SKEIN MODULES OF LINKS IN CYLINDERS OVER SURFACES

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Abstract. We define the Conway skein module $\mathcal{C}(M)$ of ordered based links in a 3-manifold M. This module gives rise to $\mathcal{C}(M)$-valued invariants of usual links in M. We determine a basis of the $\mathbb{Z}[z]$-module $\mathcal{C}(\Sigma \times [0,1])/\mathrm{Tor}(\mathcal{C}(\Sigma \times [0,1]))$ where Σ is the real projective plane or a surface with boundary. For cylinders over the Möbius strip or the projective plane we derive special properties of the Conway skein module, among them a refinement of a theorem of Hartley and Kawauchi about the Conway polynomial of strongly positive amphicheiral knots in S^3. In addition, we determine the Homfly and Kauffman skein modules of $\Sigma\times[0,1]$ where Σ is an oriented surface with boundary.

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Introduction

In 1969 J. H. Conway found a normalized version $\tilde{\nabla}$ of the Alexander polynomial of a link. It satisfies a so-called skein relation on certain triples of links differing only by local modifications, namely

$$\tilde{\nabla}(\quad) - \tilde{\nabla}(\quad) = z\tilde{\nabla}(\quad) \tag{1}$$

(see [Ale],[Con]). The Conway polynomial also satisfies $\tilde{\nabla}(O_n) = \delta_{1n}$, where O_n is the trivial link with n components and δ_{ij} is 1 if $i = j$ and 0 otherwise. The skein relation together with $\tilde{\nabla}(O_1) = 1$ uniquely determine the isotopy invariant $\tilde{\nabla}(L)$ for every link L. More precisely, the so-called skein module $\mathcal{C}(S^3)$ generated over $\mathbb{Z}[z^{\pm 1}]$ by isotopy classes of links modulo the skein relation is free with basis O_1. The Jones, Homfly and Kauffman polynomials of links in S^3 ([Jon], [HOM], [Kau]) can be characterized by similar skein relations. In 1988 J. Przytycki and V. Turaev generalized the Homfly and Kauffman polynomials to links in a solid torus and determined the corresponding skein modules. Since then, various versions of skein modules related to these polynomials have been studied ([Pr2]), among them the Homfly skein module of links in a handlebody ([Pr1]).
In 1990 Vassiliev invariants were introduced ([Vas]). They can be characterized as follows: every link invariant v with values in an abelian group can recursively be extended to an invariant of singular links by $v(\quad) = v(\quad) - v(\quad)$. Vassiliev invariants v of degree n of links are those invariants that vanish on singular links

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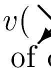
with $n+1$ double points. If the 3-manifold is not orientable, then there exists no preferred choice of a sign in $v($$) = \pm v($$) \mp v($$)$, but it still makes sense to speak of Vassiliev invariants of degree n. Let $I = [0, 1]$. In 1996, Andersen, Mattes and Reshetikhin found a universal Vassiliev invariant $Z_{\Sigma \times I}$ of links in a cylinder over an orientable surface Σ with boundary ([AMR]). We use $Z_{\Sigma\times I}$ to determine the Homfly and Kauffman skein modules of $\Sigma \times I$. In comparison with [Pr1] our choice of a basis of the Homfly skein module has some technical advantages. The result seems to be new for the Kauffman skein module.

In [Lie] the invariant $Z_{\Sigma\times I}$ was extended to a universal Vassiliev invariant of links in cylinders over non-orientable surfaces with boundary and over the real projective plane P^2. The main idea that lead to this article was to extract an explicitly computable link invariant from $Z_{\Sigma\times I}$ in the non-orientable case. We define a skein module $\mathcal{C}(M)$ of a 3-manifold M using a version of the skein relation in Equation (1) for ordered based links. In Theorem 10 we establish an isomorphism of $\mathbb{Z}[z]$-modules

$$\mathcal{C}^{\#}(\Sigma \times I) \cong \mathbb{Z}[z] \otimes_{\mathbb{Z}} \mathcal{C}_0^{\#}(\Sigma \times I), \tag{2}$$

where $\Sigma = P^2$ or Σ is a non-orientable surface with boundary, $\mathcal{C}^{\#}(M)$ is the quotient of the $\mathbb{Z}[z]$-module $\mathcal{C}(M)$ by its torsion submodule, and $\mathcal{C}_0^{\#}(M)$ is the quotient of the $\mathbb{Z}$-module $\mathcal{C}(M)/(z\mathcal{C}(M))$ by its torsion submodule. We determine the structure of $\mathcal{C}_0^{\#}(M)$ for an arbitrary 3-manifold. It is isomorphic to a tensor product of a polynomial algebra and an exterior algebra. A basis of $\mathcal{C}^{\#}(\Sigma \times I)$ is given by certain descending links. We describe explicitly how to compute a Conway polynomial inducing the isomorphism in Equation (2) and how to obtain invariants of links (without order or basepoints) from the Conway skein module. When Σ is the Möbius strip X or $\Sigma = P^2$ we show that the isomorphism in Equation (2) is an isomorphism of involutive algebras (Theorem 11). In addition, we have the following theorem for the Conway polynomials $\tilde{\nabla}$ of knots in $P^2 \times I$ and in $S^2 \times I$.

Theorem 1. *For a noncontractible knot $K \subset P^2 \times I$ we have*

$$\tilde{\nabla}(p^{-1}(K)) = \tilde{\nabla}(K)^2,$$

where $p : S^2 \times I \longrightarrow P^2 \times I$ denotes the 2-fold covering map.

See Theorem 12 for an equivalent formulation of Theorem 1. For strongly positive amphicheiral knots in S^3 a theorem of Hartley and Kawauchi ([HaK]) says that the Conway polynomial is a square. Theorem 1 is a refinement of this result (see Corollary 13) because it assures that the square root of the Conway polynomial of $p^{-1}(K)$ can be computed directly by using the Conway skein relation for ordered based links in $P^2 \times I$.

The paper is organized as follows. In Section 1 we define $\mathcal{C}(M)$ and prove some of its general properties. After this section we concentrate on the case $M = \Sigma \times I$. In Section 2 we introduce decomposed surfaces and make a choice of representatives of conjugacy classes in the fundamental group of Σ. This choice is important in the definition of a descending link. Only a careful choice leads to a basis of $\mathcal{C}^{\#}(\Sigma \times I)$

over $\mathbb{Z}[z]$. In Section 3 we state the Theorems 10, 11, and 12 mentioned above. Sections 4 to 8 are devoted to the proof of these theorems. In Section 4 we show that descending knots and so-called cabled descending knots generate $\mathcal{C}(\Sigma \times I)$ as an algebra. By making computations in $\mathcal{C}^{\#}(X \times I)$ we show in Section 5 that cabled descending knots vanish in $\mathcal{C}^{\#}(\Sigma \times I)$. In Section 6 we introduce labeled ordered based chord diagrams and prove that a map $W^{\mathrm{ob}}_{\mathfrak{gl}_0}$ defined on these diagrams is compatible with certain relations. In Section 7 we compose the universal Vassiliev invariant $Z_{\Sigma \times I}$ adapted to ordered based links with $W^{\mathrm{ob}}_{\mathfrak{gl}_0}$ to prove Theorems 10 and 11. In Section 8 we prove Theorem 12 by using a map between chord diagrams induced by the covering $S^2 \times I \longrightarrow P^2 \times I$. In an appendix we determine the structure of the Homfly and Kauffman skein modules of $\Sigma \times I$ for oriented surfaces Σ with boundary.

