Denote by B the set of the following super-letters:*

$$\begin{array}{lcll} [u_{km}] & \stackrel{df}{=} & [x_kx_{k+1}\dots x_m], & 1\leq k\leq m<n; \\ [e_{km}] & \stackrel{df}{=} & [x_kx_{k+1}\dots x_{n-2}\cdot x_nx_{n-1}\dots x_m], & 1\leq k<m\leq n, \\ [e_{n-1n}] & \stackrel{df}{=} & x_n. & \end{array} \tag{98}$$

The statements given below are valid.

1. *The values of* (98) *in $U_P(\mathfrak{g})$ form the PBW-generators set.*
2. *Each of the super-letters* (98) *has infinite height in $U_P(\mathfrak{g})$.*
3. *The relations* (21) *together with the following ones form a Groebner–Shirshov system that determines the crystal basis of $U_P(\mathfrak{g})$.*

$$\begin{array}{lcll} [u_0] & \stackrel{df}{=} & [x_kx_m]=0, & 1\leq k<m-1<n,\ (k,m)\neq(n-2,n); \\ [u_1] & \stackrel{df}{=} & [u_{km}x_{k+1}]=0, & 1\leq k<m<n; \\ [u_1'] & \stackrel{df}{=} & [x_{n-2}x_n^2]=0, & \\ [u_2] & \stackrel{df}{=} & [u_{km}u_{k\,m+1}]=0, & 1\leq k\leq m<n-1; \\ [v_3] & \stackrel{df}{=} & [e_{km}x_{k+1}]=0, & 1\leq k<m\leq n,\ n-1\neq k\neq m-2; \\ [v_4] & \stackrel{df}{=} & [e_{kk+1}x_{k+2}]=0, & 1\leq k<n-2; \\ [v_4'] & \stackrel{df}{=} & [e_{n-3\,n-2}x_n]=0, & \\ [v_5] & \stackrel{df}{=} & [e_{km}e_{k\,m-1}]=0, & 1\leq k<m-1\leq n-1; \\ [v_6] & \stackrel{df}{=} & [u_{km}e_{kn}]=0, & 1\leq k\leq m<n,\ n-2\leq m. \end{array} \tag{99}$$

4. *If $p_{11}\neq 1$, then the generators x_i, their powers $x_i^t, x_i^{tl^k}$, such that p_{ii} is a primitive t-th root of 1, together with the constants $1-g$, $g\in G$ form a basis of $\mathfrak{g}_P=L(U_P(\mathfrak{g}))$. Here $l=char(\mathbf{k})$.*
5. *If $p_{11}=1$, then the elements of B and, for $l>0$, their l^k-th powers together with the constants $1-g$, $g\in G$ form a basis of $\mathfrak{g}_P$.*

In the case D_n the algebra $U_P^b(\mathfrak{g})$ can be defined by the condition that the subalgebras U_{n-1} and U_n generated, respectively, by $x_1,\dots,x_{n-1}$ and $x_1,\dots,x_{n-2},x_{n-1}'=$

x_n are quantum universal enveloping algebras of the type A_{n-1}, and by the only additional relation

$$[x_{n-1}x_n] = 0. \tag{100}$$

The existence conditions take up the form

$$\begin{gathered} p_{ii} = p_{nn} = p_{11},\ p_{i+1i}p_{ii+1} = p_{n-2n}p_{nn-2} = p_{11}^{-1}, \quad \text{if } 1 \le i < n, \\ p_{n-1n}p_{nn-1} = p_{ij}p_{ji} = 1, \quad \text{if } i-j > 1\&(i,j) \neq (n, n-2). \end{gathered} \tag{101}$$

LEMMA 7.24. *The brackets in* (98) *are set up by the recurrence formulae*

$$\begin{array}{lcll} [e_{km}] &=& [x_k[e_{k+1m}]], & \textit{if } 1 \le k < m-1 < n,\ k \neq n-1; \\ [e_{kk+1}] &=& [[e_{kk+2}]x_{k+1}], & \textit{if } 1 \le k < n-1. \end{array} \tag{102}$$

Proof. It is enough to use the properties 6s, 1s, and 2s. □

LEMMA 7.25. *If* $[u],[v] \in B$, *then one of the statements below is correct.*
1) $[[u][v]]$ *is not a standard nonassociative word;*
2) uv *contains a sub-word of one of the types* $u_0, u_1, u_1'u_2, v_3, v_4, v_4', v_5, v_6$;
3) $[[u][v]] \in B$.

