ON SEMISIMPLE HOPF ALGEBRAS OF DIMENSION 2^m

YEVGENIA KASHINA

Abstract. In this paper we classify all nontrivial semisimple Hopf algebras of dimension 2^{n+1} with the group of grouplikes isomorphic to $\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_2$. Moreover, we extend some results on irreducible representations from groups to semisimple Hopf algebras and prove that certain semisimple Hopf algebras, including the ones classified in this paper, satisfy the generalized power map property.

1. Introduction

Recently various classification results were obtained for finite-dimensional semisimple Hopf algebras over an algebraically closed field of characteristic 0. In this paper we classify all semisimple nontrivial (that is, non-commutative and non-cocommutative) Hopf algebras of dimension 2^{n+1} for $n \geq 4$ over an algebraically closed field of characteristic 0 with the group of grouplikes isomorphic to $\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_2$. Note that all semisimple Hopf algebras of dimension 16 were classified in [K4]. In that case there were 16 nontrivial semisimple Hopf algebras and of these 7 had the group of grouplikes $\mathbf{G}(H) \cong \mathbb{Z}_4 \times \mathbb{Z}_2$. Thus the case of the group of grouplikes $\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_2$ is one of the major classes of Hopf algebras of dimension 2^{n+1}. The above mentioned Hopf algebras are all obtained as *extensions* of $F = k\mathbb{Z}_2$ by $K = k(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_2)$. Our ability to describe all such extensions heavily depends on the structure of K, and in particular, on the number of order 2 isomorphisms of $\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_2$.

On the other hand there are two interesting families of examples of semisimple Hopf algebras of this dimension, obtained as biproducts (see [K3] and [K5]). Duals of those Hopf algebras have $\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_2$ as a group of grouplikes, thus their duals are among the Hopf algebras, classified in this paper. Moreover, Hopf algebras from one of these families themselves have $\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_2$ as a group of grouplikes. Comparing their properties, such as multiplication in Grothendieck rings, with the results obtained in this paper, we can conclude that Hopf algebras of that family are self-dual.

In order to distinguish non-isomorphic Hopf algebras of dimension 2^{n+1} we compute Schur indicators of their irreducible representations of degree 2 (see [LM] and [KMM]). We will see that $N_{2,i}$, the number of degree 2 irreducible representations with Schur indicator equal to i (where $i = -1, 0, 1$), is a very useful invariant of the Hopf algebras described in this paper. The other important invariants used here include the group of grouplikes of the dual Hopf algebra, $\mathbf{G}(H^*)$, and the group of central grouplikes of the Hopf algebra itself, $\mathbf{G}(H) \cap Z(H)$. Note that using Schur indicators for distinguishing non-isomorphic Hopf algebras could have simplified the proof of [K4, Propositon 3.2].

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In this paper we also extend some results on representations, known in group theory, to semisimple Hopf algebras (see Section 3). For example, if a finite-dimensional Hopf algebra H has a commutative Hopf subalgebra of index p, the smallest prime divisor of $\dim H$, then every irreducible representation of H has degree 1 or p.

We also prove that semisimple Hopf algebras of dimension 2^{n+1} over a field of characteristic 0 with a commutative Hopf subalgebra of index 2 satisfy the generalized power map property $(\mathrm{Q}_{2^{n+1}})$ (see [K1] and [K2]), that is, have the exponent dividing 2^{n+1} (see [EG]). As a corollary we see that all Hopf algebras considered in this paper satisfy $(\mathrm{Q}_{2^{n+1}})$ and all semisimple Hopf algebras of dimension 16 over a field of characteristic 0 satisfy (Q_{16}).

The main result of this paper is stated in the following theorem:

Theorem 1.1. *Let k be an algebraically closed field of characteristic 0.*

1. *There are exactly 15 nonisomorphic nontrivial semisimple Hopf algebras of dimension 32 over k with the group of grouplikes isomorphic to $\mathbb{Z}_8 \times \mathbb{Z}_2$.*
2. *For $n \geq 5$ there are exactly 17 nonisomorphic nontrivial semisimple Hopf algebras of dimension 2^{n+1} over k with the group of grouplikes isomorphic to $\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_2$.*

Remark 1.2. One might presume that the number of nontrivial semisimple Hopf algebras of dimension 2^{n+1} with a given group of grouplikes of the order 2^n should grow as n grows. We see that it is not true if $\mathbf{G}(H) = \mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_2$, namely this number is always the same (equal to 17) as soon as $n \geq 5$.

Table 1

	$\mathbf{G}(H^*)$	$G \cap Z(H)$	$N_{2,0}$	$N_{2,1}$	$N_{2,-1}$	Notes
$H_{a_1:1}$ $H_{a_1:y}$	$\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}}$	$\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}$	$2^{n-2}-2$	2 1	0 1	
$H_{a_2:1}$ $H_{a_2:x^{2^{n-2}}}$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$2^{n-2}-2$	2^{n-2} 2^{n-3}	0 2^{n-3}	$n \geq 4$
$H_{b_1:1}$ $H_{b_1:y}$ $H_{b_1:x^{2^{n-2}}y}$	$(\mathbb{Z}_2)^3$ $\mathbb{Z}_2 \times \mathbb{Z}_4$ $\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	0	$3 \cdot 2^{n-3}-2$ $3 \cdot 2^{n-3}-1$ $2^{n-3}-1$	2^{n-3} $2^{n-3}-1$ $3 \cdot 2^{n-3}-1$	
$H_{b_2:1}$ $H_{b_2:y}$	$(\mathbb{Z}_2)^2 \times \mathbb{Z}_{2^{n-2}}$ $\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-2}}$	$\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}$	2^{n-2}	0 0	0 0	$n \geq 4$
$H_{b_3:1}$ $H_{b_3:y}$	$(\mathbb{Z}_2)^3$ $\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	2^{n-2}	$2^{n-2}-2$ $2^{n-3}-1$	0 $2^{n-3}-1$	$n \geq 4$
$H_{c_1:\tilde{\sigma}}$ $H_{c_1:x\tilde{\sigma}}$	$(\mathbb{Z}_2)^2 \times \mathbb{Z}_{2^{n-2}}$ $\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}}$	$\mathbb{Z}_{2^{n-1}}$	2^{n-2}	0 0	0 0	
$H_{c_2:\tilde{\sigma}}$ $H_{c_2:x^{2^{n-3}}y\tilde{\sigma}}$	$(\mathbb{Z}_2)^3$ $\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_4$	2^{n-2}	$2^{n-2}-2$ $2^{n-3}-1$	0 $2^{n-3}-1$	$n \geq 4$
H_{c_3}	$\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}$	$\mathbb{Z}_{2^{n-2}}$	$3 \cdot 2^{n-3}-2$	2	0	$n \geq 5$
H_{c_4}	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_4$	$3 \cdot 2^{n-3}-2$	2^{n-3}	0	$n \geq 5$

Remark 1.3. All semisimple Hopf algebras of dimension 2^{n+1} with $\mathbf{G}(H) = G \cong \mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_2$ are described in Section 4. The data required to distinguish them is obtained in Sections 4 and 5 and is gathered in Table 1.

Remark 1.4. In Section 4 we also compute some multiplication rules in the Grothendieck rings of Hopf algebras under consideration. We see that irreducible representations of the same degree commute with each other. One can completely describe the structure of the Grothendieck rings of these Hopf algebras and use it for classification purposes (as it was done in [K4]), however we do not need that, since other invariants completely distinguish our Hopf algebras.

Remark 1.5. The values of $N_{2,1}$ and $N_{2,-1}$, given for H_{a_1} in the table, are obtained only for the case $n \geq 4$. If $n = 3$ they trade places, that is, $N_{2,1} = N_{2,-1} = 1$ for $H_{a_1:1}$ and $N_{2,1} = 2$, $N_{2,-1} = 0$ for $H_{a_1:x^{2^{n-2}}}$.

2. Preliminaries

Everywhere in this paper we will assume that k is an algebraically closed field of characteristic 0 and H is a semisimple Hopf algebra.

First we will need the following definition, which was introduced in [Se].

Definition 2.1. Let $K_0(H)^+$ denote the abelian semigroup of all equivalence classes of representations of H with the addition given by a direct sum. Then its enveloping group $K_0(H)$ has the structure of an ordered ring with involution * and is called the Grothendieck ring.

The structure of $K_0(H)$ was described in [NR]. The multiplication in this ring is defined as follows: let $[\pi_1]$ and $[\pi_2]$ denote the classes of representations equivalent to π_1 and π_2, then $[\pi_1] \bullet [\pi_2]$ is the class of the representation $(\pi_1 \otimes \pi_2) \circ \Delta$; the unit of this ring is the class $[\varepsilon]$ and $[\pi]^*$ is the equivalence class of the dual representation $^t(\pi \circ S)$ defined by $\langle {}^t(\pi \circ S(h))(f), v\rangle = \langle f, (\pi \circ S(h))(v)\rangle$. The equivalence classes of irreducible representations of H form a basis of $K_0(H)$. Moreover, $[\pi_1]^* = [\pi_2]$ if and only if $[\varepsilon]$ is a summand in the basis decomposition of $[\pi_1] \bullet [\pi_2]$ (necessarily of multiplicity 1). For simplicity of notation we will write π instead of $[\pi]$ for elements of $K_0(H)$.

Definition 2.2 ([Ma1])**.** A short sequence

$$K \overset{i}{\hookrightarrow} H \overset{\pi}{\twoheadrightarrow} F$$

of finite-dimensional Hopf algebras and Hopf algebra maps i and π is called *exact*, if i is injective, π is surjective and $K = \{h \in H | \sum h_1 \otimes \pi(h_2) = h \otimes 1_F\}$, where $K \subset H$ via i. In this case we say that H is an *extension* of F by K.

Definition 2.3. Let K and F be Hopf algebras equipped with a weak action $\rightharpoonup$: $F \otimes K \to K$, a weak coaction $\rho : F \to F \otimes K$, a cocycle $\sigma : F \otimes F \to K$ and a dual cocycle $\theta : F \to K \otimes K$. The *bicrossed product* $K \#_\sigma^\theta F$ is a Hopf algebra with $K \otimes F$ as an underlying vector space and the following multiplication and comultiplication:

$$(x\#a)(y\#b) = \sum x(a_1 \rightharpoonup y)\,\sigma(a_2, b_1)\,\#a_3 b_2 \tag{2.1}$$

$$\Delta(x\#a) = \sum x_1\theta_i(a_1)\#(a_2)^0 \otimes x_2\theta^i(a_1)(a_2)^1\#a_3 \tag{2.2}$$

where

$$\rho(a) = \sum a^0 \otimes a^1$$
$$\theta(a) = \sum_i \theta_i(a) \otimes \theta^i(a)$$

For the necessary conditions on $\rightharpoonup$, ρ, σ, and τ see [A, 3.1].

The following proposition is contained in works of Masuoka (see [Ma2] and [Ma3]). We give the proof for completeness.

Proposition 2.4. *Let H be a finite-dimensional semisimple Hopf algebra over an algebraically closed field of characteristic* 0*. Assume that H has a commutative Hopf subalgebra $K = (kG)^*$ of index p, the smallest prime divisor of* $\dim H$*. Then $H = K\#_\sigma^\theta F$, a bicrossed product of K and $F = k\langle t\rangle = k\mathbb{Z}_p$ with an action $\rightharpoonup: F\otimes K \to K$, trivial coaction, a cocycle $\sigma : F \otimes F \to K$ and a dual cocycle $\theta : F \to K \otimes K$, and the action by t is a Hopf algebra automorphism of K.*

Proof. K is normal in H by [KoMa, Proposition 2]. Thus we have a short exact sequence of Hopf algebras

$$K \overset{i}{\hookrightarrow} H \overset{\pi}{\twoheadrightarrow} F \tag{2.3}$$

where $\dim F = p$. By [Z, Theorem 2] $F = k\langle t\rangle = k\mathbb{Z}_p$. Since H is finite-dimensional, short exact sequence (2.3) is cleft by [Sch, Theorem 2.2]. Therefore $H = K\#_\sigma^\theta F$ with a weak action $\rightharpoonup: F \otimes K \to K$, a weak coaction $\rho : F \to F \otimes K$, a cocycle $\sigma : F \otimes F \to K$ and a dual cocycle $\theta : F \to K \otimes K$ (see [A, 3.1.12] or [H]). Since K is commutative and F is cocommutative, $\rightharpoonup$ is an action and ρ is a coaction. By [Ma4, Lemma 1.1] ρ is trivial and the action by t is a Hopf algebra automorphism of K. □

We will also need the notion of a Schur indicator of an irreducible representation of a Hopf algebra. It was shown in [LM] that a version of the Frobenius-Schur theorem holds for Hopf algebras, and thus that the Schur indicator of an irreducible representation of H is well-defined. Namely, the following is true:

Theorem 2.5. *Let H be a semisimple Hopf algebra over an algebraically closed field k. If k has characteristic $p \neq 0$, assume in addition that $p \neq 2$ and that H^* is semisimple. Let Λ be an integral of H with $\varepsilon(\Lambda) = 1$, and set $\Lambda^{[2]} := \sum_{(\Lambda)} \Lambda_1\Lambda_2$. For an irreducible character χ of H corresponding to the irreducible representation V_χ, define $\nu(\chi) := \chi\left(\Lambda^{[2]}\right)$. Then the following properties hold:*

1. *$\nu(\chi) = 0$, 1 or -1, $\forall\chi \in Irr(H)$,*
2. *$\nu(\chi) \neq 0$ if and only if $V_\chi \cong V_\chi^*$. Moreover $\nu(\chi) = 1$ (respectively -1) if and only if V_χ admits a symmetric (resp. skewsymmetric) non-degenerate bilinear H-invariant form.*
3. *Considering $S \in End(V)$, $TrS = \sum_{\chi\in Irr(H)} \nu(\chi)\chi(1_H)$.*

As for groups, $\nu(\chi)$ is called the *Schur indicator* of χ. For $i=-1,0,1$ let $N_{s,i}$ be the number of s-dimensional irreducible representations with Schur indicator equal to i.

