ABELIAN SOLITONS

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Abstract. We describe a new algebraically completely integrable system, whose integral manifolds are co-elliptic subvarieties of Jacobian varieties. This is a multi-periodic extension of the Krichever-Treibich-Verdier system, which consists of elliptic solitons.

The goal of this work is to generalize the theory of elliptic solitons, which was developed by A. Treibich and J.-L. Verdier based on earlier works by H. Airault, H.P. McKean and J. Moser [AMcKM], and I.M. Krichever [K]. Elliptic solitons are a subclass of algebro-geometric KP solutions (cf. §**1**), namely those that are elliptic functions in the x variable. Viewed as Jacobians which contain special configurations of elliptic curves, the elliptic solitons are dense in the classical topology [CPP].

We pose the problem of describing the moduli of abelian solitons, that is, KP-solitons whose first $k \geq 1$ flow variables are tangent to some proper abelian subvariety of the Jacobian. We explore several possible constructions of such abelian solitons. Most of these actually fail, cf. the appendix. We do succeed in constructing, for any natural number k and elliptic curve C, an infinite sequence of families of co-elliptic solitons, which are periodic along k-dimensional abelian subvarieties of $(k+1)$-dimensional Jacobians, with quotients isogenous to the given elliptic curve C. We show (Theorem 2.1) how these families of co-elliptic solitons organize into algebraically integrable systems (aci), which can be identified as nonlinear subsystems of Markman's system of meromorphic Higgs bundles.

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§1 Abelian solitons

We fix a smooth point p on a curve $\widetilde{C}$ of genus g, use another smooth point p_0 to define the Abel map $A_{\widetilde{C}}$ on the smooth part of $\widetilde{C}$ (but this choice is not intrinsic to the problem) and study the image of a neighborhood of $A_{\widetilde{C}}(p)$ in Jac$\widetilde{C}$. This allows us to define the g vector fields which are the flows of the KP hierarchy, cf. e.g. [SW] for a definition of the hierarchy.

1.1 The osculating spaces. We describe these both with and without choice of a local parameter near p. Such a parameter is traditionally denoted by z^{-1}, with $z^{-1}(p) = 0$, and extends through the disk $|z^{-1}| \leq 1$ [SW]. Let $\omega_1, \dots, \omega_g$ be a basis of $H^0(\widetilde{C}, \Omega_{\widetilde{C}})$ and $\omega_i = \sum_{j=0}^{\infty} a_j^{(i)} z^{-j} d(z^{-1})$ the local expansion near p. The vectors $U_{j+1} = (a_j^{(1)}, \dots, a_j^{(g)})$ osculate the curve $A(\widetilde{C})$ at $A(p)$ to order $j+1$, for instance U_1 is a tangent vector since $A(q) = (\int_{p_0}^{q} \omega_i)_{1 \leq i \leq g}$. They generate translation-invariant vector fields on Jac$(\widetilde{C})$ which are called the "KP flows." Without choosing z^{-1}, the curve and the point only determine the flag of hyperosculating vector spaces $< U_1 > \subset < U_1, U_2 > ... \subset \mathbf{T} := H^0(\widetilde{C}, \Omega^1_{\widetilde{C}})^*$.

Our goal will be to 'integrate' the first k flows in the sense of finding a k-dimensional abelian subvariety whose tangent space at every point contains $< U_1, U_2, ..., U_k >$.

1.2.0 Elliptic solitons. By definition [TV1], these are KP solutions which are elliptic in the first variable; this means that the 1-parameter subgroup generated by U_1 is an elliptic curve C.

1.2.1 Tangential covers. Let $\widetilde{C}, C$ be (projective, integral) curves of (arithmetic) genus > 0, $p \in \widetilde{C}$, $q \in C$ smooth points and $\pi : (\widetilde{C}, p) \to (C, q)$ a finite pointed morphism. The choice of a smooth point in $\widetilde{C}$ and C determines Abel maps $A_{\widetilde{C}}$ and A_C; if π^* denotes pull-back of line bundles, the composition $\pi^* \circ A_C$ sends C to Jac$(\widetilde{C})$. Then, π is called a tangential cover if $\pi^* \circ A_C(C)$ and $A_{\widetilde{C}}(\widetilde{C})$ are tangent at the origin of Jac$(\widetilde{C})$.

[TV2] shows that elliptic solitons correspond to tangential covers of an elliptic curve (though not in a one-to-one manner; extra data are needed in both directions).

1.3.0 Abelian solitons. Motivated by 1.2, we say that $\widetilde{C}$ gives rise to a k-abelian soliton if Jac$(\widetilde{C})$ contains an abelian subvariety P of dimension $k < g$ and the flows $U_1, \dots, U_k$ are tangent to P.

