Herrero's Approximation Problem for Quasidiagonal Operators

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Abstract

Let T be a quasidiagonal operator on a separable Hilbert space. It is shown that there exists a sequence of operators $\{T_n\}$ such that $dim(C^*(T_n)) < \infty$ and $\|T - T_n\| \to 0$ if and only if $C^*(T)$ is exact.

Dedicated to Yuriko Hoshiya and Isaac Seiji Brown.

1 Introduction

Let H be a separable Hilbert space and $B(H)$ denote the bounded operators on H. $T \in B(H)$ is called block diagonal if there exists an increasing sequence of finite rank projections $P_1 \leq P_2 \leq \ldots$ such that $P_nT - TP_n = 0$ for all n and $P_n \to 1_H$ (s.o.t.). $T \in B(H)$ is called *quasidiagonal* if it is the norm limit of a sequence of block diagonal operators.

Herrero asked whether every quasidiagonal operator is the norm limit of operators T_n such that the C^*-algebra generated by each T_n is finite dimensional. This is equivalent to asking if quasidiagonal operators are always norm limits of block diagonal operators each of which is comprised of blocks of bounded dimension. (See [Vo2,section 5] for a nice discussion of this problem, [DHS] for equivalent formulations and [Vo2], [Br] for the basic theory of quasidiagonal (sets of) operators.) A negative answer was first obtained by S. Szarek (cf. [Sz]). Using probabalistic methods he was able to show the existence of a quasidiagonal operator for which there was an operator *theoretic* obstruction to such approximations. In [Vo1] Voiculescu observed an operator *algebraic* obstruction to the existence of such approximations which we now recall.

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Assume $T_n \to T$ (in norm) and $dim(C^*(T_n)) < \infty$ for all n. Let $E_n : B(H) \to C^*(T_n)$ be conditional expectations (which exist by finite dimensionality). It follows that $E_n(x) \to x$ (in norm) for every $x \in C^*(T)$ and hence the inclusion $C^*(T) \hookrightarrow B(H)$ is a nuclear map. Thanks to the work of E. Kirchberg, this latter condition is equivalent to saying that $C^*(T)$ is an exact C^*-algebra (cf. [Wa]). Hence exactness provides a natural operator algebraic obstruction to Herrero's approximation question. In this note we observe that this is the only obstruction.

Theorem 1.1 *If T is a quasidiagonal operator on a separable Hilbert space, then T is the norm limit of operators $\{T_n\}$ such that $dim(C^*(T_n)) < \infty$ for every n if and only if $C^*(T)$ is exact.*

This gives an affirmative answer to [DHS, Problem 4.3] and is an immediate consequence of the following answer to a question of Dadarlat (cf. [Da]).

Theorem 1.2 *Let A be an exact C^*-algebra and $\psi : A \to B(H)$ be a *-homomorphism with H separable. Then $\psi(A)$ is a quasidiagonal set of operators if and only if for each $\varepsilon > 0$ and finite set $\mathcal{F} \subset A$ there exists a finite dimensional C^*-subalgebra $C \subset B(H)$ with $\psi(\mathcal{F}) \subset^{\varepsilon} C$ (i.e. for each $a \in \mathcal{F}$ there exists $c \in C$ with $\|\psi(a) - c\| < \varepsilon$).*

Dadarlat proved Theorem 1.2 under the additional assumption that $\psi(A)$ contains no nonzero compact operators (cf. [Da, Thm. 6]). We will deduce the general case from Dadarlat's result and a technical generalization of Voiculescu's noncommutative Weyl-von Neumann Theorem.

2 Proofs of Main Results

If A is a separable, unital C^*-algebra and $\varphi : A \to B(H)$ (with H separable and infinite dimensional) is a unital completely positive map then we say that φ is a *faithful representation modulo the compacts* if $\pi \circ \varphi : A \to Q(H)$ is a *-monomorphism, where π is the quotient map onto the Calkin algebra. In this situation we define constants $\eta_\varphi(a)$ by

$$\eta_\varphi(a) = 2\max(\|\varphi(a^*a) - \varphi(a^*)\varphi(a)\|^{1/2}, \|\varphi(aa^*) - \varphi(a)\varphi(a^*)\|^{1/2})$$

for every $a \in A$. The following lemma is the generalization of Voiculescu's Theorem that we will need. The main idea in the proof is essentially due to Salinas (see the proofs of [Sa, Thm. 2.9] and [DHS, Thm. 4.2]). This result also appears in [Br], but we include the proof for the reader's convenience.