3. Induced representations

Definition 3.1. Let K be a Hopf subalgebra of H. If V is a left K-module then $V\uparrow^H = H\otimes_K V$ is a left H-module, called the *induced* module. If W is a left H-module we may restrict the action to give the *restricted* left K-module, which we denote by $W\downarrow_K$. If ψ and π are representations, corresponding to V and W respectively, we write $\psi\uparrow^H$ and $\pi\downarrow_K$ for the induced and restricted representations.

Definition 3.2. Let H be a semisimple algebra. Define the form $\langle\ ,\ \rangle : K_0(H)\times K_0(H)\to\mathbb{Z}$ by $\langle\pi,\rho\rangle=\delta_{[\pi],[\rho]}$ for irreducible H-representations π and ρ, and extend it linearly to $K_0(H)\times K_0(H)$.

The following lemma generalizes the facts, known in group theory (see [JL, Chapter 26]).

Lemma 3.3. *Let K be a commutative Hopf subalgebra of a semisimple Hopf algebra H and π be an irreducible representation of H. Then $\deg\pi\le[H:K]$.*

Proof. Consider $\pi\downarrow_K$. It can be written as a sum of irreducible representations of K (all of which have degree 1, since K is commutative). Then $\langle\pi\downarrow_K,\psi\rangle=1$, where ψ is some irreducible representation of K. By the Nakayama relations [B, Proposition 2.8.3 (i)], $\left\langle\pi,\psi\uparrow^H\right\rangle=\langle\pi\downarrow_K,\psi\rangle=1$. Therefore, since π is irreducible, it is a summand of representation $\psi\uparrow^H$, which has degree $[H:K]$. Thus $\deg\pi\le[H:K]$. □

Proposition 3.4. *Assume H has a commutative Hopf subalgebra K of index p, the smallest prime divisor of $\dim H$. Let π be an irreducible representation of H. Then $\deg\pi=1$ or p and in the case of $\deg\pi=p$, π is induced from some irreducible representation of K.*

Proof. By Proposition 2.4 $H=K\#_\sigma^\theta F$ and therefore, by [MW, Theorem 3.4], H is of Frobenius type. Thus $\deg\pi$ divides $\dim H$ and therefore, since by Lemma 3.3 $\deg\pi\le[H:K]=p$, $\deg\pi=1$ or p. Now assume $\deg\pi=p$. Let ψ be the irreducible representation of K from the proof of Lemma 3.3. Then π is a summand of representation $\psi\uparrow^H$ and, since $\deg\pi=\deg\psi\uparrow^H=p$, $\pi=\psi\uparrow^H$. □

Remark 3.5. Given a p-dimensional irreducible representation π of H as in Corollary 3.4, with $\pi=\psi\uparrow^H$ for some irreducible representation ψ of K, we may give an explicit formula for π. Let $V_\psi=kv$ be a left K-module, corresponding to the K-representation ψ. Then $H\otimes_K V_\psi$ is a left H-module, corresponding to the H-representation π. Write $F=k\langle t\rangle\cong k\mathbb{Z}_p$. The set $\left\{1\otimes v,\bar t\otimes v,\bar t^2\otimes v,\dots,\bar t^{p-1}\otimes v\right\}$ forms the basis of $H\otimes_K V_\psi$, where $\bar t=1\#t$. Let $c\in K$. Then

$$c\cdot\left(\bar t^i\otimes v\right) = c\bar t^i\otimes v=\bar t^i\left(t^{-i}\rightharpoonup c\right)\otimes v=\bar t^i\otimes\left(t^{-i}\rightharpoonup c\right)\cdot v$$

$$\begin{aligned}
\overline{t}\cdot\left(\overline{t}^{i}\otimes v\right) &= \overline{t}^{i+1}\otimes v \qquad \text{if } i\neq p-1\\
\overline{t}\cdot\left(\overline{t}^{p-1}\otimes v\right) &= \overline{t}^{p}\otimes v=\sigma\left(t,t\right)\sigma\left(t^{2},t\right)\cdots\sigma\left(t^{p-1},t\right)\otimes v\\
&= 1\otimes\sigma\left(t,t\right)\sigma\left(t^{2},t\right)\cdots\sigma\left(t^{p-1},t\right)\cdot v
\end{aligned}$$

Therefore

$$\begin{aligned}
\pi\left(c\right) &= \mathrm{Diag}\left(\psi\left(\mathrm{c}\right),\psi\left(\mathrm{t}^{-1}\rightharpoonup\mathrm{c}\right),\ldots,\psi\left(\mathrm{t}^{-\mathrm{p}+1}\rightharpoonup\mathrm{c}\right)\right)\\
&= \mathrm{Diag}\left(\psi\left(\mathrm{c}\right),\left(\mathrm{t}\rightharpoonup\psi\right)\left(\mathrm{c}\right),\ldots,\left(\mathrm{t}^{\mathrm{p}-1}\rightharpoonup\psi\right)\left(\mathrm{c}\right)\right)\\
\pi\left(\overline{t}\right) &= \begin{pmatrix} 0 & 0 & 0 & \psi\left(\sigma\left(t,t\right)\cdots\sigma\left(t^{p-1},t\right)\right)\\ 1 & 0 & 0 & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 1 & 0\end{pmatrix}
\end{aligned}$$

4. Hopf algebras of dimension 2^{n+1} with $\mathbf{G}\left(H\right)=\mathbb{Z}_{2^{n-1}}\times\mathbb{Z}_{2}$

In this Section we describe all possible nontrivial semisimple Hopf algebra of dimension 2^{n+1} with group of grouplikes isomorphic to $\mathbb{Z}_{2^{n-1}}\times\mathbb{Z}_{2}$. We also compute their groups o central grouplikes, groups of grouplikes of their duals, and their irreducible representations.

Let H be a nontrivial semisimple Hopf algebra of dimension 2^{n+1} with $\mathbf{G}\left(H\right)=G\cong\mathbb{Z}_{2^{n-1}}\times\mathbb{Z}_{2}$. Then by Proposition 2.4 H becomes a bicrossed product $K\#_{\sigma}^{\theta}F$ of $K=kG\cong\left(kG\right)^{*}$ and $F=k\left\langle t\right\rangle\cong k\mathbb{Z}_{2}$ with an action $\rightharpoonup: F\otimes K\to K$, trivial coaction, a cocycle $\sigma:F\otimes F\to K$ and a dual cocycle $\theta:F\to K\otimes K$, and the action by t is an order 2 Hopf algebra automorphism of K.

Let $G=\left\langle x\right\rangle\times\left\langle y\right\rangle$ with $\left|x\right|=2^{n-1}$ and $\left|y\right|=2$. Then the dual basis of $K\cong K^{*}$ is given by

$$e_{pq}=\frac{1}{2^{n}}\left(\sum_{k=1}^{2^{n-1}}\omega^{pk}x^{k}\right)\left(1+\left(-1\right)^{q}y\right),\qquad p=0,\ldots,2^{n-1};q=0,1$$

where ω is a primitive 2^{n-1}-th root of unity. Then

$$\begin{aligned}
\Delta_{H}\left(e_{pq}\right) &= \Delta_{K}\left(e_{pq}\right)=\sum_{\substack{p_1+p_2\equiv p \mod 2^{n-1}\\ q_1+q_2\equiv q \mod 2}} e_{p_1q_1}\otimes e_{p_2q_2}\\
\Delta_{H}\left(\overline{t}\right) &= \theta\left(t\right)\overline{t}\otimes\overline{t}
\end{aligned}$$

where $\overline{t}=1\#t$. Dualizing (2.3) we get another extension

$$F^{*}\overset{\pi^{*}}{\hookrightarrow}H^{*}\overset{i^{*}}{\twoheadrightarrow}K^{*}$$

and as in [Ma2, 2.11] or [Ma3, 2.1], since k is algebraically closed, there exist units $\bar{x}$ and $\bar{y}\in H^{*}$, such that $\bar{x}^{2^{n-1}}=\bar{y}^{2}=1_{H^{*}}$, $\left\langle e_{pq},\bar{x}^{i}\bar{y}^{j}\right\rangle=\delta_{ip}\delta_{jq}$ and $\alpha=\bar{y}^{-1}\bar{x}^{-1}\bar{y}\bar{x}\in F^{*}=k\left\{e_{0},e_{1}\right\}$, where $\left\{e_{r}\right\}$ is a dual basis of $\left\{t^{r}\right\}$, $r=0,1$. $\varepsilon\left(\alpha\right)=\varepsilon\left(\bar{y}^{-1}\bar{x}^{-1}\bar{y}\bar{x}\right)=1$

and therefore $\alpha = e_0 + \xi e_1$. The right action $\rho^* : F^* \otimes K^* \to F^*$ is trivial, thus F^* lies in the center of H^*.

$$\bar{x} = \bar{y}^2\bar{x} = \bar{y}\bar{x}\bar{y}\alpha = \bar{x}\bar{y}\alpha\bar{y}\alpha = \bar{x}\bar{y}^2\alpha^2 = \bar{x}\alpha^2$$

Thus $\alpha^2 = 1$ and therefore $\xi = \pm 1$.

$$\left\langle \Delta_H\left(\bar{t}\right), \bar{x}^i\bar{y}^je_k \otimes \bar{x}^p\bar{y}^qe_r \right\rangle = \left\langle \bar{t}, \bar{x}^i\bar{y}^je_k\bar{x}^p\bar{y}^qe_r \right\rangle = \delta_{kr}\left\langle \bar{t}, \bar{x}^{i+p}\bar{y}^{j+q}\alpha^{jp}e_k \right\rangle = \xi^{jp}\delta_{k1}\delta_{r1}$$

On the other hand

$$\left\langle \Delta_H\left(\bar{t}\right), \bar{x}^i\bar{y}^je_k \otimes \bar{x}^p\bar{y}^qe_r \right\rangle = \left\langle \theta\left(t\right)\bar{t} \otimes \bar{t}, \bar{x}^i\bar{y}^je_k \otimes \bar{x}^p\bar{y}^qe_r \right\rangle = \left\langle \theta\left(t\right), \bar{x}^i\bar{y}^j \otimes \bar{x}^p\bar{y}^q \right\rangle \delta_{k1}\delta_{r1}$$

Therefore

$$\theta\left(t\right) = \sum_{ijpq} \xi^{jp} e_{ij} \otimes e_{pq}$$

and since H should be nontrivial (non-cocommutative), $\theta\left(t,t\right)$ is nontrivial, and thus $\xi = -1$ and

$$\theta\left(t\right) = \sum_{ijpq} \left(-1\right)^{jp} e_{ij} \otimes e_{pq} = \frac{1}{2}\left(\left(1+y\right) \otimes 1 + \left(1-y\right) \otimes x^{2^{n-2}}\right)$$

Let $v = \sigma\left(t,t\right) \in K$. Then by the properties of the cocycle v is a unit and

$$t \rightharpoonup v = v \tag{4.1}$$

Multiplication in H gives us

$$\bar{t}^2 = v \tag{4.2}$$

$$\bar{t}e_{ij} = \left(t \rightharpoonup e_{ij}\right)\bar{t} \tag{4.3}$$

Moreover, if a unit $v \in K$ satisfies (4.1), (4.2) and (4.3), we can define a cocycle σ by $\sigma\left(1,1\right) = \sigma\left(1,t\right) = \sigma\left(t,1\right) = 1$ and $\sigma\left(t,t\right) = v$. Write $v = \sigma\left(t,t\right) = \sum c_{i,j}e_{i,j}$ then $c_{0,0} = \varepsilon\left(v\right) = 1$ and $c_{i,j} \neq 0$, since v is a unit, and

$$\Delta\left(\bar{t}^2\right) = \Delta\left(v\right) = \Delta\left(\sum c_{i,j}e_{i,j}\right) = \sum c_{i+p,j+q}e_{i,j} \otimes e_{p,q}$$

On the other hand, if we write

$$t \rightharpoonup e_{p,q} = e_{\alpha_1(p,q),\alpha_2(p,q)}$$

we get

$$\begin{aligned}
\Delta\left(\bar{t}\right)\Delta\left(\bar{t}\right) &= \sum_{ijpq} \left(-1\right)^{jp} e_{ij}\bar{t} \otimes e_{pq}\bar{t} \sum_{ijpq} \left(-1\right)^{jp} e_{ij}\bar{t} \otimes e_{pq}\bar{t} \\
&= \sum_{ijpq} \left(-1\right)^{jp} e_{ij} \otimes e_{pq} \sum_{ijpq} \left(-1\right)^{jp} e_{\alpha_1(i,j),\alpha_2(i,j)}\bar{t}^2 \otimes e_{\alpha_1(p,q),\alpha_2(p,q)}\bar{t}^2 \\
&= \sum_{ijpq} \left(-1\right)^{jp} e_{ij} \otimes e_{pq} \sum_{ijpq} \left(-1\right)^{\alpha_2(i,j)\alpha_1(p,q)} e_{ij}\bar{t}^2 \otimes e_{pq}\bar{t}^2 \\
&= \sum_{ijpq} \left(-1\right)^{jp+\alpha_2(i,j)\alpha_1(p,q)} c_{ij}c_{pq}e_{ij} \otimes e_{pq}
\end{aligned}$$

Thus for H to be a bialgebra we should have

$$c_{i+p,j+q} = (-1)^{jp+\alpha_2(i,j)\alpha_1(p,q)} c_{i,j} c_{p,q} \tag{4.4}$$

$\rightharpoonup$ is a nontrivial action on K, since otherwise $H \cong K^t[\mathbb{Z}_2]$ as an algebra, and thus H is commutative. The action by t is a Hopf algebra map and therefore $t \rightharpoonup G = G$ and $f_t : G \to G$ defined by $f_t(g) = t \rightharpoonup g$ is a group automorphism of order 2. Since f_t is a group automorphism, $|f_t(x)| = 2^{n-1}$ and $|f_t(y)| = 2$ and therefore

$$\begin{aligned} f_t(x) &= x^p y^q, \qquad p \text{ is odd}, q = 0, 1 \\ f_t(y) &= x^r y, \qquad r = 0, 2^{n-2} \end{aligned}$$

Now, since f_t has order 2, $f_t^2(x) = x$.