1.3.1 Lemma. *A covering* $\pi: (\widetilde{C}, p) \to (C, q)$ *with a ramification point of index* $g - h + 1$ *at* p*, where* $h =$ *genus* C*, gives rise to a* $(g-h)$*-abelian soliton for* $P =$Prym(π)*. More generally,* $\mathrm{Jac}(\widetilde{C}) \sim A_1 \oplus A_2$ *with* A_i *an abelian subvariety of dimension* k_i *is an abelian soliton with respect to* $A_2 \Leftrightarrow \pi_{A_1}$ *of* $A_{\widetilde{C}}$ *has a ramification point of index* $k_2 + 1$ *at the origin.*

Note that we adopted [H]'s terminology, according to which the ramification index is the local-ring valuation $e_p(\tau)$ where τ is a local parameter at q, so that there is ramification exactly when the index is > 1. This lemma is simply a restatement of condition 1.3.0 on the osculating flag (1.1) at $A_{\widetilde{C}}(p)$ and $A_C(q)$ under the pull-back π^*. Indeed, $\mathrm{Jac}(\widetilde{C}) \sim \mathrm{Jac}(C) \oplus P$, where $P =$Prym(π) is the Prym variety of the cover, namely the connected component through the origin of the kernel of the norm map. Again up to isogeny, we call π_1 and π_2 the projections to $\mathrm{Jac}(C)$ and P, respectively. Then the osculating flag to $A_{\widetilde{C}}(\widetilde{C})$ is tangential to $P \Leftrightarrow$ the differential of the projection π_1 vanishes on the flag $\Leftrightarrow$ the projection π_1 restricted to $A_{\widetilde{C}}(\widetilde{C})$ is ramified at the origin; and an abelian soliton for $P \Leftrightarrow \pi_1$ has a ramification point of index $\dim P + 1$ at the origin. This situation is somewhat 'dual' to the elliptic soliton $\widetilde{C} \to C$, which indeed is unramified at p.

1.4 Coelliptic solitons. Abelian solitons constructed from covers seem to be quite rare, due to numerical constraints, cf. **Appendix**. Here are the families we could find. They give connected components of the "moduli space" of abelian solitons.

Construction. Fix integers $g \geq 2$ and $n \geq g$, and an elliptic curve C. The Hurwitz formula implies that a map $\widetilde{C} \to C$ of degree n will be ramified at a total of $2g - 2$ points (counting multiplicities), where g is the genus of $\widetilde{C}$. For the coelliptic solitons, we bring $g - 1$ of these ramification points together, i.e. we consider the family of all covers $\widetilde{C} \to C$ of degree n with one ramification point where g sheets come together, and generically $g - 1$ additional simple ramification points. Note that by Riemann's existence theorem (cf. e.g. [Mu p. 15]) such a cover is determined, up to finite choices, by specifying the location of the g branch points (=images of the ramification points) in C. Since the elliptic curve C has a 1-parameter group of automorphisms, we can fix the location of the multiplicity $g - 1$ branch point at the origin $q \in C$. Our remaining family of covers is $g - 1$ dimensional: it is quasi-finite over the $(g-1)$-st symmetric product of C. By Riemann's existence theorem, any such cover is algebraic. In fact, it is an abelian soliton: by Lemma 1.3.1, $\widetilde{C}$ osculates

P to order $g-1 = \dim P$ where P is the Prym variety of the cover.

Remark. The same idea won't give higher-than-coelliptic abelian solitons, for if $\widetilde{C} \to C$ is an n sheeted cover with branch pattern as above, then $g(\widetilde{C}) = n(h-1)+g$ where $h = g(C)$, $\dim P = (n-1)h - n + g$ and $\widetilde{C}$ osculates P to order $g-1$. We have an abelian soliton if $g - 1 \geq (n-1)h - n + g$ i.e. $(n-1)(h-1) \leq 0$, so either $n = 1$ or $h = 1$ again.

§2 Integrable systems

2.1 Theorem. *For every $n \geq g \geq 2$, the family of degree n, genus g coelliptic solitons constructed in §**1.4** is birationally equivalent to an algebraically completely integrable system with $g-1$ commuting Hamiltonians and $(g-1)$-dimensional Prym varieties P as Liouville tori.*

We prove the theorem by identifying our family of coelliptic solitons as the symplectic reduction of a well-known aci system, namely Markman's system of meromorphic Higgs bundles. We start in §**2.1** by reviewing Markman's system. Its total space X parametrizes semistable meromorphic Higgs bundles (E, ϕ) on a curve C with given structure group G (which we take to be either $GL(n)$ or $SL(n)$), polar divisor $D \subset C$, and conjugacy class O of the residue $Res_D(\phi)$ of ϕ along D. The base B parametrizes appropriate spectral covers $\widetilde{C} \to C$, and the Hamiltonian map $H : X \to B$ sends a meromorphic Higgs bundle (E, ϕ) to its spectrum $\widetilde{C}$ defined by the characteristic polynomial of ϕ.

For a general Hamiltonian function $h : B \to \mathbf{C}$ (or a collection of commuting Hamiltonians $h : B \to \mathbf{C}^k$), it is *not* possible to reduce a given algebraically integrable system $H : X \to B$ to one whose base is the hypersurface (or subvariety) $\{h = 0\} \subset B$. We explain this difficulty in §**2.2**, and analyze the conditions under which such symplectic reduction *can* be done in our algebraic setting.

In §**2.3** we construct (generically) an embedding of our family of coelliptic solitons into Markman's system. The image satisfies the conditions of **2.2**, so it can be identified with a symplectic reduction of Markman's system, completing the proof.

