ON THE SECOND MOMENT FOR PRIMES IN AN ARITHMETIC PROGRESSION

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Abstract. Assuming the Generalized Riemann Hypothesis, we obtain a lower bound within a constant factor of the conjectured asymptotic result for the second moment for primes in an individual arithmetic progression in short intervals. Previous results were averaged over all progression of a given modulus. The method uses a short divisor sum approximation for the von Mangoldt function, together with some new results for binary correlations of this divisor sum approximation in arithmetic progressions.

1. Introduction and Statement of results

In this paper we calculate a lower bound, of the same order of magnitude as conjectured, for the second moment of primes in an arithmetic progression. Specifically we examine

$$I(x,h,q,a) := \int_x^{2x} \left(\psi(y+h;q,a) - \psi(y;q,a) - \frac{h}{\phi(q)}\right)^2 dy \tag{1.1}$$

where

$$\psi(x;q,a) = \sum_{\substack{n\le x \\ n\equiv a (\mathrm{mod}\ q)}} \Lambda(n), \tag{1.2}$$

and Λ is the von Mangoldt function. We will take

$$(a,q)=1, \qquad x\ge 2, \qquad 1\le q\le h\le x, \tag{1.3}$$

(other ranges not being interesting). We shall assume the truth of the Generalized Riemann Hypothesis (GRH), which implies, in particular,

$$E(x;q,a) := \psi(x;q,a) - \frac{x}{\phi(q)} \ll x^{\frac{1}{2}} \log^2 x \ , \quad (q\le x). \tag{1.4}$$

The idea of our method originates from the work of Goldston [3] for the case of all primes, corresponding in the present formulation to $q=1$. An improved and generalized version of this result appeared in [5] as

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Theorem A. *Assume GRH. Then for any $\epsilon > 0$ and $1 \le \frac{h}{q} \ll \frac{x^{\frac{1}{3}}}{q^{\epsilon}\log^3 x}$ we have*

$$\sum_{\substack{a(\mathrm{mod}\ q)\\(a,q)=1}} I(x,h,q,a) \ge \frac{1}{2}xh\log\left(\left(\frac{q}{h}\right)^3 x\right) - O(xh(\log\log x)^3). \tag{1.5}$$

Moreover, for almost all q with $h^{3/4}\log^5 x \le q \le h$ we have

$$I(x,h,q) \sim xh\log(\frac{xq}{h}). \tag{1.6}$$

For an individual arithmetic progression Özlük [11] proved unconditionally

Theorem B. *For $1 \le q \le (\log x)^{1-\delta}$, and $h \le (\log x)^c$ (δ and c are any fixed positive numbers) satisfying $q \le h$, we have*

$$I(x,h,q,a) > (\frac{1}{2}-\epsilon)\frac{xh}{\phi(q)}\log x \tag{1.7}$$

for any ϵ and $x \ge X(\epsilon,c)$.

We shall see below that the GRH implies a result of the type in Theorem B for much wider ranges of q and h. An asymptotic estimate for $I(x,h,q,a)$ in certain ranges was shown by Yıldırım [12] to be implied by GRH and a pair correlation conjecture for the zeros of Dirichlet's L-functions.

Theorem C. *Assume GRH. Let α_1, α_2, η be fixed and satisfying $0 < \eta < \alpha_1 \le \alpha_2 \le 1$, and let $\delta = x^{-\alpha}$ where $\alpha_1 \le \alpha \le \alpha_2$. Assume, as $x \to \infty$, uniformly for*

$$q \le \min(x^{\frac{1}{2}}\delta^{\frac{1}{2}}\log^A x, \delta^{-1}x^{-\eta}) \qquad (q\text{: prime or } 1) \tag{1.8}$$

and

$$\frac{x^{\alpha_1}}{\phi(q)}\log^{-3} x \le T \le \phi(q)x^{\alpha_2}\log^3 x \tag{1.9}$$

that, for $(a,q)=1$,

$$\sum_{\chi_1,\chi_2(\mathrm{mod}\ q)} \overline{\chi_1}(a)\chi_2(a) \sum_{\substack{0<\gamma_1,\gamma_2\le T\\ L(\frac{1}{2}+i\gamma_1,\chi_1)=0\\ L(\frac{1}{2}+i\gamma_2,\chi_2)=0}} x^{i(\gamma_1-\gamma_2)}\frac{4}{4+(\gamma_1-\gamma_2)^2} \sim \phi(q)\frac{T}{2\pi}\log qT. \tag{1.10}$$

Then

$$\int_x^{2x}\left(\psi(u+u\delta;q,a) - \psi(u;q,a) - \frac{u\delta}{\phi(q)}\right)^2 du \sim \frac{3}{2}\frac{\delta x^2}{\phi(q)}\log\frac{q}{\delta} \tag{1.11}$$

uniformly for $x^{-\alpha_2} \le \delta \le x^{-\alpha_1}$ *and* q *as in (1.8).*

It was also shown in [12] that the left-hand side of (1.10) is $\sim \phi(q)\frac{T}{2\pi}\log x$ for $1 \le q \le x^{\frac{1}{2}}\log^{-3} x$ when $\frac{x}{q}\log x \le T \le e^{x^{\frac{1}{4}}}$. These asymptotic values are what the diagonal terms ($\chi_1 = \chi_2$) would contribute, so the assumption (1.10) is a way of expressing that the zeros of different Dirichlet L-functions are uncorrelated. Theorem C is a generalization of one half of a result of Goldston and Montgomery [4] for the case $q = 1$, where an equivalence between the pair correlation conjecture for $\zeta(s)$ and the second moment for primes was established. Since the argument in [4] works reversibly, a suitable converse to Theorem C is also provable. The restriction to prime q was made in order to avoid the presence of imprimitive characters. The formula (1.11) involving differences $u\delta$ which vary with u can be converted to a formula involving a fixed-difference h.

Our main result is the following theorem.

Theorem 1. *Assume GRH. Then for any* $\epsilon > 0$ *and*

$$q \le h \le (xq)^{\frac{1}{3}-\epsilon}, \tag{1.12}$$

we have

$$I(x,h,q,a) \ge \frac{1}{2}\frac{xh}{\phi(q)}\log(\frac{xq}{h^3}) - O(\frac{xh}{\phi(q)}(\log\log 3q)^3). \tag{1.13}$$

Notice that the conditions in (1.12) imply that both h and q are $\ll x^{\frac{1}{2}-\epsilon}$.

