A REMARK ON QUIVER VARIETIES AND WEYL GROUPS

ANDREA MAFFEI

ABSTRACT. In this paper we define an action of the Weyl group on the quiver varieties $M_{m,\lambda}(d,v)$ with generic (m,λ). To do it we describe a set of generators of the projective ring of a quiver variety. We also prove connectedness for the smooth quiver variety $M(d,v)$ and normality for $M_0(d,v)$ in the case of a quiver of finite type and $d-v$ a regular weight.

In [13, 14] Nakajima defined quiver varieties and show how to use them to give a geometric construction of integrable representation of Kac-Moody algebras. Luckily these varieties can be used also to give a geometric construction of representations of Weyl groups. In [9], Lusztig constructed a representation of the Weyl group on the homology of quiver varieties. His construction is similar to the construction of Springer representations. In [14], Nakajima gave an construction of isomorphism $\Phi_{\sigma,\zeta}(d,v) : \mathfrak{M}_\zeta(d,v) \longrightarrow \mathfrak{M}_{\sigma\zeta}(d,\sigma(v-d)+d)$ in the case of a quiver of finite type. His construction was analytic and relies on a description of quiver varieties as moduli spaces of instantons on ALE spaces. The main result of this paper is a direct and algebraic construction of these isomorphism which works for a general quiver without simple loops. To do it we also describe a set of generators of the algebra of covariant functions.

The paper is organized as follows. In the first section we fix the notation and we give the definition of a quiver variety: $M_{m,\lambda}(d,v)$ where m,λ are two parameter, d is a weight of the algebra associated to the quiver and v an element of the root lattice. We are interested to quiver varieties as algebraic varieties but to explain one of the applications we need to give also the hyperKähler construction of a quiver variety. We use a result of Migliorini [11] to explain the connection between the two constructions.

Algebraic quiver varieties are defined as the **Proj** scheme of a ring of covariants. In the second section we describe a set of generators of this ring. In a special case which is not directly to Nakajima's quiver varieties we are also able to give a more precise results and to describe a basis of the vector space of χ-covariants functions.

In the third section we use this description to generalize a construction of Lusztig [9]. Namely for any element of the Weyl group we construct an isomorphism Φ_σ between $M_{m,\lambda}(d,v)$ and $M_{\sigma m,\sigma\lambda}(d,\sigma(v-d)+d)$ if m,λ are generic.

In the fourth section, following Nakajima [13], we show how to use the action constructed in section 3 (and the connection between the hyperKähler construction and the algebraic construction) to describe an action of the Weyl group on the homology

The research was supported by the MSRI, Berkeley.

of a class of quiver varieties. This action is different from the one constructed by Lusztig in [9].

In the fifth section we give a result which reduce the study of geometric and algebraic properties of quiver varieties $M_{0,0}(d,v)$ to the case $d-v$ dominant.

In the sixth section we prove the normality of the quiver variety $M_0(d,v)$ and the connectedness of $M(d,v)$ in the case of a quiver of finite type and $d-v$ a regular weight.

1. Notations and definitions

In this section we give the definition of quiver varieties. Except some minor change all definition are due to Nakajima [13, 14].

1.1. **The graph.** Let (I,H) be a finite oriented graph: I is the set of vertices that we suppose of cardinality n, H the set of arrows and the orientation is given by the two maps

$$h \longmapsto h_0 \text{ and } h \longmapsto h_1$$

from H to I. We suppose also that:

1. $\forall\, h \in H \quad h_0 \neq h_1$,
2. an involution $h \mapsto \bar{h}$ of H without fixed points and satisfying $\bar{h}_0 = h_1$ is fixed,
3. a map $\varepsilon : H \longrightarrow \{-1,1\}$ is given such that $\varepsilon(\bar{h}) = -\varepsilon(h)$. We define $\Omega = \{h \in H \,:\, \varepsilon(h) = 1\}$ and $\overline{\Omega} = \{h \in H \,:\, \varepsilon(h) = -1\}$.

Observe that given a symmetric graph without loops is always possible to define ε and an involution $^-$ as above.

1.2. **The Cartan matrix and the Weyl group.** Let A be the matrix whose entries are the numbers

$$a_{ij} = card\{h \in H \,:\, h_0 = i \text{ and } h_1 = j\}.$$

We define a generalized symmetric Cartan matrix by $C = 2I - A$. Following [7] an X,Y-regular root datum $(I, X, X^\vee, <\,,>)$ with Cartan matrix equal to C is defined in the following way:

1. $X^\vee$ and X are finitely generated free abelian groups,
2. $<\,,>: X \times X^\vee \longrightarrow \mathbb{Z}$ is a perfect bilinear pairing,
3. two linearly independent sets $\Pi = \{\alpha_i \,:\, i \in I\} \subset X$ and $\Pi^\vee = \{\alpha_i^\vee \,:\, i \in I\} \subset X^\vee$ are fixed and we set $Q = \langle \Pi \rangle$ and $Q^\vee = \langle \Pi^\vee \rangle$,
4. $<\alpha_i\,, \alpha_j^\vee> = c_{ij}$,
5. (nonstandard) $\operatorname{rank} X = \operatorname{rank} X^\vee = 2n - \operatorname{rank} C$,
6. (nonstandard) a linearly independent set $\{\omega_i \,:\, i \in I\}$ of X such that $<\omega_i, \alpha_j^\vee> = \delta_{ij}$ is fixed.

Once C is given it is easy to construct a data as above. We call $\mathfrak{h}$ the complexification of $X^\vee$ and we observe that through the bilinear pairing $<\,,>$ we can identify $\mathfrak{h}^*$ with the complexification of X. We observe also that the triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ is a realization of the Cartan matrix C ([5] pg.1).

The Weyl group W attached to C is defined as the subgroup of $Aut(X) \subset GL(\mathfrak{h}^*)$ generated by the reflections

$$s_i : x \longmapsto x - <x, \alpha_i^\vee> \alpha_i. \tag{1}$$

Observe that the dual action is given by $s_i(y) = y - <\alpha_i, y> \alpha_i^\vee$ and that the lattices Q and $Q^\vee$ are stable for these actions. So the annihilator $\overset{\circ}{Q}{}^\vee = \{x \in X \; : <x, y> = 0 \; \forall y \in Q^\vee\}$ is also stable by W and we can consider the action of Won the lattice $P = X \,/\, \overset{\circ}{Q}{}^\vee \simeq \mathrm{Hom}_{\mathbb{Z}}(Q, \mathbb{Z})$ and we call $x \mapsto \overline{x}$ the projection from X to P. We observe also that this projection is an isomorphism from the lattice $\widetilde{P}$, that is not W-stable, spanned by $\{\omega_i \,:\, i \in I\}$ and P. Finally we observe that

$$\overline{\alpha}_i = \sum_{j \in I} c_{ij} \overline{\omega}_j.$$

1.3. d, v **and the space of all matrices.** For the exposition it will be useful to identify the set I with the set of integers $\{1, \dots, n\}$.

Let $d = (d_1, \dots, d_n)$ and $v = (v_1, \dots, v_n)$ be two n-tuples of integers. We also think of d, v as elements of X in the following way:

$$d = \sum_{i \in I} d_i \omega_i \;\text{ and }\; v = \sum_{i \in I} v_i \alpha_i \,; \tag{2}$$

and through this identification we define an action of W on v. We define also $v^\vee = \sum_{i \in I} v_i \alpha_i^\vee \in Q^\vee$. Once d, v are fixed we fix complex vector spaces D_i and V_i of dimensions d_i and v_i and we define the following spaces of maps:

$$S_\Omega(d, v) = \bigoplus_{i \in I} \mathrm{Hom}(D_i, V_i) \oplus \bigoplus_{h \in \Omega} \mathrm{Hom}(V_{h_0}, V_{h_1}), \tag{3a}$$

$$S_{\overline{\Omega}}(d, v) = \bigoplus_{i \in I} \mathrm{Hom}(V_i, D_i) \oplus \bigoplus_{h \in \overline{\Omega}} \mathrm{Hom}(V_{h_0}, V_{h_1}), \tag{3b}$$

$$S(d, v) = S_\Omega(d, v) \oplus S_{\overline{\Omega}}(d, v). \tag{3c}$$

More often, when it will not be ambiguous we will write $S_\Omega, S_{\overline{\Omega}}$ and S instead of $S_\Omega(d, v), S_{\overline{\Omega}}(d, v)$ and $S(d, v)$.

For each $h \in H$ (resp. $i \in I$) we define the projection B_h (resp. γ_i and δ_i) from S to $\mathrm{Hom}(V_{h_0}, V_{h_1})$ (resp. $\mathrm{Hom}(D_i, V_i)$ and $\mathrm{Hom}(V_i, D_i)$) with respect to the decomposition described in (3).

When an element s of S is fixed we will often write B_h (resp. γ_i, δ_i) instead of $B_h(s)$ (resp. $\gamma_i(s)$ and $\delta_i(s)$). We will also use γ for $(\gamma_1, \dots, \gamma_n)$, δ for $(\delta_1, \dots, \delta_n)$ and B for $(B_h)_{h \in H}$ and often we will write an element of S as a triple (B, γ, δ).

Once D_i, V_i and an element s of S are fixed we define also:

$$T_i = D_i \oplus \bigoplus_{h \,:\, h_1 = i} V_{h_0}\,, \tag{4a}$$

$$a_i = a_i(s) = (\delta_i(s), (B_{\bar{h}}(s))_{h \,:\, h_1 = i}) : V_i \longrightarrow T_i\,, \tag{4b}$$

$$b_i = b_i(s) = (\gamma_i(s), (\varepsilon(h) B_h(s))_{h \,:\, h_1 = i}) : T_i \longrightarrow V_i\,. \tag{4c}$$

We will identify the dual of space of the $\mathbb{C}$-linear maps $\mathrm{Hom}(E,F)$ between two finite dimensional vector spaces with $\mathrm{Hom}(F,E)$ through the pairing $<\varphi,\psi>=\mathrm{Tr}(\varphi\circ\psi)$. So we can describe S also as $S_\Omega \oplus S_\Omega^* = T^* S_\Omega$ and we observe that a natural symplectic structure ω is defined over S by

$$\omega((s_\Omega, s_{\overline{\Omega}}),(t_\Omega, t_{\overline{\Omega}})) = <s_\Omega, t_{\overline{\Omega}}> - <t_\Omega, s_{\overline{\Omega}}> .$$

1.4. **Hermitian structure on S.** We suppose now that the spaces D_i, V_i are endowed with hermitian metrics. So we can speak of the adjoint φ^* of a linear map between these spaces, and we have a positive definite hermitian structure h on S with explicit formula:

$$\begin{aligned} h((B,\gamma,\delta),(\widetilde{B},\widetilde{\gamma},\widetilde{\delta})) &= \sum_{h\in H} \mathrm{Tr}(B_h \widetilde{B}_h^*) + \sum_{i\in I} \mathrm{Tr}(\gamma_i \widetilde{\gamma}_i^* + \widetilde{\delta}_i^* \delta_i) \qquad (5)\\ &= \sum_{i\in I} \mathrm{Tr}(a_i \widetilde{a}_i^* + \widetilde{b}_i^* b_i) \end{aligned}$$

and an associated real and closed symplectic form $\omega_I(s,t) = \mathrm{Re}\, h(\mathrm{i}s,t) - \mathrm{Im}\, h(s,t)$.

1.5. **Group actions and moment maps.** We can define an action of the groups $G = GL(V) = \prod GL(V_i)$ and $GL(D) = \prod GL(D_i)$ on the set S in the following way:

$$g(B_h,\gamma_i,\delta_i) = (g_{h_1} B_h g_{h_0}^{-1}, g_i\gamma_i, \delta_i g_i^{-1}) \qquad \text{for } g=(g_i)\in GL(V), \qquad (6)$$

$$g(B_h,\gamma_i,\delta_i) = (B_h, \gamma_i g_i^{-1}, g_i\delta_i) \qquad \text{for } g=(g_i)\in GL(D). \qquad (7)$$

Observe that these actions commute and that ω is $GL(V)$ invariant. Moreover if $U = U(V) = \prod U(V_i)$ is the group of unitary transformations in $GL(V)$ the real symplectic form ω_I is $U(V)$ invariant.

Define $\mu, \mu_I \colon S \longrightarrow \mathfrak{g} = \oplus gl(V_i)$ by the following explicit formulas:

$$\mu_i(B,\gamma,\delta) = \sum_{h\in H\,:\,h_1=i} \varepsilon(h) B_h B_{\bar h} + \gamma_i\delta_i = b_i a_i\,,$$

$$\mu_{I,i}(B,\gamma,\delta) = \frac{\mathrm{i}}{2}\left(\sum_{h\in H\,:\,h_1=i} B_h B_h^* - B_{\bar h}^* B_{\bar h} + \gamma_i\gamma_i^* - \delta_i^*\delta_i\right) = \frac{\mathrm{i}}{2}(b_i b_i^* - a_i^* a_i),$$

If we identify $\mathfrak{g}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g},\mathbb{C})$ (resp. $\mathfrak{u}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{u},\mathbb{R})$) with $\mathfrak{g} = \oplus gl(V_i)$ (resp. $\mathfrak{u}$) through the pairing $<(x_i),(y_i)>= \sum_i \mathrm{Tr}(x_i y_i)$, we can observe that μ is a moment map for the action of G on the symplectic manifold (S,ω) and that μ_I is a moment map for the action of U on the symplectic manifold (S,ω_I). It is common to group all these moment maps together and to define an hyperKähler moment map

$$\widetilde{\mu} = (\mu_I,\mu) : S \longrightarrow \mathfrak{u}\oplus\mathfrak{g} = (\mathbb{R}\oplus\mathbb{C})\otimes_{\mathbb{R}} \mathfrak{u}.$$

1.6. **Quiver varieties as hyperKähler quotients.** Let $\zeta_i = (\xi_i,\lambda_i) \in \mathbb{R}\oplus\mathbb{C}$ and $\zeta = (\zeta_1,\dots,\zeta_N)$. We define:

$$\mathfrak{L}_\zeta(d,v) = \{s\in S \,:\, \mu(s) - \lambda_i \mathrm{Id}_{V_i} = 0 \text{ and } \mu_{I,i}(s) - \mathrm{i}\xi_i \mathrm{Id}_{V_i} = 0\}.$$

We observe that $\mathfrak{L}_\zeta(d,v)$ is stable for the action of $U(V)$, so, at least as a topological Hausdorff space we can define the *quiver variety of type* ζ as

$$\mathfrak{M}_\zeta(d,v)=\mathfrak{L}_\zeta(d,v)/U(V).$$

It will be convenient to define also $\mathfrak{M}_\zeta(d,v)=\varnothing$ if $d,v\in\mathbb{Z}^n$ and there exists i such that $v_i<0$ or $d_i<0$ for some i. We call $\mathfrak{Z}=\mathbb{R}^n\oplus\mathbb{C}^n$ and we observe that we can identify it to $(\mathbb{R}\oplus\mathbb{C})\otimes_{\mathbb{Z}}P$ through:

$$(\xi_1,\dots,\xi_n,\lambda_1,\dots,\lambda_n)\longleftrightarrow\sum_{i\in I}(\xi_i,\lambda_i)\overline{\omega}_i. \tag{8}$$

In particular we consider an action of the Weyl group W on $\mathfrak{Z}$ through this identification.

Remark 1. There is a surjective map from $\mathfrak{Z}$ to $Z_U(\mathfrak{u})\oplus Z_G(\mathfrak{g})$:

$$(\xi_1,\dots,\xi_n,\lambda_1,\dots,\lambda_n)\longrightarrow\sum_{i\in I}(\mathrm{i}\xi_i,\lambda_i)\mathrm{Id}_{V_i}$$

Observe that $\mathfrak{L}_\zeta$ is the fiber of $\widetilde{\mu}$ over the image of ζ in $Z_U(\mathfrak{u})\oplus Z_G(\mathfrak{g})$.

Remark 2. If $v,d\geq 0$ define: $I^*=\{i\in I\,:\,v_i\neq 0\}$, $H^*=\{h\in H\,:\,h_0,h_1\in I^*\}$, $\varepsilon^*=\varepsilon\big|_{H^*}$, $v^*=(v_i)_{i\in I^*}$, $d^*=(d_i)_{i\in I^*}$, and $\zeta^*=(\zeta_i)_{i\in I^*}$ then it is clear that

$$\mathfrak{L}_{\zeta^*}(d^*,v^*)\simeq\mathfrak{L}_\zeta(d,v)\ \text{ and }\ \mathfrak{M}_{\zeta^*}(d^*,v^*)\simeq\mathfrak{M}_\zeta(d,v).$$

Except for the last equivalence which is trivial in our case the following is a general well known fact ([12] ch. 8).

Lemma 3. *Let $s\in\mathfrak{L}_\zeta$ then*

$$d\widetilde{\mu}_s \textit{ is surjective}\iff d\mu_s\textit{ is surjective}\iff d\mu_I\textit{ is surjective}\iff$$
$$\iff\dim\mathrm{Stab}_G\{s\}=0\iff\mathrm{Stab}_G\{s\}=\{1_G\}$$

Definition 4. If $u\in\mathbb{Z}^n=Q^\vee$ $A\subset Q^\vee$ we define

$$\mathcal{H}_u=\{\zeta=(\xi,\lambda)\in\mathfrak{Z}\,:<\xi\,,u^\vee>=<\lambda\,,u^\vee>=0\}\ \text{ and }\ \mathcal{H}_A=\bigcup_{u\in A}H_u.$$

Let now $U_v=\{u\in\mathbb{N}^n-\{0\}$ such that $0\leq u_i\leq v_i\}$ and $\mathcal{H}=\mathcal{H}_{U_v}$. $\mathcal{H}$ is a union of a finite number of real subspace of $\mathfrak{Z}$ of codimension 3.

Lemma 5 (Nakajima, [13])**.** *If $\zeta\in\mathfrak{Z}-\mathcal{H}$ and $\widetilde{\mu}(s)=\zeta$ then* $\mathrm{Stab}_G\{s\}=1_G$.

As a consequence of the above lemma and general results on on hyperKähler manifolds (for example [3] or [4]) we obtain the following corollary.

Corollary 6. *If $\zeta\in\mathfrak{z}-\mathcal{H}$ then if it is not empty $\mathfrak{M}_\zeta(d,v)$ is a smooth hyperKähler manifold of real dimension $2<2d-v,v^\vee>$.*

1.7. **Geometric invariant theory and moment map.** In this section we explain the relation between the moment map and the GIT quotient proved by Kempf, Ness [6], Kirwan [12] and others. To be more precise we need a generalization of their results in the case of an action on an affine variety proved by Migliorini [11].

Let X be an affine variety over $\mathbb{C}$ and G a reductive group acting on X. We can assume that X is a closed sub-variety of a vector space V where G acts linearly. Let h be an hermitian form on V invariant by the action of a maximal compact group U of G and define a real U-invariant symplectic form on V by

$$\eta(x,y) = \operatorname{Re} h(\mathrm{i}x, y).$$

Then we can define a moment map $\nu : V \longrightarrow \mathfrak{u}^* = \operatorname{Hom}_{\mathbb{R}}(\mathfrak{u}, \mathbb{R})$:

$$<\nu(x), u>= \tfrac{1}{2}\eta(u\cdot x, x).$$

We observe that the real symplectic form η restricted to a complex sub-manifold is always non degenerate and that μ restricted to the non singular locus of X is a moment map for the action of U on X.

Now let χ be a multiplicative character of G. We observe that for all $g \in U$ we have $|\chi(g)| = 1$ so $\mathrm{i}\, d\chi : u \longrightarrow \mathbb{R}$. In particular we can think to $\mathrm{i}\, d\chi$ as an element of $\mathfrak{u}^*$. Moreover we observe that it is invariant by the dual adjoint action, hence it makes sense to consider the quotient:

$$\mathfrak{M} = \nu^{-1}(i\, d\chi)/U.$$

As we saw our variety are a particular case of this construction.

On the other side we can consider the GIT quotient. Let us remind the definition. If φ is a character of G we consider the line bundle $L_\varphi = V \times \mathbb{C}$ on V with the following G-linearization:

$$g(x,z) = (g\cdot x, \varphi(g)z).$$

An invariant section of L_φ is determined by an algebraic function $f : V \longrightarrow \mathbb{C}$ such that $f(gx) = \varphi(g)f(x)$ for all $g \in G$ and $x \in V$. We use the same symbol L_φ also for the restriction of L_φ to X.