§2.1 Markman's system

Let C be a (complete, non-singular) algebraic curve with canonical bundle ω_C, and let D be an effective divisor on C. A meromorphic Higgs bundle on C is a pair (E, ϕ) where E is a rank n vector bundle on C

and $\phi : E \to E \otimes \omega_C(D)$ is an endomorphism of E taking values in meromorphic differentials on C with poles on D. There is a moduli space $\mathcal{H}iggs^n_{C,D}$ parametrizing Higgs bundles on C satisfying an appropriate stability condition, cf. [M, DM1].

A Higgs bundle (E, ϕ) determines a spectral cover $\widetilde{C}$, namely the subscheme of $Tot(\omega_C(D))$ defined by the characteristic polynomial of ϕ, where $Tot(\omega_C(D))$ is the total space of the line bundle $\omega_C(D)$ on C. The natural projection of $Tot(\omega_C(D))$ restricts to a map of degree n, $\pi : \widetilde{C} \to C$. The Higgs bundle determines also a sheaf L on $\widetilde{C}$, generically a line bundle, parametrizing the eigenspaces of the endomorphism ϕ. Conversely, the Higgs bundle (E, ϕ) can be recovered from the spectral data $(\widetilde{C}, L)$: E is the direct image $\pi_*(L)$, while ϕ is π_* of multiplication by the tautological section, cf. [BNR,M]. In particular, there is a well-defined morphism $H : \mathcal{H}iggs^n_{C,D} \to B$, where B is the vector space parametrizing spectral covers, and the fiber over a generic $\widetilde{C}$ is $\mathrm{Pic}(\widetilde{C})$.

The crucial point, proved in [M], is that this is a Poisson integrable system. In particular, $\mathcal{H}iggs^n_{C,D}$ has a Poisson structure, so it is foliated by (algebraically) symplectic submanifolds. The restriction of H to each such symplectic submanifold makes it into an integrable system in the usual, symplectic, sense.

In fact, Markman gives an explicit description of these symplectic leaves. In case the polar divisor D is reduced, such a leaf is determined by specifying the conjugacy classes in the Lie algebra $gl(n)$ of the residues $Res\phi$ of ϕ at each point $q \in D$. (When D is not reduced, one must consider instead coadjoint orbits in more complicated Lie algebras.)

There is a variant of the meromorphic Higgs bundles for each reductive group G: the vector bundle E is replaced by a G-bundle, and ϕ is now in $H^0(C, ad(E) \otimes \omega_C(D))$. The case $G = GL(n)$ is the one above. For $G = SL(n)$, we get a subsystem of the above given by two conditions: the trace of ϕ (i.e. the first coefficient in the characterstic polynomial of ϕ) must vanish, and the norm of L must be the trivial line bundle $\mathcal{O}_C$, i.e. L must be in $\mathrm{Prym}(\widetilde{C}/C)$ instead of $\mathrm{Pic}(\widetilde{C})$.

In case C is elliptic, we can identify ω_C with $\mathcal{O}_C$. If D consists of $d > 0$ distinct points, it is easy to see explicitly that the number of Hamiltonians for the $GL(n)$ system (i.e. the dimension of the base B) is $dn(n+1)/2$. Of these, $dn-1$ are Casimirs, while the remaining $1+dn(n-1)/2$ Hamiltonians induce non-zero flows on generic symplectic leaves. The generic spectral cover $\widetilde{C}$ is of genus $1 + dn(n-1)/2$, and the generic symplectic leaf has dimension $2 + dn(n-1)$ and is an algebraically integrable system. The total space has dimension $1 + dn^2$.

Indeed, the equation of a spectral cover $\widetilde{C}$ is of the form:

$$\sum_{i=0}^{n} a_i t^{n-i} = 0,$$

where a_i is an arbitrary element of the di-dimensional vector space $H^0(C, \mathcal{O}_C(iD))$ for $i > 0$, and $a_0 = 1$. This gives the number of Hamiltonians. The n-sheeted covering map $\widetilde{C} \to C$ is ramified at $dn(n-1)$ points, giving the genus. In terms of the Higgs field ϕ, the Casimirs are the symmetric functions of the residues $Res_q(\phi)$ at points $q \in D$. There are thus dn of them, subject to one relation coming from the residue theorem; hence the $dn - 1$ independent Casimirs. These Casimirs can also be seen in terms of the coefficients a_i: they are just the leading terms, or images of the n a_i's in $H^0(C, \mathcal{O}_C(iD))/H^0(C, \mathcal{O}_C((i-1)D))$. For $i \geq 2$ this is the same as the d-dimensional space $H^0(D, \mathcal{O}_C(iD))$, but for $i = 1$ it is a codimension 1 subspace of $H^0(D, \mathcal{O}_C(D))$.

For the $SL(n)$ system, these dimensions are slightly modified: the dimension of B is now $d(n-1)(n+2)/2$ (the drop of d comes from setting $a_1 = 0$); the number of Casimirs drops by $d - 1$, to $d(n-1)$; and the dimension of the Liouville tori drops by only 1: $\mathrm{Pic}(\widetilde{C})$ is replaced by $\mathrm{Prym}(\widetilde{C}/C)$. The dimension of a generic symplectic leaf drops by 2 to become $dn(n-1)$, and the dimension of the total space drops by $d+1$, to $d(n^2-1)$.