The proof of the theorem uses some new results on the function $\lambda_R(n)$ used as an approximation for the von Mangoldt function in our earlier work. Propositions 2, 3, and 4 embody these results, and we expect they will have further applications to other problems.

2. Preliminaries

We shall need the following in our calculations. Let

$$f(n,x,h) = \int_{[x,2x]\cap[n-h,n)} 1\,dy = \begin{cases} n-x, & \text{for } x \le n < x+h \\ h, & \text{for } x+h \le n \le 2x \\ 2x-n+h, & \text{for } 2x < n \le 2x+h \\ 0, & \text{elsewhere.} \end{cases} \tag{2.1}$$

Lemma 1. *For real numbers* a_n *and* b_n *we have*

$$\int_x^{2x}\left(\sum_{y<n\le y+h} a_n\right)\left(\sum_{y<m\le y+h} b_m\right)dy = \sum_{x<n\le 2x+h} a_n b_n f(n,x,h)$$
$$+ \sum_{0<k\le h}\left(\sum_{x<n\le 2x+h-k}(a_n b_{n+k} + a_{n+k}b_n) f(n,x,h-k)\right).$$

Lemma 2. *Let* $C(x) = \sum_{n\le x} c_n$. *Then we have*

$$\sum_{x<n\le 2x+h} c_n f(n,x,h) = \int_{2x}^{2x+h} C(u)\,du - \int_{x}^{x+h} C(u)\,du.$$

Lemma 1 and Lemma 2 were proved in [5]. We take this opportunity to correct a minor error in Lemma 1 of [5]. In that lemma an extraneous term $h(c_{x+h} - c_{2x})$ was incorrectly included and should be removed. This term then contributed an unnecessary error term in equations (2.7),(2.14), and (2.15) of [5]. However these same error terms correctly occurred for a different reason in equation (2.9) so that starting with equation (2.16) these error terms were correctly included in the rest of [5].

Calling $\Delta\psi = \psi(y+h;q,a) - \psi(y;q,a)$ for brevity, we have from (1.1)

$$I(x,h,q,a) = \int_x^{2x} (\Delta\psi)^2\,dy - \frac{2h}{\phi(q)}\int_x^{2x} (\Delta\psi)\,dy + \frac{h^2 x}{\phi^2(q)}.$$

By the above lemmas and (1.4) we obtain

$$\begin{aligned}\int_x^{2x} (\Delta\psi)\,dy &= \sum_{\substack{x<n\le 2x+h\\ n\equiv a(\bmod q)}} \Lambda(n) f(n,x,h)\\ &= \frac{xh}{\phi(q)} + \int_{2x}^{2x+h} E(y;q,a)\,dy - \int_x^{x+h} E(y;q,a)\,dy\\ &= \frac{xh}{\phi(q)} + O(hx^{\frac12}\log^2 x),\end{aligned}$$

so that

$$I(x,h,q,a) = \int_x^{2x} (\Delta\psi)^2\,dy - \frac{xh^2}{\phi^2(q)} + O(\frac{x^{\frac12}h^2\log^2 x}{\phi(q)}). \tag{2.2}$$

The integral $\int(\Delta\psi)^2$ leads to sums of the sort $\sum\Lambda(n)\Lambda(n+k)$ which are in the territory of the twin prime conjecture. In the uninteresting case $1\le h\le q$, only the sum $\sum\Lambda^2(n)$ is present, giving easily the evaluation

$$I(x,h,q,a) = \frac{xh}{\phi(q)}\log x - \frac{xh}{\phi(q)} - \frac{xh^2}{\phi^2(q)} + O(x^{\frac12}h\log^3 x), \quad (h\le q\le x). \tag{2.3}$$

Now let $\lambda_R(n)$ be any arithmetical function, and set

$$\psi_R(y;q,a) = \sum_{\substack{n\le y\\ n\equiv a(\bmod q)}} \lambda_R(n)\ ; \qquad \Delta\psi_R = \psi_R(y+h;q,a) - \psi_R(y;q,a). \tag{2.4}$$

Trivially $\int(\Delta\psi-\Delta\psi_R)^2\ge 0$, so that

$$\int_x^{2x}(\Delta\psi)^2\,dy \ge 2\int_x^{2x}(\Delta\psi)\cdot(\Delta\psi_R)\,dy-\int_x^{2x}(\Delta\psi_R)^2\,dy. \tag{2.5}$$

We apply Lemma 1 and Lemma 2 to these integrals to obtain

$$\begin{aligned}\int_x^{2x}(\Delta\psi)\cdot(\Delta\psi_R)\,dy \;&=\; \sum_{\substack{x<n\le 2x+h\\ n\equiv a(\bmod q)}}\Lambda(n)\lambda_R(n)f(n,x,h) \qquad\qquad (2.6\,\mathrm{a})\\ &\quad+\sum_{\substack{0<k\le h\\ q|k}}\sum_{\substack{x<n\le 2x+h-k\\ n\equiv a(\bmod q)}}[\Lambda(n)\lambda_R(n+k)+\Lambda(n+k)\lambda_R(n)]f(n,x,h-k)\\ &=\left(\int_{2x}^{2x+h}-\int_x^{x+h}\right)\sum_{\substack{n\le u\\ n\equiv a(\bmod q)}}\Lambda(n)\lambda_R(n)\,du\\ &\quad+\int_{2x}^{2x+h}\sum_{0<|j|\le\frac{u-2x}{q}}\sum_{\substack{N_1<n\le N_2\\ n\equiv a(\bmod q)}}\lambda_R(n)\Lambda(n+jq)\,du\\ &\quad-\int_x^{x+h}\sum_{0<|j|\le\frac{u-x}{q}}\sum_{\substack{N_1<n\le N_2\\ n\equiv a(\bmod q)}}\lambda_R(n)\Lambda(n+jq)\,du\,, \qquad\qquad (2.6\,\mathrm{b})\end{aligned}$$