Given a rational action of G on $\mathbb{C}$-vector space A we define

$$A_{\varphi,n} = \{a \in A \,:\, g\cdot a = \varphi^{-n}(g)a \text{ for all } g \in G\},$$

$$A_\varphi = \bigoplus_{n=0}^{\infty} A_{\varphi,n} \qquad \text{as a graded vector space.}$$

Hence we have that $H^0(X, L_\varphi)^G = \mathbb{C}[X]_{\varphi,1}$. We observe that if I is the ideal of algebraic function on V vanishing on X then

$$H^0(X, L_\varphi)^G = \frac{H^0(V, L_\varphi)^G}{I_{\varphi,1}}.$$

This last fact can be proved easily for example averaging a φ equivariant function f on X in the following way:

$$\tilde{f}(v) = \int_U \varphi^{-1}(u) f(u\cdot v)\, du.$$

Definition 7. A point x of X is said to be *χ-semi-stable* if there exist $n > 0$ and $f \in H^0(X, L_\chi^{\otimes n})^G$ such that $f(x) \neq 0$. We observe that by the remark above a point of X is χ-semi-stable if and only if is χ-semi-stable as a point of V. We call X_χ^{ss} (resp. V_χ^{ss}) the open subset of χ-semi-stable points of X (resp. V).

Proposition 8 ([12, 15]). *There exists a good quotient of X_χ^{ss} by the action of G and we have that*

$$X_\chi^{ss}//G = \operatorname{Proj} \mathbb{C}[X]_\chi.$$

Moreover $\operatorname{Proj} \mathbb{C}[X]_\chi$ *is a finitely generated $\mathbb{C}$-algebra and a natural projective map*

$$\pi : X_\chi^{ss}//G \longrightarrow X//G = \operatorname{Spec} \mathbb{C}[X]^G$$

is defined.

In the case of $\chi \equiv 1$ the following fact is well known:

$$\operatorname{Proj} \mathbb{C}[X]_\chi = \operatorname{Spec} \mathbb{C}[X]^G = \nu^{-1}(0)/U.$$

The following result is less well known, and its proof requires some adjustment of the classical proof for the case $\chi \equiv 1$ (see for example an appendix of [11] or [10]).

Proposition 9 (Migliorini, [11]). *Let $x \in X$ then*

$$\exists g \in G \, : \, \nu(gx) = \mathrm{id}\chi \iff Gx \text{ is a closed orbit in } X_\chi^{ss}.$$

Proposition 10 (Migliorini, [11]). *The inclusion $\nu^{-1}(\mathrm{id}\chi) \subset X_\chi^{ss}$ induces an homeomorphism*

$$\nu^{-1}(\mathrm{id}\chi)/U \simeq X_\chi^{ss}//G.$$

1.8. **Quiver varieties as algebraic varieties.** If $m = (m_1, \dots, m_n) \in \mathbb{Z}^N$ we define a character χ_m of G_v by $\chi_m = \prod_{i \in I} \det_{GL(V_I)}^{m_i}$.

If $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ and $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$ then we define the varieties:

$$\Lambda_\lambda(d, v) = \{s \in S \, : \, \mu_i(s) - \lambda_i \mathrm{Id}_{V_i} = 0 \text{ for all } i\},$$
$$\Lambda_{m,\lambda}(d, v) = \{s \in \Lambda_\lambda(d, v) \, : \, s \text{ is } \chi_m - \text{semi-stable}\}.$$

and the associated *quiver varieties*

$$M_{m,\lambda}(d, v) = \Lambda_{m,\lambda}(d, v)//G_v \text{ and}$$
$$M_\lambda^0(d, v) = M_{0,\lambda}(d, v) = \Lambda_\lambda(d, v)//G_v$$

We call $p_{m,\lambda}^{d,v} \colon \Lambda_{m,\lambda}(d, v) \longrightarrow M_{m,\lambda}(d, v)$ the quotient map. Observe that the inclusion $\Lambda_{m,\lambda}(d, v) \subset \Lambda_\lambda(d, v)$ induces a projective morphism

$$\pi_{m,\lambda}^{d,v} : M_{m,\lambda}(d, v) \longrightarrow M_\lambda^0(d, v).$$

Finally it will be convenient to define $M_{m,\lambda}(d, v) = \varnothing$ if $d, v \in \mathbb{Z}^n$ and $v_i < 0$ or $d_i < 0$ for some i.

We identify $\mathbb{Z}^n$ with P and $Z = \mathbb{C}^n$ with $\mathbb{C} \otimes_\mathbb{Z} P$ through

$$(m_1, \dots, m_n) \longrightarrow \sum m_i \overline{\omega}_i \quad \text{and} \quad (\lambda_1, \dots, \lambda_n) \longrightarrow \sum \lambda_i \overline{\omega}_i.$$

Remark 11. As in 1 we have a surjective map from Z to $Z(\mathfrak{g})$ and $\Lambda_\lambda(d,v)$ is the fiber of μ over the image of λ in $Z(\mathfrak{g})$.

Remark 12. Remark 2 holds without changes also in this case.

Remark 13. Observe that $P \oplus Z \subset \mathfrak{Z}$. Observe also that the map $m \longrightarrow \chi_m$ define a surjective morphism from P to $\mathrm{Hom}(G_v, \mathbb{C}^*)$ and that the following diagram commute:

$$\begin{tikzcd} P \arrow[r] \arrow[d, hook] & \mathrm{Hom}(G, \mathbb{C}^*) \arrow[d] & \ni & \chi \arrow[d] \\ \mathbb{R}^n \arrow[r] & Z(\mathfrak{u}) \simeq (\mathfrak{u}^*)^U & \ni & \mathrm{id}\chi \end{tikzcd}$$

In particular we can apply 10 to the action of G_v on $\Lambda_\lambda(d,v)$ and we obtain:

$$\mathfrak{M}_{(m,\lambda)}(d,v) \simeq M_{m,\lambda}(d,v).$$

Proposition 14. *Let $(m,\lambda) \notin \mathcal{H}$ and $s \in \Lambda_{m,\lambda}(d,v)$ then* $\mathrm{Stab}_{G_v}(s) = \{1\}$.

Proof. As we have already claimed it is enough to prove $\dim \mathrm{Stab}_G(s) = \{1\}$. We know that there is a good quotient of $\Lambda_{m,\lambda}(d,v)$ so it is enough to prove that any closed orbit has maximal dimension. By Proposition 9 if $G_v s$ is closed in $\Lambda_{m,\lambda}(d,v)$ then there exists $g \in G_v$ such that $\mu_I(gs) = \mathrm{id}\chi$. The thesis follows now form $(m,\lambda) \notin \mathcal{H}$ and Lemma 5. $\square$

Corollary 15. *If $(m,\lambda) \notin \mathcal{H}$ and $M_{m,\lambda}(d,v) \neq \varnothing$ then it is a smooth algebraic variety of dimension $<v^\vee, 2d-v>$.*

1.9. **Path algebra and b-path algebra.** To describe functions on quiver varieties we need some notation about the path algebra.

Definition 16. A *path* α in our graph is a sequence $h^{(m)} \dots h^{(1)}$ such that $h^{(i)} \in H$ and $h^{(i)}_1 = h^{(i+1)}_0$ for $i = 1, \dots, m-1$. We define also $\alpha_0 = h^{(1)}_0$, $\alpha_1 = h^{(m)}_1$ and we say that the length of α is m. If $\alpha_0 = \alpha_1$ we say that α is a closed path. We consider also the empty paths $\varnothing_i$ for $i \in I$ and we define $(\varnothing_i)_0 = (\varnothing_i)_1 = i$. The product of path is defined in the obvious way.

A *b-path* $[\beta]$ in our graph is a sequence $[i^{r_{m+1}}_{m+1} \alpha^{(m)} i^{r_m}_m \dots \alpha^{(1)} i^{r_1}_1]$, that we write between square brackets such that $i_j \in I$, $\alpha^{(j)}$ are B-path, $r_j \in \mathbb{N}$ and $\alpha^{(j)}_0 = i_j$ and $\alpha^{(j)}_1 = i_{j+1}$ for $j = 1, \dots, m$. We consider also the "empty" b-paths indexed by elements of I: $[\varnothing_i]$. We define $[\beta]_0 = i_1$, $[\beta]_1 = i_{m+1}$ and $[\varnothing_i]_0 = [\varnothing_i]_1 = i$. The length of $[\beta]$ is $\sum_{j=1}^{m+1} r_j + \sum_{j=1}^{m} \mathit{length}(\alpha^j)$ and the product of b-paths is defined in the obvious way:

$$[\beta] \cdot [\beta'] = \begin{cases} 0 & \text{if } [\beta']_1 \neq [\beta]_0 \\ [\beta\beta'] & \text{if } [\beta']_1 = [\beta]_0 = i \end{cases}$$

Given a path $\alpha = h^{(m)} \dots h^{(1)}$ and a b-path $\beta = [i_{m+1}^{r_{m+1}} \alpha^{(m)} \dots \ \dots i_1^{r_1}]$ we define an evaluation of α and β on S in the following way: if $s = (B, \gamma, \delta) \in S$ then

$$\begin{aligned}
&\varnothing_i(s) = 0 \in \mathrm{Hom}(V_i, V_i) \ \text{ and } \ [\varnothing_i](s) = 0 \in \mathrm{Hom}(V_i, V_i),\\
&\alpha(s) = B_{h^{(m)}} \circ \dots \circ B_{h^{(1)}} \in \mathrm{Hom}(V_{\alpha_0}, V_{\alpha_1}),\\
&\beta(s) = (\gamma_{i_{m+1}} \circ \delta_{i_{m+1}})^{r_{m+1}} \circ \alpha^{(m)}(s) \circ (\gamma_{i_m} \circ \delta_{i_m})^{r_m} \circ \dots \circ\\
&\qquad \circ \dots \circ \alpha^{(1)}(s) \circ (\gamma_{i_1} \circ \delta_{i_1})^{r_1} \in \mathrm{Hom}(D_{\beta_0}, D_{\beta_1}).
\end{aligned}$$

The path algebra $\mathcal{R}$ is the vector space spanned by paths with the product induced by the product of paths. If $i, j \in I$ we say that an element in $\mathcal{R}$ is of type (i, j) if it is in the linear span of the paths with source in i and target in j.

The b-path algebra $\mathcal{Q}$ is the vector space spanned by b-paths with the product induced by the product of b-paths described above. If $i, j \in I$ we say that an element in $\mathcal{R}$ is of type (i, j) if it is in the linear span of the b-paths with source in i and target in j.

Remark 17. We observe that the evaluation on S is a morphism of algebra from $\mathcal{R}$ to the algebra defined by the morphisms of the category of vector spaces. Moreover if f is of type (i, j) we observe that $f(s) \in \mathrm{Hom}(V_i, V_j)$.

2. Generators of the projective ring of a quiver variety

In this section we want to describe a set of generators of the graded ring $\mathbb{C}[S]_\chi$ and by consequence of the projective ring of a quiver variety $\mathbb{C}[\Lambda_\lambda]_\chi$. More precisely we will give a set of generators as $\mathbb{C}[S]^G$-module of its l-homogeneous part: $\mathbb{C}[S]_{\chi,l}$. This result is a generalization of the one obtained by Lusztig in the case of invariants: $\chi \equiv 1$. First of all recall his result.

Theorem 18 (Lusztig, [8] theorem 1.3)**.** *The ring $\mathbb{C}[S]^G$ is generated by the polynomials:*

$$s \longmapsto \mathrm{Tr}\,(\alpha(s)) \quad \textit{and} \quad s \longmapsto \varphi\,(\delta_{\beta_1}(s)\beta(s)\gamma_{\beta_0}(s))$$

for α a closed path, β a path and $\varphi \in (Hom(D_{\beta_0}, D_{\beta_1}))^*$.

2.0.1. *Determinants.* To describe our result we do first some general remark. Forget for a moment our quiver, and suppose to have a finite set of finite dimensional vector spaces $X_1, \dots, X_k$ of dimensions $u_1, \dots, u_k$ and a pair of nonnegative integers (m_i^+, m_i^-) for each of them. Finally let m^+, m^- two nonnegative integers such that

$$N = \sum_{i=1}^k m_i^+ u_i + m^+ = \sum_{i=1}^k m_i^- u_i + m^-,$$

and two vector spaces M^+ and M^- of dimension m^+, m^-. Construct the vector spaces:

$$Y = \bigoplus_{i=1}^k \mathbb{C}^{m_i^-} \otimes X_i \oplus M^-, \qquad Z = \bigoplus_{i=1}^k \mathbb{C}^{m_i^+} \otimes X_i \oplus M^+$$

and observe that $\dim Y = \dim Z = N$. Define an action of the general linear group $GL(X_i)$ of X_i on Y by

$$g_i \cdot (\sum_{j=1}^{k} v_j \otimes x_j + m) = \sum_{j\neq i} v_j \otimes x_j + m + v_i \otimes g_i x_i,$$

and also a similar action on Z. Hence the vector space $\mathrm{Hom}(Y, Z)$ acquires a natural structure of $G_X = \prod_{i=1}^{k} GL(X_i)$ module. If we choose an isomorphism σ between $\mathrm{Hom}(\bigwedge^N Y, \bigwedge^N Z)$ and $\mathbb{C}$ we can define a function det on $\mathrm{Hom}(Y, Z)$ by

$$det(A) = \sigma\left(\bigwedge^n A\right).$$

For simplicity we do not emphasize the role of σ on this definition, so strictly speaking, det is a function defined only up to a nontrivial constant factor. We observe also that $\bigwedge^n Y \simeq (\bigwedge^{u_i} X_1)^{\otimes m_1^-} \otimes \cdots \otimes (\bigwedge^{u_k} X_k)^{\otimes m_k^-} \otimes \bigwedge^{m^-} M^-$ (and similarly for Z) so an isomorphism σ is determined if we choose orientations, or basis, of X_j, M^+, M^-. Finally observe that for any $g = (g_j) \in G_X$ we have

$$\det(g \cdot A) = \prod_{i=1}^{k} (det_{GL(X_i)}(g_i))^{m_i^+ - m_i^-} \ \det(A).$$

2.0.2. *Description of generators.* We go back now to our quiver and we describe a set of covariant polynomials on S. Any character χ of the group $G_v = GL(V)$ is of the form $\chi = \chi_m = \prod_{i\in I} det^{m_i}_{GL(V_i)}$. We fix such a character and we define

$$\begin{aligned} I^+ &= \{i \in I \,:\, m_i > 0\} \text{ and } \widetilde{m}_i = m_i \text{ if } i \in I^+,\\ I^0 &= \{i \in I \,:\, m_i = 0\} \text{ and } \widetilde{m}_i = 0 \text{ if } i \in I^0,\\ I^- &= \{i \in I \,:\, m_i < 0\} \text{ and } \widetilde{m}_i = -m_i \text{ if } i \in I^-. \end{aligned}$$

We use now the construction explained in 2.0.1 in the case $X_i = V_i$ and $m_i^+ - m_i^- = m_i$. We choose ordered sets $A = (a_1, \dots, a_{m^-}) \subset (\bigcup D_i)^{m^-}$ and $B = (b_1, \dots, b_{m^+}) \subset (\bigcup D_i^*)^{m^+}$ and we define a function $I : A, B \longrightarrow I$ by $a \in D_{I(a)}$, $b \in D^*_{I(b)}$. In the framework described above it is then possible to set $M^- = \bigoplus_{i=1}^{m^-} \mathbb{C}_{a_i}$ and $M^+ = \bigoplus_{i=1}^{m^+} \mathbb{C}_{b_i}$. In particular we have

$$Y = \bigoplus_{i\in I} \bigoplus_{h=1}^{m_i^-} V_i^{(h)} \oplus \bigoplus_{i=1}^{m^-} \mathbb{C}_{a_i}, \qquad Z = \bigoplus_{i\in I} \bigoplus_{k=1}^{m_i^+} V_i^{[k]} \oplus \bigoplus_{i=1}^{m^+} \mathbb{C}_{b_i}$$

where $V_i^{(l)}, V_i^{[l]}$ are isomorphic copies of V_i. We choose now elements of the b-path algebra as follows:

1. for any $i, j \in I$ and for any $1 \leq h \leq m_i^-$, $1 \leq k \leq m_j^+$ we choose an element $\alpha^{i,h}_{j,k}$ of the b-path algebra of type (i, j),
2. for any $i \in I$, $1 \leq h \leq m_i^-$ and for any $1 \leq l \leq m^+$ we choose an element $\alpha^{i,h}_{l}$ of the b-path algebra of type $(i, I(b_l))$,

3. for any $1 \leq l \leq m^-$ and for any $j \in I$, $1 \leq k \leq m_j^+$ we choose an element $\alpha_{j,k}^l$ of the b-path algebra of type $(I(a_l), j)$,
4. for any $1 \leq l \leq m^-$ and for any $1 \leq l' \leq m^-$ we choose an element $\alpha_{l'}^l$ of the b-path algebra of type $(I(a_l), I(b_{l'}))$.

We call such a data $\Delta = (\{(m_i^+, m_i^-)\}_{i \in I}, (m^+, m^-), A, B, \alpha_{j,k}^{i,h}, \alpha_l^{i,h}, \alpha_{j,k}^l, \alpha_{l'}^l)$ a χ-data and we attach to it a χ-covariant function on S:

$$f_\Delta(s) = \det\left(\Psi_\Delta(s)\right)$$

where Ψ_Δ is a linear map from Y to Z defined by

$$\begin{aligned}
[\Psi_\Delta]_{V_j^{[k]}}^{V_i^{(h)}}(s) &= \alpha_{j,k}^{i,h}(s), \\
[\Psi_\Delta]_{\mathbb{C}_{b_l}}^{V_i^{(h)}}(s) &= b_l \circ \delta_{I(b_l)} \circ \alpha_l^{i,h}(s), \\
[\Psi_\Delta]_{V_j^{[k]}}^{\mathbb{C}_{a_l}}(s) &= \alpha_{j,k}^l(s) \circ \gamma_{I(a_l)}\big|_{\mathbb{C}a_l}, \\
[\Psi_\Delta]_{\mathbb{C}_{b_{l'}}}^{\mathbb{C}_{a_l}}(s) &= b_{l'} \circ \delta_{I(b_{l'})} \circ \alpha_{l'}^l(s) \circ \gamma_{I(a_l)}\big|_{\mathbb{C}a_l}.
\end{aligned}$$

The function f_Δ are a set of generators as $\mathbb{C}[S]^G$-module of $\mathbb{C}[S]_{\chi,1}$, but we will need to define a smaller set of generators. To define this set we give a notion of good Δ.

Definition 19. A data Δ as above is said to be χ-*good* if it satisfies the following conditions:

1. $m_i^+ + m_i^- = \widetilde{m}_i$ for all $i \in I$,
2. $\alpha_{l'}^l = 0$ for all l, l',
3. α_*^* is an element of the path algebra (and not just an element of the b-path algebra which is obviously bigger),
4. $card\{(j,k) : \alpha_{j,k}^{i,h} \neq 0\} + card\{l : \alpha_l^{i,h} \neq 0\} \leq v_i$ for all i, h,
5. $card\{(i,h) : \alpha_{j,k}^{i,h} \neq 0\} + card\{l : \alpha_{j,k}^l \neq 0\} \leq v_j$ for all j, k,
6. for all l there exists at most one pair (i,h) such that $\alpha_l^{i,h} \neq 0$,
7. for all l there exists at most one pair (j,k) such that $\alpha_{j,k}^l \neq 0$.

For the applications the only important point will be the first one.

Proposition 20. *The set of polynomials f_Δ with Δ χ-good generates $\mathbb{C}[S]_{\chi,1}$ as a $\mathbb{C}[S]^{G_v}$-module.*

Remark 21. Prof. Weyman said me that in the case $D = 0$ a similar proposition has been proved by him and Derksen for arbitrary characteristic in the case of a quiver without loops: [2].

2.1. **Some remark on the invariant theory of $GL(n)$.** If V is a finite dimensional representation of a linearly reductive Lie group G and S is a simple representation of S we write $V[S]$ for the S-isotipic component of type S of V.

We now fix n and we make some remark on the representations of $GL(n)$. To any partition of height less or equal to n we associate an irreducible representation of $GL(n)$ in the usual way. If we multiply these representations by a power of the inverse of determinant representation we obtain a complete list of irreducible representations of $GL(n)$. If λ is a partition we call λ the transpose partition as usual and we define $\lambda^{op} = (\lambda_1 - \lambda_n, \lambda_1 - \lambda_{n-1}, \dots, \lambda_1 - \lambda_1)$. We call δ the determinant representation of $GL(n)$ and $\varepsilon = 1^n$ the associated partition. Finally we call V the natural representation.