§2.2 Symplectic reduction in an algebraic setting

In general, it is not possible to reduce a given algebraically completely integrable system to a smaller one. This is precisely for the same reason that not every KP soliton is an abelian soliton: a general straight line flow on an abelian variety is ergodic, so it is not tangent to any abelian subvariety.

Let $H : X \to B$ be an aci system, and $h : B \to \mathbf{C}$ a Hamiltonian function, determining a hypersurface $B_0 := \{h = 0\} \subset B$ and a straight line flow along the Liouville torus A_b above each point $b \in B_0$. Locally we *can* always find a symplectic reduction, e.g. by intersecting $H^{-1}(B_0)$ with another hypersurface transversal to the flow. But even if we can form this symplectic reduction globally, it will not usually give a reduced algebraically integrable system with base B_0. The problem is that this second hypersurface will intersect each Liouville torus A_b in a hypersurface which is unlikely to be a subtorus; a generic abelian variety does not contain *any* proper subtori.

So, we want to know when there is an aci system whose base is B_0; the general fiber $\overline{A}_b$ should be an abelian variety, the quotient of A_b by the flow. This is possible in two cases:

(1) The general A_b is an abelian variety, isogenous to a product $\overline{A}_b \times C$ for some abelian variety $\overline{A}_b$ and elliptic curve C; the flow is tangent to C.

(2) The general A_b is a $\mathbf{C}$ or $\mathbf{C}^*$-extension of an abelian variety $\overline{A}_b$, and the flow is along the $\mathbf{C}$ or $\mathbf{C}^*$ fiber.

Both situations do arise. As an example of (1), consider Markman's system for $GL(n)$. The Liouville tori are the Jacobians of the spectral covers, $\mathrm{Jac}(\widetilde{C})$, and are isogenous to $\mathrm{Prym}(\widetilde{C}/C) \times C$. Let h be any linear function of $a_1 \in H^0(C, \mathcal{O}_C(D))$, viewed as a function on B. It is a Casimir if and only if it is zero on the 1-dimensional subspace $H^0(C, \mathcal{O}_C) \subset H^0(C, \mathcal{O}_C(D))$. If it is not, it induces a non-zero flow which is tangent to the elliptic factor C. The symplectic reduction is an algebraically integrable system, in fact it is just Markman's system for $SL(n)$.

Let us consider the case that the system $H : X \to B$ consists of the Jacobians of a family of curves $\mathrm{Jac}(\widetilde{C_b}), b \in B$, as in the $GL(n)$ Markman system. When the curve $\widetilde{C_b}$ becomes singular, the fiber of the integrable system can become either the generalized Jacobian $\mathrm{Jac}(\widetilde{C_b})$, which parametrizes all line bundles on $\widetilde{C_b}$, or perhaps some (partial) compactification such as the compactified Jacobian of [AK] which parametrizes all rank one torsion free sheaves on $\widetilde{C_b}$. We will use only the open and non-singular part, $\mathrm{Jac}(\widetilde{C_b})$. A *nonlinear* reduction of type (2) arises if we take $h : B \to \mathbf{C}$ to be the discriminant of $\sum_{i=0}^{n} a_i t^{n-i} = 0$, the polynomial on B which vanishes on the hypersurface $B_0 \subset B$ parametrizing singular spectral covers. For general $b \in B_0$ the spectral cover $\widetilde{C_b}$ has a node or a cusp, so its normalization is a curve $\widetilde{C_b}'$ of genus $g' = g - 1$, where $g = 1 + dn(n-1)/2$ is the genus of the generic spectral cover (over a point not in B_0). It follows that the (generalized, uncompactified) Jacobian, $\mathrm{Jac}(\widetilde{C_b})$, is a bundle over the ordinary Jacobian $\mathrm{Jac}(\widetilde{C_b}')$ of the normalization, with fiber $\mathbf{C}^*$ or $\mathbf{C}$. The Hamiltonian vector field corresponding to this h is tangent to the $\mathbf{C}^*$ or $\mathbf{C}$ fibers. So the symplectic reduction exists globally; it is an algebraically symplectic space, fibered over the discriminant hypersurface B_0 with general fiber $\mathrm{Jac}(\widetilde{C_b}')$. The general result is:

2.2 Theorem. *Let $X \to B$ be an algebraically completely integrable system for which X is algebraically symplectic, the dimension of B is $\widetilde{g}$, and the generic fiber A_b is the Jacobian of a non-singular curve $\widetilde{C_b}$ of genus $\widetilde{g}$. Let B_g be a g-dimensional irreducible component of the subvariety of B parametrizing curves of geometric genus $\leq g$, and let $B_g^0 \subset B_g$ be an open subset which is non-singular and such that for $b \in B_g^0$ the curve $\widetilde{C_b}$ is integral (=reduced and irreducible) of geometric genus g. Then the family of Jacobians $\mathrm{Jac}(\nu(\widetilde{C_b}))$ of the normalizations $\nu(\widetilde{C_b})$ of the curves $\widetilde{C_b}$ for $b \in B_g^0$ inherits from X the structure of an algebraically completely integrable system.*