Lemma 2.1 *Let A be a separable, unital C^*-algebra and $\varphi : A \to B(H)$ be a faithful representation modulo the compacts. If $\sigma : A \to B(K)$ is any faithful, unital, representation such that $\sigma(A)$ contains no nonzero compact operators then there exist unitaries $U_n : H \to K$ such that*

$$\limsup_{n\to\infty} \|\sigma(a) - U_n\varphi(a)U_n^*\| \leq \eta_\varphi(a)$$

for every $a \in A$.

Proof. Note that by the usual version of Voiculescu's Theorem it suffices to show that *there exists* a representation σ satisfying the conclusion of the theorem since all such representations are approximately unitarily equivalent.

Let $\rho : A \to B(L)$ be the Stinespring dilation of φ; i.e. ρ is a unital representation of A and there exists an isometry $V : H \to L$ such that $\varphi(a) = V^*\rho(a)V$, for all $a \in A$. Let $P = VV^* \in B(L)$ and $P^\perp = 1_L - P$. A routine calculation shows that for every $a \in A$,

$$\|P^\perp\rho(a)P\| \leq \|\varphi(a^*a) - \varphi(a^*)\varphi(a)\|^{1/2}.$$

Now write $L = PL \oplus P^\perp L$ and decompose the representation ρ accordingly. That is, consider the matrix decomposition

$$\rho(a) = \begin{pmatrix} \rho(a)_{11} & \rho(a)_{12} \\ \rho(a)_{21} & \rho(a)_{22} \end{pmatrix},$$

where $\rho(a)_{21} = P^\perp\rho(a)P$ and $\rho(a)_{12} = \rho(a^*)_{21}^*$.

Consider the Hilbert space $P^\perp L \oplus PL$ and the representation $\rho' : A \to B(P^\perp L \oplus PL)$ given in matrix form as

$$\rho'(a) = \begin{pmatrix} \rho(a)_{22} & \rho(a)_{21} \\ \rho(a)_{12} & \rho(a)_{11} \end{pmatrix}.$$

Now using the obvious identification of the Hilbert spaces

$$PL \oplus \big(\bigoplus_{\mathbb{N}} P^{\perp}L \oplus PL\big) \quad \text{and} \quad \bigoplus_{\mathbb{N}} L = \bigoplus_{\mathbb{N}} (PL \oplus P^{\perp}L)$$

a standard calculation shows that

$$\|\rho(a)_{11} \oplus \rho'^{\infty}(a) - \rho^{\infty}(a)\| \leq \eta_{\varphi}(a)$$

for all $a \in A$, where $\rho'^{\infty} = \oplus_{\mathbb{N}}\rho'$ and $\rho^{\infty} = \oplus_{\mathbb{N}}\rho$. Note also that $\rho(a)_{11} = V\varphi(a)V^*$.

Now, let C be the linear space $\varphi(A) + \mathcal{K}(H)$. Note that C is actually a separable, unital C^*-subalgebra of $B(H)$ with $\pi(C) = A$ where $\pi : B(H) \to Q(H)$ is the quotient map onto the Calkin algebra. By Voiculescu's Theorem we have that $\iota \oplus \rho'^{\infty} \circ \pi$ is approximately unitarily equivalent modulo the compacts to ι, where $\iota : C \hookrightarrow B(H)$ is the inclusion. Let $W_n : H \to H \oplus (\oplus_{\mathbb{N}}(P^{\perp}L \oplus PL))$ be unitaries such that

$$\|\varphi(a) \oplus \rho'^{\infty}(a) - W_n\varphi(a)W_n^*\| \to 0$$

for all $a \in A$.

Let

$$\tilde{V} : H \oplus (\bigoplus_{\mathbb{N}}(P^{\perp}L \oplus PL)) \to \bigoplus_{\mathbb{N}} L$$

be the unitary $V \oplus 1$ (again using the obvious identification of $PL \oplus \big(\oplus_{\mathbb{N}} (P^{\perp}L \oplus PL)\big)$ and $\oplus_{\mathbb{N}}L$). Note that $\tilde{V}(\varphi(a) \oplus \rho'^{\infty}(a))\tilde{V}^* = V\varphi(a)V^* \oplus \rho'^{\infty}(a) = \rho(a)_{11} \oplus \rho'^{\infty}(a)$. We complete the proof by defining

$$K = \oplus_{\mathbb{N}}L, \quad \sigma = \rho^{\infty} = \oplus_{\mathbb{N}}\rho, \quad U_n = \tilde{V}W_n : H \to \oplus_{\mathbb{N}}L = K. \quad \square$$