Proof. The formulae (102) coincides with (49) at $k \neq n-1$ up to replacing e by w. The inequality $e_{km} > e_{rs}$ is set up by the same conditions, $k < r \vee (k = r\&m < s)$, as the inequality $w_{km} > w_{rs}$ does. Likewise $u_{km} > e_{rs}$ is set up by the same condition, $k \le r$, as $u_{km} > w_{rs}$ does. Therefore Lemmas 7.8, 7.9, 7.10 remain valid with e in place of w :

$$\begin{array}{lll} [[e_{km}][e_{rs}]] & \text{is standard} \ \Leftrightarrow & s \ge m > k+1 = r \vee (s < m\&r = k); \\ [[u_{km}][e_{rs}]] & \text{is standard} \ \Leftrightarrow & k = r \vee k = m < r; \\ [[e_{km}][u_{rs}]] & \text{is standard} \ \Leftrightarrow & r = k+1 < m \vee r = k+1 = m = s. \end{array} \tag{103}$$

By looking over all of these possibilities we get the lemma statement. □

LEMMA 7.26. *If a super-word* W *equals one of the super-letters* (99) *or* $[v]^h$, $[v] \in B$, $h \ge 1$ *then its constitution does not equal the constitution of any super-word in less than* W *super-letters from* B.

Proof. The proof is similar to the one of Lemma 7.5 with the tableaux

$$\begin{array}{lllll} [u_{km}]^h, & [u_{km}x_{k+1}], & [u_{km}u_{k\,m+1}] & & \deg_k(u) \le \deg_{m+1}(u); \\ [e_{km}]^h, & [e_{km}x_{k+1}], & [e_{km}e_{k\,m-1}], & m < n & 2\deg_k(u) \le \deg_{m-1}(u); \\ [e_{kn}]^h, & [e_{kn}x_{k+1}], & [e_{kn}e_{k\,n-1}] & & \deg_k(u) \le \deg_{m-1}(u); \end{array}$$

$$\begin{array}{ll} [e_{kk+1}x_{k+2}] & \deg_k(u) = 0; \\ [e_{n-3n-2}x_n] & \deg_{n-3}(u) = 0; \\ [u_{k\,n-2}e_{kn}] & \deg_k(u) \le \deg_{n-1}(u) + \deg_n(u); \\ [u_{k\,n-1}e_{kn}] & \deg_k(u) \le \deg_n(u). \end{array} \tag{104}$$

LEMMA 7.27. *If* $y = x_i$, $m-1 \neq i > k$ *or* $y = x_i^2$, $m-1 = i > k$ *then*

$$e_{km}y \equiv_{k+1} 0. \tag{105}$$

Proof. If $i < m-1$, $m \neq n$, or $m = n$, $i < n-2$, then with the help of (35) and (100) it is possible to permute y to the left beyond x_n and then to use Lemma 7.2 for U_{n-1}.

If $m = n$, $i = n-2$ then we may use Lemma 7.2 for U_n.

If $y = x_i^2$, $m - 1 = i > k$ then for $m < n$ by the above case we get

$$e_{km}y = e_{k\,m+1}x_m x_{m-1}^2 = \underline{e_{k\,m+1}x_{m-1}}(\alpha x_m x_{m-1} + \beta x_{m-1}x_m) \equiv_{k+1} 0. \tag{106}$$

For $m = n$ we have $e_{kn}x_{n-1}^2 = \alpha \underline{u_{k\,n-2}x_{n-1}^2} x_n \equiv_{n-1} 0$ since the underlined part belongs to U_{n-1}.

If $y = x_i$, $i = m > k$ then for $m = n$ we may use Lemma 7.2 applied to U_n; for $m = n-1$ we may use the same lemma applied to U_{n-1} provided that beforehand we permute x_n with y by (100); for $m < n-1$ we may first rewrite $e_{km}y = e_{k\,m+1}y_1$, where $y_1 = x_m^2$, and then use (106) with $m+1$ in place of m.