Proof. Inductively, we find a nested sequence of subvarieties:

$$B_g \subset B_{g+1} \subset \ldots \subset B_{\widetilde{g}} = B$$

such that each B_i is irreducible, i-dimensional, is a component of the subvariety of B parametrizing curves of geometric genus $\leq i$, and contains a non empty open subset B_i^0 which is non-singular and such that for $b \in B_i^0$ the curve $\widetilde{C_b}$ is integral of geometric genus i. For each i the inclusion $B_i \subset B_{i+1}$ corresponds to a reduction of type (2) above, so inductively we find that the locus of Jacobians $\mathrm{Jac}(\nu(\widetilde{C_b}))$ of the normalizations $\nu(\widetilde{C_b})$ of the curves $\widetilde{C_b}$ for $b \in B_i^0$, for each i, inherits from X the structure of an algebraically completely integrable system as claimed. QED

Note. In cases such as Markman's $SL(n)$ system, the two types of reduction must be combined: first we reduce the $GL(n)$ system, whose fibers are Jacobians, to a base B_g parametrizing curves of geometric genus g; then we reduce the Jacobians to Pryms over the traceless locus.

§**2.3 Embedding coelliptic solitons in Markman's system**

As we saw in §2.1, the generic symplectic leaves in Markman's system are quite large - their dimension is on the order of n^2. The coelliptic integrable system which we are trying to build is much smaller - it has total dimension $2g-2$, with $g \leq n$. We are going to show that the generic Markman symplectic leaf contains a g-dimensional family of integral curves whose normalizations are the genus-g, degree-n coelliptic solitons over a given elliptic curve C, occurring once each. By Theorem 2.2 then, each such symplectic leaf produces an algebraically integrable system structure on the coelliptic solitons.

Let us first consider the case that $n > g$. We will embed the coelliptic solitons into a symplectic leaf of Markman's system for the group $G = SL(n)$ and the polar divisor D consisting of the single point $q \in C$. Let $\pi : \widetilde{C} \to C$ be the degree n map of a coelliptic soliton. The fiber $\pi^{-1}(q)$ can be written as $gp + S$ where the effective divisor $S = p_1 + \ldots + p_{n-g}$ consists, for generic $\widetilde{C}$, of $n-g$ distinct points p_i. The ramification divisor of π can similarly be written as $R = (g-1)p + r_1 + \ldots + r_{g-1}$ with $g-1$ points r_j distinct from p. For notational convenience we also fix a non-zero differential dz on C. This allows us to identify the structure sheaf $\mathcal{O}_C$ with the canonical sheaf ω_C. Via the residue at q it also gives a trivialization of the fiber of $\mathcal{O}_C(q)$ at q.

In order to fix the symplectic leaf, we need to choose the conjugacy class O_q of $Res_q(\phi)$. This amounts to choosing a $g \times g$ conjugacy class O_p at p plus a complex number k_i (=1×1 conjugacy class) at each p_i. For O_p we take a single $g \times g$ Jordan block with some complex number k on the diagonal and 1's above. For the k_j we take $n-g$ arbitrary numbers, distinct from k and from each other, whose sum is $-k$. The corresponding spectral covers have above $q \in C$ one ramificaton point at p, where g sheets come together, plus $n-g$ separate sheets at the points p_j of the divisor S. Whenever such a curve is integral of geometric genus g, its normalization is automatically a coelliptic soliton. We need to prove the converse, namely, that each coelliptic soliton occurs, and exactly once, among the normalizations of these spectral covers. We need:

2.3 Lemma. *For a generic coelliptic soliton $\widetilde{C}$, we have $h^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(gq)) = 1$.*

Proof. Since $\omega_{\widetilde{C}} = \mathcal{O}_{\widetilde{C}}(R) = \mathcal{O}_{\widetilde{C}}((g-1)p + r_1 + \ldots + r_{g-1})$, the claim follows via Riemann-Roch:

$$h^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(gq)) = 1 + h^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(r_1 + \ldots + r_{g-1} - p))$$

from the fact that on a generic $\widetilde{C}$ the line bundle $\mathcal{O}_{\widetilde{C}}(r_1 + \ldots + r_{g-1})$ has a unique global section, which vanishes at the r_i but not at p. QED

Now a map of $\widetilde{C}$ to $Tot(\mathcal{O}_C(q))$ compatible with the cover map π is given by a section $f \in H^0(\widetilde{C}, \pi^*(\mathcal{O}_C(q)))$. We claim that there is a unique section f such that:

• The image curve is traceless, i.e. it is a spectral curve in the $SL(n)$ subsystem.

• In terms of our trivialization of the fiber of $\mathcal{O}_C(q)$ at q, this map sends the points p_j to k_j. (It then automatically sends p to k, by the residue theorem. In particular, it separates the points in the fiber of π over q. So if the original $\widetilde{C}$ is integral, so will its image be in $Tot(\mathcal{O}_C(q))$.)