Proof of Theorem 1.2. We may assume that A is separable unital and π is a unital *-homomorphism. Since quotients of exact C^*-algebras are again exact it suffices to prove that if $A \subset B(H)$ is an exact C^*-algebra and a quasidiagonal set of operators then for each finite set $\mathcal{F} \subset A$ and $\varepsilon > 0$ there exists a finite dimensional subalgebra $C \subset B(H)$ such that $\mathcal{F} \subset^{\varepsilon} C$. In fact, it suffices to show that $\mathcal{F} \subset^{\varepsilon} C$ where C is an approximately finite dimensional (AF) C^*-subalgebra of $B(H)$.

Let $P_1 \leq P_2 \leq P_3 \ldots$ be an increasing sequence of finite rank projections which converge to the identity operator in the strong operator topology and such that $\|aP_n - P_n a\| \to 0$ for all $a \in A$. Let $B \subset Q(H)$ be the image of A in the Calkin algebra. Then we have the natural exact sequence $0 \to \mathcal{K}(H) \to A + \mathcal{K}(H) \to B \to 0$. Since $\mathcal{K}(H)$ and B are exact and the quotient map $A + \mathcal{K}(H) \to B$ is locally liftable (since A is exact), it follows from [Ki2, Prop. 7.1] that $A+\mathcal{K}(H)$ is also exact. Since the compact operators are actually nuclear it follows from [EH], that this short exact sequence is semisplit; i.e. there exists a unital completely positive splitting $\varphi : B \to A+\mathcal{K}(H)$. Define $\varphi_n(a) = (1-P_n)\varphi(a)(1-P_n)$ and note that each φ_n is also a completely positive splitting. Moreover (as is well known) the maps φ_n are asymptotically multiplicative and hence B is a quasidiagonal (QD) C^*-algebra. Let $H_n = (1 - P_n)H$. Note that each H_n has finite codimension in H.

Let $\rho : B \to B(K)$ be a faithful representation such that $\rho(B)$ contains no nonzero compact operators. By Dadarlat's result ([Da, Thm. 6]), we can find a sequence of finite dimensional C^*-subalgebras $D_n \subset B(K)$ such that each element of $\rho(B)$ is the norm limit of a sequence taken from the algebras D_n. Since the maps φ_n are asymptotically multiplicative faithful representations modulo the compacts (when regarded as taking values in $B(H_n)$) and each H_n has finite codimension in H it follows from Lemma 2.1 that we can find a sequence of finite dimensional C^*-subalgebras $E_n \subset Q(H)$ such that each element of B is the norm limit of a sequence taken from the algebras E_n. Since extensions of AF algebras are again AF the proof of the theorem is completed by defining C_n to be the pullbacks of the E_n. $\Box$

In [Sa] N. Salinas defined $Ext_{qd}(B)$ for a QD C^*-algebra as follows. A unital *-monomorphism $\rho : B \to Q(H)$ defines an element of $Ext_{qd}(B)$ if the pullback of $\rho(B)$ in $B(H)$ is a quasidiagonal set of operators. It is easy to see that this notion in invariant under strong (unitary) equivalence of extensions (i.e. gives a well defined subset of $Ext(B)$). When B is nuclear, $Ext_{qd}(B)$ is a subgroup of $Ext(B)$ (cf. [Sa, Cor. 2.10]). Kirchberg has given examples of exact QD C^*-algebras B and *-monomorphisms $\rho : B \to Q(H)$ which define noninvertible elements of $Ext_{qd}(B)$ (cf. [Ki1]). The proof of Theorem 1.2 above easily yields a new characterization (different than Arveson's characterization as those *-monomorphisms which admit a completely positive lifting) of the invertible elements of $Ext_{qd}(B)$ when B is exact.

Corollary 2.2 *Let B be a separable unital exact QD C^*-algebra and $\rho : B \to Q(H)$ be a *-monomorphism such that $[\rho] \in Ext_{qd}(B)$. Then $[\rho]$ is invertible if and only if for each finite subset $\mathcal{F} \subset B$ and $\varepsilon > 0$ there exists* *a finite dimensional C^*-subalgebra $C \subset Q(H)$ such that $\rho(\mathcal{F}) \subset^{\varepsilon} C$.*

Proof. If $[\rho]$ is invertible then there exists a completely positive splitting and hence the proof of Theorem 1.2 shows that $\rho(B)$ admits such approximants. On the other hand, if $\rho(B)$ admits such approximants then the argument in the introduction shows that ρ is a nuclear map and hence, by the Choi-Effros Lifting Theorem, is liftable (hence invertible). □

We end this note with some examples of operators which generate exact C^*-algebras.