1. The Conway skein module

Throughout this paper we work in the PL-category. Let M be a connected 3-manifold equipped with an oriented subset U homeomorphic to a closed 3-ball D^3. We denote the fundamental group of M with respect to a basepoint in U by $\pi_1(M)$. The first Stiefel-Whitney class induces a homomorphism $\sigma : \pi_1(M) \longrightarrow \mathbb{Z}/2$ called *orientation character* of M. Explicitly, if a local orientation at the basepoint stays the same when we push it around $w \in \pi_1(M)$, then we have $\sigma(w) = 0$, and $\sigma(w) = 1$ otherwise.

Ordered based links in M are links in M with a linear order of their components and with a basepoint in U on each component. In particular, every component of an ordered based link meets U. The basepoints are not allowed to leave U during an isotopy (resp. homotopy) of ordered based links. The homotopy class of a based knot K can canonically be considered as an element $\alpha(K) \in \pi_1(M)$. We denote $\sigma(\alpha(K))$ simply by $\sigma(K)$. Notice that $\sigma(K)$ does not depend on the basepoint of K. We will regard the empty link $\emptyset$ as an ordered based link because this will simplify slightly the statement of our results.

In Figure 1 parts of ordered based links are shown. The numbers 1 and 2 near the basepoints in this figure indicate that the corresponding component is first or second in the order of this link.

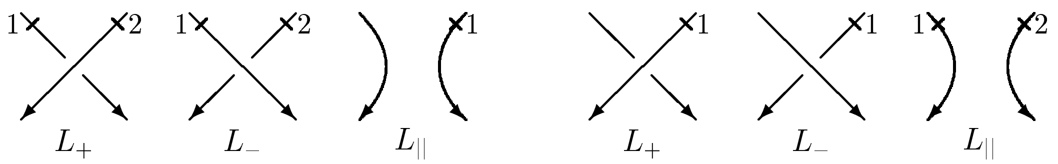


Figure 1. Skein triples of ordered based links

Ordered based links $(L_+, L_-, L_{||})$ form a *skein triple* if they differ inside of a ball $D^3 \subset U$ as one of the two triples $(L_+, L_-, L_{||})$ in Figure 1, they coincide on the outside of this ball, and the order and basepoints of the remaining components also coincide. We say that L_+ and L_- are related by changing a crossing and $L_{||}$ is obtained from L_+ or L_- by splicing a crossing.

Definition 2. *Let $\mathcal{C}(M)$ be the $\mathbb{Z}[z]$-module generated by isotopy classes of ordered based links and the Relations (Ord), (Bas) and $(Skein)$ below.*

(Ord) : $L = (-1)^{\sigma(K_1)\sigma(K_2)}L'$, where K_1, K_2 are two components of an arbitrary ordered based link L that are neighbors in the order of the components of L. The ordered based link L' is equal to L except that the order of K_1 and K_2 is interchanged.

(Bas) : $L = (-1)^{\sigma(a)\sigma(b)}L'$, where L and L' are ordered based links that are equal except for one basepoint on a component K of L and K' of L'. The element $a \in \pi_1(\Sigma \times I)$ is given by the curve described by the basepoint of K when it is pushed along K until it coincides with the basepoint of K', and $b = \alpha(K')a^{-1}$.

$(Skein)$: $L_+ - L_- = zL_{||}$, where $(L_+, L_-, L_{||})$ is a skein triple of ordered based links.

Let the homomorphism of rings $\varphi : \mathbb{Z}[z] \longrightarrow \mathbb{Z}[z]$ be given by $\varphi(p)(z) = p(-z)$. We turn $\mathcal{C}$ into a functor in the sense of the following proposition.

Proposition 3. *Let the 3-manifolds M_i $(i = 1, 2)$ be equipped with the oriented subsets $U_i \cong D^3$, and let $i : M_1 \longrightarrow M_2$ be an embedding with $i(U_1) \subseteq U_2$.*
(1) If the orientations of $i(U_1)$ and U_2 coincide, then i induces a morphism of $\mathbb{Z}[z]$-modules $i_* : \mathcal{C}(M_1) \longrightarrow \mathcal{C}(M_2)$.
(2) If the orientations of $i(U_1)$ and U_2 are opposite, then i induces a morphism of $\mathbb{Z}$-modules $i_* : \mathcal{C}(M_1) \longrightarrow \mathcal{C}(M_2)$ satisfying $i_*(pm) = \varphi(p)i_*(m)$ for all $p \in \mathbb{Z}[z]$ and $m \in \mathcal{C}(M_1)$.

Sketch of proof. (1) is obvious.
(2) The only defining relation of $\mathcal{C}(M_1)$ that depends on the orientation of U_1 is the skein relation. Reversing the orientation of U_1 interchanges the parts L_+ and L_- of a link which can be compensated by replacing z by $-z$. This way we obtain Part (2) of the proposition by using Part (1). □

The existence of a suitable homeomorphism from M to itself implies the following corollary which explains why we do not refer explicitly to U in the name of $\mathcal{C}(M)$.

Corollary 4. *If the 3-manifold M is oriented (resp. non-orientable), then for different choices of $U_i \subset M_i = M$ with $U_i \cong D^3$ equipped with the induced (resp. an arbitrary) orientation the $\mathbb{Z}[z]$-modules $\mathcal{C}(M_i)$ are isomorphic.*

Let c be an arbitrary crossing of an ordered based link. We apply an isotopy such that c is contained in U. Then we use Relations (Ord) and (Bas) such that c becomes a crossing involved in a skein triple. The skein relation then implies that the $\mathbb{Z}$-module $\mathcal{C}_0(M) = \mathcal{C}(M)/(z\mathcal{C}(M))$ is generated by homotopy classes of ordered based links. We turn a disjoint union of ordered based links $L = L_1 \cup L_2$ into an ordered based link by extending the order of the components of L_i to L by $K_1 < K_2$ for based knots $K_i \subset L_i$. This operation is not well-defined on $\mathcal{C}(M)$, but it turns $\mathcal{C}_0(M)$ into an associative ring with 1-element $\emptyset$. Let $T = \mathrm{Tor}(\mathcal{C}_0(M))$ be the torsion submodule of the $\mathbb{Z}$-module $\mathcal{C}_0(M)$ which is a two-sided ideal of $\mathcal{C}_0(M)$. Define $\mathcal{C}_0^{\#}(M) = \mathcal{C}_0(M)/T$. Let us describe the structure of $\mathcal{C}_0^{\#}(M)$. Denote the conjugacy class of an element $w \in \pi_1(M)$ by $[w]$. Let

$$\mathcal{S}_1 = \{[w] \mid w \in \pi_1(M), \sigma(w) = 1\}, \tag{3}$$

$$\mathcal{S}_0^* = \{[w] \mid w \in \pi_1(M), \sigma(w) = 0 \text{ and } N(w) \subseteq \operatorname{Ker} \sigma\}, \tag{4}$$

where $N(w) = \{v \in \pi_1(M) \mid vw = wv\}$ is the normalizer of w. For a set X, let V_X be the free $\mathbb{Z}$-module with basis $\{t_x \mid x \in X\}$. Denote the symmetric algebra on a free $\mathbb{Z}$-module V by $S(V)$ and the exterior algebra by $\Lambda(V)$. For $w \in \pi_1(M)$ we denote the corresponding based knot in $\mathcal{C}_0^{\#}(M)$ by $\overline{K}_w$.

