CLASSIFICATION OF SEMISIMPLE HOPF ALGEBRAS OF DIMENSION 16.

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Abstract. In this paper we completely classify nontrivial semisimple Hopf algebras of dimension 16. We also compute all the possible structures of the Grothendieck ring of semisimple non-commutative Hopf algebras of dimension 16. Moreover, we prove that non-commutative semisimple Hopf algebras of dimension p^n, p is prime, cannot have a cyclic group of grouplikes.

1. Introduction.

Recently various classification results were obtained for finite-dimensional semisimple Hopf algebras over an algebraically closed field of characteristic 0. The smallest dimension, for which the question was still open, was 16. In this paper we completely classify all nontrivial (i.e. non-commutative and non-cocommutative) Hopf algebras of dimension 16. Moreover, we consider all possible structures of Grothendieck rings $K_0(H)$ for semisimple non-commutative Hopf algebras of dimension 16.

Let H be a non-commutative semisimple Hopf algebra of dimension 16 over an algebraically closed field k of characteristic 0. Then irreducible representations of H of degree 1 are exactly the grouplike elements of H^*. Let $\mathbf{G}(H^*)$ denote the group of grouplikes of H^*, then $k\mathbf{G}(H^*)$ is a subHopfalgebra of H^* and thus, by Nichols-Zoeller Theorem [23], $|\mathbf{G}(H^*)| = \dim k\mathbf{G}(H^*)$ divides $\dim H^* = \dim H = 16$. Therefore by the Artin-Wedderburn Theorem, as an algebra H is isomorphic to either

$$k^{(8)} \oplus M_2(k) \oplus M_2(k) \text{ or} \tag{1.1}$$

$$k^{(4)} \oplus M_2(k) \oplus M_2(k) \oplus M_2(k) \tag{1.2}$$

$\dim Z(H)$ equals the number of summands in the Artin-Wedderburn decomposition of H, thus in the case (1.1) $\dim Z(H) = 10$ and $|\mathbf{G}(H^*)| = 8$ and in the case (1.2) $\dim Z(H) = 7$ and $|\mathbf{G}(H^*)| = 4$.

Our first result, which will be proved in the beginning of Section 3 , is the following:

Theorem 1.1. *Let H be a semisimple Hopf algebra of dimension p^n over an algebraically closed field k of characteristic 0. If $H \not\cong kC_{p^n}$ then $\mathbf{G}(H)$ is not cyclic.*

Our main result will be proved in Section 9:

Theorem 1.2. *Let k be an algebraically closed field of characteristic 0. Then there are exactly 16 nonisomorphic nontrivial semisimple Hopf algebras of dimension 16.*

1. *$\mathbf{G}(H)$ is abelian of order 8 if and only if $\mathbf{G}(H^*)$ is abelian of order 8. There are 11 Hopf algebras with such a group of grouplikes.*

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2. *If H has a nonabelian group of grouplikes then $\mathbf{G}(H) = D_8$ and $\mathbf{G}(H^*) = C_2 \times C_2$. There are 2 Hopf algebras with a nonabelian group of grouplikes.*
3. *There are 3 Hopf algebras with $\mathbf{G}(H) = C_2 \times C_2$. Two of them are dual to the Hopf algebras with a nonabelian group of grouplikes and one of them is selfdual.*

Remark 1.1. A part of Theorem 1.2, saying that if H has a nonabelian group of grouplikes then $\mathbf{G}(H) = D_8$ and $\mathbf{G}(H^*) = C_2 \times C_2$ can also be obtained as a corollary to a theorem of Natale [21], and Proposition 3.1. This theorem states that if $\mathbf{G}(H)$ is nonabelian then H^* has 4 central grouplikes.

One method of constructing a new Hopf algebra from a known one H is to twist the comultiplication of H by a 2-pseudo-cocycle $\Omega \in H \otimes H$ (or a 2-cocycle $J \in H \otimes H$). The new Hopf algebra is denoted H_Ω (or H_J). The next theorem summarizes the results of Sections 5 and 6:

Theorem 1.3. *Let H be a semisimple Hopf algebra of dimension 16 over an algebraically closed field k of characteristic 0. Then there are exactly 7 possible structures of the Grothendieck ring $K_0(H)$. Moreover*

1. *$\mathbf{G}(H^*)$ is abelian if and only if the Grothendieck ring of H is commutative. Then*
   (a) *If $|\mathbf{G}(H^*)| = 8$, as algebras $K_0(H) \otimes_{\mathbb{Z}} k \cong k^{(10)}$.*
   (b) *If $|\mathbf{G}(H^*)| = 4$, as algebras $K_0(H) \otimes_{\mathbb{Z}} k \cong k^{(7)}$.*
   (c) *$K_0(H) = K_0(kG)$, where G is one of the 9 nonabelian groups of order 16 (although only 6 of those K_0-rings are distinct).*
   (d) *H is a twisting with a 2-pseudo-cocycle of some group algebra.*
2. *If $K_0(H)$ is not commutative then*
   (a) *As algebras $K_0(H) \otimes_{\mathbb{Z}} k \cong k^{(6)} \oplus M_2(k)$.*
   (b) *H is not a twisting of a group algebra.*
   (c) *There is only one possible structure of the K_0-ring.*
   (d) *All Hopf algebras with non-commutative K_0-ring are twistings of each other.*

Remark 1.2. By Theorem 1.2 there are only 2 Hopf algebras with nonabelian $\mathbf{G}(H^*)$.

We summarize the distinct non-commutative, non-cocommutative semisimple Hopf algebras of dimension 16 in the following table. We try to distinguish nonisomorphic examples of Hopf algebras using the groups $\mathbf{G}(H)$ and $\mathbf{G}(H^*)$ and the Grothendieck rings $K_0(H)$ (defined in Section 2). Here we consider twistings of group algebras kG, where G is a nonabelian group of order 16. There are exactly nine such groups, described in [2] (see Section 4). The twistings appearing here are explained in Section 7. The coproduct $\#^\alpha$ is explained in Section 8. H_8 denotes the unique nontrivial semisimple Hopf algebra of dimension 8 (see [7] and [11]).

Remark 1.3. $H_{C:\sigma_1}$ is not triangular for the following reasons. If it were triangular then by [4, Theorem 2.1] it would be equal to a twisting with a 2-cocycle of a group algebra kG. Then by [24, Theorem 4.1] $K_0(H_{C:\sigma_1}) = K_0(kG)$ and therefore $H_{C:\sigma_1}$ would be a twisting of kD_{16} or of kQ_{16}. But by [16, Theorem 4.1], kQ_{16} doesn't have nontrivial cocycle twistings and $H_{C:1} \cong (kD_{16})_J$ is the only cocycle twisting of kD_{16}.

TABLE 1

No.	Example	$\mathbf{G}(H)$	$\mathbf{G}(H^*)$	$K_0(H)$	Notes
1	$H_{d:-1,1} \cong H_8 \otimes kC_2$	$(C_2)^3$	$(C_2)^3$	$K_{5.1} = K_0(D_8 \times C_2)$	not triang.
2	$H_{d:1,1} \cong k(D_8 \times C_2)_J$	$(C_2)^3$	$(C_2)^3$	$K_{5.1} = K_0(D_8 \times C_2)$	triangular
3	$(H_{c:\sigma_1})^*$	$(C_2)^3$	$C_2 \times C_4$	$K_{5.3} = K_0(G_5)$	
4	$(H_{b:1})^*$	$(C_2)^3$	$C_2 \times C_4$	$K_{5.3} = K_0(G_5)$	
5	$H_{c:\sigma_1}$	$C_2 \times C_4$	$(C_2)^3$	$K_{5.2} = K_0(G_7)$	
6	$H_{b:1}$	$C_2 \times C_4$	$(C_2)^3$	$K_{5.1} = K_0(D_8 \times C_2)$	
7	$H_{c:\sigma_0}$	$C_2 \times C_4$	$C_2 \times C_4$	$K_{5.4} = K_0(G_1)$	
8	$H_{a:1}$	$C_2 \times C_4$	$C_2 \times C_4$	$K_{5.3} = K_0(G_5)$	
9	$H_{a:y}$	$C_2 \times C_4$	$C_2 \times C_4$	$K_{5.3} = K_0(G_5)$	
10	$H_{b:y}$	$C_2 \times C_4$	$C_2 \times C_4$	$K_{5.3} = K_0(G_5)$	
11	$H_{b:x^2y}$	$C_2 \times C_4$	$C_2 \times C_4$	$K_{5.3} = K_0(G_5)$	
12	$H_{C:1} \cong (kD_{16})_J$	D_8	$C_2 \times C_2$	$K_{6.1} = K_0(D_{16})$	triangular
13	$H_E \cong (kG_2)_J$	D_8	$C_2 \times C_2$	$K_{6.2} = K_0(G_2)$	triangular
14	$H_{B:1} \cong ((kD_{16})_J)^*$	$C_2 \times C_2$	D_8	$K_{5.5}$	
15	$H_{B:X} \cong ((kG_2)_J)^*$	$C_2 \times C_2$	D_8	$K_{5.5}$	$kQ_8 \#^\alpha kC_2$
16	$H_{C:\sigma_1}$	$C_2 \times C_2$	$C_2 \times C_2$	$K_{6.1} = K_0(D_{16})$	not triang.

Remark 1.4. The following Hopf algebras are selfdual: $H_{d:-1,1} \cong H_8 \otimes kC_2$ (since H_8 is selfdual), $H_{c:\sigma_0}$ (since comparing K_0-rings we see that $H_{c:\sigma_0} \cong A_3^+ \cong (A_3^+)^*$, described in [8] and [9]), $H_{d:1,1} \cong k(D_8 \times C_2)_J$ and $H_{C:\sigma_1}$ (since there is no other choice for the dual).

2. Preliminaries.

First we will need the following definition, which was introduced in [28].

Definition 2.1. Let $K_0(H)^+$ denote the abelian semigroup of all equivalence classes of representations of H with the addition given by a direct sum. Then its enveloping group $K_0(H)$ has the structure of an ordered ring with involution * and is called the Grothendieck ring.

In [22] the structure of $K_0(H)$ was described for comodules; it was then translated into the language of modules in [25]. The multiplication in this ring is defined as follows: let $[\pi_1]$ and $[\pi_2]$ denote the classes of representations equivalent to π_1 and π_2, then $[\pi_1] \bullet [\pi_2]$ is the class of the representation $(\pi_1 \otimes \pi_2) \circ \Delta$; the unit of this ring is the class $[\varepsilon]$ and $[\pi]^*$ is the equivalence class of the dual representation ${}^t(\pi \circ S)$ defined by $\langle {}^t(\pi \circ S(h))(f), v\rangle = \langle f, (\pi \circ S(h))(v)\rangle$. The equivalence classes of irreducible representations of H form a basis of $K_0(H)$ and are called *basic elements*. If $[\pi_1], \ldots, [\pi_d]$ are the basic elements then $[\rho] = \sum_{i=1}^{d} \deg \pi_i [\pi_i]$ is called the *marked element*. For basic elements x and y we write

$$x \bullet y = \sum_{z \text{ - basic}} m(z, x \bullet y)\, z$$

where $m(z, x \bullet y)$ are non-negative integers. Then the following properties are true (see [22] and [25]):

$$m(z, x \bullet y) = m(x^*, y \bullet z^*) \tag{2.1}$$

$$m(1, x \bullet y^*) = \delta_{x,y} \tag{2.2}$$

$$\sum m(z, x \bullet y) \deg(z) = \deg(x \bullet y) \tag{2.3}$$

For simplicity of notation we will write π instead of $[\pi]$ for elements of $K_0(H)$. We will denote the degree 2 irreducible representations of H by π_i and the degree 1 irreducible representations of H (i.e. elements of $\mathbf{G}(H^*)$ or multiplicative characters of H) by χ_i. We denote the generators of $\mathbf{G}(H^*)$ by χ, φ and ψ. If $H = kG$ then $\mathbf{G}((kG)^*)$ is the group of multiplicative characters of G.

The following proposition can be also obtained as a corollary to [16, Proposition 2.4]:

Proposition 2.2. *Let H be a nontrivial semisimple Hopf algebra of dimension 16. Assume that there exists an element $\chi \in \mathbf{G}(H^*) \cap Z(H^*)$ of order 2 such that $\chi \bullet \pi = \pi$ for every 2-dimensional representation π of H. Then H^* has a group algebra of dimension 8 as a quotient.*

Proof. Write $G = \mathbf{G}(H^*)$. Dualizing formulas (1.1) and (1.2) we get that as coalgebras

$$\begin{aligned} H^* &= kG \oplus E_1 \oplus E_2 & \text{if } |\mathbf{G}(H^*)| = 8 \text{ or} \\ H^* &= kG \oplus E_1 \oplus E_2 \oplus E_3 & \text{if } |\mathbf{G}(H^*)| = 4 \end{aligned}$$

where E_i are simple subcoalgebras of dimension 4 and $\chi E_i = E_i$. $(\chi - 1) H^*$ is a normal Hopf ideal of H^*. Then $L = H^*/(\chi - 1) H^*$ is a Hopf algebra of dimension 8. Consider the projection $p : H^* \to L$. Since $\chi E_i = E_i$, $p(E_i) = E_i/(\chi - 1) E_i$. Therefore

$$\begin{aligned} L &= k(G/ < \chi >) \oplus p(E_1) \oplus p(E_2) & \text{if } |\mathbf{G}(H^*)| = 8 \text{ or} \\ L &= k(G/ < \chi >) \oplus p(E_1) \oplus p(E_2) \oplus p(E_3) & \text{if } |\mathbf{G}(H^*)| = 4 \end{aligned}$$

$p(E_i)$ are cosemisimple coalgebras of dimension 2, therefore each of them is spanned by two grouplikes. Thus L is spanned by 8 grouplikes and L is a group algebra. □

We will also need the notion of a twisting of a Hopf algebra (see [3], [31], [24]):

Definition 2.3. The *twisting* H_Ω of a Hopf algebra H is a Hopf algebra with the same algebra structure and counit and with comultiplication and antipode given by

$$\Delta_\Omega(h) = \Omega \Delta(h) \Omega^{-1}$$
$$S_\Omega(h) = uS(h) u^{-1}$$

for all $h \in H$, where $\Omega \in H \otimes H$ and $u \in H$ are invertible elements.

The new comultiplication Δ_Ω is coassociative if and only if Ω is a 2-*pseudo-cocycle*, that is $\partial_2(\Omega)$ lies in the centralizer of $(\Delta \otimes id)\Delta(H)$ in $H \otimes H \otimes H$, where

$$\partial_2(\Omega) = (id \otimes \Delta)\left(\Omega^{-1}\right)\left(1 \otimes \Omega^{-1}\right)(\Omega \otimes 1)(\Delta \otimes id)(\Omega).$$

Ω is called a 2-*cocycle* if $\partial_2(\Omega) = 1 \otimes 1 \otimes 1$ and in this case we will denote it by J.

Remark 2.1. By [24, Theorem 4.1] $K_0(H) \cong K_0(H_\Omega)$ as ordered rings with marked elements, and thus $\mathbf{G}(H^*) \cong \mathbf{G}((H_\Omega)^*)$.

3. Hopf algebras of dimension 16 with a commutative subHopfalgebra of dimension 8.

We apply the methods used by Masuoka in [11], [12] and [14]. Let H be a non-trivial semisimple Hopf algebra of dimension 16 with a subHopfalgebra $K = (kG)^*$ of dimension 8. Since K is a subHopfalgebra of index 2, by [10, Proposition 2] or [20, Theorem 2.1.1] K is normal in H and thus we have an exact sequence of Hopf algebras

$$K \overset{i}{\hookrightarrow} H \overset{\pi}{\twoheadrightarrow} F \tag{3.1}$$

where $F = k\langle t\rangle \cong kC_2$ and $K = (kG)^*$, which is cleft by [26] or [17]. Such a sequence is called an extension of F by K and was first studied by Kac in [6]. The construction of extensions from cohomological data was done in [18] and [1]. K is commutative and F is cocommutative and thus (F, K) form an abelian matched pair of Hopf algebras and $(G, \langle t\rangle)$ form an abelian matched pair of groups (see [30], [5], [29], [12, Section 1]). Therefore H becomes a bicrossed product $K\#_\sigma^\theta F$ with an action $\rightharpoonup: F \otimes K \to K$, a coaction $\rho : F \to F \otimes K$, a cocycle $\sigma : F \otimes F \to K$ and a dual cocycle $\theta : F \to K \otimes K$. G is a normal subgroup of the group $G \times \langle t\rangle$, arising from a matched pair $(G, \langle t\rangle)$, since G has index 2 in $G \times \langle t\rangle$. Thus ρ is trivial and the action by t is a Hopf algebra automorphism of K (see [12, Section 1]). $\rightharpoonup$ is a nontrivial action on K, since otherwise $H \cong K^t[C_2]$ as an algebra, and thus H is commutative.

Let $v = \sigma(t, t) \in K$. Then by the properties of the cocycle v is a unit and

$$t \rightharpoonup v = v \tag{3.2}$$

Multiplication in H gives us

$$\overline{t}^2 = v \tag{3.3}$$

$$\overline{t}c = (t \rightharpoonup c)\,\overline{t} \tag{3.4}$$

where $\overline{t} = 1\#t$ and $c \in K$.

Moreover, if a unit $v \in K$ satisfies (3.2), (3.3) and (3.4), we can define a cocycle σ by $\sigma(1,1) = \sigma(1,t) = \sigma(t,1) = 1$ and $\sigma(t,t) = v$.

We proceed by considering the possible G, namely C_8, $C_4 \times C_2$, $C_2 \times C_2 \times C_2$, D_8 and Q_8. Theorem 1.1 says that the first case cannot appear.

Theorem 1.1. Let's prove the statement by induction on n. When $n = 2$, by [13, Theorem 2] H is a group algebra and if $H \ncong kC_{p^2}$ then $H \cong k(C_p \times C_p)$ and $\mathbf{G}(H) \cong C_p \times C_p$.

Now assume the statement is true for $n = m$. Consider H of dimension p^{m+1}. $\dim(H^*) = p^{m+1}$ and thus, by [13, Theorem 1], there exists a central grouplike of order p in H^* and therefore H^* contains a normal subHopfalgebra $K \cong kC_p$. Thus we get a short exact sequence of Hopf algebras

$$K \overset{i}{\hookrightarrow} H^* \overset{\pi}{\twoheadrightarrow} F \tag{3.5}$$

where $F = H^*/K^+H^*$. Dualizing (3.5) we get another short exact sequence of Hopf algebras:

$$F^* \overset{\pi^*}{\hookrightarrow} H \overset{i^*}{\twoheadrightarrow} K^* \tag{3.6}$$

where $K^* \cong K \cong kC_p$ and $\dim F^* = \dim F = p^m$. Thus we get $\mathbf{G}(F^*) \subseteq \mathbf{G}(H)$ and $\mathbf{G}(F^*)$ is not cyclic unless $F^* \cong kC_{p^m}$. In the first case we are done since it implies that $\mathbf{G}(H)$ is not cyclic. In the second case, since K is normal in H^*, H^* is isomorphic as an algebra to a twisted group ring $K^t[F]$ where $F \cong F^* \cong kC_{p^m}$. It is easy to show that, since F is a group algebra of a cyclic group, $K^t[F]$ is commutative. Thus H is cocommutative and the only possible H with a cyclic group of grouplikes is $kC_{p^{m+1}}$. $\square$

3.1. Case of $\mathbf{G}(H) = C_4 \times C_2$.

We will show that there are at most 7 possible Hopf algebras of this kind. Let H be a nontrivial semisimple Hopf algebra of dimension 16 with a subHopfalgebra $K = k(C_4 \times C_2)^* \cong k(C_4 \times C_2)$. Then $\mathbf{G}(H) = G \cong C_4 \times C_2$.