where $N_1=\max(0,-jq)$ and $N_2=\min(u,u-jq)$. Similarly

$$\begin{aligned}\int_x^{2x}(\Delta\psi_R)^2\,dy \;&=\; \sum_{\substack{x<n\le 2x+h\\ n\equiv a(\bmod q)}}\lambda_R^2(n)f(n,x,h)\\ &\quad+2\sum_{\substack{0<k\le h\\ q|k}}\sum_{\substack{x<n\le 2x+h-k\\ n\equiv a(\bmod q)}}\lambda_R(n)\lambda_R(n+k)f(n,x,h-k)\\ &=\left(\int_{2x}^{2x+h}-\int_x^{x+h}\right)\sum_{\substack{n\le u\\ n\equiv a(\bmod q)}}\lambda_R^2(n)\,du\\ &\quad+2\int_{2x}^{2x+h}\sum_{0<j\le\frac{u-2x}{q}}\sum_{\substack{n\le u-jq\\ n\equiv a(\bmod q)}}\lambda_R(n)\lambda_R(n+jq)\,du\\ &\quad-2\int_x^{x+h}\sum_{0<j\le\frac{u-x}{q}}\sum_{\substack{n\le u-jq\\ n\equiv a(\bmod q)}}\lambda_R(n)\lambda_R(n+jq)\,du. \qquad (2.7)\end{aligned}$$

3. The choice of $\lambda_R(n)$ and some number-theoretic sums

As the auxiliary function we use

$$\lambda_R(n) := \sum_{r\le R} \frac{\mu^2(r)}{\phi(r)} \sum_{d|(r,n)} d\mu(d). \tag{3.1}$$

This function is known ([3], [8]) to exhibit behavior similar to $\Lambda(n)$ when considered on average in arithmetic progressions, and it has been employed in related problems ([1], [2], [5], [6]). An upper bound for $\lambda_R(n)$ is

$$|\lambda_R(n)| \le \sum_{d|n} d \sum_{\substack{r\le R\\ d|r}} \frac{1}{\phi(r)} \le \max_{r\le R}(\frac{r}{\phi(r)}) \sum_{d|n} d \sum_{\substack{r\le R\\ d|r}} \frac{1}{r} \ll d(n)\log R \log\log R. \tag{3.2}$$

To evaluate the sums which arise when (3.1) is used in (2.6) and (2.7) we shall need some lemmas. In the following p will denote a prime number.

Lemma 3. (Hildebrand [9]) *We have for each positive integer k, uniformly in $R\ge 1$,*

$$L_k(R) := \sum_{\substack{n\le R\\ (n,k)=1}} \frac{\mu^2(n)}{\phi(n)} = \frac{\phi(k)}{k}(\log R + c + v(k)) + O(\frac{w(k)}{\sqrt{R}}), \tag{3.3}$$

where

$$c := \gamma + \sum_p \frac{\log p}{p(p-1)}; \quad v(k) := \sum_{p|k} \frac{\log p}{p};$$
$$w(k) := \sum_{d|k} \frac{\mu^2(d)}{\sqrt{d}} = \prod_{p|k}(1+\frac{1}{\sqrt{p}}); \quad v(1)=0,\ w(1)=1. \tag{3.4}$$

Lemma 4. *We have*

$$v(k) \ll \log\log 3k, \tag{3.5}$$

$$\sum_{p|k} \frac{1}{\sqrt{p}} \ll \frac{\sqrt{\log k}}{\log\log 3k}, \tag{3.6}$$

and

$$g(k) := \prod_{p|k}(1+\frac{p}{p-1}) \ll 2^{\nu(k)}(\log\log 3k) \tag{3.7}$$

Proof. We show (3.7); the other inequalities can be proved similarly. Let $\nu(k)$ be the number of distinct prime factors of k, which satisfies the bound $\nu(k) \ll \dfrac{\log k}{\log\log k}$.

We have

$$\begin{aligned}\log g(k) &= \sum_{p|k} \log(2+\frac{1}{p-1}) \\ &< \nu(k)\log 2 + \sum_{p|k}\frac{1}{p} \\ &< \nu(k)\log 2 + \sum_{p\le 2\log 2k}\frac{1}{p} \\ &= \nu(k)\log 2 + \log\log\log 21k + O(1),\end{aligned}$$

where the prime number theorem and Mertens' theorem have been employed. Exponentiating both sides we obtain (3.7).

Lemma 5. *We have*

$$\sum_{\ell|k}\frac{\mu^2(\ell)}{\phi(\ell)}\log\ell = \frac{k}{\phi(k)}v(k). \tag{3.8}$$

Proof.

$$\begin{aligned}\sum_{\ell|k}\frac{\mu^2(\ell)}{\phi(\ell)}\log\ell &= \log\prod_{\ell|k}\ell^{\frac{\mu^2(\ell)}{\phi(\ell)}} = \log\prod_{p|k}p^{\sum_{\ell|k,p|\ell}\frac{\mu^2(\ell)}{\phi(\ell)}} \\ &= \log\prod_{p|k}p^{\frac{1}{\phi(p)}\sum_{\ell'|\frac{k}{p},(\ell',p)=1}\frac{\mu^2(\ell')}{\phi(\ell')}} = \log\prod_{p|k}p^{\frac{1}{\phi(p)}\frac{1}{1+\frac{1}{\phi(p)}}\prod_{p'|k}(1+\frac{1}{\phi(p')})} \\ &= \log\prod_{p|k}p^{\frac{1}{p}\frac{k}{\phi(k)}} = \frac{k}{\phi(k)}\sum_{p|k}\frac{\log p}{p}.\end{aligned}$$

Lemma 6. (Goldston [3]) *We have*

$$\sum_{r\le R}\frac{\mu^2(r)}{\phi(r)}\sum_{\substack{d|r\\(d,k)=1}}\frac{d\mu(d)}{\phi(d)} = \mathfrak{S}(k) + O(\frac{k\,d(k)}{R\phi(k)}), \tag{3.9}$$

where

$$\mathfrak{S}(k) = \begin{cases} 2C\prod_{\substack{p|k\\p>2}}\left(\frac{p-1}{p-2}\right), & \textit{if } k \textit{ is even, } k\neq 0; \\ 0, & \textit{if } k \textit{ is odd;}\end{cases} \tag{3.10}$$

with

$$C = \prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right). \tag{3.11}$$

Proof. The proof can be found in [3]; we just note that

$$\sum_{\substack{d|r\\(d,k)=1}} \frac{d\mu(d)}{\phi(d)} = \frac{\mu(r)}{\phi(r)}\mu((r,k))\phi((r,k))\,, \tag{3.12}$$

so the left-hand side of (3.9) may be expressed as

$$\sum_{r=1}^{\infty}\frac{\mu(r)\mu((r,k))\phi((r,k))}{\phi^2(r)} + O(\sum_{r>R}\frac{\mu^2(r)\mu^2((r,k))\phi((r,k))}{\phi^2(r)}). \tag{3.13}$$

Here the first sum is $\mathfrak{S}(k)$ and the error term is $\ll$ the O-term in (3.9).