If $q = 0$, $f_t^2(x) = f_t(x^p) = x^{p^2}$ and therefore $p^2 \equiv 1 \mod 2^{n-1}$.

If $q = 1, r = 0$, $f_t^2(x) = f_t(x^p y) = x^{p^2} y^p y = x^{p^2}$ (since p is odd) and therefore $p^2 \equiv 1 \mod 2^{n-1}$.

If $q = 1, r = 1$, $f_t^2(x) = f_t(x^p y) = x^{p^2} y^p x^{2^{n-2}} y = x^{p^2+2^{n-2}}$ and therefore $p^2 + 2^{n-2} \equiv 1 \mod 2^{n-1}$.

Equation $p^2 \equiv 1 \mod 2^{n-1}$ has four distinct solutions $\{1, 2^{n-1}-1, 2^{n-2}+1, 2^{n-2}-1\}$ for $n \geq 4$ (only two solutions for $n = 3$). Equation $p^2 + 2^{n-2} \equiv 1 \mod 2^{n-1}$ has four distinct solutions $\{2^{n-3}+1, 2^{n-3}-1, 2^{n-2}+2^{n-3}+1, 2^{n-2}+2^{n-3}-1\}$ for $n \geq 5$ (no solutions for $n = 3, 4$). Renaming generators we can eliminate some of these automorphisms. Finally we get that there are nine possibilities for a nontrivial t-action, which we consider below.

4.1. The action is given by

$$\begin{aligned} t &\rightharpoonup x = xy \\ t &\rightharpoonup y = y \end{aligned}$$

Then $\mathbf{G}(H) \cap Z(H) = \langle x^2 \rangle \times \langle y \rangle \cong \mathbb{Z}_{2^{n-2}} \times \mathbb{Z}_2$ and $t \rightharpoonup e_{i,j} = e_{i+2^{n-2}j,j}$. Write $v = \sigma(t,t) = \sum_{i,j} c_{i,j} e_{i,j}$. By (4.1) and (4.4)

$$c_{i,j} = c_{i+2^{n-2}j,j} \tag{4.5}$$

$$c_{i+p,j+q} = (-1)^{jp+j(p+2^{n-2}q)} c_{i,j} c_{p,q} = c_{i,j} c_{p,q} \tag{4.6}$$

Conditions (4.5) and (4.6) imply that $c_{1,0}^{2^{n-2}} = c_{0,1}^2 = 1$. Thus for $k = 0, \ldots, 2^{n-2}-1; l = 0, 1$ let $c_{1,0} = \tau^k$, where τ is a primitive 2^{n-2}-th root of unity, $c_{0,1} = (-1)^l$ and

$$\sigma(t,t) = \sum_{p,q} \tau^{pk} (-1)^{lq} e_{p,q} = \sum \tau^{pk} e_{p,q} \sum (-1)^{lq} e_{p,q} = x^{2k} y^l$$

For $k = 0, \ldots, 2^{n-2}-1; l = 0, 1$ let $H_{k,l}$ be the Hopf algebras with the structures described above with cocycles $\sigma_{k,l}(t,t) = x^{2k} y^l$. Define

$$\begin{aligned} f : H_{k+1,l+1} &\to H_{k,l} \qquad \text{by} \\ f(e_{r,s}) &= e_{r,s} \\ f(\overline{t}) &= x\overline{t} \end{aligned}$$

and extend it multiplicatively to $f\left(e_{r,s}\bar{t}\right)$. Then f is a trivial group isomorphism on $G\left(H_{k+1,l+1}\right)$ and

$$\begin{aligned} f\left(\bar{t}\right)f\left(\bar{t}\right) &= x\bar{t}x\bar{t} = x^2y\bar{t}^2 = x^2yx^{2k}y^l = x^{2(k+1)}y^{l+1} = f\left(\bar{t}^2\right) \\ f\left(\bar{t}x\right) &= f\left(xy\bar{t}\right) = xyx\bar{t} = x\bar{t}x = f\left(\bar{t}\right)f\left(x\right) \end{aligned}$$

$$\begin{aligned} \left(f\otimes f\right)\Delta\left(\bar{t}\right) &= \left(f\otimes f\right)\left(\theta\left(t\right)\bar{t}\otimes\bar{t}\right) = \theta\left(t\right)\left(f\left(\bar{t}\right)\otimes f\left(\bar{t}\right)\right) = \theta\left(t\right)\left(x\bar{t}\otimes x\bar{t}\right) \\ &= \left(x\otimes x\right)\theta\left(t\right)\left(\bar{t}\otimes\bar{t}\right) = \Delta\left(x\right)\Delta\left(\bar{t}\right) = \Delta\left(x\bar{t}\right) = \Delta\left(f\left(\bar{t}\right)\right) \end{aligned}$$

and such an f is a Hopf algebra isomorphism between $H_{k,l}$ and $H_{k+1,l+1}$. Thus there are at most two nonisomorphic Hopf algebras of this type:

1. $H_{a_1:1} = H_{0,0}$ with the trivial cocycle and $\mathbf{G}\left(H^*_{a_1:1}\right) = \langle\chi\rangle\times\langle\varphi\rangle \cong \mathbb{Z}_{2^{n-1}}\times\mathbb{Z}_2$, where $\chi\left(x\right)=\omega$, $\chi\left(y\right)=\chi\left(\bar{t}\right)=1$, $\varphi\left(x\right)=\varphi\left(y\right)=1$, $\varphi\left(\bar{t}\right)=-1$, where ω is a primitive 2^{n-1}-th root of unity. There are 2^{n-2} degree 2 irreducible representations π_m for $m=0,\dots 2^{n-2}-1$, defined by
$\pi_m\left(x\right)=\omega^m\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ $\pi_m\left(y\right)=-I$ $\pi_m\left(\bar{t}\right)=\begin{pmatrix}0&1\\1&0\end{pmatrix}$
The product of two 2-dimensional irreducible representations is
$$\pi_l\bullet\pi_m=\chi^{l+m}+\chi^{l+m}\varphi+\chi^{l+m+2^{n-2}}+\chi^{l+m+2^{n-2}}\varphi$$
In particular, the only selfdual 2-dimensional irreducible representations are π_0 and $\pi_{2^{n-3}}$.
2. $H_{a_1:y} = H_{0,1}$ with the cocycle defined by $\sigma\left(t,t\right)=y$ and $\mathbf{G}\left(H^*_{a_1:y}\right)=\langle\chi\rangle\times\langle\varphi\rangle\cong\mathbb{Z}_{2^{n-1}}\times\mathbb{Z}_2$, where $\chi\left(x\right)=\omega$, $\chi\left(y\right)=\chi\left(\bar{t}\right)=1$, $\varphi\left(x\right)=\varphi\left(y\right)=1$, $\varphi\left(\bar{t}\right)=-1$, where ω is a primitive 2^{n-1}-th root of unity. There are 2^{n-2} degree 2 irreducible representations π_m for $m=0,\dots 2^{n-2}-1$, defined by
$\pi_m\left(x\right)=\omega^m\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ $\pi_m\left(y\right)=-I$ $\pi_m\left(\bar{t}\right)=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$
The product of two 2-dimensional irreducible representations is
$$\pi_l\bullet\pi_m=\chi^{l+m}+\chi^{l+m}\varphi+\chi^{l+m+2^{n-2}}+\chi^{l+m+2^{n-2}}\varphi$$
In particular, the only selfdual 2-dimensional irreducible representations are π_0 and $\pi_{2^{n-3}}$.

4.2. Assume $n\geq 4$. The action is given by

$$\begin{aligned} t &\rightharpoonup x = x^{-1}y \\ t &\rightharpoonup y = y \end{aligned}$$

Then $\mathbf{G}\left(H\right)\cap Z\left(H\right)=\left\langle x^{2^{n-2}}\right\rangle\times\langle y\rangle\cong\mathbb{Z}_2\times\mathbb{Z}_2$ and $t\rightharpoonup e_{i,j}=e_{-i+2^{n-2}j,j}$. Write $v=\sigma\left(t,t\right)=\sum c_{i,j}e_{i,j}$. By (4.1) and (4.4)

$$c_{i,j} = c_{-i+2^{n-2}j,j} \tag{4.7}$$

$$c_{i+p,j+q} = \left(-1\right)^{jp+j\left(-p+2^{n-2}q\right)}c_{i,j}c_{p,q}=c_{i,j}c_{p,q} \tag{4.8}$$

Conditions (4.7) and (4.8) imply that $c_{1,0}^2 = c_{0,1}^2 = 1$. Thus for $k, l = 0, 1$ let $c_{1,0} = (-1)^k$, $c_{0,1} = (-1)^l$ and

$$\sigma(t,t) = \sum_{p,q} (-1)^{pk} (-1)^{lq} e_{p,q} = \sum (-1)^{pk} e_{p,q} \sum (-1)^{lq} e_{p,q} = x^{2^{n-2}k} y^l$$

For $k, l = 0, 1$ let $H_{k,l}$ be the Hopf algebras with the structures described above with cocycles $\sigma_{k,l}(t,t) = x^{2^{n-2}k} y^l$. Define

$$\begin{array}{rcll} f: H_{k,l+1} & \to & H_{k,l} & \quad \text{by} \\ f(e_{r,s}) & = & e_{r,s} & \\ f(\overline{t}) & = & x\overline{t} & \end{array}$$

and extend it multiplicatively to $f\left(e_{r,s}\overline{t}\right)$. It is easy to check that such an f is a Hopf algebra isomorphism between $H_{k,l}$ and $H_{k,l+1}$. Thus for $n \geq 4$ there are at most two nonisomorphic Hopf algebras of this type:

1. $H_{a_2:1} = H_{0,0}$ with the trivial cocycle and $\mathbf{G}\left(H_{a_2:1}^*\right) = \langle \chi \rangle \times \langle \varphi \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$, where $\chi(x) = i$, $\chi(y) = -1$, $\chi(\overline{t}) = 1$, $\varphi(x) = \varphi(y) = 1$, $\varphi(\overline{t}) = -1$. There are $2^{n-1} - 2$ degree 2 irreducible representations $\pi_{m,r}$ for $m = 1, \ldots 2^{n-2} - 1; r = 0, 1,$ defined by
$$\pi_{m,r}(x) = i^{r^2} \begin{pmatrix} \omega^m & 0 \\ 0 & \omega^{-m} \end{pmatrix} \quad \pi_{m,r}(y) = (-1)^r I \quad \pi_{m,r}(\overline{t}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
where ω is a primitive 2^{n-1}-th root of unity. The product of two 2-dimensional irreducible representations is
$$\pi_{l,s} \bullet \pi_{m,r} = \begin{cases} \chi^{r+s} + \chi^{r+s}\varphi + \chi^{r+s+2}\varphi + \chi^{r+s+2}\varphi & \text{if } l = m = 2^{n-3} \\ \chi^{r+s} + \chi^{r+s}\varphi + \pi_{l+m+2^{n-2}rs, r+s} & \text{if } l = m, l + m \neq 2^{n-2} \\ \chi^{r+s+2} + \chi^{r+s+2}\varphi + \pi_{l-m+2^{n-2}rs, r+s} & \text{if } l + m = 2^{n-2}, l \neq m \\ \pi_{l+m+2^{n-2}rs, r+s} + \pi_{l-m+2^{n-2}rs, r+s} & \text{otherwise} \end{cases}$$
where we substitute $\pi_{a,b} = \pi_{2^{n-1}c+a, b+2d} = \pi_{-a,b}$ for $c, d \in \mathbb{Z}$ if necessary. In particular, there are 2^{n-2} selfdual 2-dimensional irreducible representations, namely $\pi_{m,0}$ for $m = 1, \ldots 2^{n-2} - 1$ and $\pi_{2^{n-3},1}$.
2. $H_{a_2:x^{2^{n-2}}} = H_{1,0}$ with the cocycle defined by $\sigma(t,t) = x^{2^{n-2}}$ and $\mathbf{G}\left(H_{a_2:x^{2^{n-2}}}^*\right) = \langle \chi \rangle \times \langle \varphi \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$, where $\chi(x) = i$, $\chi(y) = -1$, $\chi(\overline{t}) = 1$, $\varphi(x) = \varphi(y) = 1$, $\varphi(\overline{t}) = -1$. There are $2^{n-1} - 2$ degree 2 irreducible representations $\pi_{m,r}$ for $m = 1, \ldots 2^{n-2} - 1; r = 0, 1,$ defined by
$$\pi_{m,r}(x) = i^{r^2} \begin{pmatrix} \omega^m & 0 \\ 0 & \omega^{-m} \end{pmatrix} \quad \pi_{m,r}(y) = (-1)^r I \quad \pi_{m,r}(\overline{t}) = \begin{pmatrix} 0 & (-1)^m \\ 1 & 0 \end{pmatrix}$$
where ω is a primitive 2^{n-1}-th root of unity. The product of two 2-dimensional irreducible representations is
$$\pi_{l,s} \bullet \pi_{m,r} = \begin{cases} \chi^{r+s} + \chi^{r+s}\varphi + \chi^{r+s+2}\varphi + \chi^{r+s+2}\varphi & \text{if } l = m = 2^{n-3} \\ \chi^{r+s} + \chi^{r+s}\varphi + \pi_{l+m+2^{n-2}rs, r+s} & \text{if } l = m, l + m \neq 2^{n-2} \\ \chi^{r+s+2} + \chi^{r+s+2}\varphi + \pi_{l-m+2^{n-2}rs, r+s} & \text{if } l + m = 2^{n-2}, l \neq m \\ \pi_{l+m+2^{n-2}rs, r+s} + \pi_{l-m+2^{n-2}rs, r+s} & \text{otherwise} \end{cases}$$

where we substitute $\pi_{a,b} = \pi_{2^{n-1}c+a,b+2d} = \pi_{-a,b}$ for $c,d \in \mathbb{Z}$ if necessary. In particular, there are 2^{n-2} selfdual 2-dimensional irreducible representations, namely $\pi_{m,0}$ for $m = 1, \ldots 2^{n-2}-1$ and $\pi_{2^{n-3},1}$.