Example 2.3 If $T \in B(H)$ is essentially normal then $C^*(T)$ is always exact (even nuclear) since extensions of nuclear algebras are nuclear. It should be noted, however, that BDF-theory already implies that an essentially normal quasidiagonal operator admits Herrero approximations since every such operator is a compact perturbation of a normal operator.

Example 2.4 Let $T \in B(H)$ be such that $T^2 = 0$. Then T is always quasidiagonal and $C^*(T)$ is always exact (in fact, nuclear). This follows from the fact that the universal C^*-algebra generated by a contractive nilpotent operator of order 2 is isomorphic to the cone over the 2×2 matrices, i.e. $C_0((0,1]) \otimes M_2(\mathbb{C})$ (see [Lo, Prop. 3.3.1]). But this algebra is nuclear and hence $C^*(T)$ is always nuclear (being a quotient of a nuclear C^*-algebra). Moreover, the image of $C_0((0,1]) \otimes M_2(\mathbb{C})$ in any representation is a quasidiagonal set of operators since every representation is approximately unitarily equivalent to a direct sum of irreducible representations (by Voiculescu's Theorem).

It is interesting to note that the case $n = 2$ is quite special for nilpotent operators. That is, there exist nilpotent operators of order 3 which are not quasidiagonal and do not generate exact C^*-algebras. Indeed, if $T \in B(H)$ and we define $X_T \in B(H \oplus H \oplus H)$ by

$$X_T = \begin{pmatrix} 0 & 1 & T \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

then $C^*(X_T) \cong C^*(T) \otimes M_3(\mathbb{C})$. Since there exist nonquasidiagonal operators T for which $C^*(T)$ is not exact it follows that there exist nonquasidiagonal nilpotent operators of order 3 which don't generate exact C^*-algebras (see [DHS, pg. 978] for other consequences of this matrix trick).

We wish to thank M. Dadarlat for pointing out our final example.

Example 2.5 Let H be a Hilbert space with orthonormal basis $\{e_i\}_{i\in\mathbb{Z}}$. Let $U \in B(H)$ be the bilateral shift unitary (i.e. $U(e_i) = e_{i+1}$) and $T \in B(H)$ be any operator with the property that each e_i is an eigenvector for T. If we define $S = TU$ then $C^*(S)$ is exact. This is because it is contained in a nuclear C^*-algebra. More precisely, the C^*-algebra generated by U and the set of *all* such T (as above) is naturally isomorphic to the crossed product C^*-algebra $l^\infty(\mathbb{Z}) \rtimes_\gamma \mathbb{Z}$ where the action γ on $l^\infty(\mathbb{Z})$ arises from the bilateral shift $i \mapsto i+1$ on $\mathbb{Z}$. Since crossed products of nuclear C^*-algebras by amenable groups are again nuclear, this shows that $C^*(S) \subset l^\infty(\mathbb{Z}) \rtimes_\gamma \mathbb{Z}$ is an exact C^*-algebra. Evidently this example extends to one sided (i.e. unilateral) shifts as well since these correspond to crossed products by endomorphisms (which still preserves nuclearity).

The example above is nothing but the class of weighted shift (both bilateral and unilateral) operators. R. Smucker first characterized the quasidiagonality of weighted shifts in terms of the associated weight sequence in [Sm]. Thus our main result gives a new proof of a result of D. Herrero who showed that all quasidiagonal weighted shifts admit Herrero approximations (see [He]). Note, however, that the present approach has the added benefit of treating n-tuples as well. That is, our results show that if $S_1, \ldots, S_k$ are weighted shifts (with respect to the same basis) and $\{S_1, \ldots, S_k\}$ is a quasidiagonal *set* of operators then for each $\varepsilon > 0$ there exist operators $\tilde{S}_1, \ldots, \tilde{S}_k$ such that $\|S_i - \tilde{S}_i\| < \varepsilon$ and $C^*(\tilde{S}_1, \ldots, \tilde{S}_k)$ is finite dimensional (since $C^*(S_1, \ldots, S_k)$ will always be exact).

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