Consider the short exact sequence

$$0 \to \mathcal{O}_{\widetilde{C}}(gp) \to \pi^*(\mathcal{O}_C(q)) \to \mathcal{O}_S \to 0,$$

where again we used the differential dz to trivialize the last sheaf. The long exact sequence gives:

$$0 \to H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(gp)) \to H^0(\widetilde{C}, \pi^*(\mathcal{O}_C(q))) \to H^0(S, \mathcal{O}_S) \to H^1(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(gp)).$$

By the Lemma, the first term is 1-dimensional. By Riemann-Roch then, the last term vanishes. So for every specification of residues $\{k_j, j = 1, \ldots, n-g\} \in H^0(S, \mathcal{O}_S)$ there is a 1- dimensional affine space of liftings $f \in H^0(\widetilde{C}, \pi^*(\mathcal{O}_C(q)))$ with these specified residues. It remains to show that among them there is a unique one which is traceless. Equivalently, we need that the trace map:

$$Tr : H^0(\widetilde{C}, \pi^*(\mathcal{O}_C(q))) \to H^0(C, \mathcal{O}_C(q))$$

is non-zero on the one-dimensional subspace
$H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(gp)) \subset H^0(\widetilde{C}, \pi^*(\mathcal{O}_C(q)))$. But the trace map can be written more naturally as:

$$Tr : H^0(\widetilde{C}, \omega_{\widetilde{C}}(S + p - \sum r_i)) \to H^0(C, \omega_C(q)).$$

Our one-dimensional subspace $H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(gp))$ is then identified with π^* of $H^0(C, \omega_C) = H^0(C, \omega_C(q))$. On this the trace map is simply multiplication by $deg(\pi) = n$, so it is non-zero as required.

Now we need to consider the remaining case, $n = g$. In this case the previous divisor S is empty, so $H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(gp)) \to H^0(\widetilde{C}, \pi^*(\mathcal{O}_C(q)))$ is an isomorphism, hence the only traceless section f is $f = 0$. But the resulting curve is not integral: it is the 0-section C with multiplicity g.

Instead, we consider an $SL(n)$ Markman system with a higher polar divisor. For example, $D = q + q'$ will do, where q' is any point of C other than q. We have $\pi^{-1}(q) = gp$, and we set $S := \pi^{-1}(q')$. In this new notation we still have an exact sequence

$$0 \to H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(gp)) \to H^0(\widetilde{C}, \pi^*(\mathcal{O}_C(q+q'))) \to H^0(S, \mathcal{O}_S) \to H^1(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(gp)).$$

By the Lemma, the first term is *one*-dimensional and the last term vanishes. The trace map

$$Tr : H^0(\widetilde{C}, \pi^*(\mathcal{O}_C(q+q'))) \to H^0(C, \mathcal{O}_C(q+q'))$$

can now be identified with:

$$Tr : H^0(\widetilde{C}, \omega_{\widetilde{C}}(S + p - \sum r_i)) \to H^0(C, \omega_C(q+q')).$$

The *one*-dimensional subspace $H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(gp))$ is identified with

$$\pi^* H^0(C, \omega_C) \subset \pi^* H^0(C, \omega_C(q+q')).$$

So as before, the trace map on it is multiplication by $n = g$, hence nonzero. So a generic choice of residue in $H^0(S, \mathcal{O}_S)$ will indeed lift to a unique traceless section f, producing the desired embedding. QED

Remarks. **(1)** We defined an aci whose integral manifolds consist of Jacobians (or Prym varieties), so the results of §7.2 in [DM1] predict the existence of a cubic $c \in H^0(B, Sym^3\mathcal{V})$, where B is the 'base' (the parameter space of $g-1$ points on a fixed elliptic curve) and $\mathcal{V}$ is the vertical tangent bundle of the Prym fibration. This too will be the reduced version of the Hitchin-Markman cubic which is identified in [DM1] and [DM2].

(2) As in [DM1 §6], we could ask whether the KP flows on the fibres are compatible with the Poisson structure of our constrained Hitchin-Markman system. The answer is the same as in [DM1] (in the affirmative, that is): since the Poisson structures were compatible before reduction, they can be both reduced and remain compatible.

Appendix

The simplest examples of abelian solitons would be fibered products $\widetilde{C} = C \times_{\mathbf{P}^1} C'$ where $\widetilde{C} \to C'$ has a branch point of order h at p and Jac $\widetilde{C} \sim$ Jac $C \oplus$ Jac C'. Indeed, we show (A.2) by using results of Treibich [T] that these are the only possibilities for tangential covers when genus $C > 1$, however we found it surprising that no such example may give an abelian soliton (A.3, A.4).

