ON A PROPERTY OF SPECIAL GROUPS

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ABSTRACT. Let G be an algebraic group defined over an algebraically closed field k of characteristic zero. We give a simple proof of the following result: if $H^1(K_0, G) = \{1\}$ for some finitely generated field extension K_0/k of transcendence degree ≥ 3 then $H^1(K, G) = \{1\}$ for every field extension K/k.

1. INTRODUCTION

Let G be an algebraic group. J.-P. Serre stated the following conjectures in [Se_2] (see also [Se_3, Chapter III]).

Conjecture I: If G is connected then $H^1(K, G) = \{1\}$ for every perfect field K of cohomological dimension ≤ 1.

Conjecture II: If G is semisimple, connected and simply connected then $H^1(K, G) = \{1\}$ for every perfect field K of cohomological dimension ≤ 2.

Conjecture I was proved by Steinberg [St_1]. Conjecture II remains open, though significant progress has been made in recent years; see [BP] and [Gi].

Our main result is a partial converse of Conjectures I and II. Recall that an algebraic group G is called *special* if $H^1(K, G) = \{1\}$ for every field K. Special groups were introduced by Serre [Se_1] and classified by Grothendieck [Gr, Section 5]; cf. [PV, Section 2.6].

Theorem 1. *Let G be an algebraic group defined over an algebraically closed field k of characteristic zero. Suppose $H^1(K, G) = \{1\}$ for some finitely generated field extension K of k of transcendence degree d.*

(a) If $d \geq 1$ then G is connected.

(b) If $d \geq 2$ then G is simply connected.

(c) If $d \geq 3$ then G is special.

Note that the cohomological dimension of K equals d; see [Se_3, Section II.4]. Thus, informally speaking, the theorem may be interpreted as saying that Conjectures I and II cannot be extended or strengthened in a meaningful way.

Our proof of Theorem 1 is rather simple: the idea is to use nontoral finite abelian subgroups of G as obstructions to the vanishing of H^1. We remark that our argument (and, in particular, the proof of Lemma 3) does not rely on canonical resolution of singularities; cf. [RY, Remark 4.4].

Ph. Gille recently showed us an alternative proof of Theorem 1, based on case by case analysis and properties of the Rost invariant. We would like to thank him, J.-L. Colliot-Thélène and R. Parimala for informative discussions.

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2. Preliminaries

Throughout this note k will denote an algebraically closed base field of characteristic zero. All fields, varieties, morphisms, algebraic groups, etc., will be assumed to be defined over k.

Let G be an algebraic group. An abelian subgroup A of G is called *toral* if A is contained in a torus of G and *nontoral* otherwise.

Lemma 2. *Let G be an algebraic group, L be a Levi subgroup of G and A be a finite abelian subgroup of L. If A is nontoral in L then A is nontoral in G.*

Proof. Assume the contrary: $A \subset T$ for some torus T of G. Since T is reductive, it lies in a Levi subgroup L_1 of G; see [OV, Theorem 6.5]. Denote the unipotent radical of G by U; then L and L_1 project isomorphically onto G/U. Since A is toral in L_1, it is toral in G/U, and hence, in L, as claimed. $\square$

Recall that a G-variety X is an algebraic variety with a G-action; X is *generically free* if G acts freely on a dense open subset of X and *primitive* if $k(X)^G$ is a field (note that X is allowed to be reducible). Elements of $H^1(K, G)$ are in 1—1 correspondence with *G-torsors* over K, i.e., birational classes of primitive generically free G-varieties X such that $k(X)^G = K$; see e.g., [Po, Section 1.3]. If X is a primitive generically free G-variety, we shall write $cl(X)$ for the class in $H^1(k(X)^G, G)$ given by X.

Our proof of Theorem 1 is based on the following result.