Lemma 7. (Goldston and Friedlander [1]) *We have*

$$\sum_{0<j\le \frac{h}{q}} (h-jq)\mathfrak{S}(jq) = \frac{h^2}{2\phi(q)} - \frac{h}{2}\log\frac{h}{q} + O(h(\log\log 3q)^3). \tag{3.14}$$

Lemma 8. (Hooley [10]) *Assuming GRH, we have*

$$\sum_{\substack{a(\mathrm{mod}\ q)\\(a,q)=1}} \max_{u\le x}|E(u;q,a)|^2 \ll x\log^4 x; \quad \mathit{for} \quad q\le x. \tag{3.15}$$

Lemma 9. *We have*

$$\sum_{r\le R}\frac{\mu^2(r)\sigma(r)}{\phi(r)} \ll R \tag{3.16}$$

$$\sum_{r\le R}\frac{\mu^2(r)\sigma_{\frac{1}{2}}(r)}{\phi(r)} \ll \sqrt{R} \tag{3.17}$$

$$\sum_{0<r\le R}\frac{rd(r)}{\phi(r)} \ll R\log 2R. \tag{3.18}$$

Proof. To prove (3.16), note

$$\begin{aligned}\sum_{r\le R}\frac{\mu^2(r)\sigma(r)}{\phi(r)} &= \sum_{r\le R}\mu^2(r)\prod_{p|r}(1+\frac{2}{p-1}) = \sum_{r\le R}\mu^2(r)\sum_{d|r}\frac{2^{\nu(d)}}{\phi(d)}\\ &= \sum_{d\le R}\frac{\mu^2(d)2^{\nu(d)}}{\phi(d)}\sum_{\ell\le\frac{R}{d}}\mu^2(\ell) \le R\sum_{d\le R}\frac{\mu^2(d)2^{\nu(d)}}{d\phi(d)}\\ &\le R\prod_p(1+\frac{2}{p(p-1)}) \ll R.\end{aligned}$$

The proof of (3.17) is similar, and (3.18) was shown in [3].

4. The proof of the Theorem

In this section we calculate the right-hand sides of (2.6) and (2.7), and so obtain our result.

Proposition 1. *Assuming GRH, we have*

$$\sum_{\substack{n\le N\\ n\equiv a(\mathrm{mod}\, q)}} \Lambda(n)\lambda_R(n) = \frac{N\log R}{\phi(q)}+\frac{cN}{\phi(q)}+O(\frac{N}{\phi(q)\sqrt{R}})+O(N^{\frac12}\log^3 N)+O(R\log N). \tag{4.1}$$

Proof. Starting from the definition (3.1) and recalling $L_k(R)$ from (3.3), we have

$$\begin{aligned}\sum_{\substack{n\le N\\ n\equiv a(\mathrm{mod}\, q)}} \Lambda(n)\lambda_R(n) &= \sum_{r\le R}\frac{\mu^2(r)}{\phi(r)}\sum_{d|r} d\mu(d) \sum_{\substack{n\le N\\ n\equiv a(\mathrm{mod}\, q)\\ d|n}} \Lambda(n)\\ &= L_1(R)\psi(N;q,a) - \sum_{r\le R}\frac{\mu^2(r)}{\phi(r)}\sum_{p|r} p\log p \sum_{\substack{k\ge 1\\ p^k\le N\\ p^k\equiv a(\mathrm{mod}\, q)}} 1.\end{aligned}$$

Here the sum over k is trivially of size $O(\frac{\log N}{\log p})$, so that by Lemma 3

$$\begin{aligned}\sum_{r\le R}\frac{\mu^2(r)}{\phi(r)}\sum_{p|r} p\log p \sum_{\substack{k\ge 1\\ p^k\le N\\ p^k\equiv a(\mathrm{mod}\, q)}} 1 &\ll \log N \sum_{r\le R}\frac{\mu^2(r)}{\phi(r)}\sum_{p|r} p\\ &= \log N \sum_{p\le R}\frac{p}{\phi(p)} \sum_{\substack{m\le \frac{R}{p}\\ (m,p)=1}} \frac{\mu^2(m)}{\phi(m)}\\ &\ll \log N\sum_{p\le R}\log\frac{R}{p} \ll R\log N.\end{aligned}$$

By the prime number theorem we obtain (4.1). In order for the main term to dominate the error terms in (4.1) we will require that

$$qR\le \frac{N}{\log N},\quad q\le \frac{N^{\frac12}}{\log^3 N}. \tag{4.2}$$

Hence the relevant contribution to (2.6 b) will be

$$\begin{aligned}&\left(\int_{2x}^{2x+h}-\int_{x}^{x+h}\right)\sum_{\substack{n\le u\\ n\equiv a(\mathrm{mod}\, q)}} \Lambda(n)\lambda_R(n)\,du \qquad (4.3)\\ &=\frac{xh}{\phi(q)}(\log R + c) + O(\frac{xh}{\phi(q)\sqrt{R}}) + O(x^{\frac12}h\log^3 x) + O(Rh\log x).\end{aligned}$$

Proposition 2. *Assuming GRH, we have for*

$$1 \le q \le h \le x, \quad qR \le x, \tag{4.4}$$

that

$$\begin{aligned}\sum_{\substack{0<k\le h\\ q|k}} \sum_{\substack{x<n\le 2x+h-k\\ n\equiv a(\bmod q)}} &[\Lambda(n)\lambda_R(n+k)+\Lambda(n+k)\lambda_R(n)]f(n,x,h-k)\\ &= \frac{xh^2}{\phi^2(q)} - \frac{xh}{\phi(q)}\log\frac{h}{q} + O(\frac{xh}{\phi(q)}(\log\log 3q)^3) + O(\frac{xh^2 d(q)}{\phi^2(q)R}\log\frac{2h}{q})\\ &\quad + O(\frac{x^{\frac12}h^{\frac32}R\log^2 x}{q^{\frac12}}) + O(\frac{x^{\frac12}h^2R^{\frac12}\log^2 x}{q}).\end{aligned} \tag{4.5}$$