If $y = x_i$, $i > m > k$ then for $i < n$ we have $e_{km}y = \alpha e_{ki+1}\underline{x_i x_{i-1} x_i} \cdot v$. Replacing the underlined word by (33) in U_{n-1}, we may use the previously considered cases: $m' - 1 = i'$, where $m' = i+1$, $i' = i$; and $i' < m' - 1$, where $m' = i+1$, $i' = i-1$. For $i = n$, and $m = n-1$ we have $e_{k\,n-1}x_n = \alpha \underline{u_{k\,n-2}x_n^2} x_{n-1}$ and one may apply Lemma 7.2 to U_n. Finally, for $i = n$ and $m < n-1$ we get

$$e_{km}x_n = \beta_1 u_{k\,n-2}x_n x_{n-1}x_{n-2}x_n \cdot v = \beta_2 u_{k\,n-2}x_{n-1}\underline{x_n x_{n-1} x_n} \cdot v =$$

$$\beta_3 \underline{u_{k\,n-2}x_{n-1}x_{n-2}} x_n^2 \cdot v + \beta_4 u_{k\,n-2}\underline{x_{n-1}x_n^2} x_{n-2} \cdot v.$$

One may apply first Lemma 7.2 for U_{n-1} to the underlined sub-word of the first term, and then, after (100), Lemma 7.2 for U_n to the second term. □

LEMMA 7.28. *If $k < s < m \leq n$ then $e_{km}e_{ks} \equiv_{k+1} \varepsilon e_{ks}e_{km}$, $\varepsilon \neq 0$.*

Proof. Let us carry out downward induction on k. The largest value of k equals $n-2$. In this case $s = n-1$, $m = n$ and we have

$$\underline{x_{n-2}x_n \cdot x_{n-2}} x_n x_{n-1} \equiv_n x_{n-2}^2 \underline{x_n^2 x_{n-1}} = \alpha \underline{x_{n-2}^2 x_{n-1}} x_n^2 \equiv_{n-1}$$
$$\beta x_{n-2}x_{n-1}\underline{x_{n-2}x_n^2} \equiv_n \varepsilon x_{n-1}x_n \cdot x_{n-1}x_{n-2}x_n. \tag{107}$$

Let us first transpose the second letter x_k of $e_{km}e_{ks}$ as far to the left as possible by (35), and then replace the onset $x_k x_{k+1} x_k$ by (36). We get

$$e_{km}e_{ks} \equiv_{k+1} \alpha x_k^2 (e_{k+1m}e_{k+1s}), \quad \alpha \neq 0. \tag{108}$$

For $k+1 < s$ it suffices to apply the inductive supposition to the word in the parentheses and then by (36) and (35) to put x_k to the proper place.

For $k+1 = s$ one may use downward induction on s. The basis of this induction, $s = n-1$, has been proved, see (107). For $k < n-3$ the inductive step on s coincides with the one of Lemma 7.14 with e in place of w since in this case the active variables x_k, x_{k+1} q-commute with x_n. If $k = n-3$ then in consideration of Lemma 7.14 the variable $x_{k+1} = x_{n-2}$ is transposed across x_n twice: in (58) and in the second word of (60).

In (58) with $k=n-3$ we have $s=n-2$, $m=n$; and (58) becomes

$$e_{n-3n}e_{n-3n-2}\equiv_{n-2}\beta e_{n-3n-1}\underline{x_{n-3}x_{n-2}x_nx_{n-2}}. \tag{109}$$

In view of Lemma 7.27, we may transform the underlined part in U_n neglecting the words starting with x_{n-2}^2 and x_n in much the same way as in (59), with x_n in place of x_{k+1}. So (109) reduces to the required form.

The second word of (60) with $k=n-3$ assumes the form $e_{n-3n}^2x_{n-2}x_{n-1}^2=e_{n-3n}\underline{x_{n-3}x_{n-2}x_nx_{n-2}}x_{n-1}^2$. By Lemma 7.2 applied to U_n, the underlined word is a linear combination of words starting with x_{n-2} and x_n. However, by Lemma 7.27 both $e_{n-3n}x_{n-2}$ and $e_{n-3n}x_n$ equal zero up to $\equiv_{n-2}$. □

Lemma 7.29. *The set B satisfies the conditions of Lemma* 4.8.

Proof. By Lemmas 7.25 and 4.7 one need show only that in $U_P^b(\mathfrak{g})$ the words (99) are linear combinations of lesser ones. The words v_6 with $m=n-2$, and u_0, u_1, u_1', u_2 have the required decomposition since they belong either to U_{n-1} or to U_n. Lemma 7.27 shows that $v_3\equiv_{k+1}0$, $v_4\equiv_{k+1}0$, $v_4'\equiv_{k+1}0$. Lemma 7.28 with $s=m-1$ yields the required representation for v_5. Consider v_6 with $m=n-1$. Let us prove by downward induction on k that

$$u_{k\,n-1}e_{kn}\equiv_{k+1}\varepsilon e_{kn}u_{k\,n-1},\quad \varepsilon\neq 0.$$