Proposition 5. *For every choice $\hat{\mathcal{S}}_0^*$ of representatives of the conjugacy classes in $\mathcal{S}_0^*$ exists an isomorphism of algebras*

$$\psi : S\left(V_{\hat{\mathcal{S}}_0^*}\right) \otimes \Lambda\left(V_{\mathcal{S}_1}\right) \longrightarrow \mathcal{C}_0^{\#}(M).$$

given by mapping $t_v \otimes 1$ to $\overline{K}_v$ ($v \in \hat{\mathcal{S}}_0^*$) and $1 \otimes t_{[w]}$ to $\overline{K}_v$ for some $v \in [w] \in \mathcal{S}_1$.

Proof. The $\mathbb{Z}$-module $\mathcal{C}_0(M)$ can be described by non-commutative generators k_w ($w \in \pi_1(M)$) and the Relations

$$\begin{aligned} (Ord') &\ :\ k_v k_w = (-1)^{\sigma(v)\sigma(w)} k_w k_v \\ (Bas') &\ :\ k_{vw} = (-1)^{\sigma(v)\sigma(w)} k_{wv} \end{aligned}$$

for all $v, w \in \pi_1(M)$. By the Relation (Bas') we have $k_a = k_b$ for all $a, b \in [a] \in \mathcal{S}_1$. The Relation (Ord') then implies that ψ is well-defined. If $\sigma(w) = 0$ and there exists $v \in \pi_1(M)$ with $vw = wv$ and $\sigma(v) = 1$, then

$$k_w = k_{vwv^{-1}} = (-1)^{\sigma(v)\sigma(wv^{-1})} k_{wv^{-1}v} = -k_w \tag{5}$$

implying $k_w = 0 \in \mathcal{C}_0^{\#}(M)$. If $\sigma(w) = 1$, then $k_w^2 = -k_w^2$ implies $k_w^2 = 0 \in \mathcal{C}_0^{\#}(M)$. Therefore ψ is surjective. Since we know a presentation of $\mathcal{C}_0(M)$ by generators and relations, it is easy to verify that an inverse map of ψ is well-defined. □

Now we fix a choice of a linearly ordered set $\hat{\mathcal{S}}^*$ consisting of representatives of $\mathcal{S}_0^* \cup \mathcal{S}_1$. Then by Proposition 5 we can equip $\mathcal{C}_0^{\#}(M)$ with the basis

$$\overline{K}_{w_1} \ldots \overline{K}_{w_n} \quad \text{where} \quad w_i \in \hat{\mathcal{S}}^*, w_i \leq w_{i+1}, (w_i = w_{i+1} \Rightarrow \sigma(w_i) = 0).$$

We define the linear map $\theta : \mathcal{C}_0^{\#}(M) \longrightarrow \mathbb{Z}$ to be equal to 1 on these basis elements. For non-orientable M the definition of θ depends on the representatives chosen for $\mathcal{S}_0^*$ and on the order chosen on $\mathcal{S}_1$.

The module $\mathcal{C}(M)$ gives rise to invariants of links in M (without order or basepoints) as follows. Given a link L, we choose a basepoint b_i on each component and pull all basepoints into U along disjoint paths β_i. Then we choose an arbitrary order of the components of L and obtain an ordered based link L_b. We denote the image of $L_b \in \mathcal{C}(M)$ under the $\mathbb{Z}$-linear projection $\mathcal{C}(M) \longrightarrow \mathcal{C}_0^{\#}(M)$ by $\overline{L}_b$. Define $v_i(L)$ and $C(L)$ by

$$\mathcal{C}(M)/z^{i+1}\mathcal{C}(M) \ni v_i(L) = \theta\left(\overline{L}_b\right)\overline{L}_b, \tag{6}$$

$$\{\{a,b\} \mid a,b\in\mathcal{C}(M)\} \ni C(L) = \begin{cases} \{\theta\left(\overline{L}_b\right)\overline{L}_b\} & \text{if } \theta\left(\overline{L}_b\right)\neq 0, \\ \{\pm\overline{L}_b\} & \text{if } \theta\left(\overline{L}_b\right)=0. \end{cases} \tag{7}$$

Proposition 6. *The maps C and v_i ($i \geq 0$) are isotopy invariants of links. In addition, the maps v_i are Vassiliev invariants of degree i.*

Proof. Let L be a link and let L_a, L_b be ordered based links obtained from L by choosing basepoints a_i and b_i on the components of L that are pulled into U along paths α_i and β_i respectively. We first assume that all the paths α_i and β_i are disjoint. Then we can pull the points b_i on L_a into U along β_i and the points a_i on L_b into U along α_i. This shows that we can pass from L_a to L_b by isotopies of ordered based links and by application of the Relations (Ord) and (Bas). The Relations (Ord) and (Bas) only influence signs and $\theta\left(\overline{L}_b\right)$ changes signs simultaneously with $\overline{L}_b$. If the paths α_i and β_i are not disjoint, then we choose basepoints c_i of L and paths γ_i disjoint to α_i and β_i and apply the argument above to the pairs (L_a, L_c) and (L_c, L_b). This shows that the maps v_i and C are well-defined isotopy invariants of links. It follows from the skein relation by the same arguments as for the usual Conway polynomial of links that the maps v_i are Vassiliev invariants of degree i (see [Vog]). □

2. Decomposed surfaces

Let us now consider the special case $M = \Sigma \times I$, where Σ is a connected compact surface and $I = [0,1]$ is an oriented interval. First assume that $\partial\Sigma \neq \emptyset$. We choose an oriented subset $B_0 \cong D^2$ of Σ such that $B_0 \cap \partial\Sigma \neq \emptyset$. Define $\mathcal{C}(\Sigma \times I)$ with respect to $U = B_0 \times I \cong D^3$. Fix the choice of a point $u \in B_0 \cap \partial\Sigma$. For the following definition we need some notation: we denote the fundamental group of Σ with respect to a basepoint in B_0 by $\pi_1(\Sigma)$. Let p_Σ and p_I be the projections from $\Sigma \times I$ to Σ and I respectively. For $x \in \Sigma \times I$ we call the value $p_I(x)$ the height of x.

Definition 7. *(1) A based knot $K \subset \Sigma \times I$ is called* descending with respect to its basepoint*, if the height of K is descending when we travel along K starting at the basepoint A and following the orientation of K until we reach a point A' with $p_\Sigma(A) = p_\Sigma(A')$ from which K leads back to A by increasing the height and by keeping the projection to Σ constant.*
(2) A based knot $K \subset \Sigma \times I$ is called a descending knot*, if K is descending with respect to its basepoint A and there exists a neighborhood $V \cong D^2$ of u in B_0 such that $(V \times I) \cap K$ consists of an interval containing A.*

The important property of descending knots is that for $w \in \pi_1(\Sigma)$ there exists up to isotopy exactly one descending knot K_w such that $\alpha(K_w) = w$. If $v, w \in \pi_1(\Sigma)$ are conjugate then one can pass from K_v to K_w by isotopies and crossing changes and by moving the basepoint. In this section we will choose a set $\hat{\mathcal{S}}$ of representatives of conjugacy classes in $\pi_1(\Sigma)$ with good properties (compare the example preceding Equation 14 for a choice with bad properties).

Definition 8. *A* decomposed surface *is a triple* $(\Sigma, (B_0, B_1, \ldots, B_k), u)$ *where* Σ *is a surface,* $I^2 \cong B_0 \subset \Sigma$ *is oriented,* $u \in B_0 \cap \partial\Sigma$*, and* $I^2 \cong B_i \subset \Sigma$ *such that* $\bigcup_{i=0}^{k} B_i = \Sigma$ *and Equations (8) and (9) are satisfied.*

$$B_0 \cap B_i \cong I \times \{0,1\} \text{ for } i \geq 1, \tag{8}$$

$$B_i \cap B_j = \emptyset \text{ for } i \neq j, i, j \geq 1. \tag{9}$$

We represent decomposed surfaces graphically as shown in Figure 2 by an example. By convention the orientation of B_0 is counterclockwise in this figure. We will also represent $\Sigma \times I$ as in Figure 2 where we assume that on B_0 the interval I directs towards the reader.