Lemma 22. 1. $L_\lambda{}^* = \delta^{-\lambda_1} \otimes L_{\lambda^{op}}$,

2. $Hom_{GL(n)}(\delta^m, L_\lambda \otimes L_\mu) = \begin{cases} \mathbb{C} & \textit{if } \lambda = \mu^{op} + (m - \mu_1)\varepsilon, \\ 0 & \textit{otherwise,} \end{cases}$

3. $Hom_{GL(n)}(\delta^m, L_\lambda \otimes L_\mu^*) = \begin{cases} \mathbb{C} & \textit{if } \lambda = \mu + m\varepsilon, \\ 0 & \textit{otherwise.} \end{cases}$

Proof. We prove only 2).

$$\begin{aligned} Hom_{GL(n)}(\delta^m, L_\lambda \otimes L_\mu) &= Hom_{GL(n)}(\delta^m \otimes L_\mu{}^*, L_\lambda) \\ &= Hom_{GL(n)}(\delta^{m-\mu_1} \otimes L_{\mu^{op}}, L_\lambda) \end{aligned}$$

If $m \geq \mu_1$ the last group is isomorphic to $Hom_{GL(n)}(L_{\mu^{op}+(m-\mu_1)\varepsilon}, L_\lambda)$ and if $m < \mu_1$ is isomorphic to $Hom_{GL(n)}(L_{\mu^{op}}; L_{\lambda+(\mu_1-m)\varepsilon})$. In any case the thesis follows. ☐

We want now to describe $\mathrm{Hom}_{GL(n)}\left(\delta^m, V^{\otimes i} \otimes (V^*)^{\otimes j}\right)$. To do it we will use Schur-duality. Remind that the irreducible representations of the groups S_m are parametrized by partitions of m and call S_λ the irreducible representation associated with λ. Consider now the action of S_m on $V^{\otimes m}$ given by permuting the factors. This action commute with the $GL(n)$ action. Schur duality asserts that the action of the group $S_m \times GL(n)$ on $V^{\otimes m}$ decomposes in the following way:

$$V^{\otimes m} = \bigoplus_{\substack{\lambda \vdash m \\ ht(\lambda) \leq n}} S_\lambda \otimes L_\lambda.$$

We describe a set of elements of $\mathrm{Hom}_{GL(n)}\left(\delta^m, V^{\otimes i} \otimes (V^*)^{\otimes j}\right)$. Let m be a non-negative integers a choose a permutation σ of $\{1, \dots, i + mn\}$. To σ we associate maps:

$$\begin{aligned} \Phi_\sigma &: \left(V^{\otimes i} \otimes (V^*)^{\otimes i}\right)^{GL(n)} \longrightarrow V^{\otimes i+mn} \otimes (V^*)^{\otimes i}[\delta^m] \\ \Psi_\sigma &: \left(V^{\otimes i} \otimes (V^*)^{\otimes i}\right)^{GL(n)} \longrightarrow V^{\otimes i} \otimes (V^*)^{\otimes i+mn}[\delta^{-m}] \end{aligned}$$

by

$$\begin{aligned} \Phi_\sigma(t \otimes s) &= \sigma(o \otimes \cdots \otimes o \otimes t) \otimes s \\ \Psi_\sigma(t \otimes s) &= t \otimes \sigma(o^* \otimes \cdots \otimes o^* \otimes s) \end{aligned}$$

where o is a nonzero vector in $\bigwedge^n V$ and o^* is a non zero vector in $\bigwedge^n V^*$ and $t \in V^{\otimes i}$, $s \in (V^*)^{\otimes i}$.

Lemma 23. *1) If $i \neq j + mn$ then*

$$Hom_{GL(N)}\left(\delta^m, V^{\otimes i} \otimes (V^*)^{\otimes j}\right) = 0.$$

2) If $m > 0$ then

$$V^{\otimes i+mn} \otimes (V^*)^{\otimes i}[\delta^m] = \sum_\sigma \operatorname{Im}\Phi_\sigma.$$

3) If $m > 0$ then

$$V^{\otimes i} \otimes (V^*)^{\otimes i+mn}[\delta^{-m}] = \sum_\sigma \operatorname{Im}\Psi_\sigma.$$

Proof. 1) follows directly from lemma 22.

2) Let $M = V^{\otimes i+mn} \otimes (V^*)^{\otimes i}[\delta^m]$ and $N = (V^{\otimes i} \otimes (V^*)^{\otimes i})^G$. N is a $S_i \times S_i$ module, M is a $S_{i+mn} \times S_i$- module and the maps $\Phi_{\mathcal{I},\sigma}$ are equivariant with respect the S_i action on $(V^*)^{\otimes i}$. In particular it is enough to prove that if λ is a partition of i, M_λ is the S_λ-isotipic component of M w.r.t. the S_i action and N_λ the S_λ-isotipic component of N w.r.t. the S_i action on $(V^*)^{\otimes i}$ then

$$M_\lambda = \sum_\sigma \Phi_\sigma(N_\lambda).$$

By point 3 of lemma22 we have that

$$\begin{aligned} M &= \bigoplus_{\substack{\lambda\vdash i\\ ht(\lambda)\leq n}} (S_{\lambda+m\varepsilon} \otimes L_{\lambda+m\varepsilon}) \otimes (S_\lambda \otimes L_\lambda)^* [\delta^m] \\ &= \bigoplus_{\substack{\lambda\vdash i\\ ht(\lambda)\leq n}} S_{\lambda+m\varepsilon} \otimes S_\lambda \otimes \left(\delta^m \otimes (L_\lambda \otimes {L_\lambda}^*)^G\right) \end{aligned}$$

In particular $M_\lambda = S_{\lambda+m\varepsilon} \otimes S_\lambda$ is an irreducible representation of $S_{i+mn} \times S_i$. Observe $\sum_\sigma \Phi(N_\lambda)$ is a $S_{i+mn} \times S_i$-submodule of M_λ and that it is clearly nonzero. So $M_\lambda = \sum_\sigma \Phi_\sigma(N_\lambda)$ as claimed.

The proof of 3) is equal to the previous one. □

We want now to give a slightly different formulation of the lemma above. Let $M = V^{\otimes i} \otimes (V^*)^{\otimes j}$ we want to describe $M^*_{\delta^m} = \{\varphi \in M^* : g \cdot \varphi = \delta^{-m}(g)\varphi\}$. Of course this problem is completely equivalent to the previous one. What we want to do is to reformulate in a more convenient way for our purposes the description of a set of generators of $M^*_{\delta^m}$. Let $m \geq 0$ and choose $\mathcal{I} = \{I_1, \dots, I_m\}$ a collection of m disjoint subsets of $\{1, \dots, i + mn\}$ of cardinality n. Let $I_j = \{i_{j1} < \dots < i_{jn}\}$ and $\{1, \dots, i + mn\} - \bigcup \mathcal{I} = \{j_1 < \dots < j_i\}$. To $\mathcal{I}$ and to a permutation $\sigma \in S_i$ we associate elements

$$\phi_{\mathcal{I},\sigma} \in \left(V^{\otimes i+mn} \otimes (V^*)^{\otimes i}\right)^*_{\delta^m} \quad \text{and} \quad \psi_{\mathcal{I},\sigma} \in \left(V^{\otimes i} \otimes (V^*)^{\otimes i+mn}\right)^*_{\delta^{-m}}$$

defined by

$$\phi_{\mathcal{I},\sigma}(v_1\otimes\dots v_{i+mn}\otimes\varphi_1\dots\varphi_i)=\prod_{j=1}^{m}<o^*,v_{j1}\wedge\dots\wedge v_{jn}>\cdot\prod_{h=1}^{i}<v_{j_h},\varphi_{\sigma_h}>$$

$$\psi_{\mathcal{I},\sigma}(v_1\otimes\dots v_i\otimes\varphi_1\dots\varphi_{i+mn})=\prod_{j=1}^{m}<o,\varphi_{j1}\wedge\dots\wedge \varphi_{jn}>\cdot\prod_{h=1}^{i}<v_{\sigma_h},\varphi_{j_h}>$$

where o is a nonzero vector in $\bigwedge^n V$ and o^* is a non zero vector in $\bigwedge^n V^*$.

Lemma 24. *1) If $i\neq j+mn$ then $(V^{\otimes i}\otimes(V^*)^{\otimes j})^*_{\delta^m}=0$.*
2) If $m\geq 0$ then $(V^{\otimes i+mn}\otimes(V^*)^{\otimes i})^*_{\delta^m}$ is generated by the functions $\phi_{\mathcal{I},\sigma}$.
3) If $m\geq 0$ then $(V^{\otimes i}\otimes(V^*)^{\otimes i+mn})^*_{\delta^{-m}}$ is generated by the functions $\psi_{\mathcal{I},\sigma}$.

Proof. The proof is clear by the previous lemma. □

2.2. **A special case.** In this section we prove a special case of Proposition 20 in which we are able to give a more precise result. To simplify the exposition of the proof of Proposition 20 we will also prove another lemma.

Here and in the following we will use polarization. If V is finite dimensional vector space then we can define a map

$$\wp:(V^{\otimes n})^*\longrightarrow S^n(V^*)\subset\mathbb{C}[V]\ \text{ through }\ \wp(\varphi)(v)=\varphi(v\otimes\dots\otimes v).$$

Lemma 25. *$\wp$ is surjective, moreover if V is a finite dimensional representation of a reductive group Γ, and χ is a character of Γ then*

$$\wp((V^{\otimes n})^*_\chi)=S^n(V^*)_\chi$$

where E_χ is the isotipic component of type χ^{-1} of a G module E.

Lemma 26. *For $i=1,\dots,n$ let Γ_i be a reductive group, χ_i be a character of Γ_i and E_i be a f.d.representation of Γ_i. Let $\Gamma=\prod\Gamma_i$, then $E=\otimes_iE_i$ is a representation of Γ and $\chi=\prod\chi_i$ is a character of Γ. Then*

$$E^*_\chi=(E_1)^*_{\chi_1}\otimes\dots\otimes(E_n)^*_{\chi_n}.$$

Let J^+, J^- be two sets of indexes and define $\widetilde{J}^+=\{0\}\coprod J^+$, $\widetilde{J}^-=\{0\}\coprod J^-$ and $J=J^+\times J^-$, $\widetilde{J}=\widetilde{J}^+\times\widetilde{J}^- -\{(0,0)\}$. For each $i\in\widetilde{J}^+$ (resp. $j\in\widetilde{J}^+$) choose a vector space Y_i (resp. X_j) and define $X=\bigoplus_{j\in J^-}X_j$ and $Y=\bigoplus_{i\in J^+}Y_i$. Consider the group

$$G_{XY}=\prod_{i\in J^+}GL(Y_i)\times\prod_{j\in J^-}GL(X_j)$$

and its character $c=\prod_{i\in J^+}\det_{GL(Y_i)}\times\left(\prod_{j\in J^-}\det_{GL(X_j)}\right)^{-1}$.

We fix a matrix $r=(r_{ij})_{i\in\widetilde{J}^+,j\in\widetilde{J}^-}$ of integers such that $r_{i0}=1=r_{0j}$ for all i,j and $r_{00}=-1$ and we consider the vector spaces:

$$H^{XY}=H=\bigoplus_{(i,j)\in\widetilde{J}}\operatorname{Hom}(X_j,Y_i)^{\oplus r_{ij}}\ \text{ and }\ H_0^{XY}=H_0=\bigoplus_{(i,j)\in J}\operatorname{Hom}(X_j,Y_i)$$

where we adopt the convention $E^{\oplus n} = \mathbb{C}^n = 0$ if $n < 0$. When the spaces X, Y will be clear from the context we will write H and H_0 instead of H^{XY} and H_0^{XY}. We fix a basis e^{ij}_m of $\mathbb{C}^{r_{ij}}$ so we have a canonical identification

$$H = \bigoplus_{(i,j)\in\widetilde{J}} \mathrm{Hom}(X_j, Y_i) \otimes \mathbb{C}^{r_{ij}}. \tag{9}$$

We want to study c-equivariant polynomials on H. If we choose two finite dimensional vector spaces $\tilde{A}, \tilde{B}$, linear maps $\alpha : \tilde{A} \longrightarrow X_0$, $\beta : Y_0 \longrightarrow \tilde{B}$, and elements $\varphi_{ij} \in (\mathbb{C}^{r_{ij}})^*$ for all i, j then we can define a map $\Phi_{\varphi,\alpha,\beta} : H \longrightarrow H_0 \oplus \mathrm{Hom}(\tilde{A}, Y) \oplus \mathrm{Hom}(X, \tilde{B}) \subset \mathrm{Hom}(X \oplus \tilde{A}, Y \oplus \tilde{B})$ by

$$\Phi_{\varphi,\alpha,\beta}(\sum_{(i,j)\in\widetilde{J}} A_{ij} \otimes v_{ij}) = \sum_{(i,j)\in J} \varphi_{ij}(v_{ij})A^{ij} + \sum_{i\in J^+} A^{i0}\circ\alpha + \sum_{j\in J^-} \beta\circ A^{0j} \tag{10}$$

where $A_{ij} \otimes v_{ij} \in \mathrm{Hom}(X_j, Y_i) \otimes \mathbb{C}^{r_{ij}}$.

The following is a special version of 20.

Lemma 27. *$\mathbb{C}[H]_c$ is generated as a vector space by the following functions:*

$$s \longmapsto \det\bigl(\Phi_{\varphi,\alpha,\beta}(s)\bigr)$$

where $\Phi_{\varphi,\alpha,\beta} : H \longrightarrow H_0$ is as above.

2.2.1. *The special case.* We will study an even more special case in which we are able to prove a better result that I find nice. In the above setting suppose that $X_0 = Y_0 = 0$ and that $r_{ij} = 1$ for all i, j.

Define the following set of matrices:

$$\mathcal{S}_n = \{S = (s_{ij}) \in \mathbb{N}^{J^+\times J^-} \ : \sum_{i,j} s_{ij} = n\}$$

$$\mathcal{S}^{XY} = \mathcal{S} = \{S = (s_{ij}) \in \mathbb{N}^{J^+\times J^-} \ : \sum_{j} s_{ij} = \dim Y_i \ \forall i \in J^+ \quad \text{and} \ \sum_i s_{ij} = \dim X_j \ \forall j \in J^- \ \}$$

As for H we will write $\mathcal{S}$ when the spaces X_j, Y_i will be clear from the context. Observe that $\mathcal{S} = \varnothing$ if $\sum_j \dim X_j \neq \sum_i \dim Y_i$ and that if $N = \sum \dim X_j = \sum \dim Y_i$ then $\mathcal{S} \subset \mathcal{S}_N$. For each $card(J^+) \times card(J^-)$ matrix $S = (s_{ij})$ we consider $\varphi_{ij} \in \mathbb{C}^*$ given by $\varphi_{ij}(\lambda) = s_{ij}\lambda$ and we define

$$\Phi_S = \Phi_{\varphi,0,0} \ \text{ and } \ f_S = f_S^{XY} = \det(\Phi_S).$$

Proposition 28. *$\{f_S\}_{S\in\mathcal{S}^{XY}}$ is a basis of $\mathbb{C}[H]_c$.*

Proof. We have to compute $S^n(H^*)_c = (S^n(H))^*_c$ for all n. For all $S \in \mathcal{S}_n$ define

$$E_S = \bigotimes_{(i,j)\in J^+\times J^-} S^{s_{ij}}\left(\mathrm{Hom}(X_j, Y_i)\right).$$

Observe that $S^n(H)=\bigoplus_{S\in\mathcal{S}_n}E_S$ as a G-module. So $S^n(H)^*_c=\bigoplus_{S\in\mathcal{S}_n}(E_S)^*_c$. Observe now that E_S is a quotient of

$$\tilde{E}_S=\bigotimes_{(i,j)\in J^+\times J^-}(X_j^*)^{\otimes s_{ij}}\otimes Y_i^{\otimes s_{ij}}. \tag{11}$$

By the lemmas in the previous section we have that

$$(\tilde{E}_S)^*_c=\begin{cases}0 & \text{if } S\notin\mathcal{S}^{XY},\\ \mathbb{C} & \text{if } S\in\mathcal{S}^{XY}.\end{cases}$$

So in particular $(E_S)^*_c=0$ if $S\notin\mathcal{S}^{XY}$. Hence $\dim S^n(H)^*_c\leq card(\mathcal{S}^{XY})$.

The function f_S are clearly c-equivariant so the only thing that we have to prove is that they are linearly independent. To prove it we will prove a generalization of it.

If $i\in J^+$ and $j\in J^-$ let E_{ij} be the $card(J^+)\times card(J^-)$ matrix with a 1 in the (i,j) position and 0 elsewhere.

For each $i\in J^+, j\in J^-, m\in\mathbb{N}$ and $N\in\mathbb{N}$ we consider the following sentence $P_{i,j,m,N}$:

If $\sum_j X_j=N=\sum_i Y_i$ then $\{f_{S+mE_{ij}}\}_{S\in\mathcal{S}^{XY}}$ is linearly independent.

In the case $m=0$ we call this proposition $P_{0,N}$ since it does not depend on i,j and observe that $\forall N\, P_{0,N}$ is equivalent to our thesis.

For each $N\in\mathbb{N}$ we consider also the following sentence Q_N:

If $\sum_j X_j=N=\sum_i Y_i$ then $P_{i,j,m,N}$ is true for all $i\in J^+$, $j\in J^-$ and $m\in\mathbb{N}$.

First remark: $N=1$ is true.
Second remark: let $\mathcal{S}_0^{XY}=\{S\in S^{XY}\,:\,s_{ij}=0$ for all $i\in J^+$ and $j\in J^-$ such that $\dim Y_i,\dim X_j\geq 2$. Observe that $\{f_S\}_{S\in\tilde{\mathcal{S}}^{XY}}$ is linearly independent.

Now we prove Q_N by induction on N.
First step: $Q_{N-1}\Rightarrow P_{0,N}$. Suppose that there exists $c_S\in\mathbb{C}$ such that

$$\sum_{S\in\mathcal{S}^{XY}}c_Sf_S=0.$$

If $\dim X_{j_0},\dim Y_{i_0}\geq 2$ choose a nonzero element $x_{j_0}\in X_{j_0}$ (resp. $y_{i_0}\in Y_{i_0}$) and an hyper-plane $X'_{j_0}\subset X_{j_0}$ (resp. $Y'_{i_0}\subset Y_{i_0}$) such that $X_{j_0}=\mathbb{C}x_{j_0}\oplus X'_{j_0}$ (resp. $Y_{j_0}=\mathbb{C}y_{i_0}\oplus Y'_{i_0}$) and define:

$$\tilde{X}_j=\begin{cases}X_j & \text{if } j\neq j_0\\ X'_{j_0} & \text{if } j=j_0\end{cases}\quad\text{and}\quad\tilde{Y}_i=\begin{cases}Y_i & \text{if } i\neq i_0\\ Y'_{i_0} & \text{if } i=i_0\end{cases} \tag{12}$$

and define $\Psi: H^{\tilde{X}\tilde{Y}}\longrightarrow H^{XY}$ by

$$\Psi(T)\Big|_{\tilde{X}_j}=T\qquad\text{and}\qquad\Psi(T)(x_{j_0})=y_{j_0}. \tag{13}$$

Then we see that

$$
\begin{aligned}
0 = \sum_{S\in\mathcal{S}^{XY}} c_S f_S^{XY}(\Psi(T)) &= \sum_{S\in\mathcal{S}^{XY}\,:\,s_{i_0j_0}\neq 0} s_{i_0j_0} c_S f_S^{\tilde X\tilde Y}(T)\\
&= \sum_{S\in\mathcal{S}^{\tilde X\tilde Y}} (s_{i_0j_0}+1) c_{S+E_{i_0j_0}} f^{\tilde X\tilde Y}_{S+E_{i_0j_0}}(T)
\end{aligned}
$$

By induction $P_{i,j,1,N-1}$ is true for all i,j so we see that $c_S=0$ for all $S\in\mathcal{S}$ such that there exists i_0,j_0 such that $s_{i_0,j_0}\geq 1$ and $\dim X_{j_0}, \dim Y_{i_0}\geq 2$. Now we conclude by the second remark.

Second step: $Q_{N-1}\Rightarrow P_{i_0,j_0,m,N}$ if $\dim X_{j_0}, \dim Y_{i_0}\geq 2$ and $m\geq 1$. Suppose that $\sum_{S\in\mathcal{S}^{XY}} c_S f^{XY}_{S+mE_{i_0j_0}}=0$. We can construct $\tilde X_j, \tilde Y_i, \Psi$ as in the first step and we see that

$$
\begin{aligned}
0 = \sum_{S\in\mathcal{S}^{XY}} c_S f^{XY}_{S+mE_{i_0j_0}}(\Psi(T)) &= \sum_{S\in\mathcal{S}^{XY}} (s_{i_0j_0}+m) c_S f^{\tilde X\tilde Y}_{S+mE_{i_0j_0}}(T)\\
&= \sum_{S\in\mathcal{S}^{\tilde X\tilde Y}} (s_{i_0j_0}+m+1) c_{S+E_{i_0j_0}} f^{\tilde X\tilde Y}_{S+(m+1)E_{i_0j_0}}(T)
\end{aligned}
$$

and by $P_{i_0,j_0,m+1,N-1}$ we deduce $c_S=0$ for all S.