4.3. The action is given by

$$\begin{aligned} t &\rightharpoonup x = x^{-1} \\ t &\rightharpoonup y = y \end{aligned}$$

Then $\mathbf{G}(H) \cap Z(H) = \left\langle x^{2^{n-2}} \right\rangle \times \langle y \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $t \rightharpoonup e_{i,j} = e_{-i,j}$. Write $v = \sigma(t,t) = \sum c_{i,j} e_{i,j}$. By (4.1) and (4.4)

$$c_{i,j} = c_{-i,j} \tag{4.9}$$

$$c_{i+p,j+q} = (-1)^{jp+j(-p)} c_{i,j} c_{p,q} = c_{i,j} c_{p,q} \tag{4.10}$$

Conditions (4.9) and (4.10) imply that $c_{1,0}^2 = c_{0,1}^2 = 1$. Thus for $k,l = 0,1$ $c_{1,0} = (-1)^k$, $c_{0,1} = (-1)^l$ and

$$\sigma(t,t) = \sum_{p,q} (-1)^{pk} (-1)^{lq} e_{p,q} = \sum (-1)^{pk} e_{p,q} \sum (-1)^{lq} e_{p,q} = x^{2^{n-2}k} y^l$$

For $k,l = 0,1$ let $H_{k,l}$ be the Hopf algebras with the structures described above with cocycles $\sigma_{k,l}(t,t) = x^{2^{n-2}k} y^l$. Define

$$\begin{aligned} f : H_{0,0} &\to H_{1,0} \quad \text{by} \\ f(e_{r,s}) &= e_{r,r+s} \\ f(\bar{t}) &= \sum_{p,q} i^{p^2} e_{p,q} \bar{t} = \left(\frac{1+i}{2} 1 + \frac{1-i}{2} x^{2^{n-2}} \right) \bar{t} \end{aligned}$$

and extend it multiplicatively to $f(e_{r,s}\bar{t})$. Such an f is a Hopf algebra isomorphism between $H_{0,0}$ and $H_{1,0}$. Thus for there are at most three nonisomorphic Hopf algebras of this type:

1. $H_{b_1:1} = H_{0,0}$ with the trivial cocycle and $\mathbf{G}(H_{b_1:1}^*) = \langle \chi \rangle \times \langle \varphi \rangle \times \langle \psi \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, where $\chi(x) = 1$, $\chi(y) = -1$, $\chi(\bar{t}) = 1$, $\varphi(x) = -1$, $\varphi(y) = \varphi(\bar{t}) = 1$, $\psi(x) = \psi(y) = 1$, $\psi(\bar{t}) = -1$. There are $2^{n-1}-2$ degree 2 irreducible representations $\pi_{m,r}$ for $m = 1, \ldots 2^{n-2}-1; r = 0,1$, defined by
$$\pi_{m,r}(x) = \begin{pmatrix} \omega^m & 0 \\ 0 & \omega^{-m} \end{pmatrix} \quad \pi_{m,r}(y) = (-1)^r I \quad \pi_{m,r}(\bar{t}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
where ω is a primitive 2^{n-1}-th root of unity. The product of two 2-dimensional irreducible representations is
$$\pi_{l,s} \bullet \pi_{m,r} = \begin{cases} \chi^{r+s} + \chi^{r+s}\psi + \chi^{r+s}\varphi + \chi^{r+s}\varphi\psi & \text{if } l = m = 2^{n-3} \\ \chi^{r+s} + \chi^{r+s}\psi + \pi_{l+m,r+s} & \text{if } l = m, l+m \neq 2^{n-2} \\ \chi^{r+s}\varphi + \chi^{r+s}\varphi\psi + \pi_{l-m,r+s} & \text{if } l+m = 2^{n-2}, l \neq m \\ \pi_{l+m,r+s} + \pi_{l-m,r+s} & \text{otherwise} \end{cases}$$
where we substitute $\pi_{a,b} = \pi_{2^{n-1}c-a,b+2d}$ for $c,d = 0,1$ if necessary. In particular, every 2-dimensional irreducible representation is selfdual.

2. $H_{b_1:y} = H_{0,1}$ with the cocycle defined by $\sigma(t,t) = y$ and $\mathbf{G}\left(H^*_{b_1:y}\right) = \langle\chi\rangle \times \langle\varphi\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$, where $\chi(x) = 1$, $\chi(y) = -1$, $\chi\left(\overline{t}\right) = i$, $\varphi(x) = -1$, $\varphi(y) = \varphi\left(\overline{t}\right) = 1$. There are $2^{n-1}-2$ degree 2 irreducible representations $\pi_{m,r}$ for $m = 1,\ldots 2^{n-2}-1; r = 0,1$, defined by
$\pi_{m,r}(x) = \begin{pmatrix} \omega^m & 0 \\ 0 & \omega^{-m} \end{pmatrix}$ $\pi_{m,r}(y) = (-1)^r I$ $\pi_{m,r}\left(\overline{t}\right) = \begin{pmatrix} 0 & (-1)^r \\ 1 & 0 \end{pmatrix}$
where ω is a primitive 2^{n-1}-th root of unity. The product of two 2-dimensional irreducible representations is
$$\pi_{l,s} \bullet \pi_{m,r} = \begin{cases} \chi^{r+s} + \chi^{r+s+2} + \chi^{r+s}\varphi + \chi^{r+s+2}\varphi & \text{if } l = m = 2^{n-3} \\ \chi^{r+s} + \chi^{r+s+2} + \pi_{l+m,r+s} & \text{if } l = m, l+m \neq 2^{n-2} \\ \chi^{r+s}\varphi + \chi^{r+s+2}\varphi + \pi_{l-m,r+s} & \text{if } l+m = 2^{n-2}, l \neq m \\ \pi_{l+m,r+s} + \pi_{l-m,r+s} & \text{otherwise} \end{cases}$$
where we substitute $\pi_{a,b} = \pi_{2^{n-1}c-a,b+2d}$ for $c,d = 0,1$ if necessary. In particular, every 2-dimensional irreducible representation is selfdual.
3. $H_{b_1:x^{2^{n-2}}y} = H_{1,1}$ with the cocycle defined by $\sigma(t,t) = x^{2^{n-2}}y$ and$\mathbf{G}\left(H^*_{b_1:x^{2^{n-2}}y}\right)$ $= \langle\chi\rangle \times \langle\varphi\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$, where $\chi(x) = 1$, $\chi(y) = -1$, $\chi\left(\overline{t}\right) = i$, $\varphi(x) = -1$, $\varphi(y) = \varphi\left(\overline{t}\right) = 1$. There are $2^{n-1}-2$ degree 2 irreducible representations $\pi_{m,r}$ for $m = 1,\ldots 2^{n-2}-1; r = 0,1$, defined by
$\pi_{m,r}(x) = \begin{pmatrix} \omega^m & 0 \\ 0 & \omega^{-m} \end{pmatrix}$ $\pi_{m,r}(y) = (-1)^r I$ $\pi_{m,r}\left(\overline{t}\right) = \begin{pmatrix} 0 & (-1)^{m+r} \\ 1 & 0 \end{pmatrix}$
where ω is a primitive 2^{n-1}-th root of unity. The product of two 2-dimensional irreducible representations is
$$\pi_{l,s} \bullet \pi_{m,r} = \begin{cases} \chi^{r+s} + \chi^{r+s+2} + \chi^{r+s}\varphi + \chi^{r+s+2}\varphi & \text{if } l = m = 2^{n-3} \\ \chi^{r+s} + \chi^{r+s+2} + \pi_{l+m,r+s} & \text{if } l = m, l+m \neq 2^{n-2} \\ \chi^{r+s}\varphi + \chi^{r+s+2}\varphi + \pi_{l-m,r+s} & \text{if } l+m = 2^{n-2}, l \neq m \\ \pi_{l+m,r+s} + \pi_{l-m,r+s} & \text{otherwise} \end{cases}$$
where we substitute $\pi_{a,b} = \pi_{2^{n-1}c-a,b+2d}$ for $c,d = 0,1$ if necessary. In particular, every 2-dimensional irreducible representation is selfdual.

4.4. Assume $n \geq 4$. The action is given by
$$\begin{aligned} t &\rightharpoonup x = x^{2^{n-2}+1} \\ t &\rightharpoonup y = y \end{aligned}$$
Then $\mathbf{G}(H) \cap Z(H) = \langle x^2\rangle \times \langle y\rangle \cong \mathbb{Z}_{2^{n-2}} \times \mathbb{Z}_2$ and $t \rightharpoonup e_{i,j} = e_{(2^{n-2}+1)i,j}$. Write $v = \sigma(t,t) = \sum c_{i,j}e_{i,j}$. By (4.1) and (4.4)
$$c_{i,j} = c_{(2^{n-2}+1)i,j} \tag{4.11}$$
$$c_{i+p,j+q} = (-1)^{jp+\left(2^{n-2}+1\right)jp} c_{i,j}c_{p,q} = c_{i,j}c_{p,q} \tag{4.12}$$

Conditions (4.11) and (4.12) imply that $c_{1,0}^{2^{n-2}} = c_{0,1}^2 = 1$. Thus for $k = 0, \ldots, 2^{n-2} - 1; l = 0, 1$ let $c_{1,0} = \tau^k$, where τ is a primitive 2^{n-2}-th root of unity, $c_{0,1} = (-1)^l$ and

$$\sigma(t,t) = \sum_{p,q} \tau^{pk} (-1)^{lq} e_{p,q} = \sum \tau^{pk} e_{p,q} \sum (-1)^{lq} e_{p,q} = x^{2k} y^l$$

For $k = 0, \ldots, 2^{n-2} - 1; l = 0, 1$ let $H_{k,l}$ be the Hopf algebras with the structures described above with cocycles $\sigma_{k,l}(t,t) = x^{2k} y^l$. Define

$$\begin{aligned} f : H_{k+1,l} &\to H_{k,l} \qquad \text{by} \\ f(e_{r,s}) &= e_{r,s} \\ f(\bar{t}) &= x^{2^{n-3}+1}\bar{t} \end{aligned}$$

and extend it multiplicatively to $f(e_{r,s}\bar{t})$. It is easy to check that such an f is a Hopf algebra isomorphism between $H_{k,l}$ and $H_{k+1,l}$. Thus for $n \geq 4$ there are exactly two nonisomorphic Hopf algebras of this type:

1. $H_{b_2:1} = H_{0,0}$ with the trivial cocycle and $\mathbf{G}(H_{b_2:1}^*) = \langle\chi\rangle \times \langle\varphi\rangle \times \langle\psi\rangle \cong \mathbb{Z}_{2^{n-2}} \times \mathbb{Z}_2 \times \mathbb{Z}_2$, where $\chi(x) = \tau$ (τ is some primitive 2^{n-2}-th root of unity), $\chi(y) = \chi(\bar{t}) = 1$, $\varphi(x) = \varphi(y) = 1$, $\varphi(\bar{t}) = -1$, $\psi(x) = 1$, $\psi(y) = -1$, $\psi(\bar{t}) = 1$. There are 2^{n-2} degree 2 irreducible representations $\pi_{m,r}$ for $m = 0, \ldots 2^{n-3} - 1; r = 0, 1$, defined by
$$\pi_{m,r}(x) = \omega^{2m+1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \pi_{m,r}(y) = (-1)^r I \quad \pi_{m,r}(\bar{t}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
where ω is a primitive 2^{n-1}-th root of unity and $\omega^2 = \tau$. The product of two 2-dimensional irreducible representations is
$$\pi_{l,s} \bullet \pi_{m,r} = \chi^{l+m+1}\varphi^{r+s} + \chi^{l+m+1}\varphi^{r+s}\psi + \chi^{l+m+1+2^{n-3}}\varphi^{r+s} + \chi^{l+m+1+2^{n-3}}\varphi^{r+s}\psi$$
In particular, there are no selfdual 2-dimensional irreducible representations.
2. $H_{b_2:y} = H_{0,1}$ with the cocycle defined by $\sigma(t,t) = y$ and $\mathbf{G}(H_{b_2:y}^*) = \langle\chi\rangle \times \langle\varphi\rangle \cong \mathbb{Z}_{2^{n-2}} \times \mathbb{Z}_4$, where $\chi(x) = \tau$ (τ is some primitive 2^{n-2}-th root of unity), $\chi(y) = \chi(\bar{t}) = 1$, $\varphi(x) = 1$, $\varphi(y) = -1$, $\varphi(\bar{t}) = i$. There are 2^{n-2} degree 2 irreducible representations $\pi_{m,r}$ for $m = 0, \ldots 2^{n-3} - 1; r = 0, 1$, defined by
$$\pi_{m,r}(x) = \omega^{2m+1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \pi_{m,r}(y) = (-1)^r I \quad \pi_{m,r}(\bar{t}) = \begin{pmatrix} 0 & (-1)^r \\ 1 & 0 \end{pmatrix}$$
where ω is a primitive 2^{n-1}-th root of unity and $\omega^2 = \tau$. The product of two 2-dimensional irreducible representations is
$$\pi_{l,s} \bullet \pi_{m,r} = \chi^{l+m+1}\varphi^{r+s} + \chi^{l+m+1}\varphi^{r+s+2} + \chi^{l+m+1+2^{n-3}}\varphi^{r+s} + \chi^{l+m+1+2^{n-3}}\varphi^{r+s+2}$$
In particular, there are no selfdual 2-dimensional irreducible representations.