Let $G = \langle x \rangle \times \langle y \rangle$ with $|x| = 4$ and $|y| = 2$. Then the dual basis of $K \cong K^*$ is given by

$$e_{pq} = 1/8\left(1 + i^p x + i^{2p}x^2 + i^{3p}x^3\right)\left(1 + (-1)^q y\right), \qquad p = 0,1,2,3; q = 0,1$$

Then

$$\begin{aligned} \Delta_H(e_{pq}) &= \Delta_K(e_{pq}) = \sum_{\substack{p_1+p_2 \equiv p \mod 4 \\ q_1+q_2 \equiv q \mod 2}} e_{p_1q_1} \otimes e_{p_2q_2} \\ \Delta_H(\overline{t}) &= \theta(t)\,\overline{t} \otimes \overline{t} \end{aligned}$$

where $\overline{t} = 1\#t$. Dualizing (3.1) we get another extension

$$F^* \overset{\pi^*}{\hookrightarrow} H^* \overset{i^*}{\twoheadrightarrow} K^*$$

and as in [11, 2.4], [12, 2.11] or [15, 2.1], since k is algebraically closed, there exist units $\overline{x}$ and $\overline{y} \in H^*$, such that $\overline{x}^4 = \overline{y}^2 = 1_{H^*}$, $\left\langle e_{pq}, \overline{x}^i\overline{y}^j \right\rangle = \delta_{ip}\delta_{jq}$ and $\alpha = \overline{y}^{-1}\overline{x}^{-1}\overline{yx} \in F^* = k\{e_0, e_1\}$, where $\{e_r\}$ is a dual basis of $\{t^r\}$, $r = 0,1$. $\varepsilon(\alpha) = \varepsilon\left(\overline{y}^{-1}\overline{x}^{-1}\overline{yx}\right) = 1$ and therefore $\alpha = e_0 + \xi e_1$. The right action $\rho^* : F^* \otimes K^* \to F^*$ is trivial, thus F^* lies in the center of H^*.

$$\overline{x} = \overline{y}^2\overline{x} = \overline{yxy}\alpha = \overline{xy}\alpha\overline{y}\alpha = \overline{xy}^2\alpha^2 = \overline{x}\alpha^2$$

Thus $\alpha^2 = 1$ and therefore $\xi = \pm 1$.

$$\left\langle \Delta_H(\overline{t}), \overline{x}^i\overline{y}^j e_k \otimes \overline{x}^p\overline{y}^q e_r \right\rangle = \left\langle \overline{t}, \overline{x}^i\overline{y}^j e_k \overline{x}^p\overline{y}^q e_r \right\rangle = \delta_{kr}\left\langle \overline{t}, \overline{x}^{i+p}\overline{y}^{j+q}\alpha^{jp} e_k \right\rangle = \xi^{jp}\delta_{k1}\delta_{r1}$$

On the other hand

$$\left\langle \Delta_H\left(\overline{t}\right), \overline{x}^i\overline{y}^j e_k \otimes \overline{x}^p\overline{y}^q e_r\right\rangle = \left\langle \theta\left(t\right)\overline{t}\otimes\overline{t}, \overline{x}^i\overline{y}^j e_k \otimes \overline{x}^p\overline{y}^q e_r\right\rangle = \left\langle \theta\left(t\right), \overline{x}^i\overline{y}^j \otimes \overline{x}^p\overline{y}^q\right\rangle \delta_{k1}\delta_{r1}$$

Therefore

$$\theta\left(t\right) = \sum_{ijpq} \xi^{jp} e_{ij}\otimes e_{pq}$$

and since H should be non-cocommutative, $\theta\left(t,t\right)$ is nontrivial, and thus $\xi = -1$ and

$$\theta\left(t\right) = \sum_{ijpq} \left(-1\right)^{jp} e_{ij}\otimes e_{pq} = \frac{1}{2}\left(\left(1+y\right)\otimes 1 + \left(1-y\right)\otimes x^2\right)$$

Write $v = \sigma\left(t,t\right) = \sum c_{i,j}e_{i,j}$ then $c_{0,0} = \varepsilon\left(v\right) = 1$ and $c_{i,j} \neq 0$, since v is a unit, and

$$\Delta_H\left(\overline{t}^2\right) = \Delta_H\left(v\right) = \Delta_K\left(\sum c_{i,j}e_{i,j}\right) = \sum c_{i+p,j+q}e_{i,j}\otimes e_{p,q}$$

On the other hand, if we write

$$t \rightharpoonup e_{p,q} = e_{\alpha_1(p,q),\alpha_2(p,q)}$$

$$\begin{aligned}
\Delta\left(\overline{t}\right)\Delta\left(\overline{t}\right) &= \sum_{ijpq}\left(-1\right)^{jp} e_{ij}\overline{t}\otimes e_{pq}\overline{t}\sum_{ijpq}\left(-1\right)^{jp} e_{ij}\overline{t}\otimes e_{pq}\overline{t} \\
&= \sum_{ijpq}\left(-1\right)^{jp} e_{ij}\otimes e_{pq}\sum_{ijpq}\left(-1\right)^{jp} e_{\alpha_1(i,j),\alpha_2(i,j)}\overline{t}^2\otimes e_{\alpha_1(p,q),\alpha_2(p,q)}\overline{t}^2 \\
&= \sum_{ijpq}\left(-1\right)^{jp} e_{ij}\otimes e_{pq}\sum_{ijpq}\left(-1\right)^{\alpha_2(i,j)\alpha_1(p,q)} e_{ij}\overline{t}^2\otimes e_{pq}\overline{t}^2 \\
&= \sum_{ijpq}\left(-1\right)^{jp+\alpha_2(i,j)\alpha_1(p,q)} c_{ij}c_{pq}e_{ij}\otimes e_{pq}
\end{aligned}$$

Thus for H to be a bialgebra we should have

$$c_{i+p,j+q} = \left(-1\right)^{jp+\alpha_2(i,j)\alpha_1(p,q)} c_{i,j}c_{p,q} \tag{3.7}$$

Action by t is a Hopf algebra map and therefore $t \rightharpoonup G = G$ and $f_t : G\rightarrow G$ defined by $f_t\left(g\right) = t\rightharpoonup g$ is a group automorphism of order 2. There are three possibilities for such an automorphism; we consider them below:

Case a). The action is given by

$$\begin{aligned}
t &\rightharpoonup x = xy \\
t &\rightharpoonup y = y
\end{aligned}$$

Then $t \rightharpoonup e_{i,j} = e_{i+2j,j}$. Write $v = \sigma\left(t,t\right) = \sum c_{i,j}e_{i,j}$. By (3.2) and (3.7)

$$c_{i,j} = c_{i+2j,j} \tag{3.8}$$

$$c_{i+p,j+q} = c_{i,j}c_{p,q} \tag{3.9}$$

Conditions (3.8) and (3.9) imply that $c_{1,0} = (-1)^k$ and $c_{0,1} = (-1)^l$ for $k, l = 0, 1$ and

$$\sigma(t,t) = \sum_{p,q} (-1)^{kp+lq} e_{p,q} = \sum (-1)^{kp} e_{p,q} \sum (-1)^{lq} e_{p,q} = x^{2k} y^l \qquad k, l = 0, 1$$

For $k, l = 0, 1$ let $H_{k,l}$ be the Hopf algebras with the structures described above with cocycles $\sigma_{k,l}(t,t) = x^{2k} y^l$. Define

$$\begin{aligned} f : H_{k,l} &\rightarrow H_{k+1,l+1} \qquad \text{by} \\ f(e_{r,s}) &= e_{r,s} \\ f(\bar{t}) &= x\bar{t} \end{aligned}$$

and extend it multiplicatively to $f\left(e_{r,s}\bar{t}\right)$. Then f is a trivial group homomorphism on $\mathbf{G}(H_{k,l})$ and

$$\begin{aligned} f(\bar{t}) f(\bar{t}) &= x\bar{t}x\bar{t} = x^2 y \bar{t}^2 = x^2 y x^{2(k+1)} y^{(l+1)} = x^{2k} y^l = f\left(\bar{t}^2\right) \\ f(\bar{t}x) &= f\left(xy\bar{t}\right) = xyx\bar{t} = x\bar{t}x = f(\bar{t}) f(x) \\ (f \otimes f)\Delta(\bar{t}) &= (f \otimes f)\left(\theta(t)\bar{t} \otimes \bar{t}\right) = \theta(t)\left(f(\bar{t}) \otimes f(\bar{t})\right) = \theta(t)\left(x\bar{t} \otimes x\bar{t}\right) \\ &= (x \otimes x)\theta(t)\left(\bar{t} \otimes \bar{t}\right) = \Delta(x)\Delta(\bar{t}) = \Delta\left(x\bar{t}\right) = \Delta\left(f(\bar{t})\right) \end{aligned}$$

and such an f is a Hopf algebra isomorphism between $H_{k,l}$ and $H_{k+1,l+1}$. Thus there are at most two nonisomorphic Hopf algebras of this type:

1. $H_{a:1} = H_{0,0}$ with the trivial cocycle and $\mathbf{G}(H_{a:1}^*) = \langle\chi\rangle \times \langle\varphi\rangle \cong C_4 \times C_2$, where $\chi(x) = i$, $\chi(y) = \chi(t) = 1$, $\varphi(x) = \varphi(y) = 1$, $\varphi(t) = -1$. There is a degree 2 irreducible representation defined by
$\pi(x) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ $\pi(y) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $\pi(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
with the property $\pi^2 = \pi \bullet \pi = 1 + \chi^2 + \varphi + \chi^2\varphi$.
2. $H_{a:y} = H_{0,1}$ with the cocycle defined by $\sigma(t,t) = y$ and $\mathbf{G}\left(H_{a:y}^*\right) = \langle\chi\rangle \times \langle\varphi\rangle \cong C_4 \times C_2$, where $\chi(x) = i$, $\chi(y) = \chi(t) = 1$, $\varphi(x) = \varphi(y) = 1$, $\varphi(t) = -1$. There is a degree 2 irreducible representation defined by
$\pi(x) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ $\pi(y) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $\pi(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
with the property $\pi^2 = \pi \bullet \pi = 1 + \chi^2 + \varphi + \chi^2\varphi$.

Case b). The action is given by

$$\begin{aligned} t &\rightharpoonup x = x^{-1} \\ t &\rightharpoonup y = y \end{aligned}$$

Then $t \rightharpoonup e_{i,j} = e_{-i,j}$. Write $v = \sigma(t,t) = \sum c_{i,j} e_{i,j}$. By (3.2) and (3.7)

$$c_{i,j} = c_{-i,j} \tag{3.10}$$

$$c_{i+p,j+q} = c_{i,j} c_{p,q} \tag{3.11}$$

Conditions (3.10) and (3.11) imply that $c_{1,0} = (-1)^k$ and $c_{0,1} = (-1)^l$ for $k, l = 0, 1$ and

$$\sigma(t,t) = \sum_{p,q} (-1)^{kp+lq} e_{p,q} = \sum (-1)^{kp} e_{p,q} \sum (-1)^{lq} e_{p,q} = x^{2k} y^l \qquad k, l = 0, 1$$

For $k, l = 0, 1$ let $H_{k,l}$ be the Hopf algebras with the structures described above with cocycles $\sigma_{k,l}(t,t) = x^{2k} y^l$. Define

$$\begin{aligned} f : H_{0,0} &\to H_{1,0} \qquad \text{by} \\ f(e_{r,s}) &= e_{r,r+s} \\ f(\overline{t}) &= \frac{1}{2}\left((1+i)1 + (1-i)x^2\right)\overline{t} = \sum_{p=0}^{3}\sum_{q=0}^{1} i^{p^2} e_{p,q}\overline{t} \end{aligned}$$

and extend it multiplicatively to $f\left(e_{r,s}\overline{t}\right)$. Then $f\mid_{\mathbf{G}(H_{0,0})}$ is a group isomorphism $\mathbf{G}(H_{0,0}) \to \mathbf{G}(H_{1,0})$ with $f(x) = x$, $f(y) = x^2 y$ and

$$\begin{aligned} f(\overline{t}) f(\overline{t}) &= \frac{1}{4}\left((1+i)1 + (1-i)x^2\right)\overline{t}\left((1+i)1 + (1-i)x^2\right)\overline{t} \\ &= \frac{1}{4}\left((1+i)1 + (1-i)x^2\right)^2 \overline{t}^2 = \frac{1}{4}\left(2i \cdot 1 + 4x^2 - 2i \cdot 1\right)\sigma_{1,0}(t,t) \\ &= x^2 x^2 = 1 = f\left(\overline{t}^2\right) \\ f(\overline{t}x) &= f\left(x^{-1}\overline{t}\right) = \frac{1}{2}x^{-1}\left((1+i)1 + (1-i)x^2\right)\overline{t} \\ &= \frac{1}{2}\left((1+i)1 + (1-i)x^2\right)\overline{t}x = f(\overline{t}) f(x) \\ (f \otimes f)\Delta(\overline{t}) &= (f \otimes f)\left(\theta(t)\overline{t} \otimes \overline{t}\right) = (f \otimes f)\left(\sum (-1)^{q_1 p_2} e_{p_1,q_1}\overline{t} \otimes e_{p_2,q_2}\overline{t}\right) \\ &= \sum (-1)^{q_1 p_2} e_{p_1,q_1+p_1} \sum i^{p^2} e_{p,q}\overline{t} \otimes e_{p_2,q_2+p_2} \sum i^{p^2} e_{p,q}\overline{t} \\ &= \sum (-1)^{(q_1+p_1)p_2} i^{p_1^2+p_2^2} e_{p_1,q_1}\overline{t} \otimes e_{p_2,q_2}\overline{t} \\ &= \sum (-1)^{q_1 p_2} i^{p_1^2+2p_1p_2+p_2^2} e_{p_1,q_1}\overline{t} \otimes e_{p_2,q_2}\overline{t} \\ &= \sum (-1)^{q_1 p_2} i^{(p_1+p_2)^2} e_{p_1,q_1}\overline{t} \otimes e_{p_2,q_2}\overline{t} \\ &= \left(\sum i^{(p_1+p_2)^2} e_{p_1,q_1} \otimes e_{p_2,q_2}\right)\left(\sum (-1)^{s_1 r_2} e_{r_1,s_1}\overline{t} \otimes e_{r_2,s_2}\overline{t}\right) \\ &= \Delta\left(\sum i^{p^2} e_{p,q}\right)\Delta(\overline{t}) = \Delta\left(\sum i^{p^2} e_{p,q}\overline{t}\right) = \Delta\left(f(\overline{t})\right) \end{aligned}$$

and such an f is a Hopf algebra isomorphism between $H_{0,0}$ and $H_{1,0}$. There are at most three nonisomorphic Hopf algebras of this kind:

1. $H_{b:1} = H_{0,0}$ with the trivial cocycle and $\mathbf{G}(H_{b:1}^*) = \langle\chi\rangle \times \langle\varphi\rangle \times \langle\psi\rangle \cong C_2 \times C_2 \times C_2$, where $\chi(x) = -1$, $\chi(y) = \chi(t) = 1$, $\varphi(x) = \varphi(y) = 1$, $\varphi(t) = -1$, $\psi(y) = -1$, $\psi(x) = \psi(t) = 1$. There is a degree 2 irreducible representation defined by

$$\pi(x)=\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \pi(y)=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \pi(t)=\begin{pmatrix} 0 & (-1)^p \\ 1 & 0 \end{pmatrix}$$

with the property $\pi^2=\pi\bullet\pi=1+\chi+\varphi+\chi\varphi$.

2. $H_{b:y}=H_{0,1}$ with the cocycle defined by $\sigma(t,t)=y$ and $\mathbf{G}\left(H_{b:y}^*\right)=\langle\chi\rangle\times\langle\varphi\rangle\cong C_4\times C_2$, where $\chi(x)=1$, $\chi(y)=-1$, $\chi(t)=i$, $\varphi(x)=-1$, $\varphi(y)=\varphi(t)=1$. There is a degree 2 irreducible representation defined by

$$\pi(x)=\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \pi(y)=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \pi(t)=\begin{pmatrix} 0 & (-1)^p \\ 1 & 0 \end{pmatrix}$$

with the property $\pi^2=\pi\bullet\pi=1+\chi^2+\varphi+\chi^2\varphi$.

3. $H_{b:x^2y}=H_{1,1}$ with the cocycle defined by $\sigma(t,t)=x^2y$ and $\mathbf{G}\left(H_{b:x^2y}^*\right)=\langle\chi\rangle\times\langle\varphi\rangle\cong C_4\times C_2$, where $\chi(x)=1$, $\chi(y)=-1$, $\chi(t)=i$, $\varphi(x)=-1$, $\varphi(y)=\varphi(t)=1$. There is a degree 2 irreducible representation defined by

$$\pi(x)=\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \pi(y)=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \pi(t)=\begin{pmatrix} 0 & (-1)^p \\ 1 & 0 \end{pmatrix}$$

with the property $\pi^2=\pi\bullet\pi=1+\chi^2+\varphi+\chi^2\varphi$.

Case c). The action is given by

$$\begin{aligned} t &\rightharpoonup x=x \\ t &\rightharpoonup y=x^2y \end{aligned}$$

Then $t\rightharpoonup e_{i,j}=e_{i,j+i}$. Write $v=\sigma(t,t)=\sum c_{i,j}e_{i,j}$. By (3.2) and (3.7)

$$c_{i,j} = c_{i,i+j} \tag{3.12}$$

$$c_{i+p,j+q} = (-1)^{ip}\, c_{i,j}c_{p,q} \tag{3.13}$$

Conditions (3.12) and (3.13) imply that $c_{1,0}^4=c_{0,1}=1$, $c_{2,0}=-c_{1,0}^2$ Thus $c_{1,0}=i^k$ for $k=0,1,2,3$ and

$$\sigma_k(t,t)=\sum_{p,q}(-1)^{\frac{p(p-1)}{2}}\,i^{kp}e_{p,q}=x^{1-k}\left(\frac{1+i}{2}1+\frac{1-i}{2}x^2\right)$$

For $k=0,1,2,3$ let H_k be the Hopf algebras with the structures described above with cocycles σ_k. Define

$$\begin{aligned} f:H_{k+2} &\to H_k \qquad \text{by} \\ f\left(e_{p,q}\right) &= e_{p,q} \\ f\left(\overline{t}\right) &= \sum_{p,q}(-1)^{p+q}\,e_{p,q}\overline{t}=y\overline{t} \end{aligned}$$

and extend it multiplicatively to $f\left(e_{p,q}\overline{t}\right)$. Then

$$\begin{aligned} f\left(\overline{t}\right)f\left(\overline{t}\right) &= y\overline{t}y\overline{t}=x^2\overline{t}^2=x^2x^{1-(k-2)}\left(\frac{1+i}{2}+\frac{1-i}{2}x^2\right) \\ &= x^{1-k}\left(\frac{1+i}{2}+\frac{1-i}{2}x^2\right)=f\left(\overline{t}^2\right) \\ f\left(\overline{t}y\right) &= f\left(x^2y\overline{t}\right)=x^2yy\overline{t}=y\overline{t}y=f\left(\overline{t}\right)f\left(y\right) \end{aligned}$$

$$\begin{aligned}(f\otimes f)\Delta\left(\overline{t}\right) &= (f\otimes f)\left(\theta(t)\,\overline{t}\otimes\overline{t}\right)=\theta(t)\left(f\left(\overline{t}\right)\otimes f\left(\overline{t}\right)\right)=\theta(t)\left(y\overline{t}\otimes y\overline{t}\right)\\ &= (y\otimes y)\,\theta(t)\left(\overline{t}\otimes\overline{t}\right)=\Delta(y)\,\Delta\left(\overline{t}\right)=\Delta\left(y\overline{t}\right)=\Delta\left(f\left(\overline{t}\right)\right)\end{aligned}$$

and such a f is a Hopf algebra isomorphism between H_{k+2} and H_k. Thus there are exactly 2 nonisomorphic Hopf algebras of this type:

1. $H_{c:\sigma_0}=H_0$ with cocycle σ_0 defined by $\sigma(t,t)=\frac{1+i}{2}x+\frac{1-i}{2}x^3$ and $\mathbf{G}\left(H^*_{c:\sigma_0}\right)=\langle\chi\rangle\times\langle\varphi\rangle\cong C_4\times C_2$, where $\chi(x)=-1$, $\chi(y)=1$, $\chi(t)=i$, $\varphi(y)=-1$, $\varphi(x)=\varphi(t)=1$. There is a degree 2 irreducible representation defined by
$$\pi(x)=\begin{pmatrix} i & 0\\ 0 & i\end{pmatrix}\quad \pi(y)=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\quad \pi(t)=\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix}$$
with the property $\pi^2=\pi\bullet\pi=\chi+\chi^3+\chi\varphi+\chi^3\varphi$.
2. $H_{c:\sigma_1}=H_1$ with cocycle σ_1 defined by $\sigma(t,t)=\frac{1+i}{2}1+\frac{1-i}{2}x^2$ and $\mathbf{G}\left(H^*_{c:\sigma_1}\right)=\langle\chi\rangle\times\langle\varphi\rangle\times\langle\psi\rangle\cong C_2\times C_2\times C_2$, where $\chi(y)=-1$, $\chi(x)=\chi(t)=1$, $\varphi(x)=\varphi(y)=1$, $\varphi(t)=-1$, $\psi(x)=-1$, $\psi(y)=\psi(t)=1$. There is a degree 2 irreducible representation defined by
$$\pi(x)=\begin{pmatrix} i & 0\\ 0 & i\end{pmatrix}\quad \pi(y)=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\quad \pi(t)=\begin{pmatrix} \omega & 0\\ 0 & -\omega\end{pmatrix}$$
where ω is a primitive 8^{th}-root of unity, with the property $\pi^2=\pi\bullet\pi=\psi+\chi\psi+\varphi\psi+\chi\varphi\psi$.