For $k=n-1$ this equality assumes the form (100). Let $k<n-1$. Let us transpose the second letter x_k of $u_{k\,n-1}e_{kn}$ as far to the left as possible in U_{n-1}. After an application of (33) we get

$$u_{k\,n-1}e_{kn}\equiv_{k+1}\alpha x_k^2(u_{k+1n-1}e_{k+1n}),\quad \alpha\neq 0.$$

It suffices to apply the inductive supposition to the term in the parentheses, and then by (33) and (35) for U_n to move x_k to the proper place. □

Lemma 7.30. *If $p_{11}\neq 1$ then the values of $[v]^h$, where $[v]\in B$, $v\neq x_i$, $h\geq 1$ are not skew primitive, in particular they are non-zero.*

Proof. One need consider only super-letters that belong neither to U_{n-1} nor to U_n. That is $[e_{km}]$ with $m<n$. We use induction on n.

For $n=3$ the algebra of the type D_3 reduces to the algebra of the type A_3 with a new ordering of variables $x_2>x_1>x_3$. Therefore we may use Theorem A_n, after the decomposition below of e_{12} in the PBW-basis:

$$[[x_1x_3]x_2]=-p_{12}p_{32}[x_2[x_1x_3]]+\beta[x_1x_3]\cdot x_2.$$

Let $n>3$. If $k>1$ then the inductive supposition works. For $k=1$, $m>2$ we have $e_{1m}=[x_1[e_{2m}]]$, and one may repeat the arguments of Lemma 7.6 with e in place of u starting at (39). If $m=2$ then we may repeat the arguments of Lemma 7.16 with e on place of w starting at (63). □

Proof of Theorem D_n. For the first statement it will suffice to prove that all super-letters (98) are hard in $U_P^b(\mathfrak{g})$.

Since none of u_{km} contains sub-words (28), $[u_{km}]$ are hard.

Suppose $[e_{km}]$ is non-hard. By Lemmas 7.29 and 4.8 all hard super-letters belong to B. Thus, by Lemma 7.26, we get $[e_{km}]=0$. Since $\deg_n(e_{km})=\deg_{n-1}(e_{km})=1$, the equality $[e_{km}]=0$ is also valid in the algebra D' defined by the same relations as $U_P^b(\mathfrak{g})$ is, but $[x_{n-2}x_n^2]=0$ and $[x_{n-2}x_{n-1}^2]=0$. Let us equate to zero all monomials in all the defining relations of D', but $[x_{n-1}x_n]=0$. Consider the algebra R' defined by (100) and by the resulting system of monomial relations. It is easy to verify that the mentioned relations system Σ of R' is closed under the compositions. Since e_{km} contains none of leading words of Σ, the super-letter $[e_{km}]$ is non-zero in R', and so in D' too. This contradiction proves the first statement.

If $[v]^h, [v]\in B$ is of finite height then by Lemma 7.26 and the homogeneous version of Definition 4.4 we have $[v]^h=0$. For $p_{11}\neq 1$ this contradicts Lemma 7.30. In a similar manner, by Lemma 4.9, every skew primitive homogeneous element has the form $[v]^h$. This, together with Lemma 7.30, proves both the fourth statement and the second one with $p_{11}\neq 1$.

If $p_{11}=1$ then by (101) we have $p_{ii}=p_{ij}p_{ji}=1$ for all i,j. This means that the skew commutator itself is a quantum operation. Hence all elements of B are skew-primitive. These elements span a colour Lie super-algebra. Now, as in Theorem A_n, one may use the PBW theorem for colour Lie super-algebras.

For the third statement it will suffice to show that all super-letters (99) are zero in $U_P(\mathfrak{g})$. We have proved already that they are non-hard. Therefore it remains to use the homogeneous version of Definition 4.3 and Lemma 7.26. □

8. *Conclusion*

We see that in all Theorems A_n–D_n the lists of hard super-letters are independent of the parameters p_{ij}. This fact signifies that the Lalonde–Ram basis of the ground Lie algebra (see, [**26**, Figure 1]) with the skew commutator in place of the Lie operation coincides with the set of all hard super-letters. It is very interesting to clarify how general this statement is. On the one hand, this does not hold without exception for all quantum enveloping algebras since in Theorems A_n–D_n a restriction does exist. If $p_{ii}=-1$, $1\leq i<n$, $n>2$ then it is easy to see by means of the Diamond Lemma that the sets of hard super-letters are infinite. On the other hand, this is not a specific property of Lie algebras defined by the Serre relations. By the Shirshov theorem [**36**] any relation can be reduced to a linear combination of standard nonassociative words.