Third step: $Q_{N-1}\Rightarrow Q_N$. By the previous two step we have only to prove $P_{i_0,j_0,m,N}$ for $m\geq 1$ and $\dim X_{j_0}=1$ or $\dim Y_{i_0}=1$. We will suppose $\dim X_{j_0}=1$, the other case is completely similar. Suppose that $\sum_{S\in\mathcal{S}^{XY}} c_S f_{S+mE_{i_0j_0}}=0$. Set

$$\tilde{\mathcal{S}}_i = \{S\in\mathcal{S}^{XY} \,:\, s_{ij_0}=1\}$$

and observe that since $\dim X_{j_0}=1$ then $\mathcal{S}^{XY}=\coprod \tilde{\mathcal{S}}_i$. Now choose a non zero vector $x_{j_0}\in X_{j_0}$ and for all $i\in J^+$ choose a non zero vector $y_i\in Y_i$ and an hyper-plane Y_i' of Y_i such that $Y_i=\mathbb{C}y_i\oplus Y_i'$.

Now fix $i_1\neq i_0$ such that $\dim Y_{i_1}\geq 2$ and consider $\tilde J^+=J^+$ and $\tilde J^-=J^- -\{j_0\}$. For all $i\in\tilde J^+$ and for all $j\in\tilde J^-$ define:

$$\tilde X_j = X_j, \quad\text{and}\quad \tilde Y_i = \begin{cases} Y_i & \text{if } i\neq i_1,\\ Y'_{i_1} & \text{if } i=i_1.\end{cases}$$

For any $S\in\tilde{\mathcal{S}}_{i_1}$ we define also $t(S)\in\mathcal{S}^{\tilde X\tilde Y}$ by $t(S)_{ij}=s_{ij}$ for all $i\in\tilde J^+$, $j\in\tilde J^-$. $S\longmapsto t(S)$ is a bijection between $\tilde{\mathcal{S}}_{i_1}$ and $\mathcal{S}^{\tilde X\tilde Y}$: we call t^{-1} the inverse map. Finally we define $\Psi : H^{\tilde X\tilde Y}\longrightarrow H^{XY}$ as in the previous step and we observe that if $S\in\mathcal{S}$ then $f_{S+mE_{i_0j_0}}\circ\Psi=0$ if $S\notin\tilde{\mathcal{S}}_{i_1}$. Hence

$$
\begin{aligned}
0 = \sum_{S\in\mathcal{S}^{XY}} c_S f_{S+mE_{i_0j_0}}(\Psi(T)) &= \sum_{S\in\tilde{\mathcal{S}}_i} c_S f_{t(S)}(T)\\
&= \sum_{S\in\mathcal{S}^{\tilde X\tilde Y}} c_{t^{-1}(S)} f_S(T)
\end{aligned}
$$

and applying $P_{0,N-1}$ we obtain $c_S=0$ for all $S\in\tilde{\mathcal{S}}_{i_1}$ if $\dim Y_{i_1}\geq 2$ and $i_1\neq i_0$.

In a similar way we prove c_S if $S\in\tilde{\mathcal{S}}_{i_1}$ and $\dim Y_{i_1}=1$ and $i_1\neq i_0$.

Finally we observe that if $S \in \tilde{\mathcal{S}}_{i_0}$ then $f_{S+mE_{i_0j_0}} = (m+1)f_S$, hence $c_S = 0$ follows now from $P_{0,N}$ that we already know to be true. $\square$

2.2.2. *Proof of Lemma 27.* We study first $(H^{\otimes n})^*_c$ and then we apply polarization. As in the previous section we can decompose $H^{\otimes n}$ in summands of the following form:

$$E = \bigotimes_{(i,j)\in\tilde{J}} (\mathrm{Hom}(X_j, Y_i) \otimes \mathbb{C}^{r_{ij}})^{\otimes s_{ij}} = \bigotimes_{(i,j)\in\tilde{J}} (X_j^*)^{\otimes s_{ij}} \otimes Y_i^{\otimes s_{ij}} \otimes (\mathbb{C}^{r_{ij}})^{\otimes s_{ij}} \tag{14}$$

where s_{ij} are nonnegative integers such that $\sum_{i,j} s_{ij} = n$. Observe that the order of the factors is not important for us since we will apply polarization.

We can describe easily E^*_c using the lemma in the previous section. In particular a necessary and sufficient condition for the existence of c-covariants is $\sum_{i\in\tilde{J}^+} s_{ij} = \dim X_j$ for all $j \in J^-$ and $\sum_{j\in\tilde{J}^+} s_{ij} = \dim Y_i$ for all $i \in J^+$. Moreover

$$E^*_c \simeq \bigotimes_{(i,j)\in J} ((\mathbb{C}^{r_{ij}})^*)^{\otimes s_{ij}} \otimes \bigotimes_{j\in J^-} (Y_0^*)^{\otimes s_{0j}} \otimes \bigotimes_{i\in J^+} X_0^{\otimes s_{i0}}$$

To write explicit formulas we choose an order on the factors of E for example choosing a lexicographic order in $i \in \widetilde{J}^+$, $j \in \widetilde{J}^-$ and $1 \le q \le s_{ij}$:

$$E = \underbrace{X_1^* \otimes Y_1 \otimes \mathbb{C}^{r_{11}}}_{q=1} \otimes \cdots \otimes \underbrace{X_1^* \otimes Y_1 \otimes \mathbb{C}^{r_{11}}}_{q=s_{11}} \otimes \underbrace{X_1^* \otimes Y_2 \otimes \mathbb{C}^{r_{12}}}_{q=1} \otimes \cdots$$

Once we have chosen such an order we can write an element of E as linear combination elements of the form $\otimes_{(i,j,q)\in\widetilde{K}} x^{i,j,q} \otimes y^{i,j,q} \otimes v^{i,j,q}$ with $x^{i,j,q} \in X_j^*$, $y^{i,j,q} \in Y_i$ $v^{i,j,q} \in \mathbb{C}^{r_{ij}}$ and we settled $\widetilde{K} = \{(i,j,q) \in \widetilde{J} \times \mathbb{N} : 1 \le q \le s_{ij}\}$. We define also $K = \{(i,j,q) \in \widetilde{K} : (i,j) \in J\}$. Using this convention if

$$\phi = \bigotimes_{(i,j,q)\in\widetilde{K}} \phi^{i,j,q} \in \bigotimes_{(i,j,q)\in K} (\mathbb{C}^{r_{ij}})^* \otimes \bigotimes_{(0,j,q)\in\widetilde{K}} Y_0^* \otimes \bigotimes_{(i,0,q)\in\widetilde{K}} X_0 \tag{15}$$

the corresponding c equivariant linear function on E is defined on an element $s = \otimes_{(i,j,q)\in\widetilde{K}} x^{i,j,q} \otimes y^{i,j,q} \otimes v^{i,j,q}$ by

$$\phi(s) = \prod_{i\in J^+} < \bigwedge_{\overrightarrow{(i,j,q)}\in\widetilde{K}} y^{i,j,q}, o_i^* > \prod_{j\in J^-} < \bigwedge_{\overrightarrow{(i,j,q)}\in\widetilde{K}} x^{i,j,q}, o_j >$$
$$\prod_{(i,j,q)\in K} \phi^{i,j,q}(v^{i,j,q}) \prod_{(i,0,q)\in\widetilde{K}} \phi^{i,0,q}(x^{i,0,q}) \prod_{(0,j,q)\in\widetilde{K}} \phi^{0,j,q}(y^{0,j,q})$$

Now consider the group

$$\mathfrak{S} = \mathfrak{S}_1 \times \mathfrak{S}_2 \times \mathfrak{S}_3 = \prod_{(i,j)\in J} S_{s_{ij}} \times \prod_{j\in J^-} S_{s_{0j}} \times \prod_{i\in J^+} S_{s_{i0}}.$$

This group acts naturally on $\bigotimes_{(i,j)\in J} ((\mathbb{C}^{r_{i_1j_1}})^*)^{\otimes s_{ij}} \otimes \bigotimes_{j\in J^-} (Y_0^*)^{\otimes s_{0j}} \otimes \bigotimes_{i\in J^+} X_0^{\otimes s_{i0}} = E^*_c$ by permuting the factors and we observe that

$$\wp\big((\sigma_1, \sigma_2, \sigma_3)\phi\big) = \varepsilon(\sigma_2)\varepsilon(\sigma_3)\wp(\phi)$$

for all $\phi \in E_c^*$ and for all $(\sigma_1, \sigma_2, \sigma_3) \in \mathfrak{S}$. So we have that

$$\wp(E_c^*) = \wp\Big(\bigotimes_{(i,j)\in J} S^{s_{ij}}\big((\mathbb{C}^{r_{ij}})^*\big) \otimes \bigotimes_{j\in J-} \bigwedge^{s_{0j}} Y_0^* \otimes \bigotimes_{i\in J+} \bigwedge^{s_{i0}} X_0 \Big).$$

In particular since $S^m(V)$ is spanned by vectors of the form $v\otimes\cdots\otimes v$, $\wp(E_c^*)$ is spanned by the functions $\wp(\phi)$ with ϕ of the following special form:

$$\phi = \bigotimes_{(i,j)\in J} (\phi^{i,j})^{\otimes s_{ij}} \otimes \bigotimes_{j\in J^-} \phi^{0,j,1}\wedge\cdots\wedge\phi^{0,j,s_{0j}} \otimes \bigotimes_{i\in J^+} \phi^{i,0,1}\wedge\cdots\wedge\phi^{i,0,s_{i0}}. \tag{16}$$

The lemma now follows from the following claim:

Claim: For each ϕ as in (16) $\wp(\phi)$ is a linear combination of the functions $\det(\phi_{\varphi,\alpha,\beta})$.

We prove the claim as follows: we construct vector spaces A_i, B_j and $A = \bigoplus_{i\in J^+} A_i$, $B = \bigoplus_{j\in J^-} B_j$ and

$$\tilde H_0 = H_0 \oplus \bigoplus_{i\in J^+} \mathrm{Hom}(A_i, Y_i) \oplus \bigoplus_{j\in J^-} \mathrm{Hom}(X_j, B_j) \subset \mathrm{Hom}(X\oplus A, Y\oplus B) = \tilde H.$$

Observe that on $\tilde H, \tilde H_0$ there is an action of $\tilde G = G_{XY}\times G_{AB} = G_{XY}\times\prod_{i\in J^+} Gl(A_i)\times\prod_{j\in J^-} Gl(B_j)$ and we call $\tilde c$ the character of $\tilde G$ given by $(\prod_j \det_{GL(X_j)}\times\prod_i \det_{GL(A_i)})^{-1}\times\prod_i \det_{GL(Y_i)}\times\prod_j \det_{GL(B_j)}$. We have an embedding of G_{XY} in $\tilde G$ such that $\sigma^*\tilde c = c$. Observe also that by Proposition 28 we know that the $\tilde c$-covariants functions on H are generated by the functions $\det(\Phi_{\tilde S})$ with $S\in\tilde{\mathcal S}$: we put a tilde to emphasize that we have to consider also the components $\{A_i\}$ and $\{B_j\}$. Then we construct a G_{XY}-equivariant map $\rho : H \longrightarrow \tilde H_0$ such that

1. there exists a $\tilde c$-covariant function f on $\tilde H$ such that $\wp(\phi) = f\circ\rho$.
2. for all $\tilde S\in\tilde{\mathcal S}$ there exists φ,α,β as in equation (10) such that $\det(\Phi_{\tilde S})\circ\rho = \det(\Phi_{\varphi,\alpha,\beta})$.

The claim now follows by Proposition 28.

For $i\in J^+$ and $j\in J^-$ define

$$A_i = \mathbb{C}^{s_{i0}}, \qquad A = \bigoplus_{i\in J^+} A_i, \qquad B_j = \mathbb{C}^{s_{0j}}, \qquad B = \bigoplus_{j\in J^-} B_j.$$

Define also $\alpha_i : A_i \longrightarrow X_0$ and $(\beta_j)^t : B_j^* \longrightarrow Y_0^*$ by

$$\alpha_i(e_l) = \phi^{i,0,l}, \qquad \text{and} \qquad \alpha = \coprod_{i\in J^+} \alpha_i : A \longrightarrow X_0$$

$$(\beta_i)^t(e^l) = \phi^{0,j,l} \qquad \text{and} \qquad \beta^t = \coprod_{j\in J^-} B_j : B^* \longrightarrow Y_0,$$

where e_l (resp. e^l) is the canonical basis of $\mathbb{C}^m$ (resp. $(\mathbb{C}^m)^*$). We define β_i (resp. β) as the transpose of $(\beta_i)^t$ (resp. β^t). Now define $\rho^{ij} : \mathrm{Hom}(X_j, Y_i)\otimes\mathbb{C}^{r_{ij}} \longrightarrow \mathrm{Hom}(X_j, Y_i)$, $\rho^{i0} : \mathrm{Hom}(X_0, Y_i) \longrightarrow \mathrm{Hom}(A_i, Y_i)$, $\rho^{0j} : \mathrm{Hom}(X_j, Y_0) \longrightarrow \mathrm{Hom}(X_j, B_j)$ by

$$\rho^{ij}(T\otimes v) = \phi^{i,j}(v)T, \qquad \rho^{i0}(T) = T\circ\alpha_i, \qquad \rho^{0j}(T) = \beta_j\circ T,$$

and finally define $\rho = \bigoplus_{i,j\in\tilde J} \rho^{ij} : H \longrightarrow \tilde H_0$. Observe that ρ is G_{XY}-equivariant.

Observe now that $\tilde{H}_0^{\otimes n} = \bigoplus \tilde{E}_{\tilde{S}}$ where $\tilde{S} \in \tilde{\mathcal{S}}$ and $\tilde{E}_{\tilde{S}}$ is defined as in (11). In particular we choose the following summand of $\tilde{H}_0^{\otimes n}$:

$$\tilde{E} = \bigotimes_{(i,j)\in J} \mathrm{Hom}(X_j, Y_i)^{\otimes s_{ij}} \otimes \bigotimes_{j\in J^-} \mathrm{Hom}(X_j, B_j)^{\otimes s_{0j}} \otimes \bigotimes_{i\in J^+} \mathrm{Hom}(A_i, Y_i)^{\otimes s_{i0}}$$

and we observe that $(\tilde{E})^*_{\tilde{c}} = \mathbb{C}$. Choose a non zero element $\tilde{\phi} \in (\tilde{E})^*_{\tilde{c}}$ and observe that up to a scalar we have

$$\wp_{\tilde{H}}(\tilde{\phi}) \circ \rho = \wp(\phi). \tag{17}$$

To see this choose ϕ as in (16), and bases y^i_h of Y_i, x^j_k of X^*_j (and its dual basis z^j_k of X_j) . Choose also a bases ε^{ij}_m of $\mathbb{C}^{r_{ij}}$ such that $\phi^{i,j}(\varepsilon^{ij}_m) = \delta_{m,1}$ and set $A^{ij} = \rho^{ij}(s) = \sum_{h,k} a^{ij}_{hk} y^i_h \otimes x^j_k$ for $s \in H$. Then

$$\begin{aligned}
\wp(\phi)(t) = \sum_{h\in\mathcal{K}_Y, k\in\mathcal{K}_X} \prod_{i\in J^+} &< \bigwedge_{\overrightarrow{(i,j,q)\in \widetilde{K}}} a^{ij}_{h(i,j,q)k(i,j,q)} y^i_{h(i,j,q)}, o^*_i> \prod_{j\in J^-} < \bigwedge_{\overrightarrow{(i,j,q)\in \widetilde{K}}} x^j_{k(i,j,q)}, o_j> \\
&\prod_{i\in J^+} < \bigotimes_{\overrightarrow{(i,0,q)\in \widetilde{K}}} x^0_{k(i,0,q)}, \phi^{i,0,1} \wedge \cdots \wedge \phi^{i,0,s_{i0}}> \\
&\prod_{j\in J^-} < \bigotimes_{\overrightarrow{(0,j,q)\in \widetilde{K}}} a^{0j}_{h(0,j,q)k(0,j,q)} y^0_{h(0,j,q)}, \phi^{0,j,1} \wedge \cdots \wedge \phi^{0,j,s_{i0}}> \\
= \sum_{k\in\mathcal{K}_X} \prod_{i\in J^+} &< \bigwedge_{\overrightarrow{(i,j,q)\in \widetilde{K}}} A^{ij} z^j_{k(i,j,q)}, o^*_i> \prod_{j\in J^-} < \bigwedge_{\overrightarrow{(i,j,q)\in \widetilde{K}}} x^j_{k(i,j,q)}, o_j> \\
&\prod_{i\in J^+} < \bigwedge_{\overrightarrow{(i,0,q)\in \widetilde{K}}} x^0_{k(i,0,q)}, \phi^{i,0,1} \wedge \cdots \wedge \phi^{i,0,s_{i0}}> \\
&\prod_{j\in J^-} < \bigwedge_{\overrightarrow{(0,j,q)\in \widetilde{K}}} A^{0j} z^j_{k(0,j,q)}, \phi^{0,j,1} \wedge \cdots \wedge \phi^{0,j,s_{i0}}>
\end{aligned}$$

where the indexes are as follows:

$$\mathcal{K}_X = \{k : \widetilde{K} \longrightarrow \mathbb{N} \,:\, 1 \leq k(i,j,q) \leq \dim X_j\}$$
$$\mathcal{K}_Y = \{h : \widetilde{K} \longrightarrow \mathbb{N} \,:\, 1 \leq h(i,j,q) \leq \dim Y_i\}.$$

The lefthandside in (17) clearly furnishes the same expression.

Finally if we fix $\tilde{S} = (s_{MN})_{N\in\{X_j\}\cup\{A_i\} \text{ and } M\in\{Y_i\}\cup\{B_j\}} \in \tilde{\mathcal{S}}$ and we choose $\varphi_{ij} = s_{Y_iX_j}\phi^{i,j}$ and $\alpha = \coprod_{i\in J^+} s_{Y_iA_i}\alpha_i : A \longrightarrow X_0$ and $\beta = \coprod_{j\in J^-} s_{B_jX_j}\beta_j : Y_0 \longrightarrow B$ we have

$$\det(\Phi_{\tilde{S}}) \circ \rho = \det(\Phi_{\varphi,\alpha,\beta}).$$

Remark 29. The basis of $\mathbb{C}[H]_c$ we have described are different from the polarization of the natural basis of E^*_c. The relation between the two basis is given by formulas of the following types

1. If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ then

$$\det \begin{pmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{pmatrix} + \det \begin{pmatrix} b_{11} & a_{12} \\ b_{21} & a_{22} \end{pmatrix} = \det(A+B) - \det A - \det B.$$

2. If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, $C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$ and $D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$ then

$$\det \begin{pmatrix} a_{11} & b_{11} \\ a_{21} & b_{21} \end{pmatrix} \det \begin{pmatrix} c_{12} & d_{12} \\ c_{22} & d_{22} \end{pmatrix} - \det \begin{pmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{pmatrix} \det \begin{pmatrix} c_{12} & d_{11} \\ c_{22} & d_{21} \end{pmatrix} +$$
$$- \det \begin{pmatrix} a_{12} & b_{11} \\ a_{22} & b_{21} \end{pmatrix} \det \begin{pmatrix} c_{11} & d_{12} \\ c_{21} & d_{22} \end{pmatrix} + \det \begin{pmatrix} a_{12} & b_{12} \\ a_{22} & b_{22} \end{pmatrix} \det \begin{pmatrix} c_{11} & d_{11} \\ c_{21} & d_{21} \end{pmatrix} =$$
$$= - \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} + \det \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \det \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

The first type of formula correspond to the reduction of Lemma 27 to the case $r_{ij} = 1$ and $X_0 = Y_0 = 1$. The second type of formula correspond to the case of Proposition 28.