3.2. **Case of $\mathbf{G}(H)=C_2\times C_2\times C_2$.**
We will show that there are at most 4 possible Hopf algebras of this kind. Let H be a nontrivial semisimple Hopf algebra of dimension 16 with a subHopfalgebra $K=k\left(C_2\times C_2\times C_2\right)^*\cong k\left(C_2\times C_2\times C_2\right)$. Then $\mathbf{G}(H)=G\cong C_2\times C_2\times C_2$.

Let $\mathbf{G}(H)=\langle x\rangle\times\langle y\rangle\times\langle z\rangle$, where $|x|=|y|=|z|=2$. Then the dual basis of $K\cong K^*$ is given by

$$e_{p,q,r}=1/8\left(1+(-1)^p x\right)\left(1+(-1)^q y\right)\left(1+(-1)^r z\right),\qquad p,q,r=0,1$$

Then

$$\begin{aligned}\Delta_H\left(e_{p,q,r}\right) &= \Delta_K\left(e_{p,q,r}\right)=\sum_{\substack{p_1+p_2\equiv p \mod 2\\ q_1+q_2\equiv q \mod 2\\ r_1+r_2\equiv r \mod 2}} e_{p_1,q_1,r_1}\otimes e_{p_2,q_2,r_2}\\ \Delta_H\left(\overline{t}\right) &= \theta(t)\,\overline{t}\otimes\overline{t}\end{aligned}$$

where $\overline{t}=1\#t$. Dualizing (3.1) we get another extension

$$F^*\overset{\pi^*}{\hookrightarrow}H^*\overset{i^*}{\twoheadrightarrow}K^*$$

and as in [11, 2.4], [12, 2.11] or [15, 2.1], since k is algebraically closed, there exist units $\overline{x}$, $\overline{y}$ and $\overline{z}\in H^*$, such that $\overline{x}^2=\overline{y}^2=\overline{z}^2=1_{H^*}$, $\left\langle e_{p,q,r},\overline{x}^i\overline{y}^j\overline{z}^k\right\rangle=\delta_{ip}\delta_{jq}\delta_{kr}$ and $\alpha=\overline{z}^{-1}\overline{y}^{-1}\overline{zy},\beta=\overline{z}^{-1}\overline{x}^{-1}\overline{zx},\gamma=\overline{y}^{-1}\overline{x}^{-1}\overline{yx}\in F^*=k\left\{e_0,e_1\right\}$, where $\{e_r\}$ is a dual basis of $\{t^r\}$, $r=0,1$. $\varepsilon(\alpha)=\varepsilon(\beta)=\varepsilon(\gamma)=1$ and therefore $\alpha=e_0+\xi_3e_1,\beta=e_0+\xi_2e_1,\gamma=e_0+\xi_1e_1$. The right action $\rho^*:F^*\otimes K^*\to F^*$ is trivial, thus F^* lies in the center of H^*.

$$\overline{x}=\overline{y}^2\overline{x}=\overline{yxy}\gamma=\overline{xy}\gamma\overline{y}\gamma=\overline{xy}^2\gamma^2=\overline{x}\gamma^2$$

Thus $\gamma^2 = 1$ and similarly $\alpha^2 = \beta^2 = 1$. Therefore $\xi_1, \xi_2, \xi_3 = \pm 1$ and, since H^* is non-commutative, they cannot be all equal to 1.

$$\begin{aligned}\left\langle \Delta_H\left(\overline{t}\right), \overline{x}^i\overline{y}^j\overline{z}^k e_l \otimes \overline{x}^p\overline{y}^q\overline{z}^r e_s\right\rangle &= \left\langle \overline{t}, \overline{x}^i\overline{y}^j\overline{z}^k e_l \overline{x}^p\overline{y}^q\overline{z}^r e_s\right\rangle \\ &= \delta_{ls}\left\langle \overline{t}, \overline{x}^{i+p}\overline{y}^{j+q}\overline{z}^{k+r}\alpha^{kq}\beta^{kp}\gamma^{jp} e_l\right\rangle = \xi_1^{jp}\xi_2^{kp}\xi_3^{kq}\delta_{l1}\delta_{s1}\end{aligned}$$

On the other hand

$$\begin{aligned}\left\langle \Delta_H\left(\overline{t}\right), \overline{x}^i\overline{y}^j\overline{z}^k e_l \otimes \overline{x}^p\overline{y}^q\overline{z}^r e_s\right\rangle &= \left\langle \theta(t)\,\overline{t}\otimes\overline{t}, \overline{x}^i\overline{y}^j\overline{z}^k e_l \otimes \overline{x}^p\overline{y}^q\overline{z}^r e_s\right\rangle \\ &= \left\langle \theta(t), \overline{x}^i\overline{y}^j\overline{z}^k \otimes \overline{x}^p\overline{y}^q\overline{z}^r\right\rangle \delta_{k1}\delta_{r1}\end{aligned}$$

Therefore

$$\theta(t) = \sum_{ijkpqr} \xi_1^{jp}\xi_2^{kp}\xi_3^{kq} e_{i,j,k}\otimes e_{p,q,r}$$

Action by t is a Hopf algebra map and therefore $t \rightharpoonup G = G$ and $f_t : G \to G$ defined by $f_t(g) = t \rightharpoonup g$ is a group automorphism of order 2. Then, without loss of generality there is only one possibility for such an automorphism:

$$\begin{aligned} t &\rightharpoonup x = y \\ t &\rightharpoonup y = x \\ t &\rightharpoonup z = z \end{aligned}$$

Then $t \rightharpoonup e_{i,j,k} = e_{j,i,k}$.

Write $v = \sigma(t,t) = \sum c_{i,j,k}e_{i,j,k}$ then $c_{0,0,0} = \varepsilon(v) = 1$ and $c_{i,j,k} \neq 0$, since v is a unit. By formula (3.2)

$$c_{i,j,k} = c_{j,i,k} \tag{3.14}$$

For H to be a bialgebra we need $\Delta_H\left(\overline{t}^2\right) = \Delta_H\left(\overline{t}\right)\Delta_H\left(\overline{t}\right)$

$$\Delta_H\left(\overline{t}^2\right) = \Delta_H(v) = \Delta_K\left(\sum c_{i,j,k}e_{i,j,k}\right) = \sum c_{i+p,j+q,k+r}e_{i,j,k}\otimes e_{p,q,r}$$

On the other hand,

$$\begin{aligned}&\Delta_H\left(\overline{t}\right)\Delta_H\left(\overline{t}\right) = \left(\theta(t)\,\overline{t}\otimes\overline{t}\right)\left(\theta(t)\,\overline{t}\otimes\overline{t}\right) \\ =\ & \sum_{ijpq}\xi_1^{jp}\xi_2^{kp}\xi_3^{kq}e_{ijk}\otimes e_{pqr}\left(\sum_{ijpq}\xi_1^{jp}\xi_2^{kp}\xi_3^{kq}\left(t\rightharpoonup e_{ijk}\right)\otimes\left(t\rightharpoonup e_{pqr}\right)\right)\sigma(t,t)\otimes\sigma(t,t) \\ =\ & \sum \xi_1^{jp+iq}\xi_2^{kp+kq}\xi_3^{kq+kp}c_{i,j,k}c_{p,q,r}e_{i,j,k}\otimes e_{p,q,r}\end{aligned}$$

Therefore

$$c_{i+p,j+q,k+r} = \xi_1^{jp+iq}\xi_2^{kp+kq}\xi_3^{kq+kp}c_{i,j,k}c_{p,q,r} \tag{3.15}$$

Conditions (3.14) and (3.15) imply that $c_{1,0,0}^2 = c_{0,1,0}^2 = c_{0,0,1}^2 = c_{1,1,0}^2 = c_{1,1,1}^2 = 1$ and $c_{0,1,0} = c_{1,0,0}$

$$\begin{aligned} c_{1,1,0} &= \xi_1 c_{0,1,0}c_{1,0,0} = \xi_1 c_{1,0,0}^2 = \xi_1 \\ c_{1,0,1} &= c_{1,0,0}c_{0,0,1}\end{aligned}$$

$$c_{1,0,1} = c_{0,0,1}c_{1,0,0}\xi_2\xi_3$$

Thus $\xi_2\xi_3 = 1$, that is $\xi_2 = \xi_3$ and $c_{1,0,0} = c_{0,1,0} = \omega = \pm 1$ and $c_{0,0,1} = \tau = \pm 1$ and

$$\theta(t) = \sum_{ijkpqr} \xi_1^{jp}\xi_2^{kp+kq} e_{i,j,k} \otimes e_{p,q,r}$$

$$\begin{aligned}
\sigma(t,t) &= e_{0,0,0} + \tau e_{0,0,1} + \xi_1 e_{1,1,0} + \xi_1\tau e_{1,1,1} + \omega(e_{1,0,0} + e_{0,1,0} + \tau e_{1,0,1} + \tau e_{0,1,1}) \\
&= \sum \omega^{p+q} e_{pqr} \sum \xi_1^{pq} e_{pqr} \sum \tau^r e_{pqr} = (xy)^{\delta_{\omega,-1}} (1 + x + y - xy)^{\delta_{\xi_1,-1}} z^{\delta_{\tau,-1}}
\end{aligned}$$

For $\xi_1, \xi_2, \tau, \omega = \pm 1$ let $H_{d:\xi_1,\xi_2,\tau,\omega}$ be the Hopf algebras with the structures described above with cocycles $\sigma_{\xi_1,\tau,\omega}$. Then $\sigma_{\xi_1,\tau,-1}(t,t) = xy\sigma_{\xi_1,\tau,1}(t,t)$. Define

$$\begin{aligned}
f : H_{d:\xi_1,\xi_2,\tau,-1} &\to H_{d:\xi_1,\xi_2,\tau,1} \quad \text{by} \\
f(e_{p,q,r}) &= e_{p,q,r} \\
f(\bar{t}) &= x\bar{t}
\end{aligned}$$

and extend it multiplicatively to $f(e_{p,q,r}\bar{t})$. Then

$$\begin{aligned}
f(\bar{t}) f(\bar{t}) &= x\bar{t}x\bar{t} = xy\bar{t}^2 = xy\sigma_{\xi_1,\tau,1}(t,t) \\
&= \sigma_{\xi_1,\tau,-1}(t,t) = f(\sigma_{\xi_1,\tau,-1}(t,t)) = f(\bar{t}^2) \\
f(\bar{t}x) &= f(y\bar{t}) = yx\bar{t} = x\bar{t}x = f(\bar{t}) f(x) \\
f(\bar{t}y) &= f(x\bar{t}) = x^2\bar{t} = x\bar{t}y = f(\bar{t}) f(y) \\
(f \circ f)\Delta(\bar{t}) &= (f \circ f)(\theta(t)\bar{t} \otimes \bar{t}) = \theta(t)(f(\bar{t}) \otimes f(\bar{t})) = \theta(t)(x\bar{t} \otimes x\bar{t}) \\
&= (x \otimes x)\theta(t)(\bar{t} \otimes \bar{t}) = \Delta(x)\Delta(\bar{t}) = \Delta(x\bar{t}) = \Delta(f(\bar{t}))
\end{aligned}$$

and such a f is a Hopf algebra isomorphism between $H_{d:\xi_1,\xi_2,\tau,-1}$ and $H_{d:\xi_1,\xi_2,\tau,1}$. Define

$$\begin{aligned}
f' : H_{d:-1,-1,\tau,1} &\to H_{d:-1,1,\tau,1} \quad \text{by} \\
f'(e_{p,q,r}) &= e_{p+r,q+r,r} \\
f'(\bar{t}) &= \frac{1}{2}(1 + z + iy - iyz)\bar{t} = \sum i^{r^2}(-1)^{qr} e_{p,q,r}\bar{t}
\end{aligned}$$

and extend it multiplicatively to $f'(e_{p,q,r}\bar{t})$. Then $f' |_{\mathbf{G}(H_{d:-1,-1,\tau,1})}$ is a group isomorphism $\mathbf{G}(H_{d:-1,-1,\tau,1}) \to \mathbf{G}(H_{d:-1,1,\tau,1})$ with $f'(x) = xz$, $f'(y) = yz$, $f'(z) = z$ and

$$\begin{aligned}
f'(\bar{t}) f'(\bar{t}) &= \frac{1}{4}((1+z) + iy(1-z))\bar{t}((1+z) + iy(1-z))\bar{t} \\
&= \frac{1}{4}((1+z) + iy(1-z))((1+z) + ix(1-z))\bar{t}^2 \\
&= \frac{1}{8}(2 + 2z - xy(2 - 2z))(1 + x + y - xy) z^{\delta_{\tau,-1}} \\
&= \frac{1}{4}((1 - xy) + z(1 + xy))((1 - xy) + x(1 + xy)) z^{\delta_{\tau,-1}} \\
&= \frac{1}{4}(2 - 2xy + xz(2 + 2xy)) z^{\delta_{\tau,-1}} = \frac{1}{2}(1 + xz + yz - xy) z^{\delta_{\tau,-1}}
\end{aligned}$$

$$
\begin{aligned}
&= f\left(\frac{1}{2}(1+x+y-xy)\,z^{\delta_{\tau,-1}}\right) = f'\left(\bar{t}^2\right)\\
f'\left(\bar{t}x\right) &= f'\left(y\bar{t}\right) = \frac{1}{2}yz\,(1+z+iy-iyz)\,\bar{t}\\
&= \frac{1}{2}(1+z+iy-iyz)\,\bar{t}xz = f'\left(\bar{t}\right)f'(x)\\
f'\left(\bar{t}y\right) &= f'\left(x\bar{t}\right) = \frac{1}{2}xz\,(1+z+iy-iyz)\,\bar{t}\\
&= \frac{1}{2}(1+z+iy-iyz)\,\bar{t}yz = f'\left(\bar{t}\right)f'(y)
\end{aligned}
$$

$$
\begin{aligned}
\left(f'\otimes f'\right)\Delta\left(\bar{t}\right) &= \left(f'\otimes f'\right)\left(\theta(t)\,\bar{t}\otimes\bar{t}\right)\\
&= \left(f'\otimes f'\right)\left(\sum(-1)^{bp}(-1)^{cp+cq}\,e_{a,b,c}\bar{t}\otimes e_{p,q,r}\bar{t}\right)\\
&= \left(\sum(-1)^{bp}(-1)^{c(p+q)}\,e_{a+c,b+c,c}\otimes e_{p+r,q+r,r}\right)\\
&\quad\times\left(\sum i^{n^2}(-1)^{mn}\,e_{l,m,n}\bar{t}\otimes\sum i^{n^2}(-1)^{mn}\,e_{l,m,n}\bar{t}\right)\\
&= \left(\sum(-1)^{(b+c)(p+r)}(-1)^{c(p+q)}\,e_{a,b,c}\otimes e_{p,q,r}\right)\\
&\quad\times\left(\sum i^{n^2}(-1)^{mn}\,e_{l,m,n}\bar{t}\otimes\sum i^{n^2}(-1)^{mn}\,e_{l,m,n}\bar{t}\right)\\
&= \sum(-1)^{bp+cp+br+cr}(-1)^{cp+cq}\,i^{c^2}(-1)^{bc}\,i^{r^2}(-1)^{qr}\,e_{a,b,c}\bar{t}\otimes e_{p,q,r}\bar{t}\\
&= \sum(-1)^{bp+br+cq+bc+qr}(-1)^{cr}\,i^{c^2}i^{r^2}e_{a,b,c}\bar{t}\otimes e_{p,q,r}\bar{t}\\
&= \sum(-1)^{bp+br+cq+bc+qr}(-1)^{cr}\,i^{c^2}i^{r^2}e_{a,b,c}\bar{t}\otimes e_{p,q,r}\bar{t}\\
&= \sum i^{(c+r)^2}(-1)^{(b+q)(c+r)}(-1)^{bp}\,e_{a,b,c}\bar{t}\otimes e_{p,q,r}\bar{t}\\
&= \sum i^{n^2}(-1)^{mn}\sum_{\substack{l_1+l_2=l\\ m_1+m_2=m\\ n_1+n_2=n}} e_{l_1,m_1,n_1}\otimes e_{l_2,m_2,n_2}\\
&\quad\times\sum(-1)^{bp}\,e_{a,b,c}\bar{t}\otimes e_{p,q,r}\bar{t}\\
&= \sum i^{n^2}(-1)^{mn}\,\Delta\left(e_{l,m,n}\right)\Delta\left(\bar{t}\right)\\
&= \Delta\left(\sum i^{n^2}(-1)^{mn}\,e_{l,m,n}\bar{t}\right) = \Delta f'\left(\bar{t}\right)
\end{aligned}
$$

and such an f' is a Hopf algebra isomorphism between $H_{d:-1,-1,\tau,1}$ and $H_{d:-1,1,\tau,1}$. Thus we may assume that $\omega=1$ and $\xi_2=-1$. Therefore there are at most four nonisomorphic Hopf algebras $H_{d:\xi_1,\tau}$ of this kind, $H_{d:1,1}$, $H_{d:1,-1}$, $H_{d:-1,1}$ and $H_{d:-1,-1}$:

1. $H_{d:1,1}$ with the trivial cocycle and $\mathbf{G}\left(H^*_{d:1,1}\right)=\langle\chi\rangle\times\langle\varphi\rangle\times\langle\psi\rangle\cong C_2\times C_2\times C_2$, where $\chi(x)=\chi(y)=\chi(z)=1$, $\chi(t)=-1$, $\varphi(x)=\varphi(y)=-1$, $\varphi(z)=\varphi(t)=1$, $\psi(z)=-1$, $\psi(x)=\psi(y)=\psi(t)=1$. There is a degree 2 irreducible representation defined by

$\pi(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\pi(y) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $\pi(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\pi(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
with the property $\pi^2 = \pi \bullet \pi = 1 + \chi + \varphi + \chi\varphi$.

2. $H_{d:1,-1}$ with the cocycle defined by $\sigma(t,t) = z$ and $\mathbf{G}\left(H^*_{d:1,-1}\right) = \langle\chi\rangle \times \langle\varphi\rangle \cong C_4 \times C_2$, where $\chi(x) = \chi(y) = 1$, $\chi(z) = -1$, $\chi(t) = i$, $\varphi(x) = \varphi(y) = -1$, $\varphi(t) = \varphi(z) = 1$. There is a degree 2 irreducible representation defined by
$\pi(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\pi(y) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $\pi(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\pi(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
with the property $\pi^2 = \pi \bullet \pi = 1 + \chi^2 + \varphi + \chi^2\varphi$.
3. $H_{d:-1,1}$ with the cocycle defined by $\sigma(t,t) = 1 + x + y - xy$ and $\mathbf{G}\left(H^*_{d:-1,1}\right) = \langle\chi\rangle \times \langle\varphi\rangle \times \langle\psi\rangle \cong C_2 \times C_2 \times C_2$, where $\chi(x) = \chi(y) = \chi(z) = 1$, $\chi(t) = -1$, $\varphi(x) = \varphi(y) = -1$, $\varphi(z) = 1$, $\varphi(t) = i$, $\psi(z) = -1$, $\psi(x) = \psi(y) = \psi(t) = 1$. There is a degree 2 irreducible representation defined by
$\pi(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\pi(y) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $\pi(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\pi(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
with the property $\pi^2 = \pi \bullet \pi = 1 + \chi + \varphi + \chi\varphi$.
4. $H_{d:-1,-1}$ with the cocycle defined by $\sigma(t,t) = (1 + x + y - xy)\, z$ and $\mathbf{G}\left(H^*_{d:-1,-1}\right) = \langle\chi\rangle \times \langle\varphi\rangle \cong C_4 \times C_2$, where $\chi(x) = \chi(y) = 1$, $\chi(z) = -1$, $\chi(t) = i$, $\varphi(x) = \varphi(y) = -1$, $\varphi(z) = 1$, $\varphi(t) = i$. There is a degree 2 irreducible representation defined by
$\pi(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\pi(y) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $\pi(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\pi(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
with the property $\pi^2 = \pi \bullet \pi = 1 + \chi^2 + \varphi + \chi^2\varphi$.