4.5. Assume $n \geq 4$. The action is given by

$$\begin{aligned} t &\rightharpoonup x = x^{2^{n-2}-1} \\ t &\rightharpoonup y = y \end{aligned}$$

Then $\mathbf{G}(H) \cap Z(H) = \left\langle x^{2^{n-2}} \right\rangle \times \langle y \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $t \rightharpoonup e_{i,j} = e_{(2^{n-2}-1)i,j}$. Write $v = \sigma(t,t) = \sum c_{i,j} e_{i,j}$. By (4.1) and (4.4)

$$c_{i,j} = c_{(2^{n-2}-1)i,j} \tag{4.13}$$

$$c_{i+p,j+q} = (-1)^{jp+\left(2^{n-2}-1\right)jp} c_{i,j} c_{p,q} = c_{i,j} c_{p,q} \tag{4.14}$$

Conditions (4.13) and (4.14) imply that $c_{1,0}^2 = c_{0,1}^2 = 1$. Thus for $k, l = 0, 1$ $c_{1,0} = (-1)^k$, $c_{0,1} = (-1)^l$ and

$$\sigma(t,t) = \sum_{p,q} (-1)^{pk} (-1)^{lq} e_{p,q} = \sum (-1)^{pk} e_{p,q} \sum (-1)^{lq} e_{p,q} = x^{2^{n-2}k} y^l$$

For $k, l = 0, 1$ let $H_{k,l}$ be the Hopf algebras with the structures described above with cocycles $\sigma_{k,l}(t,t) = x^{2^{n-2}k} y^l$. Define

$$\begin{aligned} f : H_{k+1,l} &\to H_{k,l} \qquad \text{by} \\ f(e_{r,s}) &= e_{r,s} \\ f(\bar{t}) &= x\bar{t} \end{aligned}$$

and extend it multiplicatively to $f\left(e_{r,s}\bar{t}\right)$. It is easy to check that such an f is a Hopf algebra isomorphism between $H_{k,l}$ and $H_{k+1,l}$. Thus for $n \geq 4$ there are exactly two nonisomorphic Hopf algebras of this type:

1. $H_{b_3:1} = H_{0,0}$ with the trivial cocycle and $\mathbf{G}\left(H_{b_3:1}^*\right) = \langle \chi \rangle \times \langle \varphi \rangle \times \langle \psi \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, where $\chi(x) = 1$, $\chi(y) = -1$, $\chi(\bar{t}) = 1$, $\varphi(x) = -1$, $\varphi(y) = \varphi(\bar{t}) = 1$, $\psi(x) = \psi(y) = 1$, $\psi(\bar{t}) = -1$. There are $2^{n-1} - 2$ degree 2 irreducible representations $\pi_{m,r}$ for $m = 1, \ldots 2^{n-3}, 2^{n-2}+1, \ldots 3 \cdot 2^{n-3} - 1; r = 0, 1$, defined by
$$\pi_{m,r}(x) = \begin{pmatrix} \omega^m & 0 \\ 0 & (-1)^m \omega^{-m} \end{pmatrix} \quad \pi_{m,r}(y) = (-1)^r I \quad \pi_{m,r}(\bar{t}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
where ω is a primitive 2^{n-1}-th root of unity. The product of two 2-dimensional irreducible representations is
$$\pi_{l,s} \bullet \pi_{m,r} = \begin{cases} \chi^{r+s} + \chi^{r+s}\psi + \chi^{r+s}\varphi + \chi^{r+s}\varphi\psi & \text{if } l = m = 2^{n-3} \\ \chi^{r+s}\varphi^m + \chi^{r+s}\varphi^m\psi + \pi_{l+m,r+s} & \text{if } l = m \neq 2^{n-3} \\ \chi^{r+s}\varphi^{m+1} + \chi^{r+s}\varphi^{m+1}\psi + \pi_{l+m,r+s} & \text{if } l - m = 2^{n-2} \\ \pi_{l+m,r+s} + \pi_{l-m+2^{n-2}m,r+s} & \text{otherwise} \end{cases}$$
where we substitute $\pi_{a,b} = \pi_{2^{n-1}c+a,b+2d} = \pi_{-a+2^{n-2}a,b}$ for $c, d \in \mathbb{Z}$ if necessary. In particular, there are $2^{n-2} - 2$ selfdual 2-dimensional irreducible representations, namely $\pi_{2M,r}$ for $M = 1, \ldots 2^{n-4}, 2^{n-3}+1, \ldots 3 \cdot \ldots 2^{n-4} - 1; r = 0, 1$.
2. $H_{b_3:y} = H_{0,1}$ with the cocycle defined by $\sigma(t,t) = y$ and $\mathbf{G}\left(H_{b_3:y}^*\right) = \langle \chi \rangle \times \langle \varphi \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$, where $\chi(x) = 1$, $\chi(y) = -1$, $\chi(\bar{t}) = i$, $\varphi(x) = -1$, $\varphi(y) = \varphi(\bar{t}) = 1$. There are $2^{n-1} - 2$ degree 2 irreducible representations $\pi_{m,r}$ for $m = 1, \ldots 2^{n-3}, 2^{n-2}+1, \ldots 3 \cdot 2^{n-3} - 1; r = 0, 1$, defined by
$$\pi_{m,r}(x) = \begin{pmatrix} \omega^m & 0 \\ 0 & (-1)^m \omega^{-m} \end{pmatrix} \quad \pi_{m,r}(y) = (-1)^r I \quad \pi_{m,r}(\bar{t}) = \begin{pmatrix} 0 & (-1)^r \\ 1 & 0 \end{pmatrix}$$

where ω is a primitive 2^{n-1}-th root of unity. The product of two 2-dimensional irreducible representations is

$$\pi_{l,s}\bullet\pi_{m,r}=\begin{cases}\chi^{r+s}+\chi^{r+s+2}+\chi^{r+s}\varphi+\chi^{r+s+2}\varphi & \text{if } l=m=2^{n-3}\\ \chi^{r+s}\varphi^{m}+\chi^{r+s+2}\varphi^{m}+\pi_{l+m,r+s} & \text{if } l=m, l+m\neq 2^{n-2}\\ \chi^{r+s}\varphi^{m+1}+\chi^{r+s+2}\varphi^{m+1}+\pi_{l+m,r+s} & \text{if } l-m=2^{n-2}\\ \pi_{l+m,r+s}+\pi_{l-m+2^{n-2}m,r+s} & \text{otherwise}\end{cases}$$

where we substitute $\pi_{a,b}=\pi_{2^{n-1}c+a,b+2d}=\pi_{-a+2^{n-2}a,b}$ for $c,d\in\mathbb{Z}$ if necessary. In particular, there are $2^{n-2}-2$ selfdual 2-dimensional irreducible representations, namely $\pi_{2M,r}$ for $M=1,\dots 2^{n-4},2^{n-3}+1,\dots 3\cdot\dots 2^{n-4}-1; r=0,1$.

4.6. The action is given by

$$\begin{aligned} t &\rightharpoonup x=x\\ t &\rightharpoonup y=x^{2^{n-2}}y\end{aligned}$$

Then $\mathbf{G}(H)\cap Z(H)=\langle x\rangle\cong\mathbb{Z}_{2^{n-1}}$ and $t\rightharpoonup e_{i,j}=e_{i,j+i}$. Write $v=\sigma(t,t)=\sum c_{i,j}e_{i,j}$. By (4.1) and (4.4)

$$\begin{aligned} c_{i,j} &= c_{i,j+i}\\ c_{i+p,j+q} &= (-1)^{jp+(j+i)p}c_{i,j}c_{p,q}=(-1)^{ip}c_{i,j}c_{p,q}\end{aligned}$$

Then

$$\begin{aligned} c_{1,0} &= c_{1,1}=c_{1,0}c_{0,1}\\ c_{p,0} &= (-1)^{p-1}c_{1,0}c_{p-1,0}=(-1)^{\frac{p(p-1)}{2}}c_{1,0}^{p}\end{aligned}$$

and therefore $c_{1,0}=\omega^k$ for $k=0,\dots,2^{n-1}-1$, where ω is a primitive 2^{n-1}-th root of unity, $c_{0,1}=1$ and

$$\sigma(t,t)=\sum_{p,q}(-1)^{\frac{p(p-1)}{2}}\omega^{kp}e_{p,q}=x^{2^{n-3}-k}\left(\frac{1+i}{2}1+\frac{1-i}{2}x^{2^{n-2}}\right)$$

For $k=0,\dots,2^{n-1}-1$ let H_k be the Hopf algebras with the structures described above with cocycles $\sigma_k(t,t)=\sum_{p,q}(-1)^{\frac{p(p-1)}{2}}\omega^{kp}e_{p,q}$. Define

$$\begin{aligned} f:H_k &\to H_{k+2} \qquad \text{by}\\ f(e_{r,s}) &= e_{r,s}\\ f(\bar{t}) &= x\bar{t}\end{aligned}$$

and extend it multiplicatively to $f\left(e_{r,s}\bar{t}\right)$. It is easy to check that such an f is a Hopf algebra isomorphism between H_k and H_{k+2}. Thus there are exactly two nonisomorphic Hopf algebras of this type:

1. $H_{c_1:\tilde{\sigma}}=H_{2^{n-3}}$ with the cocycle defined by $\sigma(t,t)=\tilde{\sigma}=\frac{1+i}{2}1+\frac{1-i}{2}x^{2^{n-2}}$ and $\mathbf{G}\left(H^*_{c_1:\tilde{\sigma}}\right)=\langle\chi\rangle\times\langle\varphi\rangle\times\langle\psi\rangle\cong\mathbb{Z}_{2^{n-2}}\times\mathbb{Z}_2\times\mathbb{Z}_2$, where $\chi(x)=\tau$, $\chi(y)=\chi(\bar{t})=1$ (τ is some primitive 2^{n-2}-th root of unity), $\varphi(x)=1$, $\varphi(y)=-1$, $\varphi(\bar{t})=1$, $\psi(x)=\psi(y)=1$, $\psi(\bar{t})=-1$. There are 2^{n-2} degree 2 irreducible representations π_m for $m=0,\dots 2^{n-2}-1$, defined by

$$\pi_m(x)=\omega^{2m+1}I \quad \pi_m(y)=\begin{pmatrix}1&0\\0&-1\end{pmatrix} \quad \pi_m(\bar{t})=\begin{pmatrix}0&i\\1&0\end{pmatrix}$$

where ω is a primitive 2^{n-1}-th root of unity and $\omega^2=\tau$. The product of two 2-dimensional irreducible representations is

$$\pi_l\bullet\pi_m=\chi^{l+m+1}+\chi^{l+m+1}\varphi+\chi^{l+m+1}\psi+\chi^{l+m+1}\varphi\psi$$

In particular, there are no selfdual 2-dimensional irreducible representations.

2. $H_{c_1:x\tilde{\sigma}}=H_{2^{n-3}-1}$ with the cocycle defined by $\sigma(t,t)=x\tilde{\sigma}=\frac{1+i}{2}x+\frac{1-i}{2}x^{2^{n-2}}x$ and $\mathbf{G}\left(H^*_{c_1:x\tilde{\sigma}}\right)=\langle\chi\rangle\times\langle\varphi\rangle\cong\mathbb{Z}_{2^{n-1}}\times\mathbb{Z}_2$, where $\chi(x)=\omega^2$, $\chi(y)=1$, $\chi(\bar{t})=\omega$ (ω is some primitive 2^{n-1}-th root of unity), $\varphi(x)=1$, $\varphi(y)=-1$, $\varphi(\bar{t})=1$. There are 2^{n-2} degree 2 irreducible representations π_m for $m=0,\dots 2^{n-2}-1$, defined by

$$\pi_m(x)=\omega^{2m+1}I \quad \pi_m(y)=\begin{pmatrix}1&0\\0&-1\end{pmatrix} \quad \pi_m(\bar{t})=\begin{pmatrix}0&i\omega^{2m+1}\\1&0\end{pmatrix}$$

The product of two 2-dimensional irreducible representations is

$$\pi_l\bullet\pi_m=\chi^{l+m+1}+\chi^{l+m+2^{n-2}+1}+\chi^{l+m+1}\varphi+\chi^{l+m+2^{n-2}+1}\varphi$$

In particular, there are no selfdual 2-dimensional irreducible representations.

4.7. Assume $n\geq 4$. The action is given by

$$\begin{aligned} t &\rightharpoonup x=x^{-1}\\ t &\rightharpoonup y=x^{2^{n-2}}y\end{aligned}$$

Then $\mathbf{G}(H)\cap Z(H)=\left\langle x^{2^{n-3}}y\right\rangle\cong\mathbb{Z}_4$ and $t\rightharpoonup e_{i,j}=e_{-i,j+i}$. Write $v=\sigma(t,t)=\sum c_{i,j}e_{i,j}$. By (4.1) and (4.4)

$$\begin{aligned} c_{i,j} &= c_{-i,j+i}\\ c_{i+p,j+q} &= (-1)^{jp-(j+i)p}c_{i,j}c_{p,q}=(-1)^{ip}c_{i,j}c_{p,q}\end{aligned}$$

Then

$$\begin{aligned} (c_{0,1})^2 &= c_{0,0}=1\\ c_{1,0} &= c_{-1,1}\\ (c_{1,0})^2 &= c_{1,0}c_{-1,1}=-c_{0,1}\\ c_{p,q} &= (-1)^{p-1}c_{1,0}c_{p-1,0}c_{0,q}=(-1)^{\frac{p(p-1)}{2}+q}c_{1,0}^{p+2q}\end{aligned}$$

and therefore $c_{1,0}=i^k$ for $k=0,1,2,3$, where i is a primitive 4-th root of unity, $c_{0,1}=(-1)^{k+1}$ and

$$\sigma(t,t)=\sum_{p,q}(-1)^{\frac{p(p-1)}{2}+q}i^{k(p+2q)}e_{p,q}=x^{2^{n-3}(1-k)}\left(\frac{1+i}{2}1+\frac{1-i}{2}x^{2^{n-2}}\right)y^{1+k}$$

For $k=0,1,2,3$ let H_k be the Hopf algebras with the structures described above with cocycles $\sigma_k(t,t)=\sum_{p,q}(-1)^{\frac{p(p-1)}{2}+q}i^{k(p+2q)}e_{p,q}$. Define

$$f:H_k\to H_{k+2} \qquad \text{by}$$

$$\begin{aligned} f\left(e_{r,s}\right) &= e_{r,s} \\ f\left(\bar{t}\right) &= \sum_{p,q} (-1)^q e_{p,q}\bar{t} = y\bar{t} \end{aligned}$$

and extend it multiplicatively to $f\left(e_{r,s}\bar{t}\right)$. It is easy to check that such an f is a Hopf algebra isomorphism between H_k and H_{k+2}. Thus for $n \geq 4$ there are exactly two nonisomorphic Hopf algebras of this type:

1. $H_{c2:\tilde{\sigma}} = H_1$ with the cocycle defined by $\sigma(t,t) = \tilde{\sigma} = \frac{1+i}{2}1 + \frac{1-i}{2}x^{2^{n-2}}$ and $\mathbf{G}\left(H^*_{c2:\tilde{\sigma}}\right) = \langle\chi\rangle \times \langle\varphi\rangle \times \langle\psi\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, where $\chi(x) = 1$, $\chi(y) = -1$, $\chi\left(\bar{t}\right) = 1$, $\varphi(x) = -1$, $\varphi(y) = \varphi\left(\bar{t}\right) = 1$, $\psi(x) = \psi(y) = 1$, $\psi\left(\bar{t}\right) = -1$. There are $2^{n-1} - 2$ degree 2 irreducible representations $\pi_{m,r}$ for $m = 1, \ldots 2^{n-2} - 1; r = 0, 1$, defined by
$$\pi_{m,r}(x) = \begin{pmatrix} \omega^m & 0 \\ 0 & \omega^{-m} \end{pmatrix} \quad \pi_{m,r}(y) = (-1)^r \begin{pmatrix} 1 & 0 \\ 0 & (-1)^m \end{pmatrix} \quad \pi_{m,r}(\bar{t}) = \begin{pmatrix} 0 & i^{m^2} \\ 1 & 0 \end{pmatrix}$$
where ω is a primitive 2^{n-1}-th root of unity. The product of two 2-dimensional irreducible representations is
$$\pi_{l,s} \bullet \pi_{m,r} = \begin{cases} \chi^{r+s} + \chi^{r+s}\varphi + \chi^{r+s}\psi + \chi^{r+s}\varphi\psi & \text{if } l = m = 2^{n-3} \\ \chi^{r+s+m} + \chi^{r+s+m}\psi + \pi_{l+m,r+s} & \text{if } l = m, l + m \neq 2^{n-2} \\ \chi^{r+s}\varphi + \chi^{r+s}\varphi\psi + \pi_{l-m,r+s+m} & \text{if } l + m = 2^{n-2}, l \neq m \\ \pi_{l+m,r+s} + \pi_{l-m,r+s+m} & \text{otherwise} \end{cases}$$
where we substitute $\pi_{a,b} = \pi_{2^{n-1}c+a,b+2d} = \pi_{-a,b+a}$ for $c, d \in \mathbb{Z}$ if necessary. In particular, there are $2^{n-2} - 2$ selfdual 2-dimensional irreducible representations, namely $\pi_{2M,r}$ for $M = 1, \ldots 2^{n-3} - 1; r = 0, 1$.
2. $H_{c2:x^{2^{n-3}}y\tilde{\sigma}} = H_0$ with the cocycle defined by $\sigma(t,t) = x^{2^{n-3}}y\tilde{\sigma} = \frac{1+i}{2}x^{2^{n-3}}y + \frac{1-i}{2}x^{3\cdot 2^{n-3}}y$ and $\mathbf{G}\left(H^*_{c2:x^{2^{n-3}}y\tilde{\sigma}}\right) = \langle\chi\rangle \times \langle\varphi\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$, where $\chi(x) = 1$, $\chi(y) = -1$, $\chi\left(\bar{t}\right) = i$, $\varphi(x) = -1$, $\varphi(y) = \varphi\left(\bar{t}\right) = 1$. There are $2^{n-1} - 2$ degree 2 irreducible representations $\pi_{m,r}$ for $m = 1, \ldots 2^{n-2} - 1; r = 0, 1$, defined by
$$\pi_{m,r}(x) = \begin{pmatrix} \omega^m & 0 \\ 0 & \omega^{-m} \end{pmatrix} \quad \pi_{m,r}(y) = (-1)^r \begin{pmatrix} 1 & 0 \\ 0 & (-1)^m \end{pmatrix} \quad \pi_{m,r}(\bar{t}) = \begin{pmatrix} 0 & i^{m^2+m+2r} \\ 1 & 0 \end{pmatrix}$$
where ω is a primitive 2^{n-1}-th root of unity. The product of two 2-dimensional irreducible representations is
$$\pi_{l,s} \bullet \pi_{m,r} = \begin{cases} \chi^{r+s} + \chi^{r+s+2} + \chi^{r+s}\varphi + \chi^{r+s+2}\varphi & \text{if } l = m = 2^{n-3} \\ \chi^{r+s+m} + \chi^{r+s+m+2} + \pi_{l+m,r+s} & \text{if } l = m, l + m \neq 2^{n-2} \\ \chi^{r+s}\varphi + \chi^{r+s+2}\varphi + \pi_{l-m,r+s+m} & \text{if } l + m = 2^{n-2}, l \neq m \\ \pi_{l+m,r+s} + \pi_{l-m,r+s+m} & \text{otherwise} \end{cases}$$
where we substitute $\pi_{a,b} = \pi_{a+2^{n-1}c,b+2d} = \pi_{-a,b+a}$ for $c, d \in \mathbb{Z}$ if necessary. In particular, there are $2^{n-2} - 2$ selfdual 2-dimensional irreducible representations, namely $\pi_{2M,r}$ for $M = 1, \ldots 2^{n-3} - 1; r = 0, 1$.

4.8. Assume $n \geq 5$. The action is given by

$$t \rightharpoonup x = x^{2^{n-3}+1}y$$

$$t \rightharpoonup y = x^{2^{n-2}}y$$

Then $\mathbf{G}(H) \cap Z(H) = \langle x^2 y\rangle \cong \mathbb{Z}_{2^{n-2}}$ and $t \rightharpoonup e_{i,j} = e_{i(3\cdot 2^{n-3}+1)+2^{n-2}j,j+i}$. Write $v = \sigma(t,t) = \sum c_{i,j}e_{i,j}$. By (4.1) and (4.4)

$$\begin{aligned} c_{i,j} &= c_{i(3\cdot 2^{n-3}+1)+2^{n-2}j,j+i} \\ c_{i+p,j+q} &= (-1)^{jp+(j+i)\left(\left(3\cdot 2^{n-3}+1\right)p+2^{n-2}q\right)} c_{i,j}c_{p,q} = (-1)^{ip} c_{i,j}c_{p,q} \end{aligned}$$

Then

$$\begin{aligned} (c_{0,1})^2 &= c_{0,0} = 1 \\ c_{2^{n-2},0} &= c_{2^{n-2},1}c_{0,1} = c_{0,1}c_{0,1} = 1 \\ c_{0,1} &= c_{2^{n-2},1} = -c_{2^{n-3}-1,0}c_{2^{n-3}+1,1} = -c_{2^{n-3}-1,0}c_{2^{n-3}+1,1} \\ &= -c_{2^{n-3}-1,0}c_{1,0} = c_{2^{n-3},0} \\ c_{p,q} &= (-1)^{p-1} c_{1,0}c_{p-1,0}c_{0,q} = (-1)^{\frac{p(p-1)}{2}} c_{1,0}^{p+2^{n-3}q} \end{aligned}$$

and therefore $c_{1,0} = \tau^k$ for $k = 0, \ldots, 2^{n-2}-1$, where τ is a primitive 2^{n-2}-th root of unity, $c_{0,1} = (-1)^k$ and

$$\sigma(t,t) = \sum_{p,q} (-1)^{\frac{p(p-1)}{2}} \tau^{k\left(p+2^{n-3}q\right)} e_{p,q} = x^{2^{n-3}-2k}\left(\frac{1+i}{2}1 + \frac{1-i}{2}x^{2^{n-2}}\right)y^k$$

For $k = 0, \ldots, 2^{n-2}-1$ let H_k be the Hopf algebras with the structures described above with cocycles $\sigma_k(t,t) = \sum_{p,q}(-1)^{\frac{p(p-1)}{2}} \tau^{k\left(p+2^{n-3}q\right)}e_{p,q}$. Define

$$\begin{aligned} f : H_k &\to H_{k+1} \qquad \text{by} \\ f(e_{r,s}) &= e_{r,s} \\ f(\overline{t}) &= \begin{cases} x^{1+7\cdot 2^{n-4}}\overline{t} & \text{if } n \geq 6 \\ x^3\overline{t} & \text{if } n = 5 \end{cases} \end{aligned}$$

and extend it multiplicatively to $f\left(e_{r,s}\overline{t}\right)$. It is easy to check that such an f is a Hopf algebra isomorphism between H_k and H_{k+1}. Thus for $n \geq 5$ there is exactly one Hopf algebra of this type, $H_{c_3} = H_{2^{n-4}}$ with the cocycle defined by $\sigma(t,t) = \tilde{\sigma} = \frac{1+i}{2}1 + \frac{1-i}{2}x^{2^{n-2}}$ and $\mathbf{G}\left(H_{c_3}^*\right) = \langle\chi\rangle \times \langle\varphi\rangle \cong \mathbb{Z}_{2^{n-2}} \times \mathbb{Z}_2$, where $\chi(x) = \tau$, $\chi(y) = -1$, $\chi\left(\overline{t}\right) = 1$ (τ is some primitive 2^{n-2}-th root of unity), $\varphi(x) = \varphi(y) = 1$, $\varphi\left(\overline{t}\right) = -1$. There are $3\cdot 2^{n-3}$ degree 2 irreducible representations $\pi_{m,r,u}$, where $u = 1,2; r = 0,1$, and $m = 0,\ldots 2^{n-3}-1$ for $u = 1$ and $m = 0,\ldots 2^{n-4}-1$ for $u = 2$, defined by

$$\pi_{m,r,u}(x) = \omega^{4m+2r+u}\begin{pmatrix} 1 & 0 \\ 0 & i^u \end{pmatrix} \; \pi_{m,r,u}(y) = (-1)^r\begin{pmatrix} 1 & 0 \\ 0 & (-1)^u \end{pmatrix} \; \pi_{m,r,u}(\overline{t}) = \begin{pmatrix} 0 & i^{u^2} \\ 1 & 0 \end{pmatrix}$$

where ω is a primitive 2^{n-1}-th root of unity such that $\omega^2 = \tau$. The product of two 2-dimensional irreducible representations is

$$\begin{aligned} \pi_{l,s,1} \bullet \pi_{m,r,1} &= \chi^{2(l+m)+(r+s)+2^{n-4}+1}(1+\varphi) + \pi_{l+m,r+s,2} \\ \pi_{l,s,2} \bullet \pi_{m,r,2} &= \chi^{2(l+m)+(r+s)+2}\left(1+\varphi+\chi^{2^{n-3}}+\chi^{2^{n-3}}\varphi\right) \\ \pi_{l,s,2} \bullet \pi_{m,r,1} &= \pi_{l+m-2^{n-5},r+s,1} + \pi_{l+m+2^{n-5},r+s,1} = \pi_{m,r,1} \bullet \pi_{l,s,2} \end{aligned}$$

where we substitute $\pi_{a,b,u} = \pi_{frac2^{n-3}uc+a+d,b-2d,u}$ for $c, d \in \mathbb{Z}$ if necessary. In particular, the only selfdual 2-dimensional irreducible representations are $\pi_{2^{n-5}-1,1,2}$ and $\pi_{2^{n-4}-1,1,2}$.

4.9. Assume $n \geq 5$. The action is given by

$$\begin{aligned} t &\rightharpoonup x = x^{2^{n-3}-1}y \\ t &\rightharpoonup y = x^{2^{n-2}}y \end{aligned}$$

Then $\mathbf{G}(H) \cap Z(H) = \left\langle x^{2^{n-3}}y \right\rangle \cong \mathbb{Z}_4$ and $t \rightharpoonup e_{i,j} = e_{i(3\cdot 2^{n-3}-1)+2^{n-2}j,j+i}$. Write $v = \sigma(t,t) = \sum c_{i,j}e_{i,j}$. By (4.1) and (4.4)

$$\begin{aligned} c_{i,j} &= c_{i(3\cdot 2^{n-3}-1)+2^{n-2}j,j+i} \\ c_{i+p,j+q} &= (-1)^{jp-(j+i)\left(\left(3\cdot 2^{n-3}-1\right)p+2^{n-2}q\right)} c_{i,j}c_{p,q} = (-1)^{ip} c_{i,j}c_{p,q} \end{aligned}$$

Then

$$\begin{aligned} (c_{0,1})^2 &= c_{0,0} = 1 \\ c_{4,0} &= c_{-4,0} \\ c_{8,0} &= (c_{4,0})^2 = c_{4,0}c_{-4,0} = c_{0,0} = 1 \\ c_{2,0} &= c_{-2,0} \\ c_{4,0} &= (c_{2,0})^2 = c_{2,0}c_{-2,0} = c_{0,0} = 1 \\ c_{1,0} &= c_{-1,1} \\ (c_{1,0})^2 &= c_{1,0}c_{-1,1} = -c_{0,1} \\ c_{p,q} &= (-1)^{p-1} c_{1,0}c_{p-1,0}c_{0,q} = (-1)^{\frac{p(p-1)}{2}+q} c_{1,0}^{p+2q} \end{aligned}$$

and therefore $c_{1,0} = i^k$ for $k = 0, 1, 2, 3$, where i is a primitive 4-th root of unity, $c_{0,1} = (-1)^{k+1}$ and

$$\sigma(t,t) = \sum_{p,q} (-1)^{\frac{p(p-1)}{2}+q} i^{k(p+2q)}e_{p,q} = x^{2^{n-3}(1-k)} \left(\frac{1+i}{2}1 + \frac{1-i}{2}x^{2^{n-2}}\right) y^{1+k}$$