2.3. **Proof of Proposition 20.** Choose basis $\mathcal{B}_i$ (resp. $\mathcal{B}_i^*$) of the vector spaces D_i and D_i^* and we write our vector space $S(d,v)$ in the following way:

$$S = \bigoplus_{h \in H} V_{h_0}^* \otimes V_{h_1} \oplus \bigoplus_{\substack{i \in I \\ b \in \mathcal{B}_i}} V_{i,b} \oplus \bigoplus_{i \in I, b^* \in \mathcal{B}_i^*} V_{i,b^*}^*$$

where $V_{i,b}$ (resp. V_{i,b^*}^*) is an isomorphic copy of V_i (resp. V_i^*). We fix also a character χ_m and $m_i, \widetilde{m}_i, m_i^+, m_i^-, I^+, I^-, I^0$ as in 2.0.2 and we describe first the χ_m-covariants of $S^{\otimes n}$. To do it we observe that we can decompose $S^{\otimes n}$ in the following way:

$$S^{\otimes n} = \bigoplus_{\ell} E_1^{(\ell)} \otimes \cdots \otimes E_n^{(\ell)}$$

where each $E_i^{(\ell)}$ is a representation of G of one of the following types: $V_{h_0}^* \otimes V_{h_1}$, $V_{i,b}$ or V_{i,b^*}^*. So it is enough to compute the χ-covariants of each piece $E_1^{(\ell)} \otimes \cdots \otimes E_n^{(\ell)}$. We fix one of them: $E = E_1 \otimes \cdots \otimes E_n$ and we compute E_χ^*. Let I^* be a copy of I and fix an isomorphism $i \longleftrightarrow i^*$ between the two sets. For each $j = 1, \ldots, n$ we define a subset $\mathcal{S}_j$ of $I \coprod I^*$ according to the following rule:

$$\mathcal{S}_j = \begin{cases} \{h_0^*, h_1\} & \text{if } E_j = V_{h_0}^* \otimes V_{h_1}, \\ \{i\} & \text{if } E_j = V_{i,b} \text{ for some } b \in \mathcal{B}_i, \\ \{i^*\} & \text{if } E_j = V_{i,b^*}^* \text{ for some } b^* \in \mathcal{B}_i^*. \end{cases}$$

Let now be $\mathcal{S} = \coprod_{j=1}^n \mathcal{S}_j$. An element of $\mathcal{S}$ can be thought as a couple (i,j) (or (i^*,j)) where i (or i^*) is in $\mathcal{S}_j$. We consider now a special class of partitions of $\mathcal{S}$: a collection $\mathfrak{F} = \{\mathcal{C}, \mathcal{M}_i^{(l)}$ for $i \in I$ and $1 \leq l \leq m_i\}$ of disjoint subsets of $2^{\mathcal{S}}$ is called *m-special* if:

1. $\bigcup \mathfrak{F}$ is a partition of $\mathcal{S}$,

2. $\forall C \in \mathcal{C}$ $cardC = 2$ and $\exists i \in I$, $\mathcal{S}_{j_1}, \mathcal{S}_{j_2}$ such that $i \in \mathcal{S}_{j_1}$, $i^* \in \mathcal{S}_{j_2}$ and $C = \{(i, j_1), (i^*, j_2)\}$,
3. $\forall M \in \mathcal{M}_i^{(l)}$ we have $M = \{(i, j)\}$ if $i \in I^+$ and $M = \{(i^*, j)\}$ if $i \in I^-$,
4. $card\mathcal{M}_i^{(l)} = v_i = \dim V_i$.

We can represents a special collection with an enriched graph whose vertices are the sets $\mathcal{S}_j$ and completed according with the following rules:

1. we put an arrow from $\mathcal{S}_{j_1}$ to $\mathcal{S}_{j_2}$ if there exists $C = \{(i, j_2), (i^*, j_1)\} \in \mathcal{C}$,
2. we put an indexed circle box $\bigcirc_i^l$ on $\mathcal{S}_j$ if there exists $M = \{(i, j)\} \in \mathcal{M}_i^{(l)}$
3. we put an indexed square box $\square_i^l$ on $\mathcal{S}_j$ if there exists $M = \{(i^*, j)\} \in \mathcal{M}_i^{(l)}$
4. if E_j is of type $V_{i,b}$ or V_{i,b^*}^* then we add the element b or b^* at the left of the corresponding vertex.
5. if E_j is of type $V_{h_0}^* \otimes V_{h_1}$ then we write h at the left of the corresponding vertex.

Observe that a vertex can be marked with a circle and a square but that it cannot be marked with two circles or two squares.

There is a perfect bijection between m-special collection $\mathfrak{F}$ and graphs as above such that:

1. the cardinality of vertexes marked with $\bigcirc_i^l$ is v_i for each $i \in I^+$ and $1 \leq l \leq m_i$,
2. the cardinality of vertexes marked with $\square_i^l$ is v_i for each $i \in I^-$ and $1 \leq l \leq -m_i$.

We will use the same letter $\mathfrak{F}$ to indicate the collection or the graph.

To a special collection $\mathfrak{F}$ as above we attach a function $\phi_{\mathfrak{F}}$ on E. We define it by the formula

$$\phi_{\mathfrak{F}}(e_1 \otimes \cdots \otimes e_n) = \prod_{C \in \mathcal{C}} \phi_C \cdot \prod_{i \in I^+} \prod_{l=1}^{m_i} <o_i^*, \bigwedge \mathcal{M}_i^{(l)}> \cdot \prod_{i \in I^-} \prod_{l=1}^{m_i} <o_i, \bigwedge \mathcal{M}_i^{(l)}>$$

where o_i is a non zero element in $\bigwedge^{v_i} V_i$, o_i^* is a non zero element in $\bigwedge^{v_i} V_i^*$ and

$$e_j = \begin{cases} x_j^* \in V_i^* & \text{if } E_j = V_{i,b}^*, \\ y_j \in V_i & \text{if } E_j = V_{i,b}, \\ x_j^* \otimes y_j \in V_{h0}^* \otimes V_{h_1} & \text{if } E_j = V_{h0}^* \otimes V_{h_1}, \end{cases}$$

$$\phi_C = <x_{j_1}^*, v_{j_2}> \qquad \text{if } C = \{(i^*, j_1), (i, j_2)\}$$

$$\bigwedge \mathcal{M}_i^{(l)} = y_{j_1} \wedge \cdots \wedge y_{j_{v_i}} \quad \text{if } \mathcal{M}_i^{(l)} = \{\{(i, j_1)\}, \ldots, \{(i, j_{v_i})\}\} \text{ and } i \in I^+$$

$$\bigwedge \mathcal{M}_i^{(l)} = x_{j_1}^* \wedge \cdots \wedge x_{j_{v_i}}^* \quad \text{if } \mathcal{M}_i^{(l)} = \{\{(i, j_1)\}, \ldots, \{(i, j_{v_i})\}\} \text{ and } i \in I^-$$

Finally we extend $\phi_{\mathfrak{F}}$ to all E by linearity. By the lemma above and the discussion in 2.1 we deduce easily the following lemma:

Lemma 30. *E_χ^* is generated by the functions $\phi_{\mathfrak{F}}$.*

Proposition 20 now follows from lemma 30 and the following claim: **claim:** for any special collection $\mathfrak{F}$ the function $\wp(\phi_{\mathfrak{F}})$ is a $\mathbb{C}[S]^{G_v}$-linear combination of the functions f_Δ described in 2.0.2.

We consider the connected components of the graph. There are only five possible types of paths:

1. closed paths,
2. straight paths leaving from a non boxed vertex and arriving in a non boxed vertex,
3. straight paths leaving from a non boxed vertex and arriving in a circle boxed vertex,
4. straight paths leaving from a square boxed vertex and arriving in a non boxed vertex,
5. straight paths leaving from a square boxed vertex and arriving in a circle boxed vertex.

Let now $\mathfrak{F}_0$ be the union of the connected components of the first two types and $\mathfrak{F}_1$ be the union of the remaining components. Observe that

$$\wp(\phi_{\mathfrak{F}}) = \wp(\phi_{\mathfrak{F}_0})\wp(\phi_{\mathfrak{F}_1}).$$

Observe also that $\phi_{\mathfrak{F}_0}$ is an invariant function (indeed this part of the graph corresponds to the situation studied by Lusztig in [9]). Since we are interested in generators of $\mathbb{C}[S]_{\chi,1}$ as a $\mathbb{C}[S]^G$-module, we can suppose for simplicity $\mathfrak{F} = \mathfrak{F}_1$.

Observe now that each connected component Γ of the graph of the third type and with a circle $\circ^{l}_{i_1}$ at the end, has an initial vertex which is an $\mathcal{S}_j = \{i_0^*\}$ and that is marked with $b \in \mathcal{B}^*_{i_0}$ on the left. All the other vertexes of the connected component are of type $\mathcal{S}_j = \{h_0^*, h_1\}$ and they define a path α^Γ such that $\alpha^\Gamma_0 = i_0$ and $\alpha^\Gamma_1 = i_1$. We call $b = b(\Gamma)$ and $l = L_1(\Gamma)$.

In the same way we see that:

1. each connected component Γ of the fourth type determines a path α^Γ, $b^* = b^*(\Gamma) \in \mathcal{B}^*_{\alpha^\Gamma_1}$ and $l = L_0(\Gamma)$ such that $1 \leq l \leq -m_{\alpha^\Gamma_0}$,
2. each connected component Γ of the fifth type determines a path α^Γ, $l_0 = L_0(\Gamma)$ and $l_1 = L_0(\Gamma)$ such that $1 \leq l_0 \leq -m_{\alpha^\Gamma_0}$ and $1 \leq l_1 \leq m_{\alpha^\Gamma_1}$.

Now we prove the claim in the following way, we construct X_j and Y_i as in 2.2, a groups homomorphism $\sigma : G_v \longrightarrow G_{XY}$ such that $\sigma^* c = \chi_m$, a G_v equivariant map $\rho : S \longrightarrow H$, and a G_{XY} c-covariant function f on H such that:

1. for all φ, α, β there exists a χ_m-good data such that $\det(\Phi_{\phi,\alpha,\beta}) \circ \rho = f_\Delta$,
2. $\wp(\phi_{\mathfrak{F}}) = f \circ \rho$

The claim will clearly follow.

Set

$$J^- = \{(i,l) \,:\, i \in I^- \text{ and } 1 \leq l \leq -m_i\},$$
$$J^+ = \{(i,l) \,:\, i \in I^+ \text{ and } 1 \leq l \leq m_i\}.$$

For all $(i,l) \in J^-$ choose $X_{(i,l)} = V_i$ and for each $(i,l) \in J^+$ choose $Y_{(i,l)} = V_i$. For each $(i_0,l_0) \in J^-$ and for each $(i_1,l_1) \in J^+$ define:

$$r_{(i_0,l_0)(i_1,l_1)} = card\{\text{ connected component } \Gamma \text{ of the fifth type such that } \alpha^\Gamma_0 = i_0,\ \alpha^\Gamma_1 = i_1,\ L_0(\Gamma) = l_0 \text{ and } L_1(\Gamma) = l_1\}$$

We the connected component Γ of the set in the left handside as a basis e_Γ of the vector space $\mathbb{C}^{r_{(i_0,l_0)(i_1,l_1)}}$. This basis plays the role of the basis e^{ij}_m we used to give the identification in (9).

For each Γ of the third type choose a one dimensional vector space $\mathbb{C}_{b(\Gamma)}$ and fix a generator b_Γ. For each Γ of the fourth type choose a one dimensional vector space $\mathbb{C}_{b^*(\Gamma)}$ and fix a generator b^*_Γ.

$$X_0 = \bigoplus_{\Gamma \text{ of the third type}} \mathbb{C}_{b(\Gamma)} = \bigoplus_{\Gamma \text{ of the third type}} \mathbb{C}b_\Gamma$$
$$Y_0 = \bigoplus_{\Gamma \text{ of the fourth type}} \mathbb{C}_{b^*(\Gamma)} = \bigoplus_{\Gamma \text{ of the fourth type}} \mathbb{C}b^*_\Gamma.$$

Now for each connected component Γ of the third type define $\rho^\Gamma : S \longrightarrow \mathrm{Hom}(\mathbb{C}_{b(\Gamma)}, Y_{(\alpha_1^\Gamma, L_1(\Gamma))})$ by

$$s \longmapsto \{\lambda \mapsto \alpha^\Gamma(s)\gamma_{\alpha_0^\Gamma}(b(\Gamma))\lambda\}$$

In a similar way define ρ^Γ if Γ is the fourth or of the fifth type. Finally define

$$\rho : S \longrightarrow H \quad \text{by} \quad \rho = \bigoplus_\Gamma \rho^\Gamma.$$

Define also a group homomorphism $\sigma : G_v \longrightarrow G_{XY}$ by $(\sigma(g_i))_{X_{(i_0,l_0)}} = g_{i_0}$ and $(\sigma(g_i))_{Y_{(i_1,l_1)}} = g_{i_1}$, and observe that ρ is G_v equivariant.

Now we describe $\phi \in (H^{\otimes \tilde n})^*_c$ (in general $\tilde n$ is less or equal to n) such that

$$\wp(\phi_{\mathfrak{F}})(s) = \wp(\phi)(\rho(s)). \tag{18}$$

We describe ϕ by giving a summands $\tilde E$ of $H^{\otimes \tilde n}$ as in (14) and $\phi \in \tilde E^*_c$ as in (15).To define $\tilde E$ we have to define $s_{(i_1,l_1)(i_0,l_0)}$, $s_{(i_1,l_1)0}$ and $s_{0(i_0,l_0)}$ for all $(i_1,l_1) \in J^+$ and for all $(i_0,l_0) \in J^-$. We set

$$\begin{aligned}
s_{(i_1,l_1)(i_0,l_0)} &= r_{(i_1,l_1)(i_0,l_0)} \\
s_{(i_1,l_1)0} &= card\{ \text{ connected component } \Gamma \text{ of the third type} \\
&\qquad \text{such that } \alpha_1^\Gamma = i_1 \text{ and } L_1(\Gamma) = l_1\} \\
s_{0(i_0,l_0)} &= card\{ \text{ connected component } \Gamma \text{ of the fourth type} \\
&\qquad \text{such that } \alpha_0^\Gamma = i_0 \text{ and } L_0(\Gamma) = l_0\}
\end{aligned}$$

Observe that we can choose a bijection $q \longleftrightarrow \Gamma_q$ between $\{1, \dots, s_{(i_1,l_1)(i_0,l_0)}\}$ and the set of connected component Γ of the fifth type such that $\alpha_1^\Gamma = i_1$,$L_1(\Gamma) = l_1$, $\alpha_0^\Gamma = i_0$ and $L_0(\Gamma) = l_0$. So we can define $\phi^{(i_1,l_1),(i_0,l_0),q}$ by

$$\phi^{(i_1,l_1),(i_0,l_0),q}(e_\Gamma) = \delta_{\Gamma,\Gamma_q}.$$

Observe also that we can choose a bijection $q \longleftrightarrow \Gamma_q$ between $\{1, \dots, s_{(i_1,l_1)0}\}$ and the set of connected component Γ of the third type such that $\alpha_1^\Gamma = i_1$,$L_1(\Gamma) = l_1$. So we can define $\phi^{(i_1,l_1),0,q}$ by

$$\phi^{(i_1,l_1),0,q}(b_\Gamma) = \delta_{\Gamma,\Gamma_q}.$$

In a similar way define $\phi^{0,(i_0,l_0),q}$.

Up to a sign which depends on our choices and ordering equation (18) is a tautologically satisfied.

Observe now that by lemma 27 and linearity $\mathbb{C}[H]_c$ is generated by functions $s \longmapsto \det(\Phi_{\varphi,\alpha,\beta}(s))$ where $\tilde{A}, \tilde{B}, \varphi, \alpha, \beta$ are as in (10) and moreover there exists a basis $e_1, \dots, e_{r_A}$ of $\tilde{A}$ and a basis $\tilde{e}_1, \dots, \tilde{e}_{r_B}$ of $\tilde{B}^*$ such that for all i there exist a connected component of the third type Γ_i^A such that $\alpha(e_i) = b_{\Gamma_i^A}$ and for all i there exists a connected component of the fourth type Γ_i^B such that $\tilde{e}_i(\beta(b_\Gamma^*)) = \delta_{\Gamma\Gamma_i^B}$. So it is enough to prove that if $\tilde{A}, \tilde{B}, \varphi, \alpha, \beta$ are as above then there exists a χ_m-good Δ such that

$$\det(\Phi_{\varphi,\alpha,\beta}) \circ \rho = f_\Delta.$$

We define

$$\begin{aligned}
A &= (b_{\Gamma_i^A})_{i=1,\dots,r_A} \in \Big(\bigcup D_i\Big)^{\dim \tilde{A}} \\
B &= (b^*_{\Gamma_i^B})_{i=1,\dots,r_B} \in \Big(\bigcup D_i^*\Big)^{\dim \tilde{B}} \\
\alpha_{jh}^{ik} &= \sum_{1 \le q \le s_{(j,h)(i,k)}} \phi^{(j,h),(i,k),q}(e_\Gamma)\alpha^\Gamma \\
\alpha_l^{ik} &= \delta_{i,(\alpha^{\Gamma_l^B})_0}\delta_{k,L_0(\Gamma_l^B)}\alpha^{\Gamma_l^B} \\
\alpha_{jh}^l &= \delta_{j,(\alpha^{\Gamma_l^A})_1}\delta_{h,L_1(\Gamma_l^A)}\alpha^{\Gamma_l^A}
\end{aligned}$$

The equation (18) follows now by the very definition.

3. The action of the Weyl group

For any $m \in P$ and for any $\lambda \in Z$ we defined a variety $M_{m,\lambda}(d, v)$. Observe that on both m, λ there is a natural action of the Weyl group W. We define an action of the Weyl group also on (d, v). We have already described d as an element of X and v as an element of Q. We can now define

$$\sigma(d, v) = (d, \sigma(v - d) + d).$$

Observe that $\sigma(v - d) + d \in Q$ so the definition is well given. So it make sense to consider the variety $M_{\sigma m,\sigma\lambda}(\sigma(d, v))$ or the variety $\mathfrak{M}_{\sigma\zeta}(\sigma(d, v))$ for $\zeta \in \mathfrak{Z}$.

In [13] Nakajima used analytic methods to prove, in the case of a finite Dynkin diagram, that if ζ is generic then there exists a diffeomorphism of differentiable manifolds

$$\Phi_{\sigma,\zeta} \colon \mathfrak{M}_\zeta(d, v) \longrightarrow \mathfrak{M}_{\sigma\zeta}(\sigma(d, v))$$

and moreover that $\Phi_{\sigma',\sigma\zeta} \circ \Phi_{\sigma,\zeta} = \Phi_{\sigma'\sigma,\zeta}$. In the same paper he also asserted that a similar construction could be obtained in the general case using reflection functors as indeed we are going to do.

In [9] Lusztig gave a purely algebraic construction of an isomorphism

$$M_{0,\lambda}(d, v) \simeq M_{0,s_i\lambda}(s_i(d, v))$$

whenever $\lambda_i \neq 0$. In this paper we will give a generalization of Lusztig construction.

Definition 31. If $u \in \mathbb{Z}^n = Q^\vee$ and $A \subset Q^\vee$ we define

$$H_u = \{(m,\lambda) \in P \oplus Z \ : <u,^\vee, \lambda> = <u^\vee, m> = 0\} \text{ and } H_A = \bigcup_{a \in A} H_a$$

Let $K = \max\{1, a_{ij}^2 \ : \ i,j \in I\}$. If $v \in \mathbb{Z}^n$ we define

$$\tilde{U}_v = \{u \in \mathbb{N}^I \ : \ 0 \leq u_i \leq K v_i\} \text{ and } \tilde{H}^v = H_{\tilde{U}_v}.$$

We define also

$$U_\infty = \bigcup_{i \in I} W \alpha_i^\vee \text{ and } H^\infty = H_{U_\infty}.$$

Finally we set $\mathcal{G}_v = \{(m,\lambda) \in P \times Z_G \ : \ \sigma(m,\lambda) \notin H^{\sigma \cdot v}$ for all $\sigma \in W\}$. Both of the following definition of the set $\mathcal{G}$ will be fine for us:

$$\mathcal{G} = \{(v,m,\lambda) \in Q \times P \times Z_G \ : \ (m,\lambda) \in \mathcal{G}_v\} \text{ or}$$
$$\mathcal{G} = \{(v,m,\lambda) \in Q \times P \times Z_G \ : \ (m,\lambda) \notin H^\infty\}.$$

We observe that in any case $\mathcal{G}$ is W-stable.

Proposition 32. *For all d, v, for all $\sigma \in W$ and for all (m,λ) such that $(m,\lambda,v) \in \mathcal{G}_v$ there exists an algebraic isomorphism:*

$$\Phi^{d,v}_{\sigma,m,\lambda} \colon M_{m,\lambda}\big(d,v\big) \longrightarrow M_{\sigma m,\sigma\lambda}\big(\sigma(d,v)\big).$$

Moreover this isomorphism satisfies

$$\Phi^{\sigma(d,v)}_{\tau,\sigma m,\sigma\lambda} \circ \Phi^{d,v}_{\sigma,m,\lambda} = \Phi^{d,v}_{\tau\sigma,m,\lambda}. \tag{19}$$

3.1. **Generators.** In this section we define the actions of the generators s_i of W following [9]. We fix $i \in I$ and (d,v), $\lambda \in Z$ and $m \in P$. We call $(d,v') = s_i(d,v)$, $\lambda' = s_i\lambda$ and $m' = s_i m$. Through all this section we assume $v, v' \geq 0$. For the convenience of the reader we write explicit formula in this case:

$$\lambda'_j = \lambda_j - c_{ij}\lambda_i \qquad m'_j = m_j - c_{ij} m_i \text{ for all } j$$
$$v'_i = d_i - v_i + \sum_{j \neq i} a_{ij} v_j \qquad v'_j = v_j \text{ for all } j \neq i$$

Observe that we can choose

$$D'_j = D_j \text{ for all } j \text{ and } V'_j = V_j \text{ for all } j \neq i.$$

In particular we have

$$T_i = D_i \oplus \bigoplus_{h_1 = i} V_{h_0} = T'_i$$

since we suppose that our quiver has not simple loops.