3.3. Case of $G = D_8$.

Let $G = D_8 = \langle x, y \mid x^4 = y^2 = 1, yx = x^{-1}y\rangle$. Let $\{e_{pq}\}_{p=0,1,2,3;q=0,1}$ be the basis of K, dual to the basis $\{x^p y^q\}_{p=0,1,2,3;q=0,1}$ of $K^* = kD_8$. Then

$$\Delta_H(e_{pq}) = \Delta_K(e_{pq}) = \sum_{\substack{p_1+p_2+2q_1p_2 \equiv p \mod 4 \\ q_1+q_2 \equiv q \mod 2}} e_{p_1q_1} \otimes e_{p_2q_2}$$

and it is easy to check that elements

$$X = \sum_{pq} (-1)^p e_{pq}$$
$$Y = \sum_{pq} (-1)^q e_{pq}$$

are grouplike of order 2. For $\bar{t} = 1\#t$

$$\Delta_H(\bar{t}) = \theta(t)\, \bar{t} \otimes \bar{t}$$

Dualizing (3.1) we get another extension

$$F^* \overset{\pi^*}{\hookrightarrow} H^* \overset{i^*}{\twoheadrightarrow} K^*$$

and as in [11, 2.4], [12, 2.11] or [15, 2.1], since k is algebraically closed, there exist units $\overline{x}$ and $\overline{y} \in H^*$, such that $\overline{x}^4 = \overline{y}^2 = 1_{H^*}$, $\left\langle e_{pq}, \overline{x}^i\overline{y}^j \right\rangle = \delta_{ip}\delta_{jq}$ and $\alpha = \overline{yx}^2\overline{yx}^2 \in F^* = k\{e_0, e_1\}$, where $\{e_r\}$ is a dual basis of $\{t^r\}$, $r = 0, 1$. The right action $\rho^* : F^* \otimes K^* \to F^*$ is trivial, thus F^* lies in the center of H^*.

$$\overline{x}^2 = \overline{y}^2\overline{x}^2 = \overline{yx}^2\overline{y}\alpha = \overline{x}^2\overline{y}\alpha\overline{y}\alpha = \overline{x}^2\overline{y}^2\alpha^2 = \overline{x}^2\alpha^2$$

Thus $\alpha^2 = 1$.

Consider $\beta = \overline{yxyx} \in F^* = k\{e_0, e_1\}$. $\varepsilon(\beta) = \varepsilon\left(\overline{y}^{-1}\overline{x}^{-1}\overline{yx}\right) = 1$ and therefore $\beta = e_0 + \xi e_1$. Moreover, $\overline{yxyx}^{-1} = \beta\overline{x}^2$ and

$$\overline{x} = \overline{y}^2\overline{x} = \overline{y}\beta\overline{x}^2\overline{xy} = \overline{y}\beta\overline{x}^2\overline{y}\beta\overline{x}^2\overline{x} = \overline{yx}^2\overline{yx}^3\beta^2 = \overline{y}^2\alpha\overline{x}^2\overline{x}^3\beta^2 = \overline{x}\alpha\beta^2$$

Thus $\beta^2 = \alpha^{-1} = \alpha$, implying $\beta^4 = 1$ and $\xi = \pm 1$ or $\pm i$.

$$\begin{aligned}\left\langle \Delta_H\left(\overline{t}\right), \overline{x}^i\overline{y}^j e_k \otimes \overline{x}^p\overline{y}^q e_r \right\rangle &= \left\langle \overline{t}, \overline{x}^i\overline{y}^j e_k \overline{x}^p\overline{y}^q e_r \right\rangle = \delta_{kr}\left\langle \overline{t}, \overline{x}^{i+p}\beta^{jp}\overline{x}^{2jp}\overline{y}^{j+q} e_k \right\rangle \\ &= \delta_{kr}\left\langle \overline{t}, \overline{x}^{i+p+2jp}\overline{y}^{j+q}\beta^{jp} e_k \right\rangle = \xi^{jp}\delta_{k1}\delta_{r1}\end{aligned}$$

On the other hand

$$\left\langle \Delta_H\left(\overline{t}\right), \overline{x}^i\overline{y}^j e_k \otimes \overline{x}^p\overline{y}^q e_r \right\rangle = \left\langle \theta(t)\,\overline{t} \otimes \overline{t}, \overline{x}^i\overline{y}^j e_k \otimes \overline{x}^p\overline{y}^q e_r \right\rangle = \left\langle \theta(t), \overline{x}^i\overline{y}^j \otimes \overline{x}^p\overline{y}^q \right\rangle \delta_{k1}\delta_{r1}$$

Therefore

$$\theta(t) = \sum_{ijpq} \xi^{jp} e_{ij} \otimes e_{pq}$$

It is easy to check that if $\xi = \pm i$ then $1, X, Y$ and XY are the only grouplikes of H.

Write $v = \sigma(t,t) = \sum c_{i,j}e_{i,j}$ then $c_{0,0} = \varepsilon(v) = 1$ and $c_{i,j} \neq 0$, since v is a unit and

$$\Delta_H\left(\overline{t}^2\right) = \Delta_H(v) = \Delta_K\left(\sum c_{i,j}e_{i,j}\right) = \sum c_{p+r+2rq,q+s}e_{p,q} \otimes e_{r,s}$$

On the other hand, if we write

$$t \rightharpoonup e_{p,q} = e_{\alpha_1(p,q),\alpha_2(p,q)}$$

$$\begin{aligned}\Delta\left(\overline{t}\right)\Delta\left(\overline{t}\right) &= \sum_{pqrs} \xi^{qr} e_{p,q}\overline{t} \otimes e_{r,s}\overline{t} \sum_{pqrs} \xi^{qr} e_{p,q}\overline{t} \otimes e_{r,s}\overline{t} \\ &= \sum_{pqrs} \xi^{qr} e_{p,q} \otimes e_{r,s} \sum_{pqrs} \xi^{qr} e_{\alpha_1(p,q),\alpha_2(p,q)}\overline{t}^2 \otimes e_{\alpha_1(r,s),\alpha_2(r,s)}\overline{t}^2 \\ &= \sum_{pqrs} \xi^{qr} e_{p,q} \otimes e_{r,s} \sum_{pqrs} \xi^{\alpha_2(p,q)\alpha_1(r,s)} e_{p,q}\overline{t}^2 \otimes e_{r,s}\overline{t}^2 \\ &= \sum_{pqrs} \xi^{qr+\alpha_2(p,q)\alpha_1(r,s)} c_{pq}c_{rs} e_{p,q} \otimes e_{r,s}\end{aligned}$$

Thus for H to be a bialgebra we should have

$$c_{p+r+2rq,q+s} = \xi^{qr+\alpha_2(p,q)\alpha_1(r,s)} c_{pq}c_{rs} \tag{3.16}$$

Action by t is a Hopf algebra map and therefore it induces a group automorphism $f_t : G \to G$ defined by $\langle e_{p,q}, f_t(g) \rangle = \langle t \rightharpoonup e_{p,q}, g \rangle$, which has order 2.

$f_t(x) = x$ or x^{-1} since order of x is 4. If $f_t(x) = x$ then in order for f_t to be of order 2 we should have $f_t(y) = x^2y$. If $f_t(x) = x^{-1}$ then renaming generators we are down to two choices for $f_t(y)$, namely $f_t(y) = y$ or xy. Thus there are three possibilities for the action of t; we consider them below:

Case A). The action is given by $t \rightharpoonup e_{p,q} = e_{p+2q,q}$, corresponding to

$$\begin{aligned} f_t(x) &= x \\ f_t(y) &= x^2y \end{aligned}$$

Then X and Y are central grouplikes of H. Write $v = \sigma(t,t) = \sum c_{p,q}e_{p,q}$. By (3.2) and (3.16)

$$c_{p,q} = c_{p+2q,q} \tag{3.17}$$

$$c_{p+r+2rq,q+s} = \xi^{qr+q(r+2s)}c_{p,q}c_{r,s}\xi^{2q(r+s)}c_{p,q}c_{r,s} \tag{3.18}$$

Conditions (3.17) and (3.18) imply that

$$\begin{aligned} \xi^2c_{0,1}c_{1,0} &= c_{3,1} = c_{1,1} = c_{1,0}c_{0,1} \\ \xi^2c_{0,1}c_{0,1} &= c_{0,0} = 1 \\ c_{1,0}c_{1,0} &= c_{2,0} \\ c_{2,0}c_{0,1} &= c_{2,1} = c_{0,1} \end{aligned}$$

Thus $\xi^2 = 1$, $c_{2,0} = 1$ and $c_{1,0}^2 = c_{0,1}^2 = 1$. Therefore $c_{1,0} = (-1)^k$ and $c_{0,1} = (-1)^l$ for $k,l = 0,1$ and

(3.19)

$$\sigma(t,t) = \sum(-1)^{kp}(-1)^{lq}e_{p,q} = \sum(-1)^{kp}e_{p,q}\sum(-1)^{ls}e_{r,s} = X^kY^l, \qquad k,l = 0,1$$

If $\xi = 1$ then $\bar{t}$ is a grouplike of H, if $\xi = -1$ then $\sum i^pe_{p,q}\bar{t}$ is a grouplike of H. In both cases $\mathbf{G}(H)$ is abelian of order 8 and H was described in Section 3.1 or Section 3.2.

Case B). The action is given by $t \rightharpoonup e_{p,q} = e_{-p,q}$, corresponding to

$$\begin{aligned} f_t(x) &= x^{-1} \\ f_t(y) &= y \end{aligned}$$

Then X and Y are central grouplikes of H. Write $v = \sigma(t,t) = \sum c_{p,q}e_{p,q}$. By (3.2) and (3.16)

$$c_{p,q} = c_{-p,q} \tag{3.20}$$

$$c_{p+r+2rq,q+s} = \xi^{qr-qr}c_{p,q}c_{r,s} = c_{p,q}c_{r,s} \tag{3.21}$$

Conditions (3.20) and (3.21) imply that $c_{1,0} = (-1)^k$ and $c_{0,1} = (-1)^l$ for $k,l = 0,1$ and

(3.22)

$$\sigma(t,t) = \sum(-1)^{kp}(-1)^{lq}e_{p,q} = \sum(-1)^{kp}e_{p,q}\sum(-1)^{ls}e_{r,s} = X^kY^l, \qquad k,l = 0,1$$

If $\xi = 1$ then $\bar{t}$ is a grouplike of H, if $\xi = -1$ then $\sum i^p e_{p,q}\bar{t}$ is a grouplike of H. In both cases $\mathbf{G}(H)$ is abelian of order 8 and H was described in Section 3.1 or Section 3.2. So now we will consider only $\xi = \pm i$.

For $k, l = 0, 1$ let H_{ξ, X^kY^l} be the Hopf algebras with the structures described above with cocycles $\sigma_{k,l}(t,t) = X^kY^l$. Define

$$\begin{array}{rcll} f : H_{\xi,X^kY^l} & \to & H_{-\xi,X^kY^l} & \text{by} \\ f(e_{r,s}) & = & e_{r,s} & \\ f(\bar{t}) & = & \sum i^p e_{p,q}\bar{t} & \end{array}$$

and extend it multiplicatively to $f\left(e_{r,s}\bar{t}\right)$. Then

$$\begin{aligned} f(\bar{t}) f(\bar{t}) &= \sum i^p e_{p,q}\bar{t} \sum i^p e_{p,q}\bar{t} = \sum i^p e_{p,q} \sum i^{-p} e_{p,q}\bar{t}^2 = \bar{t}^2 = f\left(\bar{t}^2\right) \\ f\left(\bar{t}e_{r,s}\right) &= f\left(e_{-r,s}\bar{t}\right) = e_{-r,s}\sum i^p e_{p,q}\bar{t} = \sum i^p e_{p,q}\bar{t}e_{r,s} = f(\bar{t}) f(e_{r,s}) \\ \Delta(f(\bar{t})) &= \Delta\left(\sum i^p e_{p,q}\bar{t}\right) = \Delta\left(\sum i^p e_{p,q}\right)\Delta(\bar{t}) = \\ &= \left(\sum i^{p_1+p_2+2q_1p_2} e_{p_1,q_1}\otimes e_{p_2,q_2}\right)\left(\sum \xi^{s_1r_2} e_{r_1,s_1}\bar{t}\otimes e_{r_2,s_2}\bar{t}\right) \\ &= \left(\sum i^{p_1+p_2+2q_1p_2}\xi^{q_1p_2} e_{p_1,q_1}\bar{t}\otimes e_{p_2,q_2}\bar{t}\right) \\ &= \sum i^{p_1+p_2}(-\xi)^{q_1p_2} e_{p_1,q_1}\bar{t}\otimes e_{p_2,q_2}\bar{t} \\ &= \sum (-\xi)^{q_1p_2} e_{p_1,q_1} f(\bar{t}) \otimes e_{p_2,q_2} f(\bar{t}) \\ &= (f\otimes f)\left(\sum (-\xi)^{q_1p_2} e_{p_1,q_1}\bar{t}\otimes e_{p_2,q_2}\bar{t}\right) = (f\otimes f)\Delta(\bar{t}) \end{aligned}$$

and such an f is a Hopf algebra isomorphism between H_{ξ,X^kY^l} and $H_{-\xi,X^kY^l}$. Thus we may assume that $\xi = i$ and write $H_{i,X^kY^l} = H_{X^kY^l}$. Define

$$\begin{array}{rcll} f' : H_{X^kY} & \to & H_{X^k} & \text{by} \\ f'(e_{r,s}) & = & e_{r+2s,s} & \\ f'(\bar{t}) & = & \sum i^q e_{p,q}\bar{t} = \left(\dfrac{1+i}{2}1 + \dfrac{1-i}{2}Y\right)\bar{t} & \end{array}$$

and extend it multiplicatively to $f'\left(e_{r,s}\bar{t}\right)$. Note that restriction $f'|_{(kD_8)^*}$ corresponds to the group automorphism f_t described in Case A) and $f'(X) = X, f'(Y) = Y$. Then

$$\begin{aligned} f'(\bar{t}) f'(\bar{t}) &= \sum i^q e_{p,q}\bar{t} \sum i^q e_{p,q}\bar{t} = \sum i^q e_{p,q} \sum i^q e_{-p,q}\bar{t}^2 \\ &= \sum (-1)^q e_{p,q} X^k = YX^k = f'\left(YX^k\right) = f'\left(\bar{t}^2\right) \\ f'\left(\bar{t}e_{r,s}\right) &= f'\left(e_{-r,s}\bar{t}\right) = e_{-r+2s,s}\sum i^q e_{p,q}\bar{t} = \sum i^q e_{p,q}\bar{t}e_{r-2s,s} = f'(\bar{t}) f'(e_{r,s}) \end{aligned}$$

There are at most two nonisomorphic Hopf algebras of this kind:

1. $H_{B:1}$ with trivial cocycle and $\mathbf{G}\left(H_{B:1}^{*}\right)=\langle\chi,\varphi\rangle\cong D_8$, where $\varphi(e_{r,s})=\delta_{r,2}\delta_{s,0}$, $\varphi\left(\overline{t}\right)=1$, $\chi(e_{r,s})=\delta_{r,2}\delta_{s,1}$, $\chi(\overline{t})=-1$. There are two degree 2 irreducible representations defined by
$$\pi_1\left(e_{1,0}\right)=\begin{pmatrix}1&0\\0&0\end{pmatrix}\quad \pi_1\left(e_{3,0}\right)=\begin{pmatrix}0&0\\0&1\end{pmatrix}\quad \pi_1\left(\overline{t}\right)=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$
$$\pi_2\left(e_{1,1}\right)=\begin{pmatrix}1&0\\0&0\end{pmatrix}\quad \pi_2\left(e_{3,1}\right)=\begin{pmatrix}0&0\\0&1\end{pmatrix}\quad \pi_2\left(\overline{t}\right)=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$
with the property $\pi_k^2=\pi_k\bullet\pi_k=1+\chi^2+\varphi+\chi^2\varphi$.
2. $H_{B:X}$ with the cocycle defined by $\sigma_X\left(t,t\right)=X$ and $\mathbf{G}\left(H_{B:X}^{*}\right)=\langle\chi,\varphi\rangle\cong D_8$, where $\chi(e_{r,s})=\delta_{r,2}\delta_{s,1}$, $\chi(\overline{t})=-1$, $\varphi(e_{r,s})=\delta_{r,2}\delta_{s,0}$, $\varphi\left(\overline{t}\right)=1$. There are two degree 2 irreducible representations defined by
$$\pi_1\left(e_{1,0}\right)=\begin{pmatrix}1&0\\0&0\end{pmatrix}\quad \pi_1\left(e_{3,0}\right)=\begin{pmatrix}0&0\\0&1\end{pmatrix}\quad \pi_1\left(\overline{t}\right)=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$
$$\pi_2\left(e_{1,1}\right)=\begin{pmatrix}1&0\\0&0\end{pmatrix}\quad \pi_2\left(e_{3,1}\right)=\begin{pmatrix}0&0\\0&1\end{pmatrix}\quad \pi_2\left(\overline{t}\right)=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$
with the property $\pi_k^2=\pi_k\bullet\pi_k=1+\chi^2+\varphi+\chi^2\varphi$.

Case C). The action is given by $t\rightharpoonup e_{p,q}=e_{-p+q,q}$, corresponding to
$$\begin{aligned}f_t\left(x\right)&=x^{-1}\\ f_t\left(y\right)&=xy\end{aligned}$$
Then Y is a central grouplike of H. Write $v=\sigma\left(t,t\right)=\sum c_{p,q}e_{p,q}$. By (3.2) and (3.16)
$$c_{p,q}=c_{-p+q,q}\tag{3.23}$$
$$c_{p+r+2rq,q+s}=\xi^{qr+q(-r+s)}c_{p,q}c_{r,s}=\xi^{qs}c_{p,q}c_{r,s}\tag{3.24}$$
Conditions (3.23) and (3.24) imply that
$$\begin{aligned}c_{0,1}&=c_{1,1}\\ c_{2,1}&=c_{3,1}\\ c_{1,0}&=c_{3,0}\\ c_{1,0}c_{0,1}&=c_{1,1}=c_{0,1}\\ c_{0,1}c_{1,0}&=c_{3,1}\\ \xi c_{0,1}c_{0,1}&=c_{0,0}=1\end{aligned}$$
Thus $c_{1,0}=1$ and $c_{0,1}=\omega^k$, where ω is a primitive 8-th root of 1 and $\xi=\omega^{-2k}$. Therefore
$$\sigma_k\left(t,t\right)=\sum\omega^{kq}e_{p,q}=\frac{1+Y}{2}+\frac{\omega^k\left(1-Y\right)}{2}\ ,\qquad k=0,\ldots,7$$
For $k=0,\ldots,7$ let H_k be the Hopf algebra with the structure described above with cocycle $\sigma_k\left(t,t\right)$. Define
$$\begin{aligned}f:H_{k+2}&\to H_k\qquad\text{by}\\ f\left(e_{p,q}\right)&=e_{p,q}\end{aligned}$$

$$f\left(\bar{t}\right) = \sum_{p,q} i^p e_{p,q}\bar{t}$$

and extend it multiplicatively to $f\left(e_{p,q}\bar{t}\right)$. Then

$$\begin{aligned}
f\left(\bar{t}\right)f\left(\bar{t}\right) &= \sum i^p e_{p,q}\bar{t}\sum i^p e_{p,q}\bar{t} = \sum i^p e_{p,q}\sum i^p e_{-p+q,q}\bar{t}^2 \\
&= \sum i^p i^{-p+q} e_{p,q}\sigma_k\left(t,t\right) = \sum i^q e_{p,q}\sum \omega^{kq}e_{p,q} \\
&= \sum \omega^{kq+2q}e_{p,q} = \sigma_{k+2}\left(t,t\right) = f\left(\sigma_{k+2}\left(t,t\right)\right) = f\left(\bar{t}^2\right) \\
f\left(\bar{t}e_{p,q}\right) &= f\left(e_{-p+q,q}\bar{t}\right) = e_{-p+q,q}\sum i^p e_{p,q}\bar{t} = \sum i^p e_{p,q}\bar{t}e_{p,q} = f\left(\bar{t}\right)f\left(e_{p,q}\right) \\
\Delta\left(f\left(\bar{t}\right)\right) &= \Delta\left(\sum i^p e_{p,q}\bar{t}\right) = \Delta\left(\sum i^p e_{p,q}\right)\Delta\left(\bar{t}\right) = \\
&= \left(\sum i^{p_1+p_2+2q_1p_2}e_{p_1,q_1}\otimes e_{p_2,q_2}\right)\left(\sum \omega^{-2ks_1r_2}e_{r_1,s_1}\bar{t}\otimes e_{r_2,s_2}\bar{t}\right) \\
&= \left(\sum i^{p_1+p_2+2q_1p_2}\omega^{-2kq_1p_2}e_{p_1,q_1}\bar{t}\otimes e_{p_2,q_2}\bar{t}\right) \\
&= \sum i^{p_1+p_2}\omega^{-2(k+2)q_1p_2}e_{p_1,q_1}\bar{t}\otimes e_{p_2,q_2}\bar{t} \\
&= \sum \omega^{-2(k+2)q_1p_2}e_{p_1,q_1}f\left(\bar{t}\right)\otimes e_{p_2,q_2}f\left(\bar{t}\right) \\
&= \left(f\otimes f\right)\left(\sum \omega^{-2(k+2)q_1p_2}e_{p_1,q_1}\bar{t}\otimes e_{p_2,q_2}\bar{t}\right) = \left(f\otimes f\right)\Delta\left(\bar{t}\right)
\end{aligned}$$

and such a f is a Hopf algebra isomorphism between H_k and H_{k+2}. Thus there are exactly two nonisomorphic Hopf algebras of this type:

1. $H_{C:1} = H_0$ with a trivial cocycle and $\xi = 1$. Then $\mathbf{G}\left(H_{C:1}\right) = \left\langle X\bar{t}, X\right\rangle \cong D_8$ and $\mathbf{G}\left(H^*_{C:1}\right) = \langle\chi\rangle\times\langle\varphi\rangle \cong C_2\times C_2$, where $\chi(e_{p,q}) = \delta_{p,2}\delta_{q,0}$, $\chi(\bar{t}) = 1$, $\varphi(e_{p,q}) = \delta_{p,0}\delta_{q,0}$, $\varphi\left(\bar{t}\right) = -1$. There are three degree 2 irreducible representations defined by
$$\pi_1\left(e_{0,1}\right) = \begin{pmatrix}1&0\\0&0\end{pmatrix}\quad \pi_1\left(e_{1,1}\right) = \begin{pmatrix}0&0\\0&1\end{pmatrix}\quad \pi_1\left(\bar{t}\right) = \begin{pmatrix}0&1\\1&0\end{pmatrix}$$
$$\pi_2\left(e_{1,0}\right) = \begin{pmatrix}1&0\\0&0\end{pmatrix}\quad \pi_2\left(e_{3,0}\right) = \begin{pmatrix}0&0\\0&1\end{pmatrix}\quad \pi_2\left(\bar{t}\right) = \begin{pmatrix}0&1\\1&0\end{pmatrix}$$
$$\pi_3\left(e_{2,1}\right) = \begin{pmatrix}1&0\\0&0\end{pmatrix}\quad \pi_3\left(e_{3,1}\right) = \begin{pmatrix}0&0\\0&1\end{pmatrix}\quad \pi_3\left(\bar{t}\right) = \begin{pmatrix}0&1\\1&0\end{pmatrix}$$
with the property $\pi_2^2 = \pi_2\bullet\pi_2 = 1+\chi+\varphi+\chi\varphi$, $\pi_1^2 = \pi_3^2 = 1+\varphi+\pi_2$.

2. $H_{C:\sigma_1}$ with cocycle σ_1 defined by $\sigma_1\left(t,t\right) = \sum\omega^q e_{p,q}$ and $\xi = \omega^{-2}$, where ω is a primitive 8-th root of 1. Then $\mathbf{G}\left(H_{C:\sigma_1}\right) = \langle X\rangle\times\langle Y\rangle \cong C_2\times C_2$ and $\mathbf{G}\left(H^*_{C:\sigma_1}\right) = \langle\chi\rangle\times\langle\varphi\rangle \cong C_2\times C_2$, where $\chi(e_{p,q}) = \delta_{p,2}\delta_{q,0}$, $\chi(\bar{t}) = 1$, $\varphi(e_{p,q}) = \delta_{p,0}\delta_{q,0}$, $\varphi\left(\bar{t}\right) = -1$. There are three degree 2 irreducible representations defined by

$$\pi_1(e_{0,1}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \pi_1(e_{1,1}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \pi_1(\overline{t}) = \begin{pmatrix} 0 & \sqrt{\omega} \\ \sqrt{\omega} & 0 \end{pmatrix}$$
$$\pi_2(e_{1,0}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \pi_2(e_{3,0}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \pi_2(\overline{t}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\pi_3(e_{2,1}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \pi_3(e_{3,1}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \pi_3(\overline{t}) = \begin{pmatrix} 0 & \sqrt{\omega} \\ \sqrt{\omega} & 0 \end{pmatrix}$$

with the property $\pi_2^2 = \pi_2 \bullet \pi_2 = 1 + \chi + \varphi + \chi\varphi$, $\pi_1^2 = \pi_3^2 = 1 + \varphi + \pi_2$.

3.4. **Case of $G = Q_8$.**
Let $G = Q_8 = \langle x, y \mid x^4 = 1, y^2 = x^2, yx = x^{-1}y \rangle$. Let $\{e_{pq}\}_{p=0,1,2,3;q=0,1}$ be the basis of K, dual to the basis $\{x^p y^q\}_{p=0,1,2,3;q=0,1}$ of $K^* = kQ_8$. Then

$$\Delta_H(e_{pq}) = \Delta_K(e_{pq}) = \sum_{\substack{p_1+p_2+2q_1(p_2+q_2)\equiv p \mod 4 \\ q_1+q_2\equiv q \mod 2}} e_{p_1q_1} \otimes e_{p_2q_2}$$

and it is easy to check that elements

$$\begin{aligned} X &= \sum_{pq} (-1)^p e_{pq} \\ Y &= \sum_{pq} (-1)^q e_{pq} \end{aligned}$$

are grouplike of order 2. For $\overline{t} = 1\#t$

$$\Delta_H(\overline{t}) = \theta(t)\,\overline{t} \otimes \overline{t}$$

Dualizing (3.1) we get another extension

$$F^* \overset{\pi^*}{\hookrightarrow} H^* \overset{i^*}{\twoheadrightarrow} K^*$$

and as in [11, 2.4], [12, 2.11] or [15, 2.1], since k is algebraically closed, there exist units $\overline{x}$ and $\overline{y} \in H^*$, such that such that $\overline{x}^4 = 1_{H^*}$, $\overline{y}^2 = \overline{x}^2$, $\left\langle e_{pq}, \overline{x}^i \overline{y}^j \right\rangle = \delta_{ip}\delta_{jq}$ and $\alpha = \overline{xyxy}^{-1} \in F^* = k\{e_0, e_1\}$, where $\{e_r\}$ is a dual basis of $\{t^r\}$, $r = 0,1$. $\varepsilon(\alpha) = \varepsilon(\overline{xyxy}^{-1}) = 1$ and therefore $\alpha = e_0 + \xi e_1$. The right action $\rho^* : F^* \otimes K^* \to F^*$ is trivial, thus F^* lies in the center of H^*. Moreover, $\overline{x}^2 = \overline{y}^2$ also lies in the center of H^*. Then

$$\begin{aligned} \overline{xyx}^{-1}\overline{y}^{-1} &= \overline{xyx}^3\overline{y}^{-1} = \overline{xyxy}^{-1}\overline{x}^2 = \alpha\overline{x}^2 \\ \overline{x}^3 &= \overline{xx}^2 = \overline{xy}^2 = \alpha\overline{x}^2\overline{yxy} = \alpha\overline{x}^2\overline{y}\alpha\overline{x}^2\overline{yx} = \alpha^2\overline{x}^4\overline{y}^2\overline{x} = \alpha^2\overline{x}^3 \end{aligned}$$

Thus $\alpha^2 = 1$ and $\xi = \pm 1$.

$$\begin{aligned} \left\langle \Delta_H(\overline{t}), \overline{x}^i\overline{y}^j e_k \otimes \overline{x}^p\overline{y}^q e_r \right\rangle &= \left\langle \overline{t}, \overline{x}^i\overline{y}^j e_k \overline{x}^p\overline{y}^q e_r \right\rangle = \delta_{kr} \left\langle \overline{t}, \overline{x}^{i+p} \left(\alpha\overline{x}^2\right)^{-jp} \overline{y}^j\overline{y}^q e_k \right\rangle \\ &= \delta_{kr} \left\langle \overline{t}, \overline{x}^{i+p+2jp+2jq}\overline{y}^{j+q-2jq}\alpha^{jp} e_k \right\rangle = \xi^{jp}\delta_{k1}\delta_{r1} \end{aligned}$$

On the other hand

$$\left\langle \Delta_H(\overline{t}), \overline{x}^i\overline{y}^j e_k \otimes \overline{x}^p\overline{y}^q e_r \right\rangle = \left\langle \theta(t)\,\overline{t} \otimes \overline{t}, \overline{x}^i\overline{y}^j e_k \otimes \overline{x}^p\overline{y}^q e_r \right\rangle = \left\langle \theta(t), \overline{x}^i\overline{y}^j \otimes \overline{x}^p\overline{y}^q \right\rangle \delta_{k1}\delta_{r1}$$

Therefore

$$\theta(t) = \sum_{ijpq} \xi^{jp} e_{ij} \otimes e_{pq}$$

If $\xi = 1$ then $\overline{t}$ is a grouplike of H, if $\xi = -1$ then $\sum i^{p+q^2} e_{p,q}\overline{t}$ is a grouplike of H. Thus $\mathbf{G}(H)$ has always order 8.

Write $v = \sigma(t,t) = \sum c_{i,j} e_{i,j}$ then $c_{0,0} = \varepsilon(v) = 1$ and $c_{i,j} \neq 0$, since v is a unit and

$$\Delta_H\left(\overline{t}^2\right) = \Delta_H(v) = \Delta_K\left(\sum c_{i,j}e_{i,j}\right) = \sum c_{p+r+2rq+2sq,q+s} e_{p,q} \otimes e_{r,s}$$

Action by t is a Hopf algebra map and therefore it induces a group automorphism $f_t : G \to G$ defined by $\langle e_{p,q}, f_t(g)\rangle = \langle t \rightharpoonup e_{p,q}, g\rangle$, which has order 2. Renaming generators we are down to two choices for f_t; we consider them below:

Case D). The action is given by $t \rightharpoonup e_{i,j} = e_{i+2j,j}$, corresponding to

$$\begin{aligned} f_t(x) &= x \\ f_t(y) &= x^2 y \end{aligned}$$

Then X and Y are central grouplikes of H. Thus $\mathbf{G}(H)$ is abelian of order 8 and H was described in Section 3.1 or Section 3.2.

Case E). The action is given by $t \rightharpoonup e_{i,j} = e_{-i+j,j}$, corresponding to

$$\begin{aligned} f_t(x) &= x^{-1} \\ f_t(y) &= xy \end{aligned}$$

Then Y is a central grouplike of H. Write $v = \sigma(t,t) = \sum c_{i,j}e_{i,j}$. By (3.2)

(3.25) $$c_{i,j} = c_{-i+j,j}$$

On the other hand, for H to be a bialgebra

$$\begin{aligned} \Delta_H\left(\overline{t}^2\right) &= \Delta_H\left(\overline{t}\right)\Delta_H\left(\overline{t}\right) = \left(\theta(t)\,\overline{t}\otimes\overline{t}\right)\left(\theta(t)\,\overline{t}\otimes\overline{t}\right) \\ &= \left(\sum_{pqrs} \xi^{rq} e_{pq} \otimes e_{rs}\right)\left(\sum_{pqrs} \xi^{rq}\,(t \rightharpoonup e_{pq}) \otimes (t \rightharpoonup e_{rs})\right)\sigma(t,t)\otimes\sigma(t,t) \\ &= \sum \xi^{rq}\xi^{(-r+s)q} c_{p,q}c_{r,s} e_{p,q}\otimes e_{r,s} = \sum \xi^{qs} c_{p,q}c_{r,s} e_{p,q} \otimes e_{r,s} \end{aligned}$$

Therefore

(3.26) $$c_{p+r+2(r+s)q,q+s} = \xi^{qs} c_{p,q} c_{r,s}$$

Conditions (3.25) and (3.26) imply that

$$\begin{aligned} c_{0,1} &= c_{1,1} \\ c_{2,1} &= c_{3,1} \\ c_{1,0} &= c_{3,0} \\ c_{1,0}c_{0,1} &= c_{1,1} = c_{0,1} \\ c_{0,1}c_{1,0} &= c_{3,1} \end{aligned}$$

$$\xi c_{0,1}c_{0,1} = c_{2,0} = c_{1,0}c_{1,0}$$

Thus $c_{1,0} = 1$ and $c_{0,1} = i^k$, where $\xi = i^{2k}$ and $k = 0, 1, 2, 3$. Therefore

$$\sigma_k(t,t) = \sum i^{kq} e_{p,q} = \frac{1+Y}{2} + \frac{i^k(1-Y)}{2}$$

Let H_k be the Hopf algebra with the structure described above with cocycle $\sigma_k(t,t)$. Define

$$\begin{aligned} f : H_k &\to H_{k+1} \qquad \text{by} \\ f(e_{p,q}) &= e_{p,q} \\ f(\bar{t}) &= \sum_{p,q} i^{p+q^2} e_{p,q}\bar{t} \end{aligned}$$

and extend it multiplicatively to $f(e_{p,q}\bar{t})$. Then

$$\begin{aligned} f(\bar{t})f(\bar{t}) &= \sum i^{p+q^2} e_{p,q}\bar{t} \sum i^{p+q^2} e_{p,q}\bar{t} = \sum i^{p+q^2} e_{p,q} \sum i^{p+q^2} e_{-p+q,q}\bar{t}^2 \\ &= \sum i^{p+q^2} i^{-p+q+q^2} e_{p,q}\sigma_{k+1}(t,t) = \sum i^{3q} e_{p,q} \sum i^{(k+1)q} e_{p,q} \\ &= \sum i^{(k+1)q+3q} e_{p,q} = \sum i^{kq} e_{p,q} = \sigma_k(t,t) = f(\sigma_k(t,t)) = f\left(\bar{t}^2\right) \\ f(\bar{t}e_{p,q}) &= f\left(e_{-p+q,q}\bar{t}\right) = e_{-p+q,q} \sum i^{p+q^2} e_{p,q}\bar{t} = \sum i^{p+q^2} e_{p,q}\bar{t}e_{p,q} = f(\bar{t}) f(e_{p,q}) \end{aligned}$$

$$\begin{aligned} \Delta f(\bar{t}) &= \Delta\left(\sum i^{p+q^2} e_{p,q}\bar{t}\right) = \Delta\left(\sum i^{p+q^2} e_{p,q}\right)\Delta(\bar{t}) = \\ &= \left(\sum i^{p_1+p_2+2q_1(p_2+q_2)+(q_1+q_2)^2} e_{p_1,q_1}\otimes e_{p_2,q_2}\right)\left(\sum i^{2(k+1)s_1r_2} e_{r_1,s_1}\bar{t}\otimes e_{r_2,s_2}\bar{t}\right) \\ &= \left(\sum i^{p_1+p_2+2q_1(p_2+q_2)+(q_1+q_2)^2+2(k+1)q_1p_2} e_{p_1,q_1}\bar{t}\otimes e_{p_2,q_2}\bar{t}\right) \\ &= \sum i^{p_1+p_2+q_1^2+q_2^2} i^{2kq_1p_2} e_{p_1,q_1}\bar{t}\otimes e_{p_2,q_2}\bar{t} \\ &= \sum i^{2kq_1p_2} e_{p_1,q_1} f(\bar{t}) \otimes e_{p_2,q_2} f(\bar{t}) \\ &= (f\otimes f)\left(\sum i^{2kq_1p_2} e_{p_1,q_1}\bar{t}\otimes e_{p_2,q_2}\bar{t}\right) = (f\otimes f)\Delta(\bar{t}) \end{aligned}$$

and such an f is a Hopf algebra isomorphism between H_k and H_{k+1}. Thus there is exactly one Hopf algebra of this type: $H_E = H_0$ with a trivial cocycle and $\xi = 1$. Then $\mathbf{G}(H_E) = \langle X\bar{t}, X\rangle \cong D_8$ and $\mathbf{G}(H_E^*) = \langle\chi\rangle \times \langle\varphi\rangle \cong C_2 \times C_2$, where $\chi(e_{p,q}) = \delta_{p,2}\delta_{q,0}$, $\chi(\bar{t}) = 1$, $\varphi(e_{p,q}) = \delta_{p,0}\delta_{q,0}$, $\varphi(\bar{t}) = -1$. There are three degree 2 irreducible representations defined by

$$\begin{aligned} &\pi_1(e_{0,1}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \pi_1(e_{1,1}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \pi_1(\bar{t}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &\pi_2(e_{1,0}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \pi_2(e_{3,0}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \pi_2(\bar{t}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &\pi_3(e_{2,1}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \pi_3(e_{3,1}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \pi_3(\bar{t}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

with the property $\pi_2^2 = \pi_2 \bullet \pi_2 = 1 + \chi + \varphi + \chi\varphi$, $\pi_1^2 = \pi_3^2 = \chi + \chi\varphi + \pi_2$.

3.5. **Summary.**

Proposition 3.1. *Let H be a nontrivial semisimple Hopf algebra of dimension* 16. *Then $\mathbf{G}(H)$ is abelian of order* 8 *if and only if $\mathbf{G}(H^*)$ is abelian of order* 8.

Proof. All nontrivial Hopf algebras with abelian groups of grouplikes were described in Sections 3.1 and 3.2 and their duals have abelian groups of grouplikes. □

Proposition 3.2. *There are exactly* 7 *nonisomorphic nontrivial semisimple Hopf algebras of dimension* 16 *with $\mathbf{G}(H) \cong C_4 \times C_2$.*

Proof. All nontrivial Hopf algebras with $\mathbf{G}(H) \cong C_4 \times C_2$ were described in Section 3.1. There are at most 7 nonisomorphic Hopf algebras with $\mathbf{G}(H) \cong C_4 \times C_2$, namely $H_{a:1}$, $H_{a:y}$, $H_{b:1}$, $H_{b:y}$, $H_{b:x^2y}$, $H_{c:\sigma_0}$, $H_{c:\sigma_1}$.

Assume f is a Hopf algebra isomorphism between Hopf algebras H_1 and H_2 with $\mathbf{G}(H_1) \cong \mathbf{G}(H_2) \cong C_4 \times C_2$. Then we get a group isomorphism

$$f\mid_{\mathbf{G}(H_1)}: \mathbf{G}(H_1) \to \mathbf{G}(H_2)$$

Write $\mathbf{G}(H_1) \cong \mathbf{G}(H_2) = \langle x \rangle \times \langle y \rangle$, where $|x| = 4$ and $|y| = 2$. Then the dual basis of $k\mathbf{G}(H_1) \cong k\mathbf{G}(H_2)$ is given by

$$e_{pq} = 1/8\left(1 + i^p x + i^{2p} x^2 + i^{3p} x^3\right)\left(1 + (-1)^q y\right), \qquad p = 0,1,2,3; q = 0,1$$

Write

$$\begin{aligned} f(e_{p,q}) &= e_{\alpha_1(p,q),\alpha_2(p,q)} \\ f^{-1}(e_{p,q}) &= e_{\beta_1(p,q),\beta_2(p,q)} \end{aligned}$$

where $\alpha_1(p,q), \beta_1(p,q) \in \{0,1,2,3\}$ and $\alpha_2(p,q), \beta_2(p,q) \in \{0,1\}$.