Proof. We have

$$\sum_{\substack{N_1<n\le N_2\\ n\equiv a(\bmod q)}} \lambda_R(n)\Lambda(n+jq) = \sum_{r\le R}\frac{\mu^2(r)}{\phi(r)}\sum_{d|r} d\mu(d) \sum_{\substack{N_1<n\le N_2\\ n\equiv a(\bmod q)\\ d|n}} \Lambda(n+jq).$$

We may write the innermost sum as

$$\sum_{\substack{N_1+jq<m\le N_2+jq\\ m\equiv a(\bmod q)\\ m\equiv jq(\bmod d)}} \Lambda(m).$$

Here $m-jq=\ell d$ for some integer ℓ, and so $a\equiv \ell d(\bmod q)$. Since $(a,q)=1$, we can include only those d's such that $(d,q)=1$. Then there is a unique b, $0<b<qd$, such that $m\equiv b(\bmod qd)$. We know $(m,q)=1$, so that $(m,d)=1$ if and only if $(j,d)=1$. Hence the innermost sum is equal to

$$\begin{aligned}\psi(N_2+jq;qd,b)-&\psi(N_1+jq;qd,b)\\ &= \frac{N_2-N_1}{\phi(qd)}E_{qd,b} + E(N_2+jq;qd,b) - E(N_1+jq;qd,b),\end{aligned}$$

where $E_{qd,b}=1$ if $(qd,b)=1$, and $E_{qd,b}=0$ if $(qd,b)>1$. Thus

$$\begin{aligned}\sum_{\substack{N_1<n\le N_2\\ n\equiv a(\bmod q)}} \lambda_R(n)\Lambda(n+jq) &= \frac{u-|j|q}{\phi(q)}\sum_{r\le R}\frac{\mu^2(r)}{\phi(r)}\sum_{\substack{d|r\\ (d,jq)=1}}\frac{d\mu(d)}{\phi(d)}\\ &+\sum_{r\le R}\frac{\mu^2(r)}{\phi(r)}\sum_{\substack{d|r\\ (d,q)=1}} d\mu(d)[E(N_2+jq;qd,b)-E(N_1+jq;qd,b)],\end{aligned} \tag{4.6}$$

where the first term on the right-hand side is the main term, its value settled by Lemma 6, and its contribution to (2.6 b) will be

$$\begin{aligned}
&\int_{2x}^{2x+h} \sum_{0<|j|\le \frac{u-2x}{q}} \frac{u-|j|q}{\phi(q)}[\mathfrak{S}(jq)+O(\frac{jqd(jq)}{R\phi(jq)})]\,du \\
&\qquad - \int_{x}^{x+h} \sum_{0<|j|\le \frac{u-x}{q}} \frac{u-|j|q}{\phi(q)}[\mathfrak{S}(jq)+O(\frac{jq\,d(jq)}{R\phi(jq)})]\,du \\
&= \frac{2x}{\phi(q)} \sum_{0<j\le \frac{h}{q}} (h-jq)\mathfrak{S}(jq) + O(\frac{xhqd(q)}{R\phi^2(q)} \sum_{0<j\le\frac{h}{q}} \frac{jd(j)}{\phi(j)}) \\
&= \frac{xh^2}{\phi^2(q)} - \frac{xh}{\phi(q)}\log\frac{h}{q} + O(\frac{xh}{\phi(q)}(\log\log 3q)^3) + O(\frac{xh^2d(q)}{\phi^2(q)R}\log\frac{2h}{q}),
\end{aligned} \tag{4.7}$$

by Lemma 7 and (3.18). For the second term in the right-hand side of (4.6), if we use (1.4) directly, we will get the upper bound $Rx^{\frac{1}{2}}\log^2 x$, by (3.16). This will lead to a contribution of $O(\frac{x^{\frac{1}{2}}h^2R}{q}\log^2 x)$ in (2.6 b). Instead, in view of the averaging over j in (2.6 b), we will use Hooley's estimate quoted as Lemma 8 above. To do this note that some of the d's may not be coprime to b, but we can discard them (from the j- and n-summations) with an error

$$\ll \sum_{\substack{0<|j|\le\frac{h}{q} \\ (j,d)>1}} \psi(3x;qd,b) \ll \sum_{0<|j|\le\frac{h}{q}} \sum_{p|d} \sum_{\substack{n\le 3x \\ p|n}} \Lambda(n) \ll \frac{h}{q}\log^2 x\,,$$

and this leads to an error of $O(\frac{h^2R}{q}\log^2 x)$ in (2.6 b). Hence the contribution to (2.6 b) from the second term on the right-hand side of (4.6) is

$$\begin{aligned}
&\ll \frac{h^2R}{q}\log^2 x + \left(\int_{2x}^{2x+h} - \int_{x}^{x+h}\right) \sum_{r\le R}\frac{\mu^2(r)}{\phi(r)} \sum_{\substack{d|r \\ (d,q)=1}} d \sum_{\substack{0<|j|\le\frac{h}{q} \\ (j,d)=1}} \max_{u\le 2x+h} |E(u;qd,b)|\,du \\
&\ll \frac{h^2R}{q}\log^2 x + h\sum_{r\le R}\frac{\mu^2(r)}{\phi(r)} \sum_{\substack{d|r \\ (d,q)=1}} d\,(\frac{h}{q})^{\frac{1}{2}}(\sum_{\substack{0<|j|\le\frac{h}{q} \\ (j,d)=1}} \max_{u\le 2x+h} |E(u;qd,b)|^2)^{\frac{1}{2}} \\
&\ll \frac{h^2R}{q}\log^2 x + \frac{h^{\frac{3}{2}}}{q^{\frac{1}{2}}} \sum_{r\le R}\frac{\mu^2(r)}{\phi(r)}\sum_{d|r} d(1+(\frac{h}{qd}))^{\frac{1}{2}}(\sum_{\substack{j \,(\mathrm{mod}\, d) \\ (j,d)=1}} \max_{u\le 3x} |E(u;qd,b)|^2)^{\frac{1}{2}}.
\end{aligned} \tag{4.8}$$