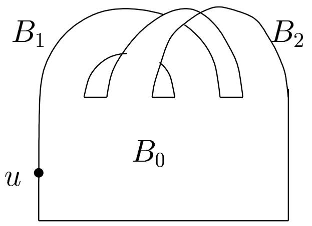
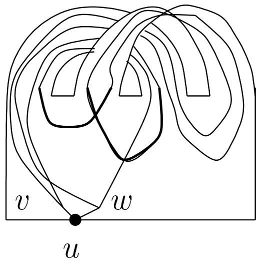


FIGURE 2. A picture of a decomposed surface

The decomposition of Σ determines generators x_i $(i = 1, \ldots, k)$ of $\pi_1(\Sigma)$ that pass through the band B_i exactly once in the clockwise sense in our pictures and that do not meet the other bands B_j $(j \geq 1, i \neq j)$. For $w \in \pi_1(\Sigma)$ we denote by $\ell(w)$ the length of the unique reduced expression of w in the generators x_i of the free group $\pi_1(\Sigma)$ and we denote the generator at position i $(1 \leq i \leq \ell(w))$ in this expression by $w(i)$. Traveling along ∂B_0 in the clockwise sense starting at u we number the $2k$ intervals $B_i \cap B_0$ $(i = 1, \ldots, k)$ from 1 to $2k$. We denote these intervals by I_ν $(\nu \in \{1, \ldots, 2k\})$. This numbering determines maps $s, d : \pi_1(\Sigma) \longrightarrow \{1, \ldots, 2k\}$ where $I_{s(w)}$ (resp. $I_{d(w)}$) is the interval that a generic representative of w intersects first (resp. last). For example, we have $s(x_1^{-1}x_2) = 3$ and $d(x_1^{-1}x_2) = 4$ for the decomposed surface in Figure 2. We define an order on elements of $\pi_1(\Sigma)$ of fixed length r such that for v, w with $s(v) < s(w)$ we have $v < w$. For the remaining pairs $(v, w) \in \pi_1(\Sigma)^2$ with $v \neq w$ we choose $m > 0$ such that $t := v(1) \ldots v(m-1) = w(1) \ldots w(m-1)$ and $x := v(m) \neq w(m) =: y$ and say that $v < w$ if

$$\sigma(t) + \rho(s(y), s(x)) + \rho(d(t), s(x)) + \rho(d(t), s(y)) \equiv 0 \bmod 2, \tag{10}$$

where $\rho(a, b)$ is 0 for $a > b$ and 1 for $a \leq b$. Notice that we have $s(x) \neq s(y)$, $d(t) \neq s(x)$, $d(t) \neq s(y)$. See Figure 3 for an example.
We can visualize the relation defined above as follows: the condition $s(v) < s(w)$ says that if the first interval where v enters a band lies to the left of the first interval where w enters a band, then we have $v < w$. For the second condition, represent v by a generic loop on Σ. We start drawing w by entering the first band to the right of v and then by running close and parallel to v through the first $m - 1$ bands.



$$
\begin{aligned}
&m = 3\\
&x = x_1^{-1}, y = x_1, t = x_1 x_2^{-1}\\
&s(x) = 3, s(y) = 1, d(t) = 2\\
&\sigma(t) + \rho(1,3) + \rho(2,3) + \rho(2,1) =\\
&1 + 1 + 1 + 0 \equiv 1 \bmod 2
\end{aligned}
$$

FIGURE 3. $x_1 x_2^{-1} x_1^{-1} = v > w = x_1 x_2^{-1} x_1$

Then w leaves the $m-1$-st band to the left of v iff $\sigma(t) = 0$. Three configurations of $(s(x), s(y), d(t)) \in \mathbb{N}^3$ and the possible ways how we can continue drawing w for $\sigma(t) = 0$ are shown in Figure 4.

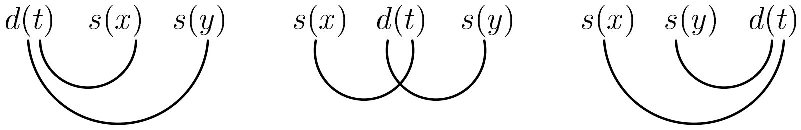


FIGURE 4. Configurations of $(s(x), s(y), d(t))$ and intersections

The remaining possibilities in Figure 4 for $\sigma(t) = 0$ are given by interchanging x and y. The case $\sigma(t) = 1$ is treated in a similar way. We see that

$$\sigma(t) + \rho(s(y), s(x)) + \rho(d(t), s(x)) + \rho(d(t), s(y) \equiv 1 \bmod 2$$

iff the part of w corresponding to y going from the interval $d(t)$ to $s(y)$ must intersect the part of v corresponding to x going from $d(t)$ to $s(x)$. It is easy to verify that the defined relation is an order relation.

An important property of $v, w \in \pi_1(\Sigma)$ with $v > w$ is the following. Assume that v and w are represented by generic loops on Σ corresponding to reduced words, we have $v(1) = w(1)$, v enters the first band at the interval $s(v)$ to the left of w, and v and w have no intersections while they are traversing bands B_i $(i \geq 1)$. Choose the smallest number n such that the part of v going from the interval $d(v(n-1))$ to $s(v(n))$ intersects the part of w going from $d(w(n-1))$ to $s(w(n))$ in an odd number of points. Then we necessarily have $v(i) = w(i)$ for all $i < n$.

Now we can choose representatives $\hat{w}$ of conjugacy classes of elements $w \in \pi_1(\Sigma)$ as follows:

$$\dot{w} = \min\{v \mid v \text{ is cyclically reduced, } [v] = [w] \text{ or } [v^{-1}] = [w]\}, \tag{11}$$

$$\hat{w} = \begin{cases} \dot{w} & \text{if } [\dot{w}] = [w], \\ \dot{w}^{-1} & \text{if } [\dot{w}^{-1}] = [w]. \end{cases} \tag{12}$$

For a decomposed surface we choose $\hat{\mathcal{S}} := \{\hat{w} \mid w \in \pi_1(\Sigma)\}$. For $\Sigma = P^2$ we obviously have to choose $\hat{\mathcal{S}} = \pi_1(P^2, *) \cong \mathbb{Z}/2$. By a descending knot K we shall mean in this case a based knot K with constant height $p_I(K)$. The isotopy class of a descending knot K is again uniquely determined by $\alpha(K) \in \pi_1(P^2, *)$.

For $i = 0, 1$ denote by f_i an orientation preserving homeomorphism $I \cong [i/2, (i+1)/2]$. By Proposition 3 we obtain isomorphisms of $\mathbb{Z}[z]$-modules

$$(\mathrm{id} \times f_i)_* : \mathcal{C}(\Sigma \times I) \longrightarrow \mathcal{C}(\Sigma \times [i/2, (i+1)/2]).$$

The identity $\Sigma \times [1/2, 1] \cup \Sigma \times [0, 1/2] = \Sigma \times I$ induces a $\mathbb{Z}[z]$-bilinear product

$$\mathcal{C}(\Sigma \times I) \times \mathcal{C}(\Sigma \times I) \cong \mathcal{C}(\Sigma \times [1/2, 1]) \times \mathcal{C}(\Sigma \times [0, 1/2]) \longrightarrow \mathcal{C}(\Sigma \times I),$$

where we extend the order of the components of ordered based links L_1, L_2 to their product $L_1 L_2$ by $K_1 < K_2$ for $K_1 \subseteq L_1$, $K_2 \subseteq L_2$. The projection $\mathcal{C}(\Sigma \times I) \longrightarrow \mathcal{C}_0(\Sigma \times I)$ becomes a homomorphism of rings.