Definition 33 (Lusztig [9])**.** Fix $\lambda \in Z_G$ and define $Z_i^\lambda(d,v)$ to be the sub-variety of $S_i(d,v) \times S_i(d,v')$ of pairs $(s,s') = \big((B,\gamma,\delta),(B',\gamma',\delta')\big)$ such that the following conditions hold:

1. $B_h(s) = B'_h(s')$ for all h such that $h_0, h_1 \neq i$,
2. $\gamma_j(s) = \gamma'_j(s')$ for all $j \neq i$,
3. $\delta_j(s) = \delta'_j(s')$ for all $j \neq i$,

4. the following sequence is exact:

$$0 \longrightarrow V_i' \xrightarrow{a_i'} T_i \xrightarrow{b_i} V_i \longrightarrow 0, \tag{20}$$

5. $a_i'(s')b_i'(s') = a_i(s)b_i(s) - \lambda_i \mathrm{Id}_{T_i}$,
6. $s \in \Lambda_\lambda(d,v)$ and $s' \in \Lambda_{\lambda'}(d,v')$.

Lemma 34. *Let $(s,s') \in S_i(d,v) \times S_i(d,v')$ and suppose that it satisfies conditions 1), 2), 3), 4) , 5) above then:*

1. $s \in \Lambda_\lambda(d,v) \iff s' \in \Lambda_{\lambda'}(d,v')$,
2. *if $\mu_j(s) - \lambda_j \mathrm{Id}_{V_j} = 0$ for all $j \neq i$ then $s \in \Lambda_\lambda(d,v)$,*
3. *if $\mu_j(s') = \lambda_j \mathrm{Id}_{V_j'}$ for all $j \neq i$ then $s' \in \Lambda_{\lambda'}(d,v')$.*

Proof. 2) We have to prove $b_i a_i - \lambda_i \mathrm{Id}_{V_i} = 0$ and by condition 4) it is enough to prove $b_i a_i b_i = \lambda_i b_i$. So $b_i a_i b_i = b_i(a_i' b_i' - \lambda_i) = \lambda_i b_i$ by conditions 4) and 5).

The proof of 3) is equal to the proof of 2). We prove the implication $\Rightarrow$ in 1). By 2) and 3) it is enough to prove that $b_j' a_j' = \lambda_j'$ for $j \neq i$.

$$\begin{aligned}
b_j' a_j' &= \sum_{h_1 = j} \varepsilon(h) B_h' B_{\bar h}' + \gamma_j' \delta_j' = \\
&= \sum_{h_1 = j, h_0 \neq i} \varepsilon(h) B_h B_{\bar h} + \gamma_j \delta_j + \sum_{h_1 = j, h_0 = i} \varepsilon(h) B_h' B_{\bar h}' = \\
&= b_j a_j + \sum_{h_1 = j, h_0 = i} \varepsilon(h) \left(B_h' B_{\bar h}' - B_h B_{\bar h} \right) \\
&= b_j a_j + \sum_{h_0 = j, h_1 = i} \left(B_{\bar h} \varepsilon(h) B_h - B_{\bar h}' \varepsilon(h) B_h' \right) \\
&= b_j a_j + \sum_{h_0 = j, h_1 = i} \left([a_i b_i]_{V_{h_0}}^{V_{h_0}} - [a_i' b_i']_{V_{h_0}}^{V_{h_0}} \right) \\
&= \lambda_j + \sum_{h_0 = j, h_1 = i} \lambda_i = \lambda_j'
\end{aligned}$$

The proof of the converse is completely analogous. ☐

Lemma 35. *Let $\lambda \in Z_G$, $(s,s') \in Z_i^\lambda(d,v)$ and α be an element of the path algebra algebra of type (α_0, α_1) then*

1. *if $\alpha_0, \alpha_1 \neq i$ there exists an element α' of the b-path algebra of type (α_0, α_1) such that $\alpha'(s') = \alpha(s)$,*
2. *if $\alpha_1 \neq i$ there exists an element α' of the b-path algebra of type (α_0, α_1) such that $\alpha'(s')\gamma_{\alpha_0}' = \alpha(s)\gamma_{\alpha_0}$,*
3. *if $\alpha_0 \neq i$ there exists an element α' of the b-path algebra of type (α_0, α_1) such that $\delta_{\alpha_1}' \alpha'(s') = \delta_{\alpha_1} \alpha(s)$,*
4. *there exists an element α' of the b-path algebra of type (α_0, α_1) such that $\delta_{\alpha_1}' \alpha'(s') \gamma_{\alpha_0}' = \delta_{\alpha_1} \alpha(s) \gamma_{\alpha_0}$,*

Proof. By induction on the length of α we can reduce the proof of this lemma to the following identities that are a consequence of condition 5) in definition 33:

$$\begin{aligned} B'_h B'_k &= \begin{cases} B_h B_k & \text{if } h \neq \bar{k} \\ B_h B_k - \lambda_i & \text{if } k = \bar{h} \end{cases} \\ \delta'_i B'_k &= \delta_i B_k \\ B'_h \gamma'_i &= B_h \gamma_i \\ \delta'_i \gamma'_i &= \delta_i \gamma_i - \lambda_i \end{aligned}$$

for h, k such that $h_0 = i = k_1$. □

Lemma 36. *Let $(s, s') \in Z_i^\lambda(d, v)$ and suppose $m_i \geq 0$ or $\lambda_i \neq 0$ then*

$$s \textit{ is } \chi_m \textit{ semistable} \iff s' \textit{ is } \chi_{m'} \textit{ semistable}$$

Proof. We prove only $\Rightarrow$. Let's do first the case $m_i \geq 0$. If s is χ_m semi-stable, then there exists $\Delta = \{A, B, \alpha^*_*\}$ m-good such that $f_\Delta(s) \neq 0$. Using the notation in 2.0.2 we have $\varphi_\Delta = \det \Psi_\Delta$ where $\Psi_\Delta : Y \longrightarrow Z$ is a linear map. In our case we can write Z as $\mathbb{C}^{m_i} \otimes V_i \oplus \widetilde{Z}$ and we observe that no V_i summands appear in Y or $\widetilde{Z}$.

Now we construct a new data $\Delta' = \{A', B', \alpha'^*_*\}$ such that $f_{\Delta'}(s') \neq 0$ and $f_{\Delta'}$ a χ'-covariant polynomial. Our strategy will be the following: we substitute each V_i with the space T_i in the space Z and we add m_i copies of V'_i to Y. Let's do it more precise: first of all the new data will not be m' good so we have to define ${m'}_j^+$ and ${m'}_j^-$:

1. ${m'}_i^+ = 0$ and ${m'}_i^- = m_i = m_i^+$,
2. ${m'}_j^- = m_j^-$ and ${m'}_j^+ = m_j^+ + a_{ij} m_i^+$ for all $j \neq i$,
3. ${m'}^- = m^-$ and ${m'}^+ = m^+ + d_i m_i^+$.

Observe that ${m'}_j^+ - {m'}_j^- = m'_j$ for all j so our data will furnish a χ' equivariant function. Moreover if we define

$$Z' = \mathbb{C}^{m_i} \otimes V_i \oplus \widetilde{Z} \quad \text{and} \quad Y' = \mathbb{C}^{m_i} \otimes T_i \oplus Y$$

we observe that they have the numbers of V'_j, $\mathbb{C}_a$, $\mathbb{C}_b$ factors specified by m'. Now we construct the new data Δ' in such a way that with respect to the decompositions above we have:

$$\begin{aligned} [\Psi_\Delta(s)]^{Y}_{\mathbb{C}^{m_i} \otimes V_i \oplus \widetilde{Z}} &= \begin{pmatrix} (\mathrm{Id} \otimes b_i) \circ \pi \\ \Phi \end{pmatrix}, \\ [\Psi_{\Delta'}(s')]^{\mathbb{C}^{m_i} \otimes V'_i \oplus Y}_{\mathbb{C}^{m_i} \otimes T_i \oplus \widetilde{Z}} &= \begin{pmatrix} \mathrm{Id} \otimes a'_i & \pi \\ 0 & \Phi \end{pmatrix}. \end{aligned}$$

If we construct a data with this property we observe that $\Psi_\Delta(s)$ is an isomorphism if and only if $\Psi_{\Delta'}(s')$ is an isomorphism. Hence $f_\Delta(s) \neq 0$ implies $f_{\Delta'}(s') \neq 0$ and the lemma is proved.

To construct the new data we choose a basis $e_1, \dots, e_{d_i}$ of D_i and we define the other elements of the data according to the following rules

1. $A' = A$,

2. if $B = (b_1, \dots, b_{m^+})$ we set $B' = (b_1, \dots, b_{m^+}, \underbrace{e_1, \dots, e_1}_{m_1\ times}, \dots, \underbrace{e_{d_i}, \dots e_{d_i}}_{m_1\ times})$,
3. $\alpha'^{j_1,h_1}_{j_2,h_2}$ for $j_1 \neq i$ and $h_2 \leq m^+_{j_2}$ is an element constructed according to case 1) in the previous lemma,
4. $\alpha'^{a}_{j_2,h_2}$ for $h_2 \leq m^+_{j_2}$ is an element constructed according to case 2) in the previous lemma,
5. $\alpha'^{j_1,h_1}_{b_l}$ for $j_1 \neq i$ and $l \leq m^+$ is an element constructed according to case 3) in the previous lemma.
6. $\alpha'^{i,h}_{b_l} = \alpha'^{i,h}_{j,k} = 0$ if $l \leq m^+$ and $k \leq m^+_j$.

In this way we guarantee that the projection of $\Psi_{\Delta'}(s')$ onto $\widetilde{Z}$ is equal to $\begin{pmatrix}0 & \Phi\end{pmatrix}$. To define the remaining part of the new data we do not give details on the indexes, but we explain how to construct it. It is clear that we can choose $\alpha'^{i,h}_{*}$ for the remaining indexes $*$ in such a way that the projection of $\Psi_{\Delta'}(s')\big|_{\mathbb{C}^{m_i}\otimes V_i'}$ on $\mathbb{C}^{m_i} \otimes T_i$ is equal to $\mathrm{Id} \otimes a_i'$. Finally we observe that a path β from V_j to V_i with $j \neq i$ has to go through a summand of T_i so there exists a path α such that $\beta(s) = b_i \circ \alpha(s)$. Now we use the previous lemma to change α with a α' such that $\beta(s) = b_i \circ \alpha'(s')$. More generally if β is an element of the path algebra of type (j, i) with $j \neq i$ then there exists an element of the b-path algebra α' such that $\beta(s) = b_i \circ \alpha'(s)$. In this way we define the elements of the b-path algebra connecting summands of Y and summands of $\mathbb{C}^{m_i} \otimes T_i$.

In the case $m_i < 0$ we proceed in a similar way: we choose Δ m-good and we have

$$Y = \mathbb{C}^{-m_i} \otimes V_i \oplus \widetilde{Y}, \quad Y' = \mathbb{C}^{-m_i} \otimes T_i \oplus \widetilde{Y}, \quad Z' = \mathbb{C}^{-m_i} \otimes V_i' \oplus Z.$$

As in the previous case we can find a new data Δ' such that:

$$[\Psi_\Delta(s)]_Z^{\mathbb{C}^{-m_i}\otimes V_i \oplus \widetilde{Y}} = \begin{pmatrix}\pi \circ (\mathrm{Id} \otimes a_i) & \Phi\end{pmatrix},$$
$$[\Psi_{\Delta'}(s')]_{\mathbb{C}^{-m_i}\otimes V_i' \oplus Z}^{\mathbb{C}^{-m_i}\otimes T_i \oplus \widetilde{Y}} = \begin{pmatrix}\mathrm{Id} \otimes b_i' & 0 \\ \pi & \Phi\end{pmatrix}.$$

Now to conclude that $\Psi_{\Delta'}(s')$ is an isomorphism if $\Psi_\Delta(s)$ is we need to know that b_i' is an epimorphism and this is not guarantee by $(s, s') \in Z_i^\lambda(d, v)$. But if $\lambda_i \neq 0$ then, since $b_i' a_i' = -\lambda_i$, we have that b_i' is surjective. □

Definition 37. Let p (resp. p') be the projections of $Z_i^\lambda(d, v)$ on $\Lambda_\lambda(d, v) \subset S(d, v)$ (resp. $\Lambda_{\lambda'}(d, v') \subset S(d, v')$). Suppose that $m_i > 0$ or $\lambda_i \neq 0$ then we define

$$Z_i^{m,\lambda} = p^{-1}\big(\Lambda_{m,\lambda}(d, v)\big) = p'^{-1}\big(\Lambda_{m',\lambda'}(d, v')\big).$$

We define also

$$G_{i,v} = \prod_{j \neq i} GL(V_j) \times GL(V_i) \times GL(V_i').$$

Observe that there are natural projections from $G_{i,v}$ to G_v and $G_{v'}$, therefore there are natural actions of $G_{i,v}$ on $S_i(d, v)$, $S_i(d, v')$. Observe that here is a natural action of $G_{i,v}$ on Z_i^λ and $Z_i^{m,\lambda}$ such that the projections p, p' are equivariant.

Lemma 38. *Let $s \in \Lambda_{\lambda,m}(d, v)$ then*

1. *if $\lambda_i \neq 0$ then b_i is epic and a_i is mono,*

2. *if* $m_i > 0$ *then* b_i *is epic,*
3. *if* $m_i < 0$ *then* a_i *is mono.*

Proof. If $\lambda_i \neq 0$ then the result is clear by $b_i a_i = \lambda_i$. Suppose now that $\lambda_i = 0$ and $m_i > 0$. Let $U_i = \operatorname{Im} b_i$ and let $V_i = U_i \oplus W_i$. Define now a one parameter subgroup $g(t)$ of G_V in the following way:

$$[g_i(t)]^{U_i \oplus W_i}_{U_i \oplus W_i} = \begin{pmatrix} 1 & 0 \\ 0 & t^{-1} \end{pmatrix} \text{ and } g_j \equiv 1 \text{ for } j \neq i$$

Since $\operatorname{Im} b_i \subset U_i$ we have that there exists the limit $\lim_{t \to 0} g(t) \cdot s = s_0$. Let now $n > 0$ and f a χ^n-covariant function on S such that $f(s) \neq 0$. Then

$$f(s_0) = \lim_{t \to 0} f(g(t) \cdot s) = \lim_{t \to 0} \det{}^{nm_i}_{GL(V_i)} f(s) = \lim_{t \to 0} t^{-nm_i \dim W_i} f(s)$$

So we must have $\dim W_i = 0$. The proof of the third case is completely similar to this one. □

Lemma 39 (see also Lusztig [9])**.** *If* $m_i > 0$ *or* $\lambda_i \neq 0$ *then*

1. $p : Z_i^{m,\lambda}(d,v) \longrightarrow \Lambda_{m,\lambda}(d,v)$ *is a principal* $GL(V'_i)$ *bundle,*
2. $p' : Z_i^{m,\lambda}(d,v) \longrightarrow \Lambda_{m',\lambda'}(d,v')$ *is a principal* $GL(V_i)$ *bundle.*

Proof. Lusztig's proof extend to this case without changes. Let's prove for example 1. We have to prove: *i.* that the action on the fiber is free, *ii.* that it is transitive. First of all we observe that by the previous lemma if $s \in \Lambda_{m,\lambda}$ then $b_i(s)$ is epic. In particular there exists $a'_i : V'_i \longrightarrow T_i$ such that sequence (20) is exact, and clearly a'_i is univoquely determined up to the action of $GL(V'_i)$, moreover this action is free. So *i.* and *ii.* reduce to the following fact: if $s \in \Lambda_{m,\lambda}$ and a'_i is such that sequence (20) is exact, then there exists a unique, b'_i such that $a'_i b'_i = a_i b_i - \lambda_i$. Since a'_i is mono the unicity is clear. To prove the existence we observe that it is equivalent to $\operatorname{Im} a'_i \supset \operatorname{Im}(a_i b_i - \lambda_i)$. But the last statement is clear since we have: $\operatorname{Im} a'_i = \ker b_i$ and $b_i(a_i b_i - \lambda_i) = 0$. □

Proposition 40. *If* $m_i > 0$ *or* $\lambda_i \neq 0$ *then the projections* p, p' *induces algebraic isomorphisms* $\bar{p}$, $\bar{p}'$*:*

$$\Lambda_{m,\lambda}(d,v)//G_v \xleftarrow[\bar{p}]{\sim} Z_i^{m,\lambda}(d,v)//G_{i,v} \xrightarrow[\bar{p}']{\sim} \Lambda_{m',\lambda'}(d,v')//G_{v'}$$

Proof. This proposition is a straightforward consequence of the previous lemma and the following general fact (see for example [12] Proposition 0.2): let G be an algebraic groups over $\mathbb{C}$ and X, Y two irreducible algebraic variety over $\mathbb{C}$; if G acts on X and $\varphi : X \longrightarrow Y$ is such that for all $y \in Y$ the fiber X_y contains exactly one G-orbit then φ is a categorical quotient. If we apply this lemma to the projection p, (resp. p') and to the group $GL(V'_i)$ (resp. $GL(V_i)$) we obtain the required result. □

We can use this proposition to define the action of the generators of the Weyl group.

Definition 41. Let $i, \lambda, m, d, v, \lambda', m', v'$ be as above, and suppose $d_j \geq 0$, $v_j, v'_j \geq 0$ for all j then we define an isomorphism of algebraic variety

$$\Phi^{d,v}_{s_i,\lambda,m} \colon M_{m,\lambda}(d,v) \longrightarrow M_{m',\lambda'}(d,v')$$

in the following way:

1. if $m_i > 0$ or $\lambda_i \neq 0$ then we set $\Phi^{d,v}_{s_i,\lambda,m} = \bar{p}'\bar{p}^{-1}$,
2. if $m_i < 0$ then we exchange the role of v, v' in the previous construction: more precisely we observe that $m'_i > 0$ so we can define $\Phi^{d,v'}_{s_i,\lambda',m'} \colon M_{m',\lambda'}(d,v') \longrightarrow M_{m,\lambda}(d,v)$ and we define $\Phi^{d,v}_{s_i,\lambda,m} = \big(\Phi^{d,v'}_{s_i,\lambda',m'}\big)^{-1}$.

Remark 42. To see that $\Phi^{d,v}_{s_i,\lambda,m}$ is univoquely defined we have to verify that if $\lambda_i \neq 0$ and $m_i < 0$ the two definitions above coincide. This fact reduces easily to the following remark: if $\lambda_i \neq 0$ then

$$(s,s') \in Z^{\lambda}_i(d,v) \iff (s',s) \in Z^{\lambda'}_i(d,v').$$

Let us prove, for example, the $\Rightarrow$ part. Since $a_i b_i = a'_i b'_i + \lambda_i = a'_i b'_i - \lambda'_i$ the only thing we have to verify is that the sequence

$$0 \longrightarrow V_i \xrightarrow{a_i} T_i \xrightarrow{b'_i} V'_i \longrightarrow 0$$

is exact. The surjectivity of b'_i and the injectivity of a_i are a consequence of $\lambda_i \neq 0$. Since $\dim T_i = \dim V_i + \dim V'_i$ we need only to prove that $b'_i a_i = 0$. Observe that $b'_i a_i = 0$ if and only if $a'_i b'_i a_i = 0$ since also a'_i is injective. Finally $a'_i b'_i a_i = (a_i b_i - \lambda_i) a_i = 0$.

3.2. **Preliminaries.** We saw how to define

$$\Phi^{d,v}_{s_i,m,\lambda} \colon M_{m,\lambda}\big(d,v\big) \longrightarrow M_{s_i(m),s_i(\lambda)}\big(s_i(d,v)\big)$$

in the case that $(\lambda_i, m_i) \neq 0$ and $d, v, s_i v \geq 0$. To define an action of the Weyl group we have now to guarantee that coxeter relations hold. We will prove these relations in the next paragraph. Before doing it we observe that we have to guarantee some conditions on m, λ such that we will be able to define $\Phi^{\sigma(d,v)}_{s_i,\sigma m,\sigma\lambda}$ for any element $\sigma \in W$: this condition will be $(m,\lambda) \in \mathcal{G}_v$ (31). We have also to say something about the case $d_i < 0$ or $v_i < 0$ for some $i \in I$.