Write $\left\{e_{pq}\overline{t^r}\right\}_{p=0,1,2,3;q=0,1;r=0,1}$ and $\left\{e_{pq}\overline{T^r}\right\}_{p=0,1,2,3;q=0,1;r=0,1}$ for the bases of H_1 and H_2, respectively. Write

$$f(\overline{t}) = \sum_{p,q,r} \lambda_{p,q,r} e_{pq} \overline{T^r}$$

Then

$$\begin{aligned} \Delta f(\overline{t}) &= \Delta\left(\sum_{p,q,r} \lambda_{p,q,r} e_{pq}\overline{T^r}\right) = \sum_{p,q} \lambda_{p,q,0}\Delta(e_{pq}) + \sum_{p,q} \lambda_{p,q,1}\Delta(e_{pq})\Delta(\overline{T}) \\ &= \sum \lambda_{p_1+p_2,q_1+q_2,0} e_{p_1q_1} \otimes e_{p_2q_2} \\ &\quad + \left(\sum \lambda_{p_1+p_2,q_1+q_2,1} e_{p_1q_1} \otimes e_{p_2q_2}\right)\left(\sum (-1)^{bc} e_{ab}\overline{T} \otimes e_{cd}\overline{T}\right) \\ &= \sum \lambda_{p_1+p_2,q_1+q_2,0} e_{p_1q_1} \otimes e_{p_2q_2} + \sum (-1)^{p_2q_1} \lambda_{p_1+p_2,q_1+q_2,1} e_{p_1q_1}\overline{T} \otimes e_{p_2q_2}\overline{T} \end{aligned}$$

$$\begin{aligned} (f \otimes f)\Delta(\overline{t}) &= (f \otimes f)\left(\sum (-1)^{p_2q_1} e_{p_1q_1}\overline{T} \otimes e_{p_2q_2}\overline{T}\right) \\ &= \sum (-1)^{p_2q_1} f(e_{p_1q_1}) \sum_{p,q,r} \lambda_{p,q,r} e_{pq}\overline{T^r} \otimes f(e_{p_2q_2}) \sum_{p,q,r} \lambda_{p,q,r} e_{pq}\overline{T^r} \\ &= \sum (-1)^{\beta_1(p_2,q_2)\beta_2(p_1,q_1)} \lambda_{p_1,q_1,r_1}\lambda_{p_2,q_2,r_2} e_{p_1q_1}\overline{T^{r_1}} \otimes e_{p_2q_2}\overline{T^{r_2}} \end{aligned}$$

Since f is a coalgebra map,

$$\Delta f\left(\bar{t}\right)=\left(f\otimes f\right)\Delta\left(\bar{t}\right)$$

and therefore $\lambda_{p_1,q_1,0}\lambda_{p_2,q_2,1}=0$ for all $p_1,p_2\in\{0,1,2,3\}$, $q_1,q_2\in\{0,1\}$. Thus either $\lambda_{p,q,0}=0$ for all $p\in\{0,1,2,3\}$, $q\in\{0,1\}$ or $\lambda_{p,q,1}=0$ for all $p\in\{0,1,2,3\}$, $q\in\{0,1\}$. In the latter case $f\left(\bar{t}\right)=\sum\lambda_{p,q,0}e_{pq}\in k\mathbf{G}\left(H_2\right)$, which contradicts the bijectivity of f. Therefore $\lambda_{p,q,0}=0$ for all $p\in\{0,1,2,3\}$, $q\in\{0,1\}$. Write $\lambda_{p,q}=\lambda_{p,q,1}$. Then

$$f\left(\bar{t}\right)=\sum_{p,q}\lambda_{p,q}e_{pq}\overline{T}$$

and so, applying ε, also

$$\lambda_{0,0}=\varepsilon\left(\bar{t}\right)=1$$

Moreover, since $\sum\left(-1\right)^{p_2q_1}\lambda_{p_1+p_2,q_1+q_2}e_{p_1q_1}\overline{T}\otimes e_{p_2q_2}\overline{T}$
$=\sum\left(-1\right)^{\beta_1(p_2,q_2)\beta_2(p_1,q_1)}\lambda_{p_1,q_1}\lambda_{p_2,q_2}e_{p_1q_1}\overline{T}\otimes e_{p_2q_2}\overline{T}$ we get

$$\lambda_{p_1+p_2,q_1+q_2}=\left(-1\right)^{p_2q_1}\left(-1\right)^{\beta_1(p_2,q_2)\beta_2(p_1,q_1)}\lambda_{p_1,q_1}\lambda_{p_2,q_2}\tag{3.27}$$

for any $p_1,p_2\in\{0,1,2,3\}$, $q_1,q_2\in\{0,1\}$.

Let $u\in k\mathbf{G}\left(H_1\right)$. Then

$$\begin{aligned}f\left(t\rightharpoonup_1 u\right)f\left(\bar{t}\right) &= f\left(\left(t\rightharpoonup_1 u\right)\bar{t}\right)=f\left(\bar{t}u\right)=f\left(\bar{t}\right)f\left(u\right)=\sum\lambda_{p,q}e_{p,q}\overline{T}f\left(u\right)\\ &= \left(t\rightharpoonup_2 f\left(u\right)\right)\sum\lambda_{p,q}e_{p,q}\overline{T}=\left(t\rightharpoonup_2 f\left(u\right)\right)f\left(\bar{t}\right)\end{aligned}$$

Thus, since $\bar{t}$ is a unit ($\bar{t}^2=\sigma\left(t,t\right)$ is a unit),

$$f\left(t\rightharpoonup_1 u\right)=t\rightharpoonup_2 f\left(u\right)\tag{3.28}$$

Let's show that Hopf algebras from types H_a, H_b and H_c cannot be isomorphic to each other. $K_0\left(H_c\right)\ncong K_0\left(H_a\right)$ or $K_0\left(H_b\right)$, thus $H_c\ncong H_a$ or H_b. If $f:H_a\to H_b$ then by formula (3.28)

$$f\left(x\right)f\left(y\right)=f\left(xy\right)=f\left(t\rightharpoonup_1 x\right)=t\rightharpoonup_2 f\left(x\right)=f\left(x\right)^{-1}$$

and therefore $f\left(y\right)=f\left(x^2\right)$, which is impossible if f is an isomorphism.

$H_{b:1}\ncong H_{b:y}$ or $H_{b:x^2y}$ and $H_{c:\sigma_0}\ncong H_{c:\sigma_1}$ since their duals have nonisomorphic groups of grouplikes. Thus there are at least 5 nonisomorphic Hopf algebras with $\mathbf{G}\left(H\right)\cong C_4\times C_2$, namely $H_{a:1}$, $H_{b:1}$, $H_{b:y}$, $H_{c:\sigma_0}$ and $H_{c:\sigma_1}$.

Now we prove that $H_{a:1}\ncong H_{a:y}$ and $H_{b:y}\ncong H_{b:x^2y}$. If f is a Hopf algebra isomorphism as before, then since $f\mid_{\mathbf{G}(H_1)}$ is a group isomorphism $f\left(x\right)\in\{x,x^{-1},xy,x^{-1}y\}$, $f\left(y\right)\in\{y,x^2y\}$ and $f\left(x^2\right)=x^2$.

If $f\left(x\right)=x^{2k+1}y^l$ and $f\left(y\right)=y$, where $k,l=0,1$ then

$$f^{-1}\left(e_{p,q}\right)=f\left(e_{p,q}\right)=e_{(2k+1)p+2lq,q}$$

and by formula (3.27)

$$\lambda_{p_1+p_2,q_1+q_2}=\left(-1\right)^{p_2q_1}\left(-1\right)^{((2k+1)p_2+2lq_2)q_1}\lambda_{p_1,q_1}\lambda_{p_2,q_2}=\lambda_{p_1,q_1}\lambda_{p_2,q_2}$$

If $f(x)=x^{2k+1}y^l$ and $f(y)=x^2y$, where $k,l=0,1$ then

$$\begin{aligned} f(e_{p,q}) &= e_{(2k+2l+1)p+2lq,p+q} \\ f^{-1}(e_{p,q}) &= e_{(2k+1)p+2lq,p+q} \end{aligned}$$

and by formula (3.27)

$$\lambda_{p_1+p_2,q_1+q_2}=(-1)^{p_2q_1}(-1)^{((2k+1)p_2+2lq_2)(q_1+p_1)}\lambda_{p_1,q_1}\lambda_{p_2,q_2}=(-1)^{p_1p_2}\lambda_{p_1,q_1}\lambda_{p_2,q_2}$$

Now assume f is a Hopf algebra isomorphism

$$f:H_{a:1}\to H_{a:y}$$

If $f(y)=x^2y$ then by formula (3.28)

$$f(x)\,y=t\rightharpoonup_2 f(x)=f(t\rightharpoonup_1 x)=f(xy)=f(x)\,f(y)=f(x)\,x^2y$$

that is $x^2=1$, which contradicts the fact that $|x|=4$. Thus $f(y)=y$ and therefore

$$\lambda_{p_1+p_2,q_1+q_2}=\lambda_{p_1,q_1}\lambda_{p_2,q_2}$$

and thus

$$\lambda_{1,0}^4=\lambda_{2,0}^2=\lambda_{0,1}^2=\lambda_{0,0}=1$$

Then

$$\begin{aligned} f(\overline{t})\,f(\overline{t}) &= \sum\lambda_{p,q}e_{pq}\overline{T}\sum\lambda_{r,s}e_{rs}\overline{T} \\ &= \sum\lambda_{p,q}e_{pq}\sum\lambda_{r,s}e_{r+2s,s}\overline{T}^2=\sum\lambda_{p,q}\lambda_{p+2q,q}e_{pq}\sigma_{H_{a:y}}(t,t) \\ &= \sum\lambda_{2(p+q),0}e_{pq}\sigma_{H_{a:y}}(t,t)=\sum\lambda_{2,0}^{p+q}e_{pq}\sigma_{H_{a:y}}(t,t) \end{aligned}$$

If $\lambda_{2,0}=1$, $f(\overline{t})\,f(\overline{t})=\sigma_{H_{a:y}}(t,t)=y\neq f\left(\overline{t}^2\right)=1$.
If $\lambda_{2,0}=-1$, $f(\overline{t})\,f(\overline{t})=x^2y\sigma_{H_{a:y}}(t,t)=x^2\neq f\left(\overline{t}^2\right)=1$.

Therefore, there is no Hopf algebra isomorphism between $H_{a:1}$ and $H_{a:y}$.

Now assume f is a Hopf algebra isomorphism

$$f:H_{b:y}\to H_{b:x^2y}$$

Then

$$\begin{aligned} f(\overline{t})\,f(\overline{t}) &= \sum\lambda_{p,q}e_{pq}\overline{T}\sum\lambda_{r,s}e_{rs}\overline{T} \\ &= \sum\lambda_{p,q}e_{pq}\sum\lambda_{r,s}e_{-r,s}\overline{T}^2=\sum\lambda_{p,q}\lambda_{-p,q}e_{pq}\sigma_{H_{b:x^2y}}(t,t) \end{aligned}$$

$f(y)=y$ is not possible, since then we have

$$\lambda_{p,q}\lambda_{-p,q}=\lambda_{0,0}=1$$

and thus

$$f(\overline{t})\,f(\overline{t})=\left(\sum e_{pq}\right)\sigma_{H_{b:x^2y}}(t,t)=\sigma_{H_{b:x^2y}}(t,t)=x^2y\neq y=f(y)=f\left(\overline{t}^2\right).$$

$f(y) = x^2 y$ is not possible, since then we get

$$\lambda_{p_1+p_2,q_1+q_2} = (-1)^{p_1 p_2} \lambda_{p_1,q_1} \lambda_{p_2,q_2}$$

so

$$\lambda_{p,q} \lambda_{-p,q} = (-1)^{p^2} \lambda_{0,0} = (-1)^{p^2}$$

and

$$f(\bar{t}) f(\bar{t}) = \left(\sum (-1)^{p^2} e_{pq}\right) \sigma_{H_{b:x^2y}}(t,t) = x^2 x^2 y = y = f\left(x^2 y\right) \neq f(y) = f\left(\bar{t}^2\right).$$

Therefore, there is no Hopf algebra isomorphism between $H_{b:y}$ and $H_{b:x^2y}$.

Thus there are exactly 7 nonisomorphic Hopf algebras with $\mathbf{G}(H) \cong C_4 \times C_2$, namely $H_{a:1}$, $H_{a:y}$, $H_{b:1}$, $H_{b:y}$, $H_{b:x^2y}$, $H_{c:\sigma_0}$ and $H_{c:\sigma_1}$. □

Proposition 3.3. *There are at least* 2 *and at most* 4 *nonisomorphic nontrivial semisimple Hopf algebras of dimension* 16 *with* $\mathbf{G}(H) \cong C_2 \times C_2 \times C_2$.

Proof. All nontrivial Hopf algebras with $\mathbf{G}(H) \cong C_2 \times C_2 \times C_2$ were described in Section 3.2. There are at most 4 nonisomorphic Hopf algebras with $\mathbf{G}(H) \cong C_2 \times C_2 \times C_2$, namely $H_{d:1,1}$, $H_{d:1,-1}$, $H_{d:-1,1}$ and $H_{d:-1,-1}$. At least two of them are not isomorphic, since $\mathbf{G}\left(H^*_{d:1,1}\right) \ncong \mathbf{G}\left(H^*_{d:1,-1}\right)$. □

Proposition 3.4. *There are exactly* 2 *nonisomorphic nontrivial semisimple Hopf algebras of dimension* 16 *with a commutative subHopfalgebra of dimension* 8 *and nonabelian* $\mathbf{G}(H^*)$. *In this case* $\mathbf{G}(H^*) \cong D_8$, $\mathbf{G}(H) \cong C_2 \times C_2$ *and* H^* *also has a commutative subHopfalgebra of dimension* 8.

Proof. All nontrivial Hopf algebras with a commutative subHopfalgebra of dimension 8 and nonabelian $\mathbf{G}(H^*)$ were described in Section 3.3, Case B). There are at most 2 of them, namely $H_{B:1}$ and $H_{B:X}$.

Let's compute all the possible 8-dimensional Hopf quotients of H_B. There is a one-to-one correspondence between hereditary subrings of $K_0(H)$ and Hopf quotients of H (see [22, Theorem 6] or [24, Proposition 3.11]). Thus H_B has 3 quotients of dimension 8 corresponding to the hereditary subrings $R_1 = \{a1 + b\varphi + c\chi^2 + d\chi^2\varphi + e\pi_1 \in K_0(H) : a,b,c,d,e \in \mathbb{Z}\}$, $R_2 = \{a1 + b\varphi + c\chi^2 + d\chi^2\varphi + e\pi_2 \in K_0(H) : a,b,c,d,e \in \mathbb{Z}\}$ and $R_3 = \{\sum_{i=0}^{4} \sum_{j=0}^{2} a_{i,j} \chi^i \varphi^j \in K_0(H) : a_{i,j} \in \mathbb{Z}\}$. They are obtained by factoring modulo normal ideals $(Y-1)H$, $(X-1)H$ and $(XY-1)H$, where X, Y and XY are central grouplikes of H. It is easy to see that $H/(Y-1)H$ is cocommutative (in fact, $H_{B:1}/(Y-1)H_{B:1} \cong kD_8$ and $H_{B:X}/(Y-1)H_{B:X} \cong kQ_8$), $H/(X-1)H$ is commutative (therefore $H/(X-1)H \cong (kD_8)^*$ since $\mathbf{G}(H^*) \cong D_8$) and $H/(XY-1)H$ is neither commutative nor cocommutative (therefore $H/(X-1)H \cong H_8$). Therefore we see that $H_{B:1} \ncong H_{B:X}$ since they have different sets of quotients. Both $H_{B:1}$ and $H_{B:X}$ have cocommutative Hopf quotients of dimension 8, kD_8 and kQ_8 respectively. Thus their duals were described in Section 3.4. In particular, $H_{B:1} \cong H^*_{C:1}$, $H_{B:X} \cong H^*_E$ and $\mathbf{G}(H^*_{C:1}) \cong \mathbf{G}(H^*_E) \cong C_2 \times C_2$. □

4. Nonabelian groups of order 16.

There are nine nonabelian groups of order 16 (see [2, 118]). The first four of them are of exponent 8, the last five of exponent 4 (we denote the quaternion group of order 8 by Q_8 and the quasiquaternion group of order 16 by Q_{16}):

1. $G_1 = \langle a, b : a^8 = b^2 = 1, ba = a^5 b\rangle$
   $\mathbf{G}\left((kG_1)^*\right) = \langle\chi\rangle \times \langle\varphi\rangle \cong C_4 \times C_2$, where $\chi(a) = i$, $\chi(b) = 1$, $\varphi(a) = 1$, $\varphi(b) = -1$. Degree 2 irreducible representations of G_1 are defined by
   $$\pi_1(a) = \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} \quad \pi_1(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
   $$\pi_2(a) = \begin{pmatrix} \omega^3 & 0 \\ 0 & -\omega^3 \end{pmatrix} \quad \pi_2(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
   where ω is a primitive 8^{th}-root of unity and $\pi_1^2 = \chi + \chi^3 + \chi\varphi + \chi^3\varphi = \pi_2^2$.
2. $G_2 = \langle a, b : a^8 = b^2 = 1, ba = a^3 b\rangle$
   $\mathbf{G}\left((kG_2)^*\right) = \langle\chi\rangle \times \langle\varphi\rangle \cong C_2 \times C_2$, where $\chi(a) = -1$, $\chi(b) = 1$, $\varphi(a) = 1$, $\varphi(b) = -1$. Degree 2 irreducible representations of G_2 are defined by
   $$\pi_1(a) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^3 \end{pmatrix} \quad \pi_2(a) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \pi_3(a) = \begin{pmatrix} \omega^5 & 0 \\ 0 & \omega^7 \end{pmatrix}$$
   $$\pi_1(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \pi_2(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \pi_3(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
   where ω is a primitive 8^{th}-root of unity, and representations satisfy the properties
   $\pi_1^2 = \chi + \chi\varphi + \pi_2 = \pi_3^2 \quad \chi \bullet \pi_1 = \pi_3 \quad \chi \bullet \pi_3 = \pi_1$
   $\pi_2^2 = 1 + \chi + \varphi + \chi\varphi \quad \varphi \bullet \pi_1 = \pi_1 \quad \varphi \bullet \pi_3 = \pi_3$
3. $G_3 = \langle a, b : a^8 = b^2 = 1, ba = a^{-1} b\rangle = D_{16}$, the dihedral group
   $\mathbf{G}\left((kG_3)^*\right) = \langle\chi\rangle \times \langle\varphi\rangle \cong C_2 \times C_2$, where $\chi(a) = -1$, $\chi(b) = 1$, $\varphi(a) = 1$, $\varphi(b) = -1$. Degree 2 irreducible representations of G_3 are defined by
   $$\pi_1(a) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^7 \end{pmatrix} \quad \pi_2(a) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \pi_3(a) = \begin{pmatrix} \omega^3 & 0 \\ 0 & \omega^5 \end{pmatrix}$$
   $$\pi_1(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \pi_2(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \pi_3(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
   where ω is a primitive 8^{th}-root of unity, and representations satisfy the properties
   $\pi_1^2 = 1 + \varphi + \pi_2 = \pi_3^2 \quad \chi \bullet \pi_1 = \pi_3 \quad \chi \bullet \pi_3 = \pi_1$
   $\pi_2^2 = 1 + \chi + \varphi + \chi\varphi \quad \varphi \bullet \pi_1 = \pi_1 \quad \varphi \bullet \pi_3 = \pi_3$
4. $G_4 = \langle a, b : a^8 = 1, b^2 = a^4, ba = a^{-1} b\rangle = Q_{16}$, the quasiquaternion group
   $\mathbf{G}\left((kG_4)^*\right) = \langle\chi\rangle \times \langle\varphi\rangle \cong C_2 \times C_2$, where $\chi(a) = -1$, $\chi(b) = 1$, $\varphi(a) = 1$, $\varphi(b) = -1$. Degree 2 irreducible representations of G_4 are defined by
   $$\pi_1(a) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^7 \end{pmatrix} \quad \pi_2(a) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \pi_3(a) = \begin{pmatrix} \omega^3 & 0 \\ 0 & \omega^5 \end{pmatrix}$$
   $$\pi_1(b) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \pi_2(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \pi_3(b) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
   where ω is a primitive 8^{th}-root of unity, and representations satisfy the properties
   $\pi_1^2 = 1 + \varphi + \pi_2 = \pi_3^2 \quad \chi \bullet \pi_1 = \pi_3 \quad \chi \bullet \pi_3 = \pi_1$
   $\pi_2^2 = 1 + \chi + \varphi + \chi\varphi \quad \varphi \bullet \pi_1 = \pi_1 \quad \varphi \bullet \pi_3 = \pi_3$

5. $G_5 = \langle a, b : a^4 = b^4 = 1, ba = a^{-1}b \rangle$
$\mathbf{G}\left((kG_5)^*\right) = \langle \chi \rangle \times \langle \varphi \rangle \cong C_4 \times C_2$, where $\chi(a) = 1$, $\chi(b) = i$, $\varphi(a) = -1$, $\varphi(b) = 1$. Degree 2 irreducible representations of G_5 are defined by
$$\pi_1(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \pi_1(b) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$
$$\pi_2(a) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \pi_2(b) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$
with the property $\pi_1^2 = 1 + \chi^2 + \varphi + \chi^2\varphi = \pi_2^2$.