Since the multiplication in $\mathcal{C}(\Sigma \times I)$ is in general not commutative we will use a standard order for products of descending knots. For this purpose we fix an arbitrary choice of an order on the set $\hat{\mathcal{S}}$. For example, for a decomposed surface we can take the lexicographical order on the alphabet $x_1, x_1^{-1}, x_2, \ldots$ using reduced expressions.

Definition 9. *An $\hat{\mathcal{S}}$-descending link is an ordered based link isotopic to a product of descending knots $K_1 \ldots K_n$ with $\alpha(K_i) \in \hat{\mathcal{S}}$ and $\alpha(K_1) \leq \ldots \leq \alpha(K_n)$.*

3. Main results about the Conway skein module

Let $\mathrm{Tor}(\mathcal{C}(M)) = \{m \in \mathcal{C}(M) \mid \exists p \in \mathbb{Z}[z] : pm = 0\}$ be the torsion submodule of $\mathcal{C}(M)$ and denote the quotient $\mathcal{C}(M)/\mathrm{Tor}(\mathcal{C}(M))$ by $\mathcal{C}^{\#}(M)$. Let $R(M)$ be the quotient of $\mathcal{C}_0^{\#}(M)$ by the two-sided ideal generated by the elements $\overline{K}_e \overline{K}_w$, where e is the neutral element in $\pi_1(M)$ and $w \in \pi_1(M)$ is arbitrary. For non-orientable M we have $R(M) = \mathcal{C}_0^{\#}(M)$ because a link L one of whose components is a trivial knot satisfies $2L = 0 \in \mathcal{C}(M)$ by the Relation (Bas) and isotopies.

We choose $\hat{\mathcal{S}}$ as in Section 2. Recall the definition of θ from page 5. With the notation from above we can state the following theorem.

Theorem 10. *Let $\Sigma = P^2$ or let Σ be a compact connected surface with boundary. Then there exists a unique $\mathbb{Z}[z]$-linear map*

$$\nabla : \mathcal{C}(\Sigma \times I) \longrightarrow \mathbb{Z}[z] \otimes_{\mathbb{Z}} R(\Sigma \times I)$$

satisfying $\nabla(L) = 1 \otimes \overline{L}$ for all $\hat{\mathcal{S}}$-descending links L with $\theta\left(\overline{L}\right) \neq 0$. The map ∇ induces an isomorphism $\mathcal{C}^{\#}(\Sigma \times I) \cong \mathbb{Z}[z] \otimes_{\mathbb{Z}} R(\Sigma \times I)$.

Theorem 10 will be proven in Section 7. In Sections 4 and 5 we prove the uniqueness of ∇. This part of the proof contains a constructive algorithm for the computation of $\nabla(L)$. In Sections 6 and 7 we prove the existence of ∇. For this part of the proof we use a universal Vassiliev invariant $Z_{\Sigma \times I}$.

As a first example, observe that for $\Sigma = I^2$ we have $\nabla(L) = \tilde{\nabla}(L)\overline{K}_e$ where $\tilde{\nabla}(L)$ is the usual Conway polynomial of L (see Equation (1)).

For a link $L \subset \Sigma \times I$ we denote the mirror image under reflection in $\Sigma \times \{1/2\}$ by L^*. For a link L with ordered components, we use the same order for the components of L^* as for their preimages. For $i = 0, 1$ denote by $|L|_i$ the number of components K of L

satisfying $\sigma(K) = i$. Then we have the following theorem about special properties of the Conway skein module for certain surfaces.

Theorem 11. *(1) For $\Sigma = I^2$, $S^1 \times I$, P^2 or the Möbius strip X the map ∇ satisfies $\nabla(L) = 1 \otimes \overline{L}$ and $\nabla(L_1 L_2) = \nabla(L_1)\nabla(L_2)$ for all $\hat{\mathcal{S}}$-descending links L, L_1, L_2.*
(2) For $\Sigma = P^2$ or $\Sigma = X$ we have $L = (-1)^{|L|_0} L^* \in \mathcal{C}^{\#}(\Sigma \times I)$.

Theorem 11 will be proven in Section 7. Notice that we do not require $\theta(L) \neq 0$ in Part (1) of the theorem. In the case where $\Sigma = I^2$ we have the well-known symmetry property $L^* = (-1)^{|L|+1} L$ similar but different to the formula in Part (2) of Theorem 11 whereas for other surfaces no formula of this type is valid.
Let L be an ordered based link in $P^2 \times I$ represented by a link in $X \times I$ as in Figure 5, where $X = P^2 \setminus D^2$ is the Möbius strip. Let L' be a link in $I^2 \times I$ as in Figure 5, where the parts of the two diagrams consisting of a box labeled T are identical.

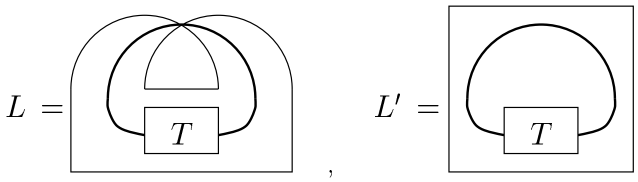


FIGURE 5. A link in $P^2 \times I$ and a link in $I^2 \times I$

If L is a knot, then we have $\nabla(L) = \tilde{\nabla}(L')\overline{K}_s$ where s denotes the unique non-trivial element of $\pi_1(P^2, *)$. In general, we can compose ∇ with the invariant $C(L)$ (see Equation (7)) and obtain

$$\nabla(C(L)) = \begin{cases} \{\tilde{\nabla}(L')\overline{K}_s\} & \text{if } L \text{ is a knot,} \\ \{\pm\tilde{\nabla}(L')\overline{K}_s\} & \text{otherwise.} \end{cases} \tag{13}$$

The example above is related to a simple special case of the following theorem.

Theorem 12. *Let K be a knot in $P^2 \times I$ with $\nabla(K) = f(z)\overline{K}_s$. Then*

$$\nabla(p^{-1}(K)) = f(z)^2 \overline{K}_e,$$

where $p : S^2 \times I \longrightarrow P^2 \times I$ denotes the 2-fold covering map.

Theorem 12 will be proven in Section 8 by using relations between Vassiliev invariants of K and $p^{-1}(K)$.
Recall that a knot K in S^3 is called strongly amphicheiral, if there exists an orientation-reversing involution i of S^3 with $i(K) = K$. One says that K is strongly positive amphicheiral, if there exists an involution i as above that preserves the orientation of K. We have the following corollary.

Corollary 13. *Let K be a strongly positive amphicheiral knot in S^3. Then $\tilde{\nabla}(K) = f(z)^2$ for some $f \in \mathbb{Z}[z]$.*

Proof. Strongly positive amphicheiral knots in S^3 are in one-to-one correspondence with knots K in $P^2 \times I$ with $\sigma(K) = 1$. To see this, recall that the set of fixed points of an orientation-reversing involution i of S^3 is either S^0 or S^2 (see [Smi]). For strongly positive amphicheiral knots K it is sufficient to consider involutions i with fixed point set $S^0 \subset S^3$ and $K \cap S^0 = \emptyset$. By a theorem of Livesay (see [Liv], [Rub]), we then have $(S^3 \setminus S^0)/i \cong P^2 \times]0,1[$. The corollary follows by applying Theorem 12 to the image of K in $(S^3 \setminus S^0)/i$. $\square$

Corollary 13 was proven directly by Kawauchi and Hartley using the Blanchfield pairing (see [HaK], [Bla], [Kaw]). Theorem 12 contains the stronger statement that $f(z)$ itself can be calculated using a skein relation. It would be interesting to have a direct skein-theoretical proof of Theorem 12.