In the case that $d_i < 0$ for some i then $M_{m,\lambda}\big(\sigma(d,v)\big) = \varnothing$ for all σ, m, λ by the very definition, so there is nothing to define.

The second trivial case is $d = v = 0$. Indeed in this case we have $M_{\sigma m,\sigma\lambda}\big(\sigma(d,v)\big) = \{0\}$ so the definition is trivial.

The other two cases are treated in the two lemmas below.

In the following we fix d such that $d_i \geq 0$ for all i. It will be convenient to define an affine action of W on Q by $\sigma \cdot v = \sigma(v-d) + d$.

Lemma 43. *Let $d \geq 0$ and $(m,\lambda) \in \mathcal{G}_v$ if there exists σ such that $\sigma \cdot v \not\geq 0$ then $M_{m,\lambda}(d,v) = \varnothing$.*

Proof. Suppose that σ is an element of minimal length such that $\sigma \cdot v \not\geq 0$ and let $l = \ell(\sigma)$. We prove the lemma by induction on l. The case $l = 0$ is trivial.
Initial step: $l = 1$. If $s_i \cdot v \not\geq 0$ then we have $0 \leq d_i + \sum a_{ij} v_j < v_i$. Hence $\dim T_i < \dim V_i$, $u = (0, \dots, \overset{i}{1}, 0, \dots) \in \tilde{U}_v$ and $(\lambda_i, m_i) \neq 0$. So $M_{m,\lambda}(d, v) = \varnothing$ by lemma 38.
Inductive step: if $l \geq 2$ then $l-1 \Rightarrow l$. Let $\sigma = \tau s_i$ with $\ell(\tau) = l-1$ and $v' = s_i \cdot v$, $\lambda' = s_i \lambda$, $m' = s_i m$. By induction $M_{m',\lambda'}(d, v') = \varnothing$ and, since $l \geq 2$, $v' \geq 0$. If $(m_i, \lambda_i) \neq 0$ then we can apply Proposition 40 and we obtain $M_{m,\lambda}(d, v) \simeq M_{m',\lambda'}(d, v') = \varnothing$. If $(m_i, \lambda_i) = 0$ then $u = (0, \dots, \overset{i}{1}, 0, \dots) \notin \tilde{U}_v$, hence $v_i = 0$. Moreover $\lambda' = \lambda$ and $m' = m$ so $(m'_i, \lambda'_i) = 0$ and $u = (0, \dots, \overset{i}{1}, 0, \dots) \notin U_{v'}$. Hence $v'_i = 0$ so $v' = v$ and $\tau v \not\geq 0$ against the minimality of σ. $\square$

Lemma 44. *Let (I, H) be connected, $(m, \lambda) \in \mathcal{G}_v$ and suppose $d \geq 0$ and $\sigma \cdot v \geq 0$ for all $\sigma \in W$. If there exists $i \in I, \sigma \in W$ such that $\sigma(m, \lambda) = (m', \lambda')$ and $(m'_i, \lambda'_i) = 0$ then $d = v = 0$.*

Proof. Without loss of generality we can assume $\sigma = 1$.
First step: $v_i = 0$. This is clear since otherwise $u = (0, \dots, \overset{i}{1}, 0, \dots) \in U_v$.
Second step: $d_i = 0$ and $v_j = 0$ for all j such that a_{ij}. Let $v' = s_i \cdot v$ and observe that $s_i \lambda = \lambda$ and $s_i m = m$. Then as in first step we have $0 = v'_i = d_i + \sum_j a_{ij} v_j$ from which the claim follows.

Let now $W' =<\{s_j \,:\, a_{ij} \neq 0 \text{ and } j \neq i\}>$. If $(d, v) \neq 0$ then there exists $j \in I$ and $\sigma \in W'$ such that $a_{ij} \neq 0$ and

$$n = d_j + \sum_{h \in I} a_{jh} \tilde{v}_h > 0.$$

where $\tilde{v} = \sigma \cdot v$. Since $(\sigma\lambda)_i = \lambda_i = 0 = m_i = (\sigma m)_i$ we can assume $\sigma = 1$. Let now $v' = s_i s_j \cdot v$, $\lambda' = s_i s_j \lambda$ and $m' = s_i s_j m$, we have:

$$\begin{array}{lll} v'_i = a_{ij} n & \lambda'_i = -a_{ij} \lambda_j & m_i = -a_{ij} m_j \\ v'_j = n & \lambda'_j = (a_{ij}^2 - 1)\lambda_j & m_j = (a_{ij}^2 - 1) m_j. \end{array}$$

Hence $u = (0, \dots, \overset{j}{a_{ij}}, 0, \dots, \overset{i}{a_{ij}^2 - 1}, 0, \dots) \in U_{v'}$ and $<u^\vee, \lambda'>=<u^\vee, m'>= 0$ against $(m, \lambda) \in \mathcal{G}_v$. $\square$

Remark 45. the analogous lemma in the case of $\mathcal{G} = \{(m, \lambda, v) \,:\, (m, grl) \notin H^\infty\}$ are more simple.

3.3. **Relations.** In this section we define an isomorphism of algebraic variety

$$\Phi^{d,v}_{\sigma,m,\lambda} \colon M_{m,\lambda}\big(d, v\big) \longrightarrow M_{\sigma m, \sigma\lambda}\big(\sigma(d, v)\big).$$

in the case $(m, \lambda) \in \mathcal{G}_v$ or $(m, \lambda) \notin H^\infty$. In the case $d \not\geq 0$ or in the case in which there exists $\sigma \in W$ such that $\sigma v \not\geq 0$ or in the case $d = v = 0$ we have seen in the previous section that there is nothing to define or that the definition is trivial. In the remaining cases we observe that for all τ, i we have $(\tau(m)_i, \tau(\lambda)_i) \neq 0$ by lemma 44. Hence we can define $\Phi^{d,v}_{\sigma,m,\lambda}$ by induction on $\ell(\sigma)$ by the formula

$$\Phi^{d,v}_{\sigma,m,\lambda} = \Phi^{\tau(d,v)}_{s_i,\tau m,\tau\lambda} \circ \Phi^{d,v}_{\tau,m,\lambda}. \tag{21}$$

Of course we have to prove that this definition is well given by checking Coxeter relations:

$$s_i^2 = \mathrm{Id}, \quad s_i s_j = s_j s_i \text{ if } a_{ij} = 0 \text{ and } s_i s_j s_i = s_j s_i s_j \text{ if } a_{ij} = 1$$

which in our situation take the following form:

$$\Phi^{s_i(d,v)}_{s_i,s_i\lambda,s_i m} \circ \Phi^{d,v}_{s_i,\lambda,m} = \mathrm{Id} \tag{22a}$$

$$\Phi^{s_j(d,v)}_{s_i,s_j\lambda,s_j m} \circ \Phi^{d,v}_{s_j,\lambda,m} = \Phi^{s_i(d,v)}_{s_j,s_i\lambda,s_i m} \circ \Phi^{d,v}_{s_i,\lambda,m} \tag{22b}$$

$$\Phi^{s_j s_i(d,v)}_{s_i,s_j s_i\lambda,s_j s_i m} \circ \Phi^{s_i(d,v)}_{s_j,s_i\lambda,s_i m} \circ \Phi^{d,v}_{s_i,\lambda,m} = \Phi^{s_i s_j(d,v)}_{s_j,s_i s_j\lambda,s_i s_j m} \circ \Phi^{s_j(d,v)}_{s_i,s_j\lambda,s_j m} \circ \Phi^{d,v}_{s_j,\lambda,m}. \tag{22c}$$

The first of the two equations is clear by the very definition and remark 42. The second equation is trivial. We need to prove the third equation. We will need the following two simple lemmas of linear algebra which proofs are trivial.

Lemma 46. *Let V, W, X, Y, Z be finite dimensional vector spaces and $\alpha, \beta, \gamma, \delta, \varepsilon, \varphi$ linear maps between them as in the diagrams below. The diagram*

$$0 \longrightarrow V \xrightarrow{\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}} W \oplus X \oplus Y \xrightarrow{\begin{pmatrix}\delta & 0 & -1\\ 0 & \varepsilon & \varphi\end{pmatrix}} Y \oplus Z \longrightarrow 0$$

is exact if and only if the diagram

$$0 \longrightarrow V \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} W \oplus X \xrightarrow{\begin{pmatrix}\varphi\delta & \varepsilon\end{pmatrix}} Z \longrightarrow 0$$

is exact and $\gamma = \delta\alpha$.

Lemma 47. *Let U, V, W, X, Y, Z be finite dimensional vector spaces and $\alpha, \beta, \gamma, \delta, \varepsilon, \varphi, \psi, \rho, \sigma$ linear maps between them as in the diagrams below such that $\psi \oplus \rho : W \oplus X \longrightarrow Z$ is an epimorphism. Then the diagram*

$$0 \longrightarrow U \xrightarrow{\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}} V \oplus W \oplus X \xrightarrow{\begin{pmatrix}\delta & 0 & 1\\ \varepsilon & \varphi & 0\\ 0 & \psi & \rho\end{pmatrix}} X \oplus Y \oplus Z \xrightarrow{\begin{pmatrix}\rho & \sigma & -1\end{pmatrix}} Z \longrightarrow 0$$

is exact if and only if $\gamma = -\delta\alpha$, $\psi = \sigma\phi$, $\rho\delta + \sigma\varepsilon = 0$ and the diagram

$$0 \longrightarrow U \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} V \oplus W \xrightarrow{\begin{pmatrix}\varepsilon & \varphi\end{pmatrix}} Y \longrightarrow 0$$

is exact.

We fix now and i,j such that $a_{ij}=1$ and we verifies (22c). Let

$$\begin{array}{ccc}
\lambda' = s_i\lambda & m' = s_i m & v' = s_i v \\
\lambda'' = s_j\lambda' & m'' = s_j m' & v'' = s_j v' \\
\lambda''' = s_i\lambda'' & m''' = s_i m'' & v''' = s_i v'' \\
\tilde\lambda = s_j\lambda & \tilde m = s_j m & \tilde v = s_j v \\
\tilde{\tilde\lambda} = s_i\tilde\lambda & \tilde{\tilde m} = s_i\tilde m & \tilde{\tilde v} = s_i\tilde v
\end{array}$$

First of all we observe that since relation (22a) holds we can assume that:

1. $\lambda_i \neq 0$ or $m_i > 0$ and $\lambda_j \neq 0$ or $m_j > 0$,
2. $\lambda'_j \neq 0$ or $m'_j > 0$ and $\tilde\lambda_i \neq 0$ or $\tilde m_i > 0$,
3. $\lambda''_i \neq 0$ or $m''_i > 0$ and $\tilde{\tilde\lambda}_j \neq 0$ or $\tilde{\tilde m}_j > 0$.

Define

$$\begin{aligned}
Z_{iji} = \{(s''',s) \in \Lambda_{m''',\lambda'''}(d,v''') \times \Lambda_{m,\lambda}(d,v) \,:\, &\exists s'' \in S(d,v''), \\
&\text{and } s' \in S(d,v') \text{ such that } (s''',s'') \in Z_i^{m'',\lambda''}(d,v''), \\
&(s'',s') \in Z_j^{m',\lambda'}(d,v') \text{ and } (s',s) \in Z_i^{m,\lambda}(d,v)\} \\
Z_{jij} = \{(s''',s) \in \Lambda_{m''',\lambda'''}(d,v''') \times \Lambda_{m,\lambda}(d,v) \,:\, &\exists \tilde{\tilde s} \in S(d,\tilde{\tilde v}), \\
&\text{and } \tilde s \in S(d,\tilde v) \text{ such that } (s''',\tilde{\tilde s}) \in Z_j^{\tilde{\tilde m},\tilde{\tilde\lambda}}(d,\tilde{\tilde v}), \\
&(\tilde{\tilde s},\tilde s) \in Z_i^{\tilde m,\tilde\lambda}(d,\tilde v) \text{ and } (\tilde s,s) \in Z_j^{m,\lambda}(d,v)\}
\end{aligned}$$

Observe that $(s''',s) \in Z_{iji} \iff p^{d,v'''}_{m''',\lambda'''}(s''') = \Phi_{s_i}\Phi_{s_j}\Phi_{s_i}(p^{d,v}_{m,\lambda}(s))$ and that $(s''',s) \in Z_{jij} \iff p^{d,v'''}_{m''',\lambda'''}(s''') = \Phi_{s_j}\Phi_{s_i}\Phi_{s_j}(p^{d,v}_{m,\lambda}(s))$. So relation (22c) is equivalent to $Z_{iji} = Z_{jij}$.

Let now $R_i = D_i \oplus \bigoplus_{h\,:\,h_1=i,h_0\neq j} V_{h_0}$, $R_j = D_i \oplus \bigoplus_{h\,:\,h_1=i,h_0\neq j} V_{h_0}$ and observe that $T_i = R_i \oplus V_j$ and $T_j = R_j \oplus V_i$. Let k be the only element of H such that $k_0 = j$ and $k_1 = i$. Let $\varepsilon = \varepsilon(k)$. Define also $A = A(s) = B_k(s)$, $B = B(s) = B_{\bar k}(s)$ and for $l = i,j$ and $\{l',l\} = \{i,j\}$ set $c_l = c_l(s) = \pi^{R_l\oplus V_{l'}}_{R_l} a_l(s)$ and $d_l = d_l(s) = b_l(s)\big|_{R_l}$.

Let now $(s,s''') \in \Lambda_\lambda(d,v)\times\Lambda_{\lambda'''}(d,v''')$ and set $A^* = A(s^*)$, $B^* = B(s^*)$, $c^*_l = c_l(s^*)$ and $d^*_l = d_l(s^*)$ for $l \in \{i,j\}$ and $* \in \{\ ,'''\}$.

If we apply lemmas 46 and 47 to our situation we obtain the following result: $(s,s''') \in Z_{iji}$ of and only there exists vector spaces V'_i, V'_j, V''_i, V''_j and linear maps $A', B', c'_i, d'_i, c'_j, d'_j, A'', B'', c''_i, d''_i, c''_j, d''_j$ such that:

1. $\dim V^*_l = v^*_l$ for $l \in \{i,j\}$ and $* \in \{','' \}$,
2. for each $* \in \{','' \}$ and $l \in \{i,j\}$ $A^* \in \mathrm{Hom}(V^*_i, V^*_j)$, $B^* \in \mathrm{Hom}(V^*_j, V^*_i)$, $c_l \in \mathrm{Hom}(V^*_l, R^*_l)$ and $d_l \in \mathrm{Hom}(R^*_l, V^*_l)$,
3. $V'''_j = V''_j$, $c'''_j = c''_j$, $d'''_j = d''_j$ and

$$\begin{array}{ll}
c'''_i d'''_i = c_i d_i - \lambda_i - \lambda_j & c'''_i B''' = c'_i B'' \\
A''' d'''_i = A'' d'_i & \varepsilon A''' B''' = \varepsilon A'' B'' - \lambda_j
\end{array}$$

4. $V_i'' = V_i'$, $c_i'' = c_i'$, $d_i'' = d_i'$ and
$$c_j'' d_j'' = c_j d_j - \lambda_i - \lambda_j \qquad c_j'' A'' = c_j A'$$
$$B'' d_j'' = B' d_j \qquad \varepsilon A'' B'' = \varepsilon A' B' + \lambda_i + \lambda_j$$
5. $V_j' = V_j$, $c_j' = c_j$, $d_j' = d_j$ and
$$c_i' d_i' = c_i d_i - \lambda_i \qquad c_i' B' = c_i B$$
$$A' d_i' = A d_i \qquad \varepsilon A' B' = \varepsilon A B - \lambda_i$$
6. $\varepsilon c_i' B'' A''' + c_i' d_i' c_i''' = 0$ and $\varepsilon A' B'' = d_j a_j''$,
7. the following diagrams are exact
$$0 \longrightarrow V_i''' \xrightarrow{\begin{pmatrix} c_i''' \\ c_j'' A''' \end{pmatrix}} R_i \oplus R_j \xrightarrow{\begin{pmatrix} Ad_i & d_j \end{pmatrix}} V_j \longrightarrow 0$$
$$0 \longrightarrow V_j'' \xrightarrow{\begin{pmatrix} c_j'' \\ c_i' B'' \end{pmatrix}} R_j \oplus R_i \xrightarrow{\begin{pmatrix} Bd_j & d_i \end{pmatrix}} V_i \longrightarrow 0$$
$$0 \longrightarrow V_i' \xrightarrow{\begin{pmatrix} c_i' \\ A' \end{pmatrix}} R_i \oplus V_j \xrightarrow{\begin{pmatrix} d_i & \varepsilon B \end{pmatrix}} V_i \longrightarrow 0$$

Remark 48. The first condition in point 6) is equivalent to $\varepsilon B'' A''' = d_i' c_i'''$. Indeed this condition is certainly sufficient. To prove the necessity observe that by the injectivity of $a_i' = (c_i' \; A')^t$ it is enough to prove $\varepsilon c_i' B'' A''' + c_i' d_i' c_i''' = 0$ and $\varepsilon A' B'' A''' + A' d_i' c_i''' = 0$. The first equation is the first condition in point 6) and the second one is a consequence of $\varepsilon A' B'' = d_j c_i'''$, $A' d_i' = A d_i$ and the exactness of the first sequence.

Remark 49. The condition $(s, s''') \in Z_{jij}$ can be expressed in a similar way. In the previous conditions we have only to change i with j and ε with $-\varepsilon$.

We will prove now $Z_{iji} \subset Z_{jij}$. To do it we suppose that $A', \dots, d_j''$ are given as above and we construct $\tilde{A}, \tilde{B}, \tilde{c}_i, \tilde{d}_i, \tilde{c}_j, \tilde{d}_j, \tilde{\tilde{A}}, \tilde{\tilde{B}}, \tilde{\tilde{c}}_i, \tilde{\tilde{d}}_i, \tilde{\tilde{c}}_j, \tilde{\tilde{d}}_j$ such that they satisfy the conditions. for $(s, s''') \in Z_{jij}$.
First step: construction of $\tilde{A}, \tilde{B}, \tilde{c}_i, \tilde{c}_j, \tilde{d}_i, \tilde{d}_j$. Choose $\tilde{s}$ such that $(\tilde{s}, s) \in Z_j^{\chi,\lambda}$ and define $\tilde{A} = A(\tilde{s})$, $\tilde{B} = B(\tilde{s})$, $\tilde{c}_l = c_l(\tilde{s})$ and $\tilde{d}_l = d_l(\tilde{s})$ for $l \in \{i, j\}$.

Now I claim that there exists unique $\tilde{\tilde{A}} : V_i''' \longrightarrow \tilde{V}_j$ and $\tilde{\tilde{B}} : \tilde{V}_j \longrightarrow V_i'''$ such that:
$$\begin{cases} \tilde{c}_j \tilde{\tilde{A}} = c_j'' A''' \\ \tilde{B} \tilde{\tilde{A}} = -\varepsilon d_i c_i''' \end{cases} \quad \text{and} \quad \begin{cases} \tilde{\tilde{A}} \tilde{\tilde{B}} = \tilde{A}\tilde{B} - \varepsilon\lambda_i - \varepsilon\lambda_j \\ c_i''' \tilde{\tilde{B}} = c_i \tilde{B} \end{cases}$$

Unicity of $\tilde{\tilde{A}}$: since the map $\tilde{a}_j = (\tilde{c}_j \quad -\varepsilon\tilde{B})^t$ is injective the unicity is clear.
Existence of $\tilde{\tilde{A}}$: to prove the existence of $\tilde{\tilde{A}}$ is enough to prove:
$$\mathrm{Im}\begin{pmatrix} c_j'' A''' \\ -\varepsilon d_i c_i''' \end{pmatrix} \subset \mathrm{Im}\begin{pmatrix} \tilde{c}_j \\ \tilde{B} \end{pmatrix} = \ker \begin{pmatrix} d_j & -\varepsilon A \end{pmatrix}.$$
So the thesis follows from $d_j c_j'' A''' + A d_i c_i''' = 0$.