6. $G_6 = \langle a, b, c : a^4 = b^2 = c^2 = 1, bab = ac \rangle$
$\mathbf{G}\left((kG_6)^*\right) = \langle \chi \rangle \times \langle \varphi \rangle \cong C_4 \times C_2$, where $\chi(a) = i$, $\chi(b) = \chi(c) = 1$, $\varphi(a) = \varphi(c) = 1$, $\varphi(b) = -1$. Degree 2 irreducible representations of G_6 are defined by
$$\pi_1(a) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \pi_1(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \pi_1(c) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\pi_2(a) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \pi_2(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \pi_2(c) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$
with the property $\pi_1^2 = 1 + \chi^2 + \varphi + \chi^2\varphi = \pi_2^2$.

7. $G_7 = \langle a, b, c : a^4 = b^2 = c^2 = 1, cbc = a^2b \rangle$
$\mathbf{G}\left((kG_7)^*\right) = \langle \chi \rangle \times \langle \varphi \rangle \times \langle \psi \rangle \cong C_2 \times C_2 \times C_2$, where $\chi(a) = -1$, $\chi(b) = \chi(c) = 1$, $\varphi(a) = \varphi(b) = -1$, $\varphi(c) = 1$, $\psi(a) = \psi(b) = 1$, $\psi(c) = -1$. Degree 2 irreducible representations of G_7 are defined by
$$\pi_1(a) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \pi_1(b) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \pi_1(c) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\pi_2(a) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \pi_2(b) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \pi_2(c) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
with the property $\pi_1^2 = \chi + \chi\varphi + \chi\psi + \chi\varphi\psi = \pi_2^2$.

8. $G_8 = \langle a, b, c : a^4 = b^2 = c^2 = 1, ba = a^{-1}b \rangle = D_8 \times C_2$
$\mathbf{G}\left((kG_8)^*\right) = \langle \chi \rangle \times \langle \varphi \rangle \times \langle \psi \rangle \cong C_2 \times C_2 \times C_2$, where $\chi(a) = \chi(b) = 1$, $\chi(c) = -1$, $\varphi(a) = -1$, $\varphi(b) = \varphi(c) = 1$, $\psi(a) = \psi(c) = 1$, $\psi(b) = -1$. Degree 2 irreducible representations of G_8 are defined by
$$\pi_1(a) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \pi_1(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \pi_1(c) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\pi_2(a) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \pi_2(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \pi_2(c) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$
with the property $\pi_1^2 = 1 + \varphi + \psi + \varphi\psi = \pi_2^2$.

9. $G_9 = \langle a, b, c : a^4 = c^2 = 1, b^2 = a^2, ba = a^{-1}b \rangle = Q_8 \times C_2$
$\mathbf{G}\left((kG_9)^*\right) = \langle \chi \rangle \times \langle \varphi \rangle \times \langle \psi \rangle \cong C_2 \times C_2 \times C_2$, where $\chi(a) = \chi(b) = 1$, $\chi(c) = -1$, $\varphi(a) = -1$, $\varphi(b) = \varphi(c) = 1$, $\psi(a) = \psi(c) = 1$, $\psi(b) = -1$. Degree 2 irreducible representations of G_9 are defined by
$$\pi_1(a) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \pi_1(b) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \pi_1(c) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\pi_2(a) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \pi_2(b) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \pi_2(c) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$
with the property $\pi_1^2 = 1 + \varphi + \psi + \varphi\psi = \pi_2^2$.

5. Computations in $K_0(H)$ in the case of $|\mathbf{G}(H^*)| = 8$.

In this case we have 8 one-dimensional irreducible representations $\chi_1 = 1_{K_0(H)}, \dots, \chi_8 \in \mathbf{G}(H^*)$ and two 2-dimensional ones π_1 and π_2. Then, since $\chi_i \bullet \chi_i^{-1} = 1_{K_0(H)}$, $\chi_i^* = \chi_i^{-1}$.

$\deg(\chi_i \bullet \pi_k) = 2$ thus there are two possibilities:

i) $\chi_i \bullet \pi_k = \chi_j + \chi_l$

ii) $\chi_i \bullet \pi_k = \pi_l$

Case i) cannot happen, since otherwise $\pi_k = \chi_i^{-1} \bullet \chi_j + \chi_i^{-1} \bullet \chi_l$ is not irreducible. Thus $\chi_i \bullet \pi_k = \pi_l$. Then it is impossible to have $\chi_i \bullet \pi_k = \pi_k$ and $\chi_i \bullet \pi_l = \pi_k$ for $k \neq l$, since otherwise $\pi_l = \chi_i^8 \bullet \pi_l = \chi_i^7 \bullet \pi_k = \pi_k$. Thus χ_i either fixes both π_1 and π_2 or interchanges them. It is easy to check that either all χ_i fix π_k, $k = 1, 2$, or half of χ_i fixes π_k and half of χ_i interchanges them.

Suppose $\chi_i \bullet \pi_k = \pi_k$, for $i = 1, \dots, 8$ and $k = 1, 2$. Then

$$1 = m(\pi_k, \chi_i \bullet \pi_k) = m(\chi_i^*, \pi_k \bullet \pi_k^*) = m\left(\chi_i^{-1}, \pi_k \bullet \pi_k^*\right)$$

Thus

$$\pi_k \bullet \pi_k^* = \sum_{i=1}^{8} \chi_i^{-1} = \sum_{i=1}^{8} \chi_i$$

but $\deg(\pi_k \bullet \pi_k^*) = 4$ and $\deg\left(\sum_{i=1}^{8} \chi_i\right) = 8$. Thus all χ_i cannot fix π_k.

Now let half of χ_i fix π_k and half of χ_i interchange them, say

$$\begin{aligned} \chi_i \bullet \pi_k &= \pi_k \text{ for } i \text{ odd} \\ \chi_i \bullet \pi_k &= \pi_l \text{ for } i \text{ even} \end{aligned}$$

if $k \neq l$. It is clear that χ_i and $\chi_i^* = \chi_i^{-1}$ fix or interchange π_k simultaneously. Then for $k \neq l$

$$\begin{aligned} 1 &= m(\pi_k, \chi_i \bullet \pi_k) = m(\chi_i^*, \pi_k \bullet \pi_k^*) = m\left(\chi_i^{-1}, \pi_k \bullet \pi_k^*\right) \text{ for } i \text{ odd} \\ 1 &= m(\pi_l, \chi_i \bullet \pi_k) = m(\chi_i^*, \pi_k \bullet \pi_l^*) = m\left(\chi_i^{-1}, \pi_k \bullet \pi_l^*\right) \text{ for } i \text{ even} \end{aligned}$$

and therefore

$$\begin{aligned} \pi_k \bullet \pi_k^* &= \chi_1 + \chi_3 + \chi_5 + \chi_7 \\ \pi_k \bullet \pi_l^* &= \chi_2 + \chi_4 + \chi_6 + \chi_8 \end{aligned}$$

There are two possibilities for the involution: either $\pi_1^* = \pi_1$ and $\pi_2^* = \pi_2$, or $\pi_1^* = \pi_2$ and $\pi_2^* = \pi_1$. It is easy to check that when $\mathbf{G}(H^*)$ is isomorphic to $C_2 \times C_2 \times C_2$ or Q_8 it does not matter which generators we choose to fix π_k and which to interchange them. In the case of $\mathbf{G}(H^*) \cong D_8$ or $C_4 \times C_2$ it matters and should give us two more nonisomorphic structures for $K_0(H)$ for each of them, but due to the results of the Section 3 we can see that in the case of $\mathbf{G}(H^*) \cong C_4 \times C_2$, π_k can be fixed only by elements of order 1 or 2 (since $(\pi_k)^2$ is either the sum of all elements of order 1 or 2, or the sum of all elements of order 4).

Now assume that $\mathbf{G}(H^*) = \langle \chi, \varphi : \chi^4 = 1, \varphi^2 = 1, \varphi\chi = \chi^{-1}\varphi \rangle \cong D_8$ or $\mathbf{G}(H^*) = \langle \chi, \varphi : \chi^4 = 1, \varphi^2 = \chi^2, \varphi\chi = \chi^{-1}\varphi \rangle \cong Q_8$. Then χ^2 is the only nontrivial central element of $\mathbf{G}(H^*)$. Since by [13, Theorem1] H^* has a nontrivial central grouplike, this grouplike should be equal to χ^2. Since χ^2 is a central grouplike of order 2, which fixes all π_k, by Proposition 2.2 H has a commutative subHopfalgebra of dimension 8. Therefore, it should have the same K_0-ring as one of the Hopf algebras described in Section 3. Thus the only possible K_0-ring structure corresponds to $\mathbf{G}(H^*) \cong D_8$ with $(\pi_k)^2 = 1 + \chi^2 + \varphi + \chi^2\varphi$ and $\pi_i^* = \pi_i$.

Now let's list all the possible ring structures of $K_0(H)$.

5.1. $\mathbf{G}(H^*) = \langle\chi\rangle \times \langle\varphi\rangle \times \langle\psi\rangle \cong C_2 \times C_2 \times C_2$ where $\pi_1^* = \pi_1$ and $\pi_2^* = \pi_2$ and
$\chi \bullet \pi_1 = \pi_2 = \pi_1 \bullet \chi \quad \psi \bullet \pi_1 = \pi_1 = \pi_1 \bullet \psi$
$\chi \bullet \pi_2 = \pi_1 = \pi_2 \bullet \chi \quad \psi \bullet \pi_2 = \pi_2 = \pi_2 \bullet \psi$
$\varphi \bullet \pi_1 = \pi_1 = \pi_1 \bullet \varphi \quad \pi_1^2 = 1 + \varphi + \psi + \varphi\psi = \pi_2^2$
$\varphi \bullet \pi_2 = \pi_2 = \pi_2 \bullet \varphi \quad \pi_1 \bullet \pi_2 = \chi + \chi\varphi + \chi\psi + \chi\varphi\psi = \pi_2 \bullet \pi_1$
Examples: $H_{b:1}$, $H_{d:1,1}$, $H_{d:-1,1}$, $k(D_8 \times C_2)$, $k(Q_8 \times C_2)$, $H_8 \otimes kC_2$.

5.2. $\mathbf{G}(H^*) = \langle\chi\rangle \times \langle\varphi\rangle \times \langle\psi\rangle \cong C_2 \times C_2 \times C_2$ where $\pi_1^* = \pi_2$ and $\pi_2^* = \pi_1$ and
$\chi \bullet \pi_1 = \pi_2 = \pi_1 \bullet \chi \quad \psi \bullet \pi_1 = \pi_1 = \pi_1 \bullet \psi$
$\chi \bullet \pi_2 = \pi_1 = \pi_2 \bullet \chi \quad \psi \bullet \pi_2 = \pi_2 = \pi_2 \bullet \psi$
$\varphi \bullet \pi_1 = \pi_1 = \pi_1 \bullet \varphi \quad \pi_1^2 = \chi + \chi\varphi + \chi\psi + \chi\varphi\psi = \pi_2^2$
$\varphi \bullet \pi_2 = \pi_2 = \pi_2 \bullet \varphi \quad \pi_1 \bullet \pi_2 = 1 + \varphi + \psi + \varphi\psi = \pi_2 \bullet \pi_1$
Examples: $H_{c:\sigma_1}$ and kG_7, where $G_7 = \langle a, b, c : a^4 = b^2 = c^2 = 1, cbc = a^2b\rangle$.

5.3. $\mathbf{G}(H^*) = \langle\chi\rangle \times \langle\varphi\rangle \cong C_4 \times C_2$ where $\pi_1^* = \pi_1$ and $\pi_2^* = \pi_2$ and
$\chi \bullet \pi_1 = \pi_2 = \pi_1 \bullet \chi \quad \varphi \bullet \pi_1 = \pi_1 = \pi_1 \bullet \varphi \quad \pi_1^2 = 1 + \chi^2 + \varphi + \chi^2\varphi = \pi_2^2$
$\chi \bullet \pi_2 = \pi_1 = \pi_2 \bullet \chi \quad \varphi \bullet \pi_2 = \pi_2 = \pi_2 \bullet \varphi \quad \pi_1 \bullet \pi_2 = \chi + \chi^3 + \chi\varphi + \chi^3\varphi = \pi_2 \bullet \pi_1$
Examples: $H_{a:1}$, $H_{a:y}$, $H_{b:y}$, $H_{b:x^2y}$, $H_{d:1,-1}$, $H_{d:-1,-1}$, kG_5 and kG_6, where $G_5 = \langle a, b : a^4 = b^4 = 1, b^{-1}ab = a^{-1}\rangle$ and $G_6 = \langle a, b, c : a^4 = b^2 = c^2 = 1, bab = ac\rangle$.

5.4. $\mathbf{G}(H^*) = \langle\chi\rangle \times \langle\varphi\rangle \cong C_4 \times C_2$ where $\pi_1^* = \pi_2$ and $\pi_2^* = \pi_1$ and
$\chi \bullet \pi_1 = \pi_2 = \pi_1 \bullet \chi \quad \varphi \bullet \pi_1 = \pi_1 = \pi_1 \bullet \varphi \quad \pi_1^2 = 1 + \chi^2 + \varphi + \chi^2\varphi = \pi_2^2$
$\chi \bullet \pi_2 = \pi_1 = \pi_2 \bullet \chi \quad \varphi \bullet \pi_2 = \pi_2 = \pi_2 \bullet \varphi \quad \pi_1 \bullet \pi_2 = \chi + \chi^3 + \chi\varphi + \chi^3\varphi = \pi_2 \bullet \pi_1$
Examples: $H_{c:\sigma_0}$, kG_1, where $G_1 = \langle a, b : a^8 = b^2 = 1, bab = a^5\rangle$.

5.5. $\mathbf{G}(H^*) = \langle\chi, \varphi : \chi^4 = 1, \varphi^2 = 1, \varphi\chi = \chi^{-1}\varphi\rangle \cong D_8$, where $\pi_1^* = \pi_1$ and $\pi_2^* = \pi_2$ and
$\chi \bullet \pi_1 = \pi_2 = \pi_1 \bullet \chi \quad \varphi \bullet \pi_1 = \pi_1 = \pi_1 \bullet \varphi \quad \pi_1^2 = 1 + \chi^2 + \varphi + \chi^2\varphi = \pi_2^2$
$\chi \bullet \pi_2 = \pi_1 = \pi_2 \bullet \chi \quad \varphi \bullet \pi_2 = \pi_2 = \pi_2 \bullet \varphi \quad \pi_1 \bullet \pi_2 = \chi + \chi^3 + \chi\varphi + \chi^3\varphi = \pi_2 \bullet \pi_1$
Examples: $H_{B:1}$, $H_{B:X}$ and $kQ_8\#^\alpha kC_2$.

Remark 5.1. Noncommutative $K_0(H)$ should have the structure 5.5.

6. Computations in $K_0(H)$ in the case of $|\mathbf{G}(H^*)| = 4$.

In this case by Theorem 1.1 $\mathbf{G}(H^*) \cong C_2 \times C_2$ and we have 4 one-dimensional irreducible representations $\chi_1 = 1_{K_0(H)}, \ldots, \chi_4 \in \mathbf{G}(H^*)$ and three two-dimensional ones π_1, π_2 and π_3. Then, since $\chi_i \bullet \chi_i = 1_{K_0(H)}$, $\quad \chi_i^* = \chi_i$. The involution is an antihomomorphism of $K_0(H)$ of order 2, thus it either fixes all π_k or interchanges two of them and fixes the third one. Assume that we always have $\pi_2^* = \pi_2$.

$\chi_i \bullet \pi_k \neq \chi_j + \chi_l$ as in the case of $|\mathbf{G}(H^*)| = 8$. Thus multiplication by χ_i permutes π_k. Since $o(\chi_i) = 1$ or 2 then each χ_i either fixes all π_k or interchanges two of them and fixes the third one. There are two possible cases:

i) $\chi_i \bullet \pi_k = \pi_k$ for $i = 1, \ldots, 4$ and $k = 1, 2, 3$. Then

$$\begin{aligned} m(\chi_i, \pi_k \bullet \pi_k^*) &= m(\pi_k, \chi_i \bullet \pi_k) = 1 \\ \pi_k \bullet \pi_k^* &= \sum_{i=1}^{4} \chi_i \quad \text{for } k = 1, 2, 3 \end{aligned} \tag{6.1}$$

By [13, Theorem 1], one of the χ_i is central of order 2 and therefore by Proposition 2.2, H has a commutative subHopfalgebra of order 8. Therefore, it should have the same K_0-ring as one of the Hopf algebras described in Section 3. But none of these K_0-rings satisfies (6.1). Therefore this case is not possible.

ii) $\chi_i \bullet \pi_k \neq \pi_k$ for some $i \in \{1, \ldots, 4\}$ and $k \in \{1, 2, 3\}$. Then, say,

$$\chi_1 \bullet \pi_k = \chi_3 \bullet \pi_k = \pi_k \text{ for } k = 1, 2, 3$$

but $\chi_2 \bullet \pi_k \neq \pi_k$, $\chi_4 \bullet \pi_k \neq \pi_k$ for some $k \in \{1, 2, 3\}$.

Assume that $\pi_1^* = \pi_3$, $\quad \pi_3^* = \pi_1$ and $\chi_2 \bullet \pi_2 \neq \pi_2$. Then

$$\begin{aligned} 1 &= m(\pi_2, \chi_i \bullet \pi_2) = m(\chi_i, \pi_2 \bullet \pi_2) \text{ for } i = 1, 3 \\ 0 &= m(\pi_2, \chi_i \bullet \pi_2) = m(\chi_i, \pi_2 \bullet \pi_2) \text{ for } i = 2, 4 \end{aligned}$$

Therefore

$$\pi_2 \bullet \pi_2^* = \chi_1 + \chi_3 + \pi_r = \pi_2 \bullet \pi_2$$

Since $(\pi_k \bullet \pi_k^*)^* = \pi_k \bullet \pi_k^*$, we get $(\pi_r)^* = \pi_r$ and and thus $\pi_r = \pi_2$, that is

$$\pi_2 \bullet \pi_2^* = \chi_1 + \chi_3 + \pi_2$$

Therefore $R = \{a\chi_1 + b\chi_3 + c\pi_2 \in K_0(H) : a, b, c \in \mathbb{Z}\}$ is a hereditary subring of $K_0(H)$ (see [24, Definition 3.10]). There is a one-to-one correspondence between hereditary subrings of $K_0(H)$ and Hopf quotients of H, that is between hereditary subrings of $K_0(H)$ and subHopfalgebras of H^* (see [22, Theorem 6] or [24, Proposition 3.11]). Thus H^* has a subHopfalgebra of dimension $1+1+4 = 6$, which contradicts Nichols-Zoeller Theorem [23].

Thus without loss of generality $\chi_2 \bullet \pi_2 = \pi_2$. Then

$$\begin{aligned} \chi_i \bullet \pi_2 &= \pi_2 \text{ for } i = 1, \ldots, 4 \\ \chi_i \bullet \pi_1 &= \pi_1 \text{ for } i = 1, 3 \\ \chi_i \bullet \pi_1 &= \pi_3 \text{ for } i = 2, 4 \\ \chi_i \bullet \pi_3 &= \pi_3 \text{ for } i = 1, 3 \\ \chi_i \bullet \pi_3 &= \pi_1 \text{ for } i = 2, 4 \end{aligned}$$

and therefore

$$\begin{aligned} 1 &= m(\pi_2, \chi_i \bullet \pi_2) = m(\chi_i, \pi_2 \bullet \pi_2) \text{ for } i = 1, \ldots, 4 \\ 0 &= m(\pi_2, \chi_i \bullet \pi_k) = m(\chi_i, \pi_k \bullet \pi_2^*) \text{ for } i = 1, \ldots, 4,\ k \neq 2 \\ 1 &= m(\pi_k, \chi_i \bullet \pi_k) = m(\chi_i, \pi_k \bullet \pi_k^*) \text{ for } i = 1, 3 \end{aligned}$$

$$\begin{aligned}
0 &= m(\pi_k, \chi_i \bullet \pi_k) = m(\chi_i, \pi_k \bullet \pi_k^*) \text{ for } i = 2,4 \\
0 &= m(\pi_3, \chi_i \bullet \pi_1) = m(\chi_i, \pi_1 \bullet \pi_3^*) \text{ for } i = 1,3 \\
1 &= m(\pi_3, \chi_i \bullet \pi_1) = m(\chi_i, \pi_1 \bullet \pi_3^*) \text{ for } i = 2,4
\end{aligned}$$

Thus we get

$$\begin{aligned}
\pi_2 \bullet \pi_2^* &= \sum_{i=1}^{4} \chi_i \\
\pi_1 \bullet \pi_1^* &= \chi_1 + \chi_3 + \pi_2 = \pi_3 \bullet \pi_3^* \\
\pi_1 \bullet \pi_3^* &= \chi_2 + \chi_4 + \pi_t \\
\pi_k \bullet \pi_2^* &= \alpha_1\pi_1 + \alpha_2\pi_2 + \alpha_3\pi_3 \text{ for } k \neq 2
\end{aligned}$$

and then

$$\begin{aligned}
1 &= m(\pi_2, \pi_k^* \bullet \pi_k) = m(\pi_k, \pi_k \bullet \pi_2^*) \text{ for } k \neq 2 \\
0 &= m(\pi_k, \pi_l^* \bullet \pi_l) = m(\pi_l, \pi_l \bullet \pi_k^*) \text{ for } k \neq 2
\end{aligned}$$

Therefore

$$\begin{aligned}
\pi_1 \bullet \pi_2^* &= \pi_1 + \pi_3 \\
\pi_3 \bullet \pi_2^* &= \pi_1 + \pi_3
\end{aligned}$$

and

$$1 = m(\pi_3, \pi_1 \bullet \pi_2^*) = m(\pi_2, \pi_3^* \bullet \pi_1) = m(\pi_2, \pi_1 \bullet \pi_3^*)$$

So, finally,

$$\pi_1 \bullet \pi_3^* = \chi_2 + \chi_4 + \pi_2$$

Now let's list all the possible ring structures of $K_0(H)$.