In the last sum as j runs through the reduced residues modulo d, b runs through those elements of the set $\{a, a+q, \ldots, a+(d-1)q\}$ which are relatively prime to

d (note that $a \equiv \ell d (\bmod q)$ and $(a,q)=1$ implies $(a,d)=1$), and this correspondence is one-to-one. This is because $m \equiv a(\bmod q)$ and $m \equiv jq(\bmod d)$ if and only if $m \equiv n_1 da + n_2 jq^2 (\bmod qd)$ where n_i satisfy $dn_1 \equiv 1(\bmod q)$, $qn_2 \equiv 1(\bmod d)$, and we have $n_1 da + n_2 jq^2 \equiv a + tq(\bmod qd)$ if and only if $j - t \equiv an_2(\bmod d)$. Hence we may replace the j-sum in (4.8) by

$$\sum_{\substack{t(\bmod d)\\(a+tq,d)=1}} \max_{1\le u\le 3x} |E(u;qd,a+tq)|^2. \tag{4.9}$$

Although the last sum is over only $\dfrac{1}{\phi(q)}$ of the reduced residue classes modulo qd, we shall use Hooley's estimate as is. One would want to get a Hooley-type estimate for (4.9) itself, thereby saving a factor of $\phi(q)$, but this seems to require some estimates for certain integrals involving pairs of L-functions. We do not follow this path now. Recall that Theorem C, which gives an asymptotic estimate for our integral already rests upon such an assumption, (1.10), about L-functions. By Lemma 8, we take $x\log^4 x$ as upper bound for (4.9) on the condition that $qR \le x$, and on applying Lemma 9 we obtain that the expression in (4.8) is

$$\ll \frac{x^{\frac12} h^{\frac32} R}{q^{\frac12}} \log^2 x + \frac{x^{\frac12} h^2 R^{\frac12}}{q} \log^2 x. \tag{4.10}$$

This completes the proof of Proposition 2.

Proposition 3. *For $(a,q)=1$ we have*

$$\sum_{\substack{n\le N\\ n\equiv a(\bmod q)}} \lambda_R^2(n) = \frac{N}{\phi(q)}[\log R + c + O(v(q)) + O(R^{-\frac12+\epsilon})] + O(R^2). \tag{4.11}$$

Proposition 4. *For $(a,q)=1$ and $j\ne 0$ we have*

$$\sum_{\substack{n\le N\\ n\equiv a(\bmod q)}} \lambda_R(n)\lambda_R(n+jq) = \frac{N}{\phi(q)}\mathfrak{S}(jq) + O\Big(\frac{Ng(q)}{\phi(q)R}\frac{jd(j)}{\phi(j)}\Big) + O(R^2). \tag{4.12}$$

Proof. The beginning of the proof of Proposition 3 may be incorporated into that of Proposition 4 upon a notational stipulation for the case $j=0$. When the positive integer t satisfies $t \mid j$, if $j=0$ we will understand that t can be any positive integer; and we will take $(t,0)=t$.

By definition (3.1),

$$\sum_{\substack{n\le N\\ n\equiv a(\bmod q)}} \lambda_R(n)\lambda_R(n+jq) = \sum_{r,\,r'\le R} \frac{\mu^2(r)\mu^2(r')}{\phi(r)\phi(r')} \sum_{\substack{d|r\\ e|r'}} d\mu(d)e\mu(e) \sum_{\substack{n\le N\\ n\equiv a(\bmod q)\\ d|n,\, e|n+jq}} 1. \tag{4.13}$$

In the innermost sum, the conditions on n, d, and e imply $(q, de) = 1$, $(d, e) \mid j$, and thus n belongs to a unique residue class modulo $[q, d, e]$. Hence we have

$$\sum_{\substack{n\le N\\ n\equiv a(\bmod q)\\ d|n,\, e|n+jq}} 1 = \frac{N}{[q,d,e]} + O(1). \tag{4.14}$$

The contribution of the $O(1)$-term in (4.14) to (4.13) is

$$\ll \sum_{r,\,r'\le R} \frac{\mu^2(r)\mu^2(r')}{\phi(r)\phi(r')} \sum_{\substack{d|r\\ e|r'}} de = \Big(\sum_{r\le R} \frac{\mu^2(r)\sigma(r)}{\phi(r)}\Big)^2 \ll R^2 \tag{4.15}$$

by (3.16), and this is where the $O(R^2)$-term in (4.11) and (4.12) comes from. Hence

$$\sum_{\substack{n\le N\\ n\equiv a(\bmod q)}} \lambda_R(n)\lambda_R(n{+}jq) = \frac{N}{q} \sum_{r,\,r'\le R} \frac{\mu^2(r)\mu^2(r')}{\phi(r)\phi(r')} \sum_{\substack{d|\frac{r}{(r,q)}\\ e|\frac{r'}{(r',q)}\\ (d,e)|j}} \mu(d)\mu(e)(d,e) + O(R^2). \tag{4.16}$$

Let $(d, e) = \delta$, $d = d'\delta$, $e = e'\delta$, so that $(d', e') = 1$. The inner sums over d and e become

$$\sum_{\substack{\delta|j\\ \delta|(\frac{r}{(r,q)},\frac{r'}{(r',q)})}} \delta \sum_{d'|\frac{r}{\delta(r,q)}} \mu(d') \sum_{\substack{e'|\frac{r'}{\delta(r',q)}\\ (e',d')=1}} \mu(e'). \tag{4.17}$$

Here the innermost sum is

$$\sum_{\substack{e'|\frac{r'}{\delta(r',q)}\\ (e',d')=1}} \mu(e') = \prod_{\substack{p|\frac{r'}{\delta(r',q)}\\ p\nmid d'}} (1+\mu(p)) = \begin{cases} 1 & \text{if } \frac{r'}{\delta(r',q)} \mid d' \ ; \\ 0 & \text{otherwise.} \end{cases} \tag{4.18}$$

Next the sum over d' becomes

$$\sum_{\substack{d'|\frac{r}{\delta(r,q)}\\ \frac{r'}{\delta(r',q)}|d'}} \mu(d') = \begin{cases} \mu\Big(\dfrac{r}{\delta(r,q)}\Big) & \text{if } \frac{r'}{(r',q)} = \frac{r}{(r,q)} \ ; \\ 0 & \text{otherwise,} \end{cases} \tag{4.19}$$

so the main term of (4.16) is

$$\frac{N}{q} \sum_{\substack{r,\,r'\le R\\ \frac{r}{(r,q)}=\frac{r'}{(r',q)}}} \frac{\mu^2(r)\mu^2(r')}{\phi(r)\phi(r')} \mu\Big(\frac{r}{(r,q)}\Big) \sum_{\delta|(\frac{r}{(r,q)},j)} \delta\mu(\delta). \tag{4.20}$$