A.1 Theorem ([T], 2.5). *If $(\widetilde{C}, p) \to (C, q)$ is a degree n tangential cover with genus $C > 1$, then $n = 2, C$ and $\widetilde{C}$ are hyperelliptic, q is a Weierstrass point of C and p is not a Weierstrass point of $\widetilde{C}$.*

A.2 Corollary. *$(\widetilde{C}, p) \to (C, q)$ is a tangential cover with genus $C = h > 1 \Leftrightarrow \widetilde{C}$ is a fibered product $C \times_{\mathbf{P}^1} C'$ with C' of genus zero, branched at say $\{0, \infty\}$, C hyperelliptic branched at $\{e_1, \ldots, e_{2h+2}\}$ and C, C' have one or two branchpoints in common.*

Proof. "$\Rightarrow$" Treibich shows that $\widetilde{C}$ is defined by a spectral polynomial $T^2 + bT + \wp$, where b is a constant, $\wp$ is a function on C with a double pole at q and regular elsewhere; if k is the function in $K(\widetilde{C})$ whose minimal polynomial is $T^2+bT+\wp$, then $\widetilde{C} \to C$ is branched where $4\wp = b^2$ and over ∞; however, since both k and $\wp$ are branched over ∞ the fiber product acquires a singularity which we resolve. If $b^2/4$ is not a branchpoint of C, then $k : \widetilde{C} \to C'$ is branched at $2(2h+1)$ points and $g(\widetilde{C}) = 2h$; if $b^2/4$ is a branchpoint of C, then $g(\widetilde{C}) = 2h - 1$. Note that Galois $(\widetilde{C}/\mathbf{P}^1)$ is the noncyclic group of order 4 and the third curve C'' is given by the function

$$\sqrt{\prod_{i=1}^{2h+1} (\wp - e_i)(\wp - \alpha)},$$

so that Jac $\widetilde{C} \sim$ Jac $C \oplus$ Jac C'': C'' is not branched over ∞, so $\widetilde{C} \to C''$ is.

"$\Leftarrow$" The general fibered product $\widetilde{C} = C \times_{\mathbf{P}^1} C'$, where $\pi : C \to \mathbf{P}^1$ is a double cover branched at S: $\{p_1, \ldots, p_{2h+2}\}$ and $\pi' : C' \to \mathbf{P}^1$ is a double cover branched at $S' = \{0, \infty\}$, has a third quotient C'' with $\pi'' : C'' \to \mathbf{P}^1$ branched at the symmetric difference $S \Delta S'$, so that $h'' = g(C'') = h + 1 - \#(S \cap S')$, $g = g(\widetilde{C}) = h + h'$. The natural decomposition of holomorphic differentials $H^0(\Omega_{\widetilde{C}}) = H^0(\Omega_C) \oplus H^0(\Omega_{C'})$ gives a corresponding decomposition of canonical spaces, $\mathbf{P}^{g-1} : \mathbf{P}(H \oplus H')$. Let $\phi_{\widetilde{C}}$ be the canonical map which sends $\widetilde{C}$ to a rational normal curve in $\mathbf{P}^{g-1}$; projecting from $\mathbf{P}H$ to $\mathbf{P}H'$ gives a map $\widetilde{C} \to \mathbf{P}^{h'-1}$ which factors through the canonical map $\phi_{C'} : C' \to \mathbf{P}^{h'-1}$, and similarly for ϕ_C. Thus, $\rho : \widetilde{C} \to C$ is a tangential cover at $p \in \widetilde{C} \Leftrightarrow \phi_{\widetilde{C}}(p) = \phi_C(\rho(p)) \in \mathbf{P}^{g-1} \Leftrightarrow \phi_{\widetilde{C}}(p) \in \mathbf{P}^{g-1} \Leftrightarrow p$ is a base point of the linear system $\rho'^* H^0(\Omega_{C'}) \subset H^0(\Omega_{\widetilde{C}}) \Leftrightarrow \rho'$ is ramified at $p \Leftrightarrow \pi(\rho(p)) \in S \cap S'$. Cases: $\#$ $(S \cap S') = 1, h' = h$, there are two such points p; $\#$ $(S \cap S') = 2, h' = h-1$, there are 4 such points p; $\#\ (S \cap S') = 0$, $\widetilde{C} \to C$ is not tangential but $\widetilde{C} \to C''$ is. QED

Remark. $\widetilde{C}$ cannot be the fibered product of more than 2 curves $C, C', C'', \ldots$ with Jac C an abelian soliton, for this would imply that $\widetilde{C} \to C'$ and $\widetilde{C} \to C''$ are both branched at the same point $p \in \widetilde{C}$.

A.3 Proposition. *Let $\pi_1 : \widetilde{C}_1 \to C$ be an abelian cover with group G of order d and let $\{G_i\}_{i \in I}$ be the lattice of subgroups of G, $\widetilde{C}_i = J_1/G_i$, $J_i = Jac\ \widetilde{C}_i$ and $P_i = J_i/(\Sigma J_k,$* for *$G_i <\neq G_k)$. Then $J_1 \sim \oplus_{i \in I} P_i$, and $P_1 = 0 \Leftrightarrow J_1 = \sum_{i>1} J_1 \Leftrightarrow$ either G is noncyclic or $g(C) = g(\widetilde{C}_1) = g_1$. The latter situation occurs $\Leftrightarrow \widetilde{C}_1$ and C are both elliptic and π unramified, or both rational and π given by $z \mapsto z^n$, some $n \in \mathbf{Z}$.*