4. Generators of $\mathcal{C}(\Sigma \times I)$

In general one can show that all sets of knots $\mathcal{K}$ for which $\{\alpha(K) \mid K \in \mathcal{K}\}$ contains a set of representatives of conjugacy classes of $\pi_1(\Sigma)$ generate a dense subalgebra of the z-adic completion $\varprojlim \mathcal{C}(\Sigma \times I)/z^n\mathcal{C}(\Sigma \times I)$ of $\mathcal{C}(\Sigma \times I)$. As we will see by the following example, the set $\mathcal{K}$ does not necessarily generate the $\mathbb{Z}[z]$-module $\mathcal{C}(\Sigma \times I)$. Choose a generator $s \in \pi_1(S^1 \times I, *)$. Denote the descending knot in $S^1 \times I^2$ belonging to s^n by K'_n. There are knots K''_n in $S^1 \times I^2$ that are a connected sum of K'_n and a Whitehead knot in $S^1 \times I^2$ such that the equation $K''_n = K'_n - zK'_{\mathrm{sgn}(n)(|n|+1)}K'_{-\mathrm{sgn}(n)}$ holds in $\mathcal{C}(S^1 \times I^2)$ (see Figure 6 for $n = 3$). Using Theorem 10 it is easy to see that the knots K''_n $(n \in \mathbb{Z})$ do not generate $\mathcal{C}(S^1 \times I^2)$ as an algebra, but $\{\alpha(K''_n) \mid n \in \mathbb{Z}\} = \pi_1(S^1 \times I)$.

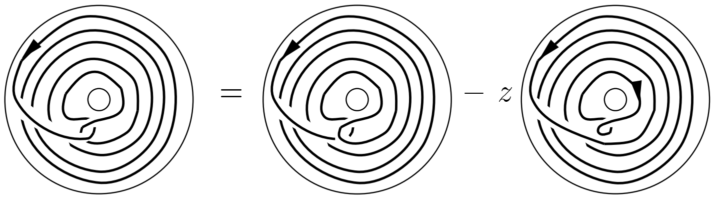


Figure 6. $K''_3 = K'_3 - zK'_4K'_{-1} \in \mathcal{C}(\Sigma \times I)$

The knots K''_n from the previous example are not descending. For a set $\hat{\mathcal{S}}'$ consisting of cyclically reduced representatives of conjugacy classes in $\pi_1(\Sigma)$, where Σ is a decomposed surface, one can show by using Theorem 10 that the descending knots K_w $(w \in \hat{\mathcal{S}}')$ generate $\mathbb{Q}(z) \otimes_{\mathbb{Z}[z]} \mathcal{C}(\Sigma \times I)$ as a $\mathbb{Q}(z)$-algebra, where $\mathbb{Q}(z)$ is the quotient field of $\mathbb{Z}[z]$. The knots K_w $(w \in \hat{\mathcal{S}}')$ do not necessarily generate $\mathcal{C}(\Sigma \times I)$ as a $\mathbb{Z}[z]$-algebra. For example, let Σ be a disc with three holes decomposed such that $s(x_i) = i$, $d(x_i) = 7 - i$ $(i = 1, 2, 3)$. By Theorem 10 and Proposition 3 there exists a basis of the $\mathbb{Z}[z]$-module $\mathcal{C}^{\#}(\Sigma \times I)$ containing the element $K^*_{x_1x_3^{-1}x_2} - K^*_{x_1x_2x_3^{-1}}$, where K^* denotes the mirror image of K with respect to $\Sigma \times \{1/2\}$. Let $\{x_2x_1x_3^{-1}, x_3^{-1}x_1x_2\} \subset \hat{\mathcal{S}}'$. If the knots K_w $(w \in \hat{\mathcal{S}}')$ would generate $\mathcal{C}(\Sigma \times I)$ we see by Theorem 10 that we could find a basis of $\mathcal{C}^{\#}(\Sigma \times I)$ containing the element $K_{x_2x_1x_3^{-1}} - K_{x_3^{-1}x_1x_2}$. But this is impossible because we can show by a computation that

$$K_{x_2x_1x_3^{-1}} - K_{x_3^{-1}x_1x_2} = (1+z^2)\left(K^*_{x_1x_3^{-1}x_2} - K^*_{x_1x_2x_3^{-1}}\right) \in \mathcal{C}(\Sigma\times I). \tag{14}$$

The following lemma says that the set $\hat{\mathcal{S}}$ chosen in Section 2 has better properties.

Lemma 14. *The descending knots K_w ($w\in\hat{\mathcal{S}}$) generate $\mathcal{C}(\Sigma\times I)$ as a $\mathbb{Z}[z]$-algebra.*

Proof. Let L be a diagram of an ordered based link. For the proof we use the following strategy: we will never increase the number of crossings of L, and make computations modulo diagrams with fewer crossings. This allows us to make crossing changes. We will prove the lemma by induction, where the induction base is given by the following arguments for link diagrams without crossings, and the induction step is given by the same arguments for link diagrams with crossings.
Case $\partial\Sigma\neq\emptyset$, Step 1: We apply an isotopy that does not increase the number of double points in $p_\Sigma(L)$ such that the projection of L to $\bigcup_{i=1}^k B_i$ consists of parallel strands connecting points in $I_{s(x_i)}$ with points in $I_{d(x_i)}$. Then we pass from L to a diagram of a product of knots by crossing changes.
Case $\partial\Sigma\neq\emptyset$, Step 2: Each component of L represents a unique word in the generators x_i of $\pi_1(\Sigma)$. We claim that we can replace each of these components by a diagram of a based knot K representing a cyclically reduced word in $\pi_1(\Sigma)$: for words that are not cyclically reduced we find a segment h of K corresponding to a cancellation such that $p_\Sigma(h)$ connects two points $P_1, P_2\in I_\nu$ for some ν and all the parts of $p_\Sigma(K)$ that go into B_0 between P_1 and P_2 intersect $p_\Sigma(h)$ (see Figure 7).

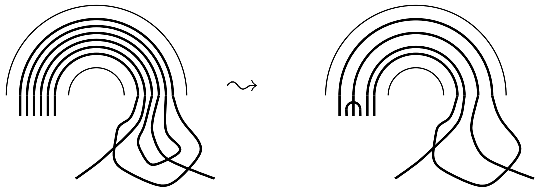


FIGURE 7. Eliminating cancellations of generators of $\pi_1(\Sigma)$

The part h can first be moved to a suitable height by making crossing changes and then be pulled back by applying an isotopy that does not increase the number of crossings as shown in Figure 7. This will remove one cancellation from the word in the generators x_i, x_i^{-1} corresponding to the knot projection. We continue by induction on the number of cancellations to pass to a cyclically reduced word.
Case $\partial\Sigma\neq\emptyset$, Step 3: Recall the definitions of $\dot{w}$ and $\hat{w}$ for $w\in\pi_1(\Sigma)$ (Equations (11) and (12)). We continue with the modification of each based knot K in L by moving the basepoint of K until $w=\hat{w}$ where $w=\alpha(K)$. By the Relation (Bas) this only contributes a factor ±1. We can further achieve that the projection A of the basepoint of K to Σ lies in $I_{s(\dot{w})}$. Assume that there are points in $p_\Sigma(K)\cap I_{s(\dot{w})}$ to the left of A in our pictures of Σ. Let P be the left neighbor of A in $p_\Sigma(K)\cap I_{s(\dot{w})}$. Notice that there may be points of $p_\Sigma(L\setminus K)\cap I_{s(\dot{w})}$ between P and A. Let v be the element of $\pi_1(\Sigma)$ that is represented by $p_\Sigma(K)$ with basepoint P and with an orientation such that at P the curve $p_\Sigma(K)$ enters the band. By the definition of