For $k = 0, 1, 2, 3$ let H_k be the Hopf algebras with the structures described above with cocycles $\sigma_k(t,t) = \sum_{p,q} (-1)^{\frac{p(p-1)}{2}+q} i^{k(p+2q)}e_{p,q}$. Define

$$\begin{aligned} f : H_k &\to H_{k+1} \qquad \text{by} \\ f(e_{r,s}) &= e_{r,s} \\ f\left(\overline{t}\right) &= x\overline{t} \end{aligned}$$

and extend it multiplicatively to $f\left(e_{r,s}\overline{t}\right)$. It is easy to check that such an f is a Hopf algebra isomorphism between H_k and H_{k+1}. Thus for $n \geq 5$ there is exactly one Hopf algebra of this type, H_{c_4} with the cocycle defined by $\sigma(t,t) = \tilde{\sigma} = \frac{1+i}{2}1 + \frac{1-i}{2}x^{2^{n-2}}$ and $\mathbf{G}\left(H_{c_4}^*\right) = \langle\chi\rangle \times \langle\varphi\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$, where $\chi(x) = i$, $\chi(y) = -1$, $\chi\left(\overline{t}\right) = 1$, $\varphi(x) = \varphi(y) = 1$, $\varphi\left(\overline{t}\right) = -1$. There are $2^{n-1}-2$ degree 2 irreducible representations $\pi_{m,r,u}$, where $u = 0, 1, 2; r = 0, 1$, and $m = 1 \ldots 2^{n-4} - 1$ for $u = 0$, $m = 0 \ldots 2^{n-3} - 1$ for $u = 1$ and $m = 0 \ldots 2^{n-5} - 1, 2^{n-4} \ldots 3 \cdot 2^{n-5} - 1$ for $u = 2$, defined by

$\pi_{m,r,u}(x) = i^{r^2}\begin{pmatrix} \omega^{4m+u} & 0 \\ 0 & i^u\omega^{-4m-u}\end{pmatrix}$ $\pi_{m,r,u}(y) = (-1)^r\begin{pmatrix} 1 & 0 \\ 0 & (-1)^u\end{pmatrix}$ $\pi_{m,r,u}(\overline{t}) = \begin{pmatrix} 0 & i^{u^2} \\ 1 & 0\end{pmatrix}$

where ω is a primitive 2^{n-1}-th root of unity. The product of two 2-dimensional irreducible representations is

$$
\begin{aligned}
\pi_{l,s,v} \bullet \pi_{m,r,0} &= \begin{cases} \chi^{r+s}\left(1+\chi^2+\varphi+\chi^2\varphi\right) & \text{if } v=0, l=m=2^{n-5} \\ \chi^{r+s}\left(1+\varphi\right)+\pi_{l+m+2^{n-4}rs,r+s,v} & \text{if } v=0, l=m\neq 2^{n-5} \\ \chi^{r+s+2}\left(1+\varphi\right)+\pi_{l-m+2^{n-4}rs,r+s,v} & \text{if } v=0, l+m=2^{n-4}, l\neq m \\ \pi_{l+m+2^{n-4}rs,r+s,v}+\pi_{l-m+2^{n-4}rs,r+s,v} & \text{otherwise}\end{cases} \\
&= \pi_{m,r,0}\bullet\pi_{l,s,v} \\
\pi_{l,s,2}\bullet\pi_{m,r,2} &= \pi_{l+m+2^{n-4}rs+1,r+s,0} + \begin{cases} \chi^{r+s+2}+\chi^{r+s+2}\varphi & \text{if } l=m \\ \chi^{r+s}+\chi^{r+s}\varphi & \text{if } l-m=2^{n-4} \\ \pi_{l-m+2^{n-4}(rs+1),r+s,0} & \text{otherwise}\end{cases} \\
\pi_{l,s,1}\bullet\pi_{m,r,1} &= \pi_{l+m+2^{n-4}rs,r+s,2} + \begin{cases} \chi^{r+s+1}+\chi^{r+s+1}\varphi & \text{if } l=m \\ \chi^{r+s-1}+\chi^{r+s-1}\varphi & \text{if } l-m=2^{n-4} \\ \pi_{l-m+2^{n-4}(rs+r+s),r+s+1,0} & \text{otherwise}\end{cases} \\
\pi_{l,s,1}\bullet\pi_{m,r,2} &= \pi_{2^{n-4}(rs+1)-l-m,r+s+1,1}+\pi_{m-l+2^{n-4}(rs+r+s),r+s+1,1} = \pi_{m,r,2}\bullet\pi_{l,s,1}
\end{aligned}
$$

where we substitute $\pi_{a,b,u} = \pi_{2^{n-3}c+a,b-2d,u}$, $\pi_{a,b,0} = \pi_{-a,b,0}$ and $\pi_{a,b,2} = \pi_{2^{n-4}-a-1,b,0}$ for $c,d\in\mathbb{Z}$ if necessary. In particular, there are 2^{n-3} selfdual 2-dimensional irreducible representations, namely $\pi_{m,0,0}$ for $m = 1\ldots 2^{n-4}-1$, $\pi_{2^{n-5},1,0}$ and $\pi_{m,1,2}$ for $m = 0\ldots 2^{n-5}-1, 2^{n-4}\ldots 3\cdot 2^{n-5}-1$.

5. Schur indicators

In this section we determine Schur indicators for 2-dimensional irreducible representations of Hopf algebras described in the previous section, and so complete Table 1.

Let H be one of the Hopf algebras considered in Section 4 with

$$\theta(t) = \sum_{ijpq}\theta^{(t)}_{i,j;p,q}e_{i,j}\otimes e_{p,q} = \sum_{ijpq}(-1)^{jp}\,e_{i,j}\otimes e_{p,q} \tag{5.1}$$

$$\sigma(t,t) = \sum_{ij}c_{i,j}e_{i,j} \tag{5.2}$$

$$t \rightharpoonup e_{i,j} = e_{\alpha_1(i,j),\alpha_2(i,j)} \tag{5.3}$$

Let us first compute $\Lambda^{[2]}$ for H. By [KMM, 3.1]

$$
\begin{aligned}
\Lambda^{[2]} &= \frac{1}{2}\left(\sum_{i,j:2i\equiv 0}e_{i,j} + \sum_{i,j:\alpha_1(i,j)\equiv -i;\alpha_2(i,j)\equiv j}\tau^{(t)}_{i,j;-i,j}c_{i,j}e_{i,j}\right) \\
&= \frac{1}{2}\left(e_{0,0}+e_{0,1}+e_{2^{n-2},0}+e_{2^{n-2},1}+\sum_{i,j:\alpha_1(i,j)\equiv -i;\alpha_2(i,j)\equiv j}(-1)^{ij}\,c_{i,j}e_{i,j}\right)
\end{aligned}
$$

Therefore

$$\begin{aligned}
\Lambda^{[2]}_{a_1:y^k} &= e_{0,0}+e_{2^{n-2},0}+\frac{1}{2}\left(e_{0,1}+e_{2^{n-2},1}\right)+\frac{(-1)^{2^{n-3}+k}}{2}\left(e_{2^{n-3},1}+e_{3\cdot 2^{n-3},1}\right)\\
\Lambda^{[2]}_{a_2:x^{2^{n-2}k}} &= e_{0,0}+e_{2^{n-2},0}+\frac{1}{2}\left(e_{0,1}+e_{2^{n-2},1}\right)+\frac{1}{2}\sum_{i=1}^{2^{n-2}-1}(-1)^{ik}\left(e_{i,0}+e_{-i,0}\right)\\
\Lambda^{[2]}_{b_1:x^{2^{n-2}k}y^l} &= e_{0,0}+e_{2^{n-2},0}+\frac{1+(-1)^l}{2}\left(e_{0,1}+e_{2^{n-2},1}\right)\\
&\quad+\frac{1}{2}\sum_{i=1}^{2^{n-2}-1}\sum_{j=0}^{1}(-1)^{ij+ik+jl}\left(e_{i,j}+e_{-i,j}\right)\\
\Lambda^{[2]}_{b_2:y^k} &= e_{0,0}+e_{2^{n-2},0}+e_{0,1}+e_{2^{n-2},1}\\
\Lambda^{[2]}_{b_3:y^k} &= e_{0,0}+e_{2^{n-2},0}+\frac{1+(-1)^k}{2}\left(e_{0,1}+e_{2^{n-2},1}\right)\\
&\quad+\frac{1}{2}\sum_{i=1}^{2^{n-3}-1}\sum_{j=0}^{1}(-1)^{jk}\left(e_{2i,j}+e_{-2i,j}\right)\\
\Lambda^{[2]}_{c_1:x^k\tilde{\sigma}} &= e_{0,0}+e_{2^{n-2},0}+e_{0,1}+e_{2^{n-2},1}\\
\Lambda^{[2]}_{c_2:x^{2^{n-3}l}y^l\tilde{\sigma}} &= e_{0,0}+e_{2^{n-2},0}+\frac{1+(-1)^l}{2}\left(e_{0,1}+e_{2^{n-2},1}\right)\\
&\quad+\frac{1}{2}\sum_{i=1}^{2^{n-3}-1}\sum_{j=0}^{1}(-1)^{li+lj}\left(e_{2i,j}+e_{-2i,j}\right)\\
\Lambda^{[2]}_{c_3} &= e_{0,0}+e_{2^{n-2},0}+\frac{1}{2}\left(e_{0,1}+e_{2^{n-2},1}\right)+\frac{1}{2}\left(e_{2^{n-3},1}+e_{3\cdot 2^{n-3},1}\right)\\
\Lambda^{[2]}_{c_4} &= e_{0,0}+e_{2^{n-2},0}+\frac{1}{2}\left(e_{0,1}+e_{2^{n-2},1}\right)+\frac{1}{2}\sum_{i=1}^{2^{n-4}-1}\left(e_{4i,0}+e_{-4i,0}\right)\\
&\quad+\frac{1}{2}\sum_{i=0}^{2^{n-4}-1}\left(e_{4i+2,1}+e_{-4i-2,1}\right)
\end{aligned}$$

Let $\pi=\psi_{p,q}\uparrow^H$, where $\psi_{p,q}$ is a representation of K, defined by $\psi_{p,q}\left(e_{i,j}\right)=\delta_{i,p}\delta_{j,q}$. Then

$$\chi_\pi\left(e_{i,j}\right)=\psi_{p,q}\left(e_{i,j}\right)+\psi_{p,q}\left(t\rightharpoonup e_{i,j}\right)=\delta_{i,p}\delta_{j,q}+\delta_{\alpha_1(i,j),p}\delta_{\alpha_2(i,j),q} \tag{5.4}$$

Using formula (5.4) we can compute Schur indicators and integers $N_{2,0}$, $N_{2,1}$ and $N_{2,-1}$ for the Hopf algebras considered in Section 4.

In $H_{a_1:1}$ if $n=3$, then $\nu\left(\chi_{\pi_m}\right)=(-1)^m$ for $m=0,1$; if $n\geq 4$ then

$$\nu\left(\chi_{\pi_m}\right)=\begin{cases}1 & \text{if } m=0,2^{n-3}\\ 0 & \text{otherwise.}\end{cases}$$

In $H_{a_1:y}$ if $n=3$, then $\nu(\chi_{\pi_m})=1$ for $m=0,1$; if $n\geq 4$ then

$$\nu(\chi_{\pi_m})=\begin{cases} 1 & \text{if } m=0 \\ -1 & \text{if } m=2^{n-3} \\ 0 & \text{otherwise.}\end{cases}$$

In $H_{a_2:x^{(2^{n-2})k}}$ for $k=0,1$, $\nu\left(\chi_{\pi_{m,r}}\right)=\begin{cases} 1 & \text{if } m=2^{n-3} \text{ and } r=1 \\ (-1)^{mk} & \text{if } r=0 \\ 0 & \text{otherwise.}\end{cases}$

In $H_{b_1:x^{(2^{n-2})k}y^l}$ for $k,l\in\{(0,0),(0,1),(1,1)\}$, $\nu\left(\chi_{\pi_{m,r}}\right)=(-1)^{mr+mk+rl}$.

In $H_{b_2:y^k}$ for $k=0,1$, $\nu(\chi_\pi)=0$ for every degree 2 irreducible representation π.

In $H_{b_3:y^k}$ for $k=0,1$, $\nu\left(\chi_{\pi_{m,r}}\right)=\begin{cases} (-1)^{rk} & \text{if } m \text{ is even} \\ 0 & \text{otherwise.}\end{cases}$

In $H_{c_1:x^k\tilde{\sigma}}$ for $k=0,1$, $\nu(\chi_\pi)=0$ for every degree 2 irreducible representation π.

In $H_{c_2:x^{(2^{n-3})l}y^l\tilde{\sigma}}$ for $l=0,1$, $\nu\left(\chi_{\pi_{m,r}}\right)=\begin{cases} (-1)^{l(m/2+r)} & \text{if } m \text{ is even} \\ 0 & \text{otherwise.}\end{cases}$

In H_{c_3} $\quad\nu\left(\chi_{\pi_{m,r,u}}\right)=\begin{cases} 1 & \text{if } (m,r,u)\in\{(2^{n-5}-1,1,2),(2^{n-4}-1,1,2)\} \\ 0 & \text{otherwise.}\end{cases}$

In H_{c_4} $\quad\nu\left(\chi_{\pi_{m,r,u}}\right)=\begin{cases} 1 & \text{if } (m,r,u)\in\{(m,0,0),(2^{n-5},1,0),(m,1,2)\} \\ 0 & \text{otherwise.}\end{cases}$

For each example the ranges of m, r, and u are given in the descriptions of the corresponding Hopf algebras in Section 4. Using that, it follows from the above computations that $N_{2,i}$ has the values as in Table 1. Therefore, comparing different invariants from Table 1, we see that Hopf algebras described in Section 4 are all non-isomorphic.

6. Generalized power map property

For $n\in\mathbb{N}$ let us consider the generalized power map

$$[n]=m^{n-1}\circ\Delta_{n-1}:H\to H$$

defined by

$$h^{[n]}=[n](h)=\sum h_1\cdots h_n$$

We say that H satisfies the generalized power map property (Q_n) if

$$h^{[n]}=[n](h)=\varepsilon(h)1$$

(see [K2]).