Proof. Since G is abelian, $H^0(\Omega_{\widetilde{C}_1})$ decomposes a G-module into 1-dimensional subspaces; the action of G on each of these factors through a cyclic quotient, so that $J_1 \sim \oplus P_i$ (sum over some i's for which G/G_i is cyclic); in particular, $P_1 \neq 0 \Rightarrow G$ cyclic. Now assume G cyclic, so that $I = \{e \in \mathbf{Z}_+ |\ e|d\}$, $|G_e| = e$, deg $(\pi_e : \widetilde{C}_e \to C) = |G/G_e| = d/e$. If none of the coverings has any branchpoints, or equivalently π_1 is unramified, then $g_e = 1 + \frac{d}{e}(h-1)$; but dim $P_e = g_e - \sum_{\substack{f|e \\ f<e}} \dim P_f$ where the sum is taken over the maximal elements of the lattice; there may be repetitions arising from chains (when e isn't square-free) so by induction dim $P_e = (h-1)\frac{d}{e}\prod_{p|\frac{d}{e}}(1-\frac{1}{p})$ for p a prime number. In particular, dim $P_1 = (h-1)d\prod_{p|d}(1-\frac{1}{p})$ is 0 iff $h = 1$. If there is one branchpoint for π_1, with stabilizer G_i say, then for $e|i$ the stabilizer of the corresponding point in $\widetilde{C}_e$ is $G_{i/e}$, so the branching contribution to g_e is $\frac{1}{2}(\frac{i}{e}-1)\frac{d}{i} = \frac{1}{2}(\frac{d}{e}-\frac{d}{i})$ and by the same analysis as above the contribution to P_1 is $\frac{d}{2}\prod_{p|i}(1-\frac{1}{p}) > 0$. All in all, dim $P_1 = (h-1)d\prod_{p|d}(1-\frac{1}{p}) + \sum_{\text{branchpoints}} \frac{d}{2}\prod_{p|i}(1-\frac{1}{p})$. This could be zero again only if $g = 0$ and π has 2 branchpoints, each totally ramified. QED

We end this section by excluding the possibility of an abelian soliton of the "simplest" type, namely: $\widetilde{C} \to C$ a Galois cover, Jac $\widetilde{C} \sim$ Jac $C \oplus A$, dim $A = n$, $A = \oplus P_e$ summed over the Pryms of coverings $\rho_e : \widetilde{C} \to C_e$ which have a total ramification of order n at a given point $p \in \widetilde{C}$. We believe this to be impossible in general (for $n > 1$), but will only address some specific situations for ease of calculation.

A.4 Proposition. *Let $\widetilde{C} \to C$ be an abelian cover with Galois group G which is an ℓ-group of exponent ℓ^n, noncyclic (for some prime number*

ℓ). Then $\widetilde{C}$ cannot be an abelian soliton with respect to a branch point $p \in \widetilde{C}$ of index ℓ.

Proof. Let $h \in G$ be an element of order ℓ such that $< h >$ is the stabilizer of p. For all $G_e \leq G$, we define $C_e = \widetilde{C}/G_e$, $\pi_e : C_e \to C$, $\rho_e : \widetilde{C} \to C_e$ and $P_e \leq \text{Jac } C_e$ to be the Prym of π_e. If $A = \oplus$ (P_e such that ρ_e is not ramified at p), then the first $\ell - 1$ osculating planes to $\widetilde{C}$ at p are contained in A, so our condition for abelian soliton is: $\ell - 1 \geq \dim A$. By the derivation obtained in the proof of A.3, dim $A = \frac{\ell-1}{\ell} \sum_{\substack{h \notin G_e \\ G/G_e \ cyclic}} [G{:}\, G_e](g - 1 + \frac{1}{2} r_e)$, where $r_e = \#$ branchpoints of ρ_e. Indeed, $P_e = 0$ unless π_e is cyclic, i.e. G/G_e is cyclic, and ρ_e is ramified at $p \Leftrightarrow h \notin G_e$. In general, a branch point of order i contributes $\frac{1}{2}[G : G_e] \prod_{\substack{q \text{ prime} \\ q|i}} (1 - \frac{1}{q})$; however, since G is an ℓ-group all branchpoints contribute equally. The condition then becomes $2 \geq \Sigma_e \frac{[G:G_e]}{\ell}(2g - 2 + r_e)$ where only integers appear and $r_e \geq 1$. Thus $g \leq 1$, for $g = 1$ there can only be one term with $r_e = 2$, $[G : G_e] = \ell$; or one with $r_e = 1, [G : G_e]/\ell = 2$; or two with $r_e = 1, [G : G_e]/\ell = 1$; similarly there are only three possible r_e when $g = 0$ and one rules them out case by case. QED

The curves constructed in 1.4 generically are not fibered products, since they have one total ramification point and the remaining ramifications are simple. When the $g - 1$ simple ramifications coincide and the difference $p_0 - p_1$ on the elliptic curve is a k torsion point (p_0, p_1 are the points on E where the map is ramified) then there is a map $E \to \mathbf{P}^1$ of degree k, totally ramified at p_0 and p_1. If k, g are coprime then $\widetilde{C}$ is the corresponding fibered product of E and $\mathbf{P}^1$, but the Galois group of $\widetilde{C} \to \mathbf{P}^1$ is cyclic, so that Jac $\widetilde{C}$ does not decompose (A.3).

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