the representative $\hat{w}$ we have $v \geq \dot{w}$. Assume first $v > \dot{w}$. We have $v(1) = \dot{w}(1)$. Therefore, following the two strands of $p_\Sigma(K)$ starting by entering a band at P and A, we follow paths p and a respectively that pass through the same bands until we find a crossing c between them. Denote that part of L by T whose projection lies in a small neighborhood of the triangle bounded by a, p, and $I_{s(\dot{w})}$ with corners P, A, and c. By crossing changes we pass to a suitable height on T and then pull back this part by an isotopy as shown in Figure 8 until the points P and A are interchanged. The projection of T arrives very close to $I_{s(\dot{w})}$ in B_0 such that no crossings with the remaining part of $p_\Sigma(L)$ can appear.

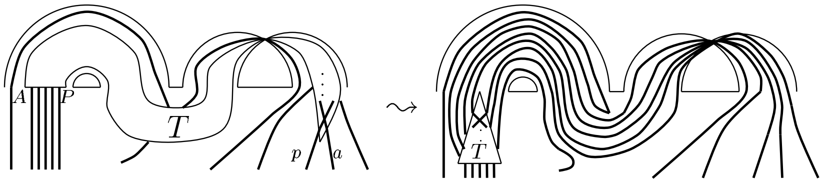


FIGURE 8. Moving the basepoint to the left in $p_\Sigma(K) \cap I_{s(\dot{w})}$

If $v = \dot{w}$, then we push the basepoint along K until its projection arrives at P. Therefore in any case this modification does not increase the number of crossings of K. This implies that by induction we can assume that there are no points in $p_\Sigma(K) \cap I_{s(\dot{w})}$ to the left of A. Then we pass from K to a descending knot by crossing changes. Since this works for all components of L we have proven the lemma in the case where $\partial\Sigma \neq \emptyset$.

Case $\Sigma = P^2$*:* We choose an open disc $D \subset P^2$. Then $P^2 \setminus D = X$ is the Möbius strip. It is easy to see that every link in $P^2 \times I$ can be represented by a link in $X \times I$ and that a descending link in $X \times I$ is isotopic to a descending link in $P^2 \times I$. Therefore we conclude by using this lemma for $\Sigma = X$. $\square$

Let Σ be a surface with nonempty boundary. Let $w \in \pi_1(\Sigma) \setminus \{e\}$. In the free group $\pi_1(\Sigma)$ the normalizer $N(w)$ is isomorphic to $\mathbb{Z}$. A generator z of $N(w)$ is characterized by the property that $w = z^k$ where $k \in \mathbb{Z}$ and $|k|$ is as large as possible. If $\sigma(z) = 1$ and $\sigma(w) = 0$, then by Proposition 5 we have $\overline{K}_w = 0 \in \mathcal{C}_0^\#(\Sigma \times I)$. Nevertheless, we may have $K_w \neq 0 \in \mathcal{C}^\#(\Sigma \times I)$. Hence it is not obvious how to calculate $\nabla(K_w)$ by using only the properties of ∇ given in Theorem 10. Our next goal will be to see why the equation in Figure 9 implies that $\nabla(K_{(x_1x_2^{-1})^2}) = -z\overline{K}_{x_1x_2^{-2}}\overline{K}_{x_1}$. In the rest of this section we will reduce this question to a problem in $\mathcal{C}^\#(X \times I)$, where X is the Möbius strip.

Fix a generator s of $\pi_1(X)$ and denote the descending knot K_{s^n} by K_n. For $v \in \pi_1(\Sigma)$ with $\sigma(v) = 1$ we have an embedding $i_v : X \times I \longrightarrow \Sigma \times I$ such that $i_v(K_1) = K_v$. Up to isotopy i_v is uniquely determined by v (since $\sigma(v) = 1$ no framing is needed). By Proposition 3 the embedding i_v induces a map between skein-modules $i_{v*} : \mathcal{C}(X \times I) \longrightarrow \mathcal{C}(\Sigma \times I)$. We can assume that i_{v*} is $\mathbb{Z}[z]$-linear by making a suitable choice of the oriented subset U of $X \times I$. For $w \in \pi_1(\Sigma) \setminus \{e\}$ there exists a unique generator u of $N(w)$ such that $w = u^n$ for $n \in \mathbb{N}$. If $\sigma(u) = 1$ and $\sigma(w) = 0$,

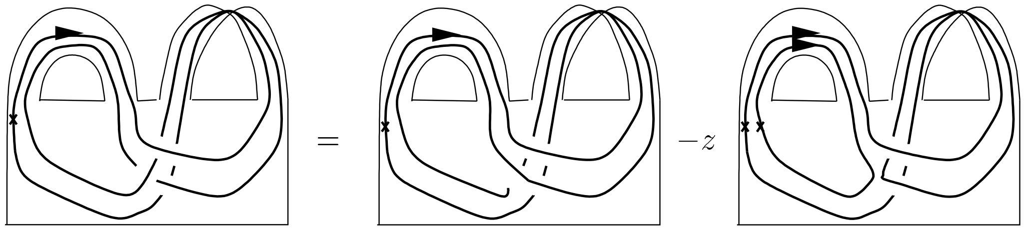


FIGURE 9. What is $\nabla(K_{(x_1x_2^{-1})^2})$?

then we define the *cabled descending knot* as $\hat{K}_w = (i_u)_*(K_n)$. We define $\hat{K}_w = K_w$ in the remaining cases. Using this notation we can rewrite the equation in Figure 9 as $K_{(x_1x_2^{-1})^2} = \hat{K}_{(x_1x_2^{-1})^2} - zK_{x_1x_2^{-2}}K_{x_1} = \hat{K}_{(x_1x_2^{-1})^2} - z\hat{K}_{x_1x_2^{-2}}\hat{K}_{x_1}$.

Lemma 15. *The knots $\hat{K}_w$ ($w \in \hat{\mathcal{S}}$) generate $\mathcal{C}(\Sigma \times I)$ as a $\mathbb{Z}[z]$-algebra.*

Proof. For $\Sigma = P^2$ there is nothing to prove. Assume that $\partial\Sigma \neq \emptyset$. Let L be a link in $\Sigma \times I$. By the first two steps of the proof of Lemma 14 we can assume that the connected components of L represent cyclically reduced words in $\pi_1(\Sigma)$. Applying the following Steps 1 and 2 to all components K of L will imply the lemma by induction on the number of crossings.