Let $H=K\#_\sigma^\theta F$, a bicrossed product of $K=(kG)^*$ and $F=k\langle t\rangle=k\mathbb{Z}_2$ with an action $\rightharpoonup: F\otimes K\to K$, trivial coaction, a cocycle $\sigma:F\otimes F\to K$ and a dual cocycle $\theta:F\to K\otimes K$. Let $|G|=s$ and let $\{e_g|g\in G\}$ be the basis of $K=(kG)^*$ dual to the basis G of kG. Write

$$\begin{aligned}\theta(t) &= \sum_{g,h\in G}\theta_{g;h}^{(t)}p_g\otimes p_h \\ \sigma(t,t) &= \sum_g \sigma_g^{(t)}p_g\end{aligned}$$

Then the following properties are satisfied:

$$\theta^{(t)}_{g;h}\theta^{(t)}_{gh;k} = \theta^{(t)}_{g;hk}\theta^{(t)}_{h;k} \tag{6.1}$$

$$\theta^{(t)}_{g;1} = \theta^{(t)}_{1;g} = 1 \tag{6.2}$$

$$\sigma^{(t)}_{g} = \sigma^{(t)}_{t\rightharpoonup g} \tag{6.3}$$

$$\sigma^{(t)}_{1} = 1 \tag{6.4}$$

$$\theta^{(t)}_{g;h}\theta^{(t)}_{t\rightharpoonup g;t\rightharpoonup h}\sigma^{(t)}_{g}\sigma^{(t)}_{h} = \sigma^{(t)}_{gh} \tag{6.5}$$

Proposition 6.1. *Let H be as described above. Then*

$$\left(p_h\overline{t}\right)^{[2s]} = \begin{cases} \sum_{g\in G}\theta^{(t)}_{g;(t\rightharpoonup g)(g(t\rightharpoonup g))^{s-1}}\theta^{(t)}_{(t\rightharpoonup g);(g(t\rightharpoonup g))^{s-1}}\theta^{(t)}_{g;(t\rightharpoonup g)(g(t\rightharpoonup g))^{s-2}}\cdots\theta^{(t)}_{g;(t\rightharpoonup g)}p_g\sigma\left(t,t\right)^{s}, & h=1 \\ 0, & h\neq 1. \end{cases}$$

Proof.

$$\begin{aligned} \left(p_h\overline{t}\right)^{[2s]} &= [2s]\left(e_g\overline{t}\right) = \sum_{g_1g_2\cdots g_{2s}=h}\theta^{(t)}_{g_1;g_2\cdots g_{2s}}\theta^{(t)}_{g_2;g_3\cdots g_{2s}}\cdots\theta^{(t)}_{g_{2s-1};g_{2s}}p_{g_1}\overline{t}p_{g_2}\overline{t}\cdots p_{g_{2s}}\overline{t} \\ &= \sum_{g_1g_2\cdots g_{2s}=h}\theta^{(t)}_{g_1;g_2\cdots g_{2s}}\theta^{(t)}_{g_2;g_3\cdots g_{2s}}\cdots\theta^{(t)}_{g_{2s-1};g_{2s}}p_{g_1}p_{t\rightharpoonup g_2}\overline{t}^2\cdots p_{g_{2s-1}}p_{t\rightharpoonup g_{2s}}\overline{t}^2 \\ &= \sum_{g_1g_2\cdots g_{2s}=h}\theta^{(t)}_{g_1;g_2\cdots g_{2s}}\theta^{(t)}_{g_2;g_3\cdots g_{2s}}\cdots\theta^{(t)}_{g_{2s-1};g_{2s}}p_{g_1}p_{t\rightharpoonup g_2}\cdots p_{g_{2s-1}}p_{t\rightharpoonup g_{2s}}\overline{t}^{2s} \\ &= \sum_{(g_1(t\rightharpoonup g_1))^s=h}\theta^{(t)}_{g_1;(t\rightharpoonup g_1)g_1\cdots(t\rightharpoonup g_1)}\theta^{(t)}_{(t\rightharpoonup g_1);g_1\cdots(t\rightharpoonup g_1)}\cdots\theta^{(t)}_{g_1;(t\rightharpoonup g_1)}p_{g_1}\sigma\left(t,t\right)^{s} \end{aligned}$$

Since $|G|=s$, $\left(g_1\left(t\rightharpoonup g_1\right)\right)^s=1$ and therefore

$$\left(p_h\overline{t}\right)^{[2s]} = \begin{cases} \sum_{g\in G}\theta^{(t)}_{g;(t\rightharpoonup g)g\cdots(t\rightharpoonup g)}\theta^{(t)}_{(t\rightharpoonup g);g\cdots(t\rightharpoonup g)}\cdots\theta^{(t)}_{g;(t\rightharpoonup g)}p_g\sigma\left(t,t\right)^{s} & \text{if } h=1 \\ 0 & \text{otherwise.} \end{cases}$$

□

Theorem 6.2. *Let H be a semisimple Hopf algebra of dimension 2^{n+1} over a field k of characteristic 0, which has a commutative Hopf subalgebra of index 2, satisfies the generalized power map property $(Q_{2^{n+1}})$, that is, for any $h\in H$*

$$(h)^{\left[2^{n+1}\right]} = \left[2^{n+1}\right](h) = \varepsilon\left(h\right)1$$

Proof. By [K1, Lemma 3], we may assume that k is algebraically closed. By Proposition 2.4 H becomes a bicrossed product $K\#^{\theta}_{\sigma}F$ of $K=kG\cong(kG)^*$ and $F=k\langle t\rangle\cong k\mathbb{Z}_2$ with an action $\rightharpoonup$: $F\otimes K\to K$, trivial coaction, a cocycle $\sigma: F\otimes F\to K$ and a dual cocycle $\theta: F\to K\otimes K$, and the action by t is an order 2 Hopf algebra automorphism of K. Therefore we can apply Proposition 6.1. If H is trivial, then

H obviously satisfies $(\mathrm{Q}_{2^{n+1}})$. If H is nontrivial then, by [K4, Theorem 1.1], $\exp G$ divides 2^{n-1}. Then

$$\left(p_g\bar{t}\right)^{\left[2^{n+1}\right]} = 0 = \varepsilon\left(p_g\bar{t}\right)1 \quad \text{if } g \neq 1$$

and, since $\left(g\left(t \rightharpoonup g\right)\right)^{2^n} = 1$,

$$\begin{aligned}
\left(p_1\bar{t}\right)^{\left[2^{n+1}\right]} &= \sum_{g\in G}\theta^{(t)}_{g;(t\rightharpoonup g)(g(t\rightharpoonup g))^{2^n-1}}\theta^{(t)}_{(t\rightharpoonup g);(g(t\rightharpoonup g))^{2^n-1}}\cdots\theta^{(t)}_{g;(t\rightharpoonup g)}p_g\sigma\left(t,t\right)^{2^n}\\
&= \sum_{g\in G}\theta^{(t)}_{g;(t\rightharpoonup g)(g(t\rightharpoonup g))^{2^{n-1}-1}}\theta^{(t)}_{(t\rightharpoonup g);(g(t\rightharpoonup g))^{2^{n-1}-1}}\cdots\theta^{(t)}_{g;(t\rightharpoonup g)}\theta^{(t)}_{(t\rightharpoonup g);1}\\
&\qquad \theta^{(t)}_{g;(t\rightharpoonup g)(g(t\rightharpoonup g))^{2^{n-1}-1}}\theta^{(t)}_{(t\rightharpoonup g);(g(t\rightharpoonup g))^{2^{n-1}-1}}\cdots\theta^{(t)}_{g;(t\rightharpoonup g)}p_g\sigma\left(t,t\right)^{2^n}
\end{aligned}$$

From formula (6.1) we obtain the following formulas:

$$\theta^{(t)}_{(t\rightharpoonup g);(g(t\rightharpoonup g))^r}\theta^{(t)}_{g((t\rightharpoonup g)g)^{r-1};(t\rightharpoonup g)} = \theta^{(t)}_{(t\rightharpoonup g);g((t\rightharpoonup g)g)^{r-1}}\theta^{(t)}_{((t\rightharpoonup g)g)^r;(t\rightharpoonup g)} \tag{6.6}$$

$$\theta^{(t)}_{g;((t\rightharpoonup g)g)^r(t\rightharpoonup g)}\theta^{(t)}_{((t\rightharpoonup g)g)^r;(t\rightharpoonup g)} = \theta^{(t)}_{g;((t\rightharpoonup g)g)^r}\theta^{(t)}_{g((t\rightharpoonup g)g)^r;(t\rightharpoonup g)} \tag{6.7}$$

Applying (6.6) and (6.7) consecutively from the left to the right, starting with $\theta^{(t)}_{(t\rightharpoonup g);g(t\rightharpoonup g)}\theta^{(t)}_{g;(t\rightharpoonup g)}$ we get that

$$\begin{aligned}
&\theta^{(t)}_{g;(t\rightharpoonup g)(g(t\rightharpoonup g))^{2^{n-1}-1}}\theta^{(t)}_{(t\rightharpoonup g);(g(t\rightharpoonup g))^{2^{n-1}-1}}\cdots\theta^{(t)}_{(t\rightharpoonup g);g(t\rightharpoonup g)}\theta^{(t)}_{g;(t\rightharpoonup g)}\\
= \ &\theta^{(t)}_{g((t\rightharpoonup g)g)^{2^{n-1}-1};(t\rightharpoonup g)}\theta^{(t)}_{g;(g(t\rightharpoonup g))^{2^{n-1}-1}}\theta^{(t)}_{(t\rightharpoonup g);g((t\rightharpoonup g)g)^{2^{n-1}-2}}\cdots\theta^{(t)}_{g;(t\rightharpoonup g)g}\theta^{(t)}_{(t\rightharpoonup g);g}
\end{aligned}$$

Moreover, $\theta^{(t)}_{g((t\rightharpoonup g)g)^{2^{n-1}-1};(t\rightharpoonup g)} = \theta^{(t)}_{(t\rightharpoonup g);g((t\rightharpoonup g)g)^{2^{n-1}-1}}$ since $g\left(\left(t \rightharpoonup g\right)g\right)^{2^{n-1}-1} = \left(t \rightharpoonup g\right)^{-1}$ and formulas (6.1) and (6.2) imply that $\theta^{(t)}_{g;g^{-1}} = \theta^{(t)}_{g^{-1};g}$. Therefore

$$\begin{aligned}
\left(p_1\bar{t}\right)^{\left[2^{n+1}\right]} &= \sum_{g\in G}\theta^{(t)}_{(t\rightharpoonup g);g((t\rightharpoonup g)g)^{2^{n-1}-1}}\theta^{(t)}_{g;(g(t\rightharpoonup g))^{2^{n-1}-1}}\cdots\theta^{(t)}_{g;(t\rightharpoonup g)g}\theta^{(t)}_{(t\rightharpoonup g);g}\\
&\qquad \theta^{(t)}_{g;(t\rightharpoonup g)(g(t\rightharpoonup g))^{2^{n-1}-1}}\theta^{(t)}_{(t\rightharpoonup g);(g(t\rightharpoonup g))^{2^{n-1}-1}}\cdots\theta^{(t)}_{(t\rightharpoonup g);g(t\rightharpoonup g)}\theta^{(t)}_{g;(t\rightharpoonup g)}\left(\sigma^{(t)}_g\right)^{2^n}p_g
\end{aligned}$$

Let us now apply formula (6.5) $2^n - 1$ times, taking into account that by formula (6.3) $\sigma^{(t)}_g = \sigma^{(t)}_{t\rightharpoonup g}$:

$$\begin{aligned}
\left(p_1\bar{t}\right)^{\left[2^{n+1}\right]} &= \sum_{g\in G}\theta^{(t)}_{(t\rightharpoonup g);g((t\rightharpoonup g)g)^{2^{n-1}-1}}\theta^{(t)}_{g;(g(t\rightharpoonup g))^{2^{n-1}-1}}\cdots\theta^{(t)}_{g;(t\rightharpoonup g)g}\\
&\qquad \theta^{(t)}_{g;(t\rightharpoonup g)(g(t\rightharpoonup g))^{2^{n-1}-1}}\theta^{(t)}_{(t\rightharpoonup g);(g(t\rightharpoonup g))^{2^{n-1}-1}}\cdots\theta^{(t)}_{(t\rightharpoonup g);g(t\rightharpoonup g)}\left(\sigma^{(t)}_g\right)^{2^n-2}\sigma^{(t)}_{g(t\rightharpoonup g)}p_g\\
&= \ldots = \sum_{g\in G}\theta^{(t)}_{(t\rightharpoonup g);g((t\rightharpoonup g)g)^{2^{n-1}-1}}\theta^{(t)}_{g;(t\rightharpoonup g)(g(t\rightharpoonup g))^{2^{n-1}-1}}\sigma^{(t)}_g\sigma^{(t)}_{(t\rightharpoonup g)(g(t\rightharpoonup g))^{2^{n-1}-1}}p_g\\
&= \sum_{g\in G}\sigma^{(t)}_1 p_g \overset{(6.4)}{=} 1 = \varepsilon\left(p_1\bar{t}\right)1
\end{aligned}$$

$\square$

Corollary 6.3. *Let H be a semisimple Hopf algebra of dimension 2^{n+1}, $n \geq 3$, over a field k of characteristic 0 with $\mathbf{G}(H) \cong \mathbb{Z}_{2^{n+1}} \times \mathbb{Z}_2$. Then H satisfies the generalized power map property ($Q_{2^{n+1}}$).*

Proof. By [K1, Lemma 3], we may assume that k is algebraically closed. If H is trivial, then H obviously satisfies ($\mathrm{Q}_{2^{n+1}}$). If H is nontrivial, it was described in Section 4 and we can apply Theorem 6.2. □

Corollary 6.4. *Every semisimple Hopf algebra H of dimension 16 over a field k of characteristic 0 satisfies the generalized power map property (Q_{16}).*

Proof. By [K1, Lemma 3], we may assume that k is algebraically closed. If H is trivial it obviously satisfies (Q_{16}). All nontrivial semisimple Hopf algebras of dimension 16 were described in [K4]; there are exactly 16 such Hopf algebras and they all have a commutative Hopf subalgebra of index 2 and therefore we can apply Theorem 6.2. □

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Department of Mathematics, Syracuse University, Syracuse, NY 13244-1150
E-mail address: ykashina@syr.edu