Since

$$\sum_{\delta|(\frac{r}{(r,q)},j)} \delta\mu(\delta) = \mu((\frac{r}{(r,q)},j))\phi((\frac{r}{(r,q)},j))\,, \tag{4.21}$$

the main term is

$$\frac{N}{q}\sum_{\substack{r,\,r'\le R\\ \frac{r}{(r,q)}=\frac{r'}{(r',q)}}} \frac{\mu^2(r)\mu^2(r')}{\phi(r)\phi(r')}\mu(\frac{r}{(r,q)})\mu((\frac{r}{(r,q)},j))\phi((\frac{r}{(r,q)},j)). \tag{4.22}$$

Writing $(r,q)=\ell$, $(r',q)=m$, $r=\ell s$, $r'=ms$ where $(s,q)=1$, (4.22) takes the form

$$\frac{N}{q}\sum_{\ell|q}\frac{\mu^2(\ell)}{\phi(\ell)}\sum_{m|q}\frac{\mu^2(m)}{\phi(m)}\sum_{\substack{s\le\min(\frac{R}{\ell},\frac{R}{m})\\ (s,q)=1}}\frac{\mu(s)}{\phi^2(s)}\mu((s,j))\phi((s,j)). \tag{4.23}$$

The $j=0$ case: We rewrite (4.23) as

$$\frac{N}{q}\sum_{\ell|q}\frac{\mu^2(\ell)}{\phi(\ell)}\sum_{m|q}\frac{\mu^2(m)}{\phi(m)}\sum_{\substack{s\le\min(\frac{R}{\ell},\frac{R}{m})\\ (s,q)=1}}\frac{\mu^2(s)}{\phi(s)}. \tag{4.24}$$

It is convenient to regard (4.24) as

$$\frac{N}{q}\sum_{\ell|q}\frac{\mu^2(\ell)}{\phi(\ell)}\sum_{m|q}\frac{\mu^2(m)}{\phi(m)}\sum_{\substack{s\le\frac{R}{\ell}\\ (s,q)=1}}\frac{\mu^2(s)}{\phi(s)} - \frac{N}{q}\sum_{\ell|q}\frac{\mu^2(\ell)}{\phi(\ell)}\sum_{\substack{m|q\\ m>\ell}}\frac{\mu^2(m)}{\phi(m)}\sum_{\substack{\frac{R}{m}<s\le\frac{R}{\ell}\\ (s,q)=1}}\frac{\mu^2(s)}{\phi(s)}. \tag{4.25}$$

For the first term of (4.25), we observe that

$$\sum_{m|q}\frac{\mu^2(m)}{\phi(m)} = \frac{q}{\phi(q)}, \tag{4.26}$$

and by Lemma 3 and Lemma 5 we have

$$\begin{aligned}
&\frac{N}{q}\sum_{\ell|q}\frac{\mu^2(\ell)}{\phi(\ell)}\sum_{m|q}\frac{\mu^2(m)}{\phi(m)}\sum_{\substack{s\le\frac{R}{\ell}\\ (s,q)=1}}\frac{\mu^2(s)}{\phi(s)}\\
&= \frac{N}{\phi(q)}(\log R + c + v(q)) - \frac{N}{q}\sum_{\ell|q}\frac{\mu^2(\ell)}{\phi(\ell)}\log\ell + O(\frac{Nw(q)}{\phi(q)\sqrt{R}}\sum_{\ell|q}\frac{\mu^2(\ell)\sqrt{\ell}}{\phi(\ell)})\\
&= \frac{N}{\phi(q)}(\log R + c) + O(\frac{N}{\phi(q)\sqrt{R}}\prod_{p|q}(1+\frac{1}{\sqrt{p}})(1+\frac{\sqrt{p}}{p-1})).
\end{aligned} \tag{4.27}$$

The last product has logarithm

$$\sum_{p|q} \log(1+\frac{1}{\sqrt{p}}) + \log(1+\frac{\sqrt{p}}{p-1}) \ \leq 2\sum_{p|q}\frac{1}{\sqrt{p}} + O(1) \ll \frac{\sqrt{\log q}}{\log\log 3q}, \tag{4.28}$$

by (3.6). Hence the first term of (4.25) is

$$\frac{N}{\phi(q)}(\log R + c) + O(\frac{N}{\phi(q)R^{\frac{1}{2}-\epsilon}}) \tag{4.29}$$

for any arbitrarily small and fixed $\epsilon > 0$. Using

$$\sum_{\substack{\frac{R}{m}<s\leq\frac{R}{\ell}\\(s,q)=1}} \frac{\mu^2(s)}{\phi(s)} = \frac{\phi(q)}{q}\log\frac{m}{\ell} + O(w(q)\sqrt{\frac{m}{R}}), \tag{4.30}$$

which is implied by Lemma 3, the second term of (4.25) is expressed as

$$\begin{aligned}\frac{N\phi(q)}{q^2}\sum_{\ell|q}\frac{\mu^2(\ell)}{\phi(\ell)}\sum_{\substack{m|q\\m>\ell}}\frac{\mu^2(m)}{\phi(m)}\log m - \frac{N\phi(q)}{q^2}\sum_{\ell|q}\frac{\mu^2(\ell)}{\phi(\ell)}\log\ell\sum_{\substack{m|q\\m>\ell}}\frac{\mu^2(m)}{\phi(m)}\\ + O(\frac{Nw(q)}{q\sqrt{R}}\sum_{\ell|q}\frac{\mu^2(\ell)}{\phi(\ell)}\sum_{\substack{m|q\\m>\ell}}\frac{\mu^2(m)}{\phi(m)}\sqrt{m}\,).\end{aligned} \tag{4.31}$$

Each term of (4.31) is majorized by deleting the restriction $m > \ell$. Then, by (4.26) and (3.8), the first two terms are each $\ll \dfrac{Nv(q)}{\phi(q)}$, and the error term is the same as that of (4.27). Hence we have shown (4.11).