6.1. $\mathbf{G}(H^*) = \langle\chi\rangle \times \langle\varphi\rangle \cong C_2 \times C_2$ where $\pi_1^* = \pi_1$, $\pi_2^* = \pi_2$ and $\pi_3^* = \pi_3$ and
$\chi \bullet \pi_1 = \pi_3 = \pi_1 \bullet \chi$ $\quad \varphi \bullet \pi_1 = \pi_1 = \pi_1 \bullet \varphi$ $\quad \pi_1 \bullet \pi_2 = \pi_1 + \pi_3 = \pi_2 \bullet \pi_1$
$\chi \bullet \pi_2 = \pi_2 = \pi_2 \bullet \chi$ $\quad \varphi \bullet \pi_2 = \pi_2 = \pi_2 \bullet \varphi$ $\quad \pi_1 \bullet \pi_3 = \chi + \chi\varphi + \pi_2 = \pi_3 \bullet \pi_1$
$\chi \bullet \pi_3 = \pi_1 = \pi_3 \bullet \chi$ $\quad \varphi \bullet \pi_3 = \pi_3 = \pi_3 \bullet \varphi$ $\quad \pi_2 \bullet \pi_3 = \pi_1 + \pi_3 = \pi_3 \bullet \pi_2$
$\pi_1^2 = 1 + \varphi + \pi_2 = \pi_3^2$ $\quad \pi_2^2 = 1 + \chi + \varphi + \chi\varphi$
Examples: $H_{C:1}$, $H_{C:\sigma_1}$, kD_{16} and kQ_{16}.

6.2. $\mathbf{G}(H^*) = \langle\chi\rangle \times \langle\varphi\rangle \cong C_2 \times C_2$ where $\pi_1^* = \pi_3$, $\pi_2^* = \pi_2$ and $\pi_3^* = \pi_1$ and
$\chi \bullet \pi_1 = \pi_3 = \pi_1 \bullet \chi$ $\quad \varphi \bullet \pi_1 = \pi_1 = \pi_1 \bullet \varphi$ $\quad \pi_1 \bullet \pi_2 = \pi_1 + \pi_3 = \pi_2 \bullet \pi_1$
$\chi \bullet \pi_2 = \pi_2 = \pi_2 \bullet \chi$ $\quad \varphi \bullet \pi_2 = \pi_2 = \pi_2 \bullet \varphi$ $\quad \pi_1 \bullet \pi_3 = 1 + \varphi + \pi_2 = \pi_3 \bullet \pi_1$
$\chi \bullet \pi_3 = \pi_1 = \pi_3 \bullet \chi$ $\quad \varphi \bullet \pi_3 = \pi_3 = \pi_3 \bullet \varphi$ $\quad \pi_2 \bullet \pi_3 = \pi_1 + \pi_3 = \pi_3 \bullet \pi_2$
$\pi_1^2 = \chi + \chi\varphi + \pi_2 = \pi_3^2$ $\quad \pi_2^2 = 1 + \chi + \varphi + \chi\varphi$
Examples: H_E and kG_2, where $G_2 = \langle a, b : a^8 = b^2 = 1, bab = a^3\rangle$.

We can now prove **Theorem 1.3**:

Proof. In Sections 5 and 6 we have described all possible Grothendieck ring structures of non-commutative semisimple Hopf algebras of dimension 16 and there are exactly 7 of them. Only one of these K_0-rings is not commutative, namely $K_{5.5}$, which corresponds to nonabelian $\mathbf{G}(H^*) \cong D_8$. Therefore, by [24, Theorem 4.1] all Hopf algebras with non-commutative K_0-ring are twistings of each other with a 2-pseudo-cocycle. Moreover, by [24, 4.5], Hopf algebras with non-commutative K_0-rings are not twistings of group algebras.

If $\mathbf{G}(H^*)$ is abelian then there are 6 possibilities for the K_0-ring structure, all of which are commutative, namely $K_{5.1} = K_0(k(D_8 \times C_2))$, $K_{5.2} = K_0(kG_7)$, $K_{5.3} = K_0(kG_5)$, $K_{5.4} = K_0(kG_1)$, $K_{6.1} = K_0(kD_{16})$ and $K_{6.2} = K_0(kG_2)$. Thus by [24, Theorem 4.1] H is a twisting of one of these group algebras with a 2-pseudo-cocycle. Since H is semisimple, $K_0(H) \otimes_{\mathbb{Z}} k$ is also semisimple by [32, Lemma 2]. Therefore, if $K_0(H)$ is commutative, as algebras $K_0(H) \otimes_{\mathbb{Z}} k \cong k^{(10)}$ when $|\mathbf{G}(H^*)| = 8$ and $K_0(H) \otimes_{\mathbb{Z}} k \cong k^{(7)}$ when $|\mathbf{G}(H^*)| = 4$. If $K_0(H)$ is not commutative, that is $K_0(H) = K_{5.5}$, it is easy to see that $\dim Z(K_0(H)) = 7$ and thus $K_0(H) \otimes_{\mathbb{Z}} k \cong k^{(6)} \oplus M_2(k)$. □

7. Twistings of group algebras with a 2-cocycle.

All nonabelian groups G, considered in this section, have an abelian subgroup $F = \{1, c, b, cb\} \cong C_2 \times C_2$. $kF \cong (kF)^*$ thus we can identify $\delta_x \in kF$ with the elements of the dual basis. Now define $J \in kF \otimes kF$ as follows:

$$\begin{aligned} J &= \delta_1 \otimes \delta_1 + \delta_1 \otimes \delta_c + \delta_1 \otimes \delta_b + \delta_1 \otimes \delta_{cb} \\ &\quad + \delta_c \otimes \delta_1 + \delta_c \otimes \delta_c + i\delta_c \otimes \delta_b - i\delta_c \otimes \delta_{cb} \\ &\quad + \delta_b \otimes \delta_1 - i\delta_b \otimes \delta_c + \delta_b \otimes \delta_b + i\delta_b \otimes \delta_{cb} \\ &\quad + \delta_{cb} \otimes \delta_1 + i\delta_{cb} \otimes \delta_c - i\delta_{cb} \otimes \delta_b + \delta_{cb} \otimes \delta_{cb} \end{aligned} \tag{7.1}$$

where

$$\begin{aligned} \delta_1 &= \frac{1}{4}(1 + c + b + cb) \\ \delta_c &= \frac{1}{4}(1 + c - b - cb) \\ \delta_b &= \frac{1}{4}(1 - c + b - cb) \\ \delta_{cb} &= \frac{1}{4}(1 - c - b + cb) \end{aligned}$$

We can rewrite J as

$$\begin{aligned} J &= \frac{1}{8}(5 \cdot 1 \otimes 1 + c \otimes 1 + b \otimes 1 + cb \otimes 1 \\ &\quad + 1 \otimes c + c \otimes c + (-1 - 2i)\, b \otimes c + (-1 + 2i)\, cb \otimes c \\ &\quad + 1 \otimes b + (-1 + 2i)\, c \otimes b + b \otimes b + (-1 - 2i)\, cb \otimes b \\ &\quad + 1 \otimes cb + (-1 - 2i)\, c \otimes cb + (-1 + 2i)\, b \otimes cb + cb \otimes cb) \end{aligned}$$

Such a J is a 2-cocycle for kF and since $J \in kG \otimes kG$, it is also a 2-cocycle for kG. Thus we can form $(kG)_J$ which is a Hopf algebra by [31], [24, 2.8]. By [31, 6.4], $(kG)_J$ is non-cocommutative if and only if $J^{-1}(\tau J)$ does not lie in the centralizer of $\Delta(kG)$ in $kG \otimes kG$. Moreover, by [24, Theorem 4.1] $K_0((kG)_J) \cong K_0(kG)$. Since J is a 2-cocycle, then by [4] $(kG)_J$ is triangular.

We now discuss Examples 2, 12 and 13 in the tables. We used GAP to compute $\mathbf{G}(H)$ in Examples 12 and 13.

Example 2. $H = (k(D_8 \times C_2))_J$, where $F = \{1, c, b, cb\} \cong C_2 \times C_2$ is a subgroup of $D_8 \times C_2 = \langle a, b, c : a^4 = b^2 = c^2 = 1, ba = a^{-1}b \rangle$ and J is given by the formula (7.1). Then

1. $\mathbf{G}(H) = \langle a^2, b, c \rangle \cong C_2 \times C_2 \times C_2$.
2. $\mathbf{G}(H^*) \cong \mathbf{G}(k(D_8 \times C_2)) \cong C_2 \times C_2 \times C_2$ and $K_0(H) \cong K_0(k(D_8 \times C_2)) \cong K_{5.1}$.

Example 12. $H = (kD_{16})_J$, where $F = \{1, a^4, b, a^4 b\} \cong C_2 \times C_2$ is a subgroup of $D_{16} = \langle a, b : a^8 = b^2 = 1, ba = a^{-1}b \rangle$ and J is given by the formula (7.1). Then

1. $\mathbf{G}(H) = \left\langle b, g = \frac{1}{2}\left(-a^2 + a^2 b + a^6 + a^6 b\right) : g^4 = b^2 = 1, bgb = g^{-1} \right\rangle \cong D_8$.
2. $\mathbf{G}(H^*) \cong \mathbf{G}((kD_{16})^*) \cong C_2 \times C_2$ and $K_0(H) \cong K_0(kD_{16}) \cong K_{6.1}$.

Example 13. $H = (kG_2)_J$, where $F = \{1, a^4, b, a^4 b\} \cong C_2 \times C_2$ is a subgroup of $G_2 = \langle a, b : a^8 = b^2 = 1, ba = a^3 b \rangle$ and J is given by the formula (7.1). Then

1. $\mathbf{G}(H) \cong D_8$.
2. $\mathbf{G}(H^*) \cong \mathbf{G}((kG_2)^*) \cong C_2 \times C_2$ and $K_0(H) \cong K_0(kG_2) \cong K_{6.2}$.

8. A construction using smash coproducts.

Let $H = kQ_8 \#^\alpha kC_2$, a smash coproduct of kQ_8 and kC_2 (see [19, 10.6.1] or [24, Proposition 3.8]), where $Q_8 = \langle a, b : a^4 = 1, b^2 = a^2, ba = a^{-1}b \rangle$, $C_2 = \{1, g\}$. H has the algebra structure of $kQ_8 \otimes kC_2$ and the following comultiplication, antipode and counit:

$$\begin{aligned}
\Delta\left(x \# \delta_{g^k}\right) &= \sum_{r+t=k} \left(x_1 \# \delta_{g^r}\right) \otimes \left(\alpha_{g^r}(x_2) \# \delta_{g^t}\right) \\
S\left(x \# \delta_{g^k}\right) &= \alpha_{g^k}(S(x)) \# \delta_{g^{-k}} \\
\varepsilon\left(x \# \delta_{g^k}\right) &= \varepsilon(x)\, \delta_{g^k,1}
\end{aligned}$$

where $\delta_1 = \frac{1}{2}(1+g)$, $\delta_g = \frac{1}{2}(1-g)$, $x \in kQ_8$ and $\alpha : G \to Aut(kQ_8)$ is defined by

$$\begin{aligned}
\alpha_1(x) &= x \\
\alpha_g(a) &= b \\
\alpha_g(b) &= a,
\end{aligned}$$

see [24, Erratum]. It follows from the above that

$$\begin{aligned}
\Delta(a\#1) &= \Delta(a\#\delta_1) + \Delta(a\#\delta_g) \\
&= a\#\delta_1 \otimes a\#\delta_1 + a\#\delta_g \otimes b\#\delta_g + a\#\delta_1 \otimes a\#\delta_g + a\#\delta_g \otimes b\#\delta_1 \\
&= a\#\delta_1 \otimes a\#1 + a\#\delta_g \otimes b\#1 \\
&= \frac{1}{2}\left(a\#1 \otimes a\#1 + a\#g \otimes a\#1 + a\#1 \otimes b\#1 - a\#g \otimes b\#1\right) \qquad (8.1)
\end{aligned}$$

and

$$\Delta(b\#1) = \frac{1}{2}\left(b\#1 \otimes b\#1 + b\#g \otimes b\#1 + b\#1 \otimes a\#1 - b\#g \otimes a\#1\right) \qquad (8.2)$$

Let's describe $\mathbf{G}(H)$, $\mathbf{G}(H^*)$ and $K_0(H)$. By straightforward computations, using (8.1) and (8.2), $\mathbf{G}(H) = \langle a^2\#1\rangle \times \langle 1\#g\rangle$. $\mathbf{G}(H^*)$ is generated by the multiplicative characters χ and φ, defined by $\chi(a) = \chi(g) = -1$, $\chi(b) = 1$ and $\varphi(a) = -1$, $\varphi(g) = \varphi(b) = 1$. Then $\chi^{-1}(a) = 1$, $\chi^{-1}(b) = \chi^{-1}(g) = -1$ and $\varphi\chi\varphi = \chi^{-1}$. Therefore $\mathbf{G}(H^*) \cong D_8$. Degree 2 irreducible representations of H are defined by

$$\pi_1(a) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \pi_1(b) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \pi_1(g) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\pi_2(a) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \pi_2(b) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \pi_2(g) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

with the property $\pi_1^2 = \pi_2^2 = 1 + \chi^2 + \varphi + \chi^2\varphi$ and therefore $K_0(H) \cong K_{5.5}$.

9. Main results.

Theorem 9.1. *Every nontrivial semisimple Hopf algebra H of dimension 16 has a commutative subHopfalgebra of dimension 8.*

Proof. By [13, Theorem 1] H^* has a central grouplike g of order 2. Thus we get a short exact sequence of Hopf algebras

$$k\langle g\rangle \overset{i}{\hookrightarrow} H^* \overset{\pi}{\twoheadrightarrow} K \tag{9.1}$$

If K is cocommutative, $K^* \subset H$ is commutative and we are done. If K is commutative, but not cocommutative, then $K^* = k\mathbf{G}(H) \subset H$ and $\mathbf{G}(H)$ is nonabelian of order 8. Applying Proposition 2.2 and results of Section 5 we get that H^* has a commutative subHopfalgebra of dimension 8. Therefore H^* was described in Section 3.3, case B), and it has a group algebra of dimension 8 as a quotient. Therefore H has a commutative subHopfalgebra of dimension 8.

Now assume that K is neither commutative nor cocommutative. Then $K \cong K^* \cong H_8$ and as algebras $H^* \cong k\langle g\rangle \#_\sigma K$, a crossed product of Hopf algebras with an action $\rightharpoonup : K \otimes k\langle g\rangle \to k\langle g\rangle$ and a cocycle $\sigma : K \otimes K \to k\langle g\rangle$. Since g is central in H^*, the action $\rightharpoonup$ is trivial. Let's prove that σ is also trivial. By [16, Theorem 4.8] $K \cong H_8$ doesn't have nontrivial right Galois objects, thus for any 2-cocycle $\alpha : K \otimes K \to k$ the crossed product ${}_\alpha K = k\#_\alpha K$ is trivial, that is $ab = \sum \alpha(a_1, b_1)\, a_2 b_2$. Write $k\langle g\rangle = k\frac{1+g}{2} \oplus k\frac{1-g}{2}$ and $\sigma(a \otimes b) = \alpha_1(a \otimes b)\frac{1+g}{2} + \alpha_2(a \otimes b)\frac{1-g}{2}$. Then for $j = 1, 2$, $\alpha_j : K \otimes K \to k$ are 2-cocycles and

$$\begin{aligned}(1\#a)(1\#b) &= \sum \sigma(a_1, b_1)\#a_2b_2 = \frac{1+g}{2}\#\sum \alpha_1(a_1, b_1)\, a_2b_2 \\ &\quad + \frac{1-g}{2}\#\sum \alpha_2(a_1, b_1)\, a_2b_2 = \frac{1+g}{2}\#ab + \frac{1-g}{2}\#ab = 1\#ab\end{aligned}$$

Therefore as algebras $H^* \cong k\langle g\rangle \otimes K$. As coalgebras $H \cong k\langle g\rangle \otimes K^* \cong k\langle g\rangle \otimes H_8$, H has 8 grouplikes and we are done by the previous argument. □

We now prove **Theorem 1.2**:

Proof. 1. Assume $\mathbf{G}(H)$ is abelian of order 8. By Theorem 1.1, $\mathbf{G}(H) \cong C_2 \times C_2 \times C_2$ or $\mathbf{G}(H) \cong C_4 \times C_2$ and by Proposition 3.1 in this case $\mathbf{G}(H^*)$ is also abelian of order 8. In Propositions 3.2 and 3.3 we have shown that there are exactly 7 nonisomorphic Hopf algebras with $\mathbf{G}(H) \cong C_4 \times C_2$ and at most 4 nonisomorphic Hopf algebras with $\mathbf{G}(H) \cong C_2 \times C_2 \times C_2$. Now we show that there are 4 distinct Hopf algebras with $\mathbf{G}(H) \cong C_2 \times C_2 \times C_2$. There are 2 nonisomorphic examples of Hopf algebras with $\mathbf{G}(H) \cong \mathbf{G}(H^*) \cong C_2 \times C_2 \times C_2$, namely $H_8 \otimes kC_2$, which is not triangular (if it were triangular, so would be H_8), and $(k(D_8 \times C_2))_J$, which is triangular (see Section 7), and 2 more nonisomorphic Hopf algebras with $\mathbf{G}(H) \cong C_2 \times C_2 \times C_2$ and $\mathbf{G}(H^*) \cong C_4 \times C_2$, namely $(H_{b:1})^*$ and $(H_{c:\sigma_1})^*$. Comparing the structures of H_8 (see [11, 2.3, 2.4, 2.8]) and $H_{d:-1,1}$ we see that $H_8 \otimes kC_2 \cong H_{d:-1,1}$, and therefore $H_{d:1,1} \cong (k(D_8 \times C_2))_J$.

2. Assume that $\mathbf{G}(H)$ is nonabelian. Then, by Theorem 9.1, H has a commutative subHopfalgebra of dimension 8. By Proposition 3.4 $\mathbf{G}(H) \cong D_8$, $\mathbf{G}(H^*) = C_2 \times C_2$ and there are exactly 2 such Hopf algebras, $H_{C:1} \cong H^*_{B:1}$ and $H_E \cong H^*_{B:X}$. Comparing their K_0-rings with K_0-rings of examples described in Section 7 we see that $H_{C:1} \cong (kD_{16})_J$ and $H_E \cong (kG_2)_J$.
3. Assume that $\mathbf{G}(H)$ is abelian of order 4. By Theorem 1.1, $\mathbf{G}(H) \cong C_2 \times C_2$. By Theorem 9.1, H has a commutative subHopfalgebra of dimension 8 and therefore it was described in Section 3. There are exactly 3 Hopf algebras with this group of grouplikes: two of them, $H_{B:1} \cong H^*_{C:1}$ and $H_{B:X} \cong H^*_E$, have $\mathbf{G}(H)^* \cong D_8$ and one of them, $H_{C:\sigma_1}$ has $\mathbf{G}(H)^* \cong C_2 \times C_2$ and therefore should be selfdual. Comparing the quotients of H_B and $kQ_8\#^\alpha kC_2$ we see that $H_{B:X} \cong kQ_8\#^\alpha kC_2$.

□

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