Lemma 3. *(*[RY, Lemma 4.3]*) Let G be an algebraic group, A be a nontoral finite abelian subgroup of G and X be a generically free primitive G-variety. Suppose A fixes a smooth point of X. Then $cl(X) \neq 1$ in $H^1(k(X)^G, G)$.*

3. Construction of a nontrivial torsor

Lemma 4. *Let A be an abelian group of rank r and let K be a finitely generated field extension of k of transcendence degree $d \geq r$. Then there exists an A-variety Y such that (i) $k(Y)^A = K$ and (ii) Y has a smooth A-fixed point.*

Proof. Since k is algebraically closed, A has a faithful r-dimensional representation V_1. Let V_2 be the trivial $(d-r)$-dimensional representation of A, and $V = V_1 \oplus V_2$. Then the (geometric) quotient V/A is isomorphic to the affine space $\mathbb{A}^d$. Denote the origin of V by 0, and its image in V/A by $\overline{0}$.

Let Y_0 be an affine variety over k such that $k(Y_0) = K$; $\dim(Y_0) = d$. Let $y_0 \in Y_0$ be a smooth point. Identifying $V/A = \mathbb{A}^d$, we can find a dominant projection $f \colon Y_0 \longrightarrow V/A$ such that $f(y_0) = \overline{0}$ and f is étale at y_0.

Now set $Y = Y_0 \times_{V/A} V$; the A-action on Y is induced from V. The natural projection $Y \longrightarrow Y_0$ is a rational quotient map for this action; see, e.g., [R, Lemma 2.16(a)]. Thus Y satisfies (i). To prove (ii), set $y = (y_0, 0)$; y is fixed by A. The morphism $Y \longrightarrow V$ is obtained from f by a base change, and hence, is étale at y; the smoothness of V implies then that Y is smooth at y. Thus, $y \in Y$ is a smooth point fixed by A. $\square$

Proposition 5. *Let G be an algebraic group, A be a nontoral abelian subgroup of G of rank r, and K/k be a field extension of transcendence degree d. If $d \geq r$ then $H^1(K, G) \neq \{1\}$.*

Proof. Choose an A-variety Y and a smooth A-fixed point $y \in Y$, as in Lemma 4. We claim that the image of $cl(Y)$ under the natural map $H^1(K,A) \longrightarrow H^1(K,G)$ is nontrivial. Indeed, recall that the image of $cl(Y)$ in $H^1(K,G)$ is $cl(X)$, where

$$X = G *_A Y = (G \times Y)/A$$

is the (geometric) quotient for the A-action on $G \times Y$ given by $a(g,y') = (ga^{-1}, ay')$; see [PV, Section 4.8]. By [PV, Proposition 4.22], $G *_A Y$ is smooth at $x = (1_G, y)$ since Y is smooth at y. Moreover, x is an A-fixed point of X; thus Lemma 3 tells us that $cl(X) \neq 1$ in $H^1(K,G)$, as claimed. □

4. Proof of Theorem 1

In view of Proposition 5 it is sufficient to show that G contains a nontoral finite abelian subgroup A, where

(a′) $\mathrm{rank}(A) = 1$, if G is not connected,

(b′) $\mathrm{rank}(A) \leq 2$, if G is not simply connected and

(c′) $\mathrm{rank}(A) \leq 3$, if G is not special.

Moreover, in view of Lemma 2 we only need to prove (a′), (b′), and (c′) under the assumption that G is reductive (otherwise we may replace G by its Levi subgroup).

Proof of (a′): Write $G = FG_0$, where G_0 is the identity component of G and F is a finite group; see [V, Proposition 7]. Since G is disconnected, F is not contained in G_0. Choose $a \in F - G_0$ and set $A = <a>$. Then A is cyclic, finite (because $a \in F$) and nontoral (because every torus of G in contained in G_0), as desired.

Proof of (b′): In view of (a′) we may assume without loss of generality that G is connected. Now the desired conclusion follows from [St_2, Theorem 2.27].

Proof of (c′): Suppose G is not special. By [Se_4, 1.5.1], G has a torsion prime p, and by [St_2, Theorem 2.28] G has a nontoral elementary p-abelian subgroup A of rank ≤ 3. see also [Se_4, 1.3]. □

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