Corollary 8.1. *If $\mathfrak{g}$ is defined by the only relation $f=0$, where f is a linear combination of standard nonassociative words, then the set of all hard in $U_P(\mathfrak{g})$ super-letters coincides with the Hall–Shirshov basis of $\mathfrak{g}$ with the skew commutator in place of the Lie operation.*

Proof. The only relation $f^*=0$ forms a Groebner–Shirshov system since, according to 1s, none of onsets of its leading word, say w, coincides with a proper terminal of w. Consequently, a super-letter $[u]$ is hard if and only if u does not contain w as a sub-word. We see that this criteria is independent of p_{ij} as well. □

Furthermore, the third statement of Theorem A_n shows that $U_P^b(\mathfrak{g})$ can be defined by the following relations in the PBW-generators $X_u = [u]$.

$$\begin{array}{lll} [X_u, X_v] = 0, & u > v, & [[u][v]] \notin B \\ [X_u, X_v] = X_{uv}, & & [[u][v]] \in B. \end{array} \tag{110}$$

This is an argument in favour of considering the super-letters PBW-generators $\mathbf{k}[G]$-module as a quantum analogue of a Lie algebra. However in the cases B_n, C_n, D_n the defining relations became more complicated. For example,

$$\begin{array}{lll} B_n: & [[u_{k\,n-1}][w_{kn}]] = \alpha[u_{kn}]^2, & \alpha \neq 0 \text{ if } p_{nn} \neq 1; \\ C_n: & [[u_{k\,n-2}][v_{k\,n-1}]] = \alpha[v_k] + \beta[u_{kn}] \cdot [u_{k\,n-1}], & \beta \neq 0 \text{ if } p_{11} \neq 1; \\ D_n: & [[u_{k\,n-2}][e_{k\,n-1}]] = \alpha[e_{kn}] \cdot [u_{k\,n-1}], & \alpha \neq 0 \text{ if } p_{11} \neq \pm 1. \end{array} \tag{111}$$

It is far more interesting that for $p_{11} \neq 1$ the algebra $\mathfrak{g}_P$ turns out to be very simple in structure. Only unary quantum operations can be non-zero. Other ones may be defined, but due to the homogeneity their values equal zero. In particular, if $p_{11}^{[t]} \neq 0$ then without exception all quantum operations have zero values. This provides reason enough to consider $U_P(\mathfrak{g}) = U(\mathfrak{g}_P)$ as an algebra of 'commutative' quantum polynomials. Certainly it is very interesting to elucidate to what extent this statement is still retained for the quantum universal enveloping algebras of homogeneous components of other Kac–Moody algebras defined by the Gabber–Kac relations (11). Also it is interesting to investigate the structure of other 'commutative' quantum polynomial algebras. For example, one may note that if a semi-group generated by $p_{ij}p_{ji}$ does not contain 1, then $G\langle x_1, \dots, x_n\rangle$ itself is a 'commutative' quantum polynomial algebra merely since in this case there exists no non-zero quantum operation at all. In another extreme case when $p_{ij}p_{ji} = 1$ for all i, j, the 'commutative' quantum variables commute by $x_i x_j = p_{ij} x_j x_i$.

In a similar manner, the Drinfeld–Jimbo enveloping algebra can be considered as a 'quantum' Weyl algebra of (skew) differential operators (see Sec. 6). The resulting 'quantum' Weyl algebra is simple in the following sense.

Corollary 8.2. *Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra of the infinite series. If $q^{[m]} \neq 0$, $m \geq 2$ then every non-zero Hopf ideal I of the Drinfeld–Jimbo enveloping algebra contains all generators x_i, x_i^-.*

Proof. By the Heyneman–Radford theorem, the ideal I has a non-zero skew primitive element, say a. According to Lemma 6.2 and Theorems A_n–C_n, the element a is either a constant, $\alpha(1-g)$, or proportional to one of the elements x_i, x_i^-. In the former case I contains all x_i with $\chi^i(g) \neq 1$ since $x_i a - \chi^i(g) a_i x_i = \alpha(1 - \chi^i(g))x_i$. Here the equality $\chi^i(g) = 1$ can not be valid for all i since $\chi^i(g_j) = q^{-d_i a_{ij}}$ (see, Example 4 of Section 2) and the columns of the Cartan matrix are linearly independent. In the latter case (and now in the former one as well) we get $[x_i, x_i^-] = \varepsilon_i(1 - g_i^2) \in I$, i.e. as above I contains all elements $y = x_i^{\pm}$ with $1 \neq \chi^y(g_i^2) = q^{\pm 2 d_j a_{ij}}$. Since the Coxeter graph is connected, I contains all x_i, x_i^-. □

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