The $j \neq 0$ case: Since

$$\sum_{\substack{s=1\\(s,q)=1}}^{\infty}\frac{\mu(s)}{\phi^2(s)}\mu((s,j))\phi((s,j)) = \prod_{\substack{p|j\\p\nmid q}}(1+\frac{1}{p-1})\prod_{p\nmid jq}(1-\frac{1}{(p-1)^2}) = \mathfrak{S}(jq)\frac{\phi(q)}{q}, \tag{4.32}$$

by (3.10) and (3.11), (4.23) is

$$\frac{N}{\phi(q)}\mathfrak{S}(jq) + O(\frac{N}{q}\sum_{\ell|q}\frac{\mu^2(\ell)}{\phi(\ell)}\sum_{m|q}\frac{\mu^2(m)}{\phi(m)}\sum_{s>\min(\frac{R}{\ell},\frac{R}{m})}\frac{\mu^2(s)\mu^2((s,j))\phi((s,j))}{\phi^2(s)}). \tag{4.33}$$

The sum over s was encountered before in (3.13) and majorized as in (3.9), so the O-term in (4.33) is

$$\begin{aligned}&\ll \frac{N}{qR}\frac{j\,d(j)}{\phi(j)}\sum_{\ell|q}\frac{\mu^2(\ell)}{\phi(\ell)}\sum_{m|q}\frac{\mu^2(m)}{\phi(m)}\max(\ell,m)\\ &\ll \frac{Ng(q)}{\phi(q)R}\frac{j\,d(j)}{\phi(j)}.\end{aligned} \tag{4.34}$$

This completes the proof of Proposition 4.

We now return to the proof of the theorem. The relevant contributions to (2.7) are

$$\sum_{\substack{x<n\le 2x+h\\ n\equiv a(\mathrm{mod}\ q)}} \lambda_R^2(n) f(n,x,h) = \frac{xh}{\phi(q)}(\log R + c + O(v(q)) + O(R^{-\frac{1}{2}+\epsilon})) + O(hR^2), \tag{4.35}$$

and

$$\sum_{\substack{0<k\le h\\ q|k}} \sum_{\substack{x<n\le 2x+h-k\\ n\equiv a(\mathrm{mod}\ q)}} \lambda_R(n)\lambda_R(n+k) f(n,x,h-k) \tag{4.36}$$

$$= \frac{x}{\phi(q)} \sum_{0<j\le \frac{h}{q}} \mathfrak{S}(jq) + O(\frac{xh}{R\phi(q)} g(q) \sum_{j\le \frac{h}{q}} \frac{jd(j)}{\phi(j)})$$

$$= \frac{1}{2}\frac{xh^2}{\phi^2(q)} - \frac{1}{2}\frac{xh}{\phi(q)}\log\frac{h}{q} + O(\frac{xh}{\phi(q)}(\log\log 3q)^3) + O(\frac{xh^2 g(q)}{Rq\phi(q)}\log\frac{2h}{q}) + O(\frac{h^2R^2}{q}).$$

Now we put together equations (2.2), (2.5), (2.6), (2.7), (4.3), (4.5), (4.35), (4.36), subject to (1.3), (4.2) and $qR^2 \le x$, to obtain

$$\begin{aligned} I(x,h,q,a) \ge & \frac{xh}{\phi(q)}\log\frac{Rq}{h} + O(\frac{xh}{\phi(q)}(\log\log 3q)^3) + O(\frac{xh^2}{R\phi(q)}(\frac{d(q)}{\phi(q)} + \frac{g(q)}{q})) \\ & + O(\frac{x^{\frac{1}{2}} h^{\frac{3}{2}} R \log^2 x}{q^{\frac{1}{2}}}) + O(\frac{x^{\frac{1}{2}} h^2 R^{\frac{1}{2}} \log^2 x}{\phi(q)}) + O(\frac{h^2R^2}{q}). \end{aligned} \tag{4.37}$$

Recall that $d(q)$ and $g(q)$ are both $\ll q^\epsilon$. Here we pick

$$R = (\frac{x}{hq\log^4 x})^{\frac{1}{2}}. \tag{4.38}$$

This choice of R makes all the error terms $o(\frac{xh}{q})$ provided that $h \le (xq)^{\frac{1}{3}-\epsilon}$.

References

1. J. B. Friedlander and D. A. Goldston, *Variance of distribution of primes in residue classes*, Quart. J. Math. Oxford (2) **47** (1996), 313–336.
2. J. B. Friedlander and D. A. Goldston, *Note on a variance in the distribution of primes*, Number Theory in Progress (Proc. of the Schinzel conference, Zakopane, Poland 1997), de Gruyter, 1999, pp. 841-848.
3. D. A. Goldston, *A lower bound for the second moment of primes in short intervals*, Expo. Math. **13** (1995), 366–376.
4. D. A. Goldston and H. L. Montgomery, *Pair correlation of zeros and primes in short intervals*, Analytic Number Theory and Diophantine Problems, Birkhaüser, Boston, 1987, pp. 183–203.

5. D. A. Goldston and C. Y. Yıldırım, *Primes in short segments of arithmetic progressions*, Canad. J. Math. **50 (3)** (1998), 563–580.
6. D. A. Goldston, S. M. Gonek , A. E. Özlük, C. Snyder, *On the pair correlation of zeros of the Riemann zeta-function*, to appear in Proc. London Math. Soc..
7. S. Graham, *An asymptotic estimate related to Selberg's sieve*, J. Number Theory **10** (1978), 83–94.
8. D. R. Heath-Brown, *The ternary Goldbach problem*, Revista Mat. Iberoamericana **1** (1985), 45–59.
9. A. Hildebrand, *Über die punktweise Konvergenz von Ramanujan-Entwicklungen zahlentheoretischer Funktionen*, Acta Arithmetica **XLIV** (1984), 109–140.
10. C. Hooley, *On the Barban-Davenport-Halberstam theorem: VI*, J. London Math. Soc. (2) **13** (1976), 57–64.
11. A. E. Özlük, *On the irregularity of distribution of primes in an arithmetic progression over short intervals*, Bull. Tech. Univ. Istanbul **40** (1987), 255-264.
12. C. Y. Yıldırım, *The pair correlation of zeros of Dirichlet L-functions and primes in arithmetic progressions*, Manuscripta Math. **72** (1991), 325–334.

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