Let now $\tilde{\tilde{a}}_i = (c_i''' \;\; \tilde{\tilde{A}})^t$. I claim that $\tilde{\tilde{a}}_i$ is injective and that $\operatorname{Im} \tilde{\tilde{a}}_i = \ker(d_i \;\; \varepsilon \tilde{B}) = \ker \tilde{b}_i$. First of all observe that since $\tilde{m}_i > 0$ or $\lambda_i \neq 0$, $\tilde{b}_i$ is surjective. Observe also that

$$\begin{pmatrix} \tilde{c}_j & 0 \\ 0 & \mathrm{Id}_{V_i'''} \end{pmatrix} \circ \begin{pmatrix} \tilde{\tilde{A}} \\ c_i''' \end{pmatrix} = \begin{pmatrix} c_j'' A''' \\ c_i''' \end{pmatrix}.$$

So $\tilde{\tilde{a}}_i$ is injective as claimed. Now since $\dim R_i + \dim \tilde{V}_j = \dim V_i''' + \dim V_i$ to prove the last part of the claim it is enough to check that $\tilde{b}_i \tilde{\tilde{a}}_i = 0$. Indeed

$$\tilde{b}_i \tilde{\tilde{a}}_i = d_i c_i''' + \varepsilon \tilde{B} \tilde{\tilde{A}} = 0.$$

Unicity of $\tilde{\tilde{B}}$: this is a consequence of $\tilde{\tilde{a}}_i$ injective.
Existence of $\tilde{\tilde{B}}$: As for the existence of $\tilde{\tilde{A}}$ this is equivalent to

$$\operatorname{Im} \begin{pmatrix} c_i \tilde{B} \\ \tilde{A}\tilde{B} - \varepsilon\lambda_i - \varepsilon\lambda_j \end{pmatrix} \subset \operatorname{Im} \begin{pmatrix} c_i''' \\ \tilde{\tilde{A}} \end{pmatrix} = \ker \begin{pmatrix} d_i & \varepsilon \tilde{B} \end{pmatrix}.$$

So the thesis follows from $\varepsilon \tilde{B}\tilde{A}\tilde{B} - \lambda_i \tilde{B} - \lambda_j \tilde{B} + d_i c_i \tilde{B} = 0$.

Finally we set

$$\begin{aligned} \tilde{\tilde{V}}_i &= V_i''' & \tilde{\tilde{c}}_i &= c_i''' & \tilde{\tilde{d}}_i &= d_i''' \\ \tilde{\tilde{V}}_i &= \tilde{V}_j & \tilde{\tilde{c}}_j &= \tilde{c}_j & \tilde{\tilde{d}}_j &= \tilde{d}_j. \end{aligned}$$

The verification of all the conditions is now straightforward.

The inclusion $Z_{jij} \subset Z_{iji}$ can be proved similarly and equation (19) is clear by definition. So Proposition 32 is proved.

4. A representation of the Weyl group

In this section, following Nakajima [13], we show how to use the above action to construct an action of the Weyl group on the homology of quiver varieties. Maybe this action is related with the one constructed by Slodowy in the case of flag varieties ([16], ch.4).

First we recall some general about the action of the Weyl group. Let $Z^\vee = Q^\vee \otimes_{\mathbb{Z}} \mathbb{C}$ and $Z = Q \otimes_{\mathbb{Z}} P$. On Z, $Z^\vee$ there is a natural action of W.

Lemma 50. *For all $u \in Z^\vee$ the set Wu is discrete.*

Lemma 51. *Consider the action of W on $\mathbb{P}(Z^\vee)$. If $p \in \mathbb{P}(Z^\vee)$ then*

$$\overline{Wp} \textit{ is countable.}$$

If $p \in \mathbb{P}(Z^\vee)$ we define $H_p = \{x \in P \otimes_{\mathbb{Z}} \mathbb{C} : <x, p> = 0\}$

Lemma 52. *If $p \in \mathbb{P}(Z^\vee)$ then*

$$\overline{WH_p} = \bigcup_{q \in \overline{Wp}} H_q.$$

We define $\mathcal{H} = \overline{W\mathcal{H}_v \cup \mathcal{H}_U}$ and $\mathcal{R} = \mathfrak{z} - \mathcal{H}$. By the previous lemmas $\mathcal{H}$ is the union of a countable number of real codimension 3 subspaces in $\mathfrak{Z}$ and in particular $\mathcal{R}$ is simply connected. We need also the following definition

$$K = \{u \in \mathbb{Z}^I \,:\, -\sum u_i\alpha_i \text{ is dominant and } \operatorname{supp} u \text{ is connected } \}$$
$$P_0 = \{p \in P \,:\, p \text{ is dominant and } <u^\vee, p> \geq 2 \text{ for all } u \in K\}.$$

Now we choose d, v such that $\bar{d} = \bar{v}$.

Lemma 53. *If $\bar{d} \in P_0$ then $\tilde{\mu}$ is surjective and is a locally trivial bundle over $\mathcal{R}$.*

Proof. By Proposition 10.5 and Corollary 10.6 in [14] there exists a closed orbit Gs in $\Lambda_0(d, v)$ with trivial stabilizer. Then by Proposition 9 there exists $t \in Gs$ such that $\tilde{\mu}(t) = 0$ and by lemma 5 and 3 $d\tilde{\mu}_t$ is surjective. Now the surjectivity follows by homogeneity.

The local triviality over $\mathcal{R}$ follows also from lemma 5 and 3. ☐

Now consider

$$\begin{aligned}
R &= \{\lambda \in Z \,:\, (0, \lambda) \notin \mathcal{R}\},\\
\Lambda(d, v) &= \{(\lambda, s) \in Z \times S \,:\, s \in \Lambda_\lambda\},\\
M(d, v) &= \Lambda(d, v)//G_v \ \text{ and } \ p : M(d, v) \longrightarrow Z \text{ the projection}\\
\mathfrak{L}(d, v) &= \{(\zeta, s) \in \mathfrak{Z} \times S \,:\, s \in \mathfrak{L}_\zeta\},\\
\mathfrak{M}(d, v) &= \mathfrak{L}(d, v)/U(V) \ \text{ and } \ \tilde{p} : \mathfrak{M}(d, v) \longrightarrow \mathfrak{Z} \text{ the projection}
\end{aligned}$$

We have the following commutative diagram

$$\begin{tikzcd}
(\lambda, s) \arrow[d] & \in & M(d, v) \arrow[r, "p"] \arrow[d, "\imath_M"] & Z \arrow[d] & \ni & \lambda \arrow[d] \\
(0, \lambda, s) & \in & \mathfrak{M}(d, v) \arrow[r, "\tilde{p}"] & \mathfrak{Z} & \ni & (0, \lambda)
\end{tikzcd}$$

By Proposition 10 the diagram is a pull back and by lemma 53 p and $\tilde{p}$ are locally trivial over R, $\mathcal{R}$. We call $M_R = p^{-1}(R)$ and $\mathfrak{M}_\mathcal{R} = \tilde{p}^{-1}(\mathcal{R})$.

Now consider the complex $\mathcal{F} = Rp_*(\mathbb{Z}_{M_R}) = \imath_M^{-1} R\tilde{p}_*(\mathbb{Z}_{\mathfrak{M}_\mathcal{R}})$ which is cohomologically a locally constant complex. We observe now that $\Pi_1(\mathcal{R})$ is trivial so $R\tilde{p}_*(\mathbb{Z}_{\mathfrak{M}_\mathcal{R}})$ is isomorphic to cohomologically constant complex on $\mathcal{R}$ so it is $\mathcal{F}$ on R. In particular for any $x, y \in R$ we have a canonically isomorphism

$$\psi^i_{x,y} : H^i(\mathcal{F}_x) \longrightarrow H^i(\mathcal{F}_y).$$

Now observe that by Proposition 32 there is an action on W on R, M_R and that p is equivariant with respect to this action. So we can define a W action on $H^i(M_{0,\lambda}(d, v), \mathbb{Z})$ by

$$\sigma(c) = \psi^i_{\sigma\lambda,\lambda} \circ H^i(\Phi^{d,v}_{\sigma,0,\lambda})(c)$$

for any $\sigma \in W$. To verify that this is an action we have only to verify that

$$\psi^i_{\sigma\lambda^2,\lambda^1} \circ H^i(\Phi^{d,v}_{\sigma,0,\lambda^2})\psi^i_{\lambda^1,\lambda^2}(c) = \psi^i_{\sigma\lambda^1,\lambda^1} \circ H^i(\Phi^{d,v}_{\sigma,0,\lambda^1})(c).$$

Since R is connected and $H^i(M,\mathbb{Z})$ is discrete this is clear. S o we have proved the following corollary.

Corollary 54. *If $d=v$ and $(0,\lambda)\in\mathcal{R}$ then there is an action of W on $H^i(M_{0,\lambda}(d,v),\mathbb{Z})$.*

Remark 55. If $m_+=(1,\dots,1)$ and $\lambda=0$ it is easy to see that $d\mu_s$ is surjective for all $s\in\Lambda_{m_+,0}(d,v)$. Then by lemma 53 there is a canonical isomorphism $H_*(M_{m_+,0}(d,v))\simeq H_*(M_{0,\lambda}(d,v))$ if $(0,\lambda)\in\mathcal{R}$. So by Nakajima's Theorem (Theorem 10.2 [14]) it is natural to make the following conjecture:

Conjecture 56. *Let $top=\frac{1}{2}\dim H_*(M_{0,\lambda}(d,v))$ then*

$$H^{top}\big(M_{(0,\lambda)}(d,v),\mathbb{C}\big)\simeq\big(L_d\big)_0$$

where $\big(L_d\big)_0$ is the 0-weight space of the Kac-Moody algebra associated to the quiver of highest weight $\sum_i d_i\bar{\omega}_i$.

5. Reduction to the dominant case

As a consequence of Proposition 32 we see that if $(m,\lambda)\in\mathcal{G}_v$ then there exists $\sigma\in W$ and $v'=\sigma\cdot v$ such that $d-v'$ is dominant and $M_{\sigma m,\sigma\lambda}(d,v')\simeq M_{m,\lambda}(d,v)$. We generalize now this result to arbitrary λ.

On Q we consider the following order: $v'\le v$ if and only if $v'_i\le v_i$.

We consider now the following construction: let $v'\le v$ and fix an embedding $V'_i\hookrightarrow V_i$ and a complement U_i of V' in V_i, then we can define a map $\widetilde{\jmath}\colon S(d,v')\longrightarrow S(d,v)$ through:

$$\widetilde{\jmath}(B',\gamma',\delta')=\left(\begin{pmatrix}B'&0\\0&0\end{pmatrix},\begin{pmatrix}\gamma'\\0\end{pmatrix},\begin{pmatrix}\delta'&0\end{pmatrix}\right)\tag{23}$$

where the matrices of the new triple represents the maps described through the decomposition $V_i=V'_i\oplus U_i$.

Suppose now that $(m_i,\lambda_i)=0$ for all i such that $v'_i\neq v_i$. Then it is easy to see that this map restrict to a map $\jmath_r:\Lambda_{m,\lambda}(d,v')\longrightarrow\Lambda_{m,\lambda}(d,v)$ and so induces a map $\jmath^{v'}_v=\jmath\colon M_{0,\lambda}(d,v')\longrightarrow M_{0,\lambda}(d,v)$.

Lemma 57. *$\jmath$ is a closed immersion*

Proof. We prove that the map $\jmath^\sharp:\mathbb{C}[\Lambda_\lambda(d,v)]^{G(v)}\longrightarrow\mathbb{C}[\Lambda_\lambda(d,v')]^{G(v')}$ is surjective. By proposition 18 this follows by the following two identities:

$$\operatorname{Tr}\left(\alpha\left(\jmath(s)\right)\right)=\operatorname{Tr}\left(\alpha(s)\right)\quad\text{and}\quad\beta\left(\jmath(s)\right)=\beta(s)$$

for each B-path α and for each admissible path β. ☐

Lemma 58. *If $2v_i>d_i+\sum_{j\in I}a_{ij}v_j$ and $v'=v-\alpha_i$ then $\jmath$ is an isomorphism of algebraic varieties*

Proof. It's enough to prove that $\jmath$ is surjective. Let $s=(B,\gamma,\delta)\in\Lambda_0(d,v)$ and consider the sequence (see (4) for the notation) :

$$T_i\xrightarrow{\ b_i\ }V_i\xrightarrow{\ a_i\ }T_i.$$

Since $b_i a_i = 0$ and $2\dim V_i > \dim T_i$ we have that b_i is not surjective or that a_i is not injective.

Suppose that b_i is not surjective, then up to the action of G_v we can assume that $\operatorname{Im} b_i \subset v_i'$. Then, for $t \in \mathbb{C}^*$ consider $g_t = (g_{j,t}) \in G_v$ with

$$g_i = \begin{pmatrix} \mathrm{Id}_{v_i'} & 0 \\ 0 & t^{-1} \end{pmatrix} \quad \text{and} \quad g_j = \mathrm{Id}_{V_j} \text{ for } j \neq i.$$

Then

1. $g_{i,t} B_h = B_h$ if $h_1 = i$ and $g_i \gamma_i = \gamma_i$, since $\operatorname{Im} B_h, \operatorname{Im} \gamma_i \subset \operatorname{Im} b_i \subset v_i'$,
2. $\exists \lim_{t\to 0} B_h g_{i,t}^{-1} = B_h$ if $h_0 = i$ and $\delta_i g_i^{-1} = \delta_i$

So $\exists \lim_{t\to 0} g_t s = s'$ and it is clear that $s' \in \widetilde{\jmath}(\Lambda_0(d, v'))$ and that $p_0(s) = p_0(s') \in \operatorname{Im} \jmath$.

If b_j is surjective and a_i is not injective the argument is similar. □

Proposition 59. *For all λ and for all $d \geq 0, v \geq 0$ there exists v' and $\sigma \in W$ such that $\bar{d} - \bar{v}'$ is dominant and*

$$M_{0,\sigma\lambda}(d, v') \simeq M_{0,\lambda}(d, v).$$

Proof. We prove this proposition by induction on the order $\leq$ on Q.
First step: $v = 0$. If $v = 0$ we can take $v' = v$ and $\sigma = 1$.
Inductive step. If $d-v$ is not dominant then there exists i such that $2v_i > d_i + \sum a_{ij} v_j$.

If $\lambda_i \neq 0$ we observe that $s_i v = v' < v$ (that is $v' \leq v$ and $v' \neq v$) and that $M_{s_i m, s_i \lambda}(d, v') \simeq M_{m,\lambda}(d, v)$ and so we can apply the inductive hypothesis.

If $\lambda_i = 0$ we apply the previous lemma and the inductive hypothesis. □

6. On normality and connectedness of quiver variety in the finite type case

In this section we restrict our attention to the case of quiver varieties of finite type and to the case $m = (1, \ldots, 1)$ and $\lambda = 0$ and we fix d, v. By remark 2 we can assume without loss of generality that $v_i > 0$ for all i. We would like to prove the following conjecture:

Conjecture 60. *$M_{m,0}(d, v)$ is connected and $M_{0,0}(d, v)$ is normal.*

Remark 61. If $\tilde{m} = (m_1, \ldots, m_n) \in \mathbb{N}_+^I$ it is easy to see that $\Lambda_{\tilde{m},0} = \Lambda_{m,0}$ ao in particular $M_{m,0}(d, v)$ is smooth. Instead $M_{0,0}$ is a cone so it is clearly connected.

By proposition 59it is enough to prove the theorem in the case $d - v$ dominant. Unfortunately I'm not able to prove the conjecture only in the case $d - v$ regular: $<d - v, \alpha_i>> 0$ for all i. To prove the conjecture in this case we will use the following stratification introduced by Lusztig in [9].

Definition 62. For any $s \in S$ and $i \in I$ let

$$V_i^+ = V_i^+(s) = \sum_{\alpha \text{ a } B\text{-path}\,:\,\alpha_1 = i} \operatorname{Im}(\alpha(s)\gamma_{\alpha_0})$$

If $v' = (v_1', \ldots, v_n') \in \mathbb{N}^n$ we define

$$\Lambda^{v'} = \{s \in \Lambda_0(d, v) \,:\, \dim V_i^+(s) = v_i'\}.$$

Observe that $\Lambda^v = \Lambda_{m_+,0}(d,v)$. To prove our result we will use the following lemma of Lusztig.

Lemma 63 (Lusztig: [9] Proposition 4.5 and Proposition 5.3)**.** *If $0 \leq v'_i \leq v_i$ for each i then*

$$\dim \Lambda^{v'}(d,v) = \dim S - \sum_{i\in I} \dim gl(V_i) - <(v-v')^\vee, d-v> -\tfrac{1}{2} <(v-v')^\vee, v-v'>$$

Our result follows trivially from the following lemma.

Lemma 64. *1) If $d-v$ is dominant then $\Lambda_{0,0}(d,v)$ is a complete intersection.*
2) If $d-v$ is regular then $\Lambda_{0,0}$ is normal and irreducible and $\Lambda_{m_+,0}(d,v)$ is connected.

Proof. Observe that $\Lambda_{0,0}(d,v) = \mu^{-1}(0)$ so each irreducible component of $\Lambda_{0,0}(d,v)$ must have dimension at least $\dim S - \sum_i \dim gl(V_i) = \delta_V$.

Suppose now that $d-v$ is dominant. By Nakajima's theorem ([14] Theorem 10.2) $M_{m_+,0}$ is not empty. Observe also that by Proposition 14 $\Lambda_{m_+,0}(d,v)$ is a smooth subset of $\Lambda_{0,0}(d,v)$ of dimension δ_V.

It is well known that $\Lambda_{m_+,0}(d,v) = \Lambda^v$. Hence

$$\Lambda_{0,0}(d,v) - \Lambda_{m_+,0}(d,v) = \bigcup_{v'\leq v \text{ and } v'\neq v} \Lambda^{v'}.$$

By the lemma above we have that if $v' \leq v$ and $v' \neq v$ $\dim \Lambda^{v'} < \delta_V$. So $\Lambda_{m_+,0}(d,v)$ must be dense in $\Lambda_{0,0}(d,v)$ and $\Lambda_{0,0}(d,v)$ is a complete intersection. Moreover if $d-v$ is regular we have that $\dim \Lambda^{v'} < \delta_V - 1$ so the singular locus has codimension at least two and normality and irreducibility follows. Finally by our discussion it is clear that if $\Lambda_{m_+,0}(d,v)$ is disconnected then $\Lambda_{0,0}(d,v)$ is not irreducible. □

Remark 65. In the lemma we can substitute $\Lambda_{m_+,0}(d,v)$ with any other subset *Reg* of regular points in $\Lambda(d,v)$. In this way is indeed possible to improve a little bit the theorem but Crawley-Boevey explained me that this strategy cannot work in general because there are cases where $d-v$ is dominant and $\Lambda_{0,0}(d,v)$ is not normal. It should be also pointed out that Crawley-Boevey proved the connectedness in complete generality ([1]). He said me that is also able to prove normality for a much bigger class of quiver varieties.

References

[1] W. Crawley-Boevey, *Geometry of the moment map for representations of quivers*, preprint available at `http://www.amsta.leeds.ac.uk/~pmtwc`.
[2] H. Derksen and J. Weyman, *Semi-invariants of quivers and saturation for Littlewood-Richardson coefficients*, to appear in J. Amer. Math. Soc.
[3] N. Hitchin, *HyperKähler manifolds*, Séminaire Bourbaky: Vol 1991/1992, Astérisque, vol. 206, 1992, pp. 137–166.
[4] N. Hitchin, A. Karlhede, U. Lindström, and M.Roček, *HyperKähler metrics and supersymmetry*, Comm. Math. Phys. **108** (1987), 535– 589.
[5] V. Kac, *Infinite dimensional Lie algebras*, third ed., Cambridge University Press, 1990.
[6] G. Kempf and L. Ness, *The length of vectors in representation spaces*, Algebraic geometry. Proc. Summer Meeting Univ. Copenhagen 1978, LNM, vol. 732, Springer, 1979, pp. 233–243.

[7] G. Lusztig, *Introduction to quantum groups*, Progress in Mathematichs, vol. 110, Birkäuser, 1993.

[8] ———, *On quiver varieties*, Advances in Math. **136** (1998), 141–182.

[9] ———, *Quiver varieties and Weyl group actions*, preprint, 1999.

[10] A. Maffei, *Quiver varieties*, Ph.D. thesis, Università di Roma "La Sapienza", 1999.

[11] L. Migliorini, *Stability of homogeneous vector bundles*, Bollettino dell'Unione Matematica Italiana B(7) **10** (1996), 963–990.

[12] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, third ed., Ergebenisse der Mathematik und ihrer Grenzgebiete, vol. 34, Springer, 1994.

[13] H. Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. Jour. **76** (1994), 365–416.

[14] ———, *Quiver varieties and Kac-Moody algebras*, Duke Math. Jour. **91** (1998), 515 – 560.

[15] P. Newstead, *Introduction to moduli problems and orbit spaces*, Tata Lectures, vol. 51, Springer, 1978.

[16] P. Slodowy, *Four lecture on simple groups and singularities*, Communications of the mathematical institute, Rijksuniversiteit Utrecht, 1980.

Mathematical Science Research Institute, Berkeley
E-mail address: amaffei@mat.uniroma1.it