Step 1 ("Collect all starting points on the left"): Choose a maximal $k > 0$ such that $\dot{w} = (a_1 \dots a_n)^k \in \pi_1(\Sigma)$ where $w = \alpha(K)$ and $a_i \in \{x_j^{\pm 1}\}$. Assume that $\sigma(a_1 \dots a_n) = 1$ (otherwise there is nothing to do). Consider the k points A_i on K such that $p_\Sigma(A_i) \in I_{s(a_1)}$ and $p_\Sigma(K)$ with each of these points represents $\hat{w}$. We want to move all the points $p_\Sigma(A_i)$ to the left side in $p_\Sigma(K) \cap I_{s(a_1)}$ without increasing the number of crossings of L: if this is not already the case, then choose j such that the point P to the left of $A := p_\Sigma(A_j)$ in $p_\Sigma(K) \cap I_{s(a_1)}$ is not one of the points $p_\Sigma(A_i)$. Let $(b_1 \dots b_n)^k$ be the element of $\pi_1(\Sigma)$ represented by $p_\Sigma(K)$ with respect to P and oriented such that it leaves B_0 at P. Since we use this orientation of $p_\Sigma(K)$ the word $b_1 \dots b_n$ is obtained from $a_1 \dots a_n$ by a cyclic permutation.
The minimality of $\dot{w}$ and the maximality of k implies $a_1 \dots a_n < b_1 \dots b_n$. We have $b_1 = a_1$. Therefore, we can follow $p_\Sigma(K)$ on two strands along $b_1 \dots b_m = a_1 \dots a_m$ ($m < n$) starting at P and A until we find a crossing c between these two strands. As in Step 3 of the proof of Lemma 14 we make crossing changes and then pull back this crossing and possibly some other parts of $L \setminus K$ by an isotopy until the points P and A are interchanged (see Figure 8). With this modification we do not increase the number of crossings of L and the point A has moved one step to the left in $p_\Sigma(K) \cap I_{s(a_1)}$. We will now verify that no other point $A' = p_\Sigma(A_{j'})$ has moved to the right during this operation: assume that we reach A' after starting at P and traveling along $b_1 \dots b_{m'}$ with $0 < m' \leq m$. Then for $x = a_1 \dots a_{m'}$ we either have $a_1 \dots a_n = xyx$ or $a_1 \dots a_n = xyx^{-1}$. We can exclude the second case because this would imply $x = a_1 \dots a_{m'} = b_1 \dots b_{m'} = x^{-1}$ which is impossible. By the minimality of $\dot{w} = (xyx)^k$ and the maximality of k we have $xyx < yxx$ and $xyx < xxy$. From $xyx < yxx$ we deduce $xy < yx$. Then $xyx < xxy$ and $yx > xy$ implies $\sigma(x) = 1$. Therefore, starting at P and A and traveling along x, the first strand reaches $I_{s(a_1)}$

to the right of the second one. This means that by pulling back the crossing c we move the two points A and A' to the left (see Figure 10 for an example). We can continue by induction until all points $p_\Sigma(A_i)$ are on the left side of $p_\Sigma(K) \cap I_{s(a_1)}$.

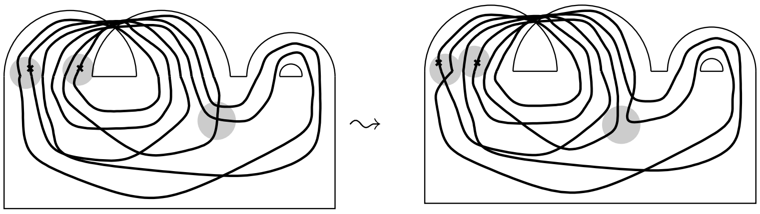


FIGURE 10. Applying step 1 to the leftmost point of $p_\Sigma(K) \cap I_1$

Step 2 ("move component into cabling position"): Successively for $i = 2, \dots, n$, we consider all parts s_ν of K such that the projections $p_\Sigma(s_\nu)$ connects some interval I_j with $I_{s(a_i)}$ inside of B_0. By crossing changes we move all the s_ν to a suitable height and then push the part T containing all crossings between these strands across the band at $I_{s(a_i)}$. This isotopy can be chosen in the following way such that the total number of crossings of L does not increase: after being pushed across the band the part T arrives very close to $I_{d(a_i)} \times I$ and the strands that now enter into $I_{s(a_i)}$ without intersections in their projection to Σ can be moved into a position such that they coincide with original parts of T except in neighborhoods of the crossings and except for orientations.
After having done the previous modification for $i = 2, \dots, n$ we make crossing changes such that $K = i_{a_1 \dots a_n *}(K')$, where K' is a knot in $X \times I$. Applying the arguments in the proof of Lemma 14 for $\Sigma = X$ to K' we see that K is equal to $i_{a_1 \dots a_n *}(K_k) = \hat{K}_w$ modulo diagrams with fewer crossings. □

In a diagram of a product of knots we can permute two components modulo diagrams with fewer crossings. By Relation (Ord) this allows to express $2K_w^2$ $(\sigma(w) = 1)$ by diagrams with fewer crossings. This can be used in the calculation of $\nabla(L)$ by working with $\mathbb{Z}[1/2][z] \otimes R(\Sigma \times I)$. In order to prove that ∇ takes values in $\mathbb{Z}[z] \otimes R(\Sigma \times I)$ we have to be more careful. Since the presentation of $\mathcal{C}_0(\Sigma \times I)$ by generators and relations (see the proof of Proposition 5) shows that $K_w^2 \not\equiv 0 \mod z$ we pass to a quotient of $\mathcal{C}(\Sigma \times I)$. Let J be the two-sided ideal of $\mathcal{C}(\Sigma \times I)$ generated by elements $i_*(K_1^2)$ for all embeddings $i : X \times I \longrightarrow \Sigma \times I$ as in Propositon 3. Then we have the following lemma.

Lemma 16. *The $\mathbb{Z}[z]$-module $\mathcal{C}(\Sigma \times I)/J$ is generated by the following set of links:*

$$\left\{ \hat{K}_{w_1} \dots \hat{K}_{w_n} \mid n \in \mathbb{N}, w_i \in \hat{\mathcal{S}}, w_i \leq w_{i+1}, (w_i = w_{i+1} \Rightarrow \sigma(w_i) = 0) \right\}.$$

Proof. The proof of Lemma 15 shows that the $\mathbb{Z}[z]$-module $\mathcal{C}(\Sigma \times I)$ is generated by the links $\hat{K}_{w_1} \dots \hat{K}_{w_n}$ with $n \in \mathbb{N}$, $w_i \in \hat{\mathcal{S}}$ and $w_i \leq w_{i+1}$. Consider links L of the form $L'K'K''L''$ where L' and L'' are ordered based links and $K' = K'' = \hat{K}_w$ with $\sigma(w) = 1$. For $\sigma(w) = 1$ we always have $K_w = \hat{K}_w$. As shown in the proof of Lemma 14 we may assume that in a diagram of L the projections of the components K' and K'' of L represent cyclically reduced words in $\pi_1(\Sigma)$. Furthermore, we may assume that the projections A' and A'' of the basepoints of K' and K'' lie in $I_{s(\dot{w})}$. When we follow the two strands of $p_\Sigma(K' \cup K'')$ starting by entering a band at A' and A'', we follow paths a' and a'' respectively that pass through the same bands until we find a crossing c between them. We pull this crossing back as shown in Figure 8. We continue with the modification of L as in Step 2 of the proof of Lemma 15. We obtain that L is equal to $L' i_*(K_1^2) L''$ modulo diagrams with fewer crossings, where we have $\alpha(i_*(K_1)) = w$ and where we can choose the embedding $i : X \times I \longrightarrow \Sigma \times I$ such that $i_*(K_1)$ is a knot descending with respect to its basepoint (but in general not a descending knot). The link $L' i_*(K_1^2) L''$ lies in the ideal J. We obtain the lemma by induction on the number of crossings. □

5. Generators of $\mathcal{C}^\#(X \times I)$

In this section we will show by making computations in $\mathcal{C}^\#(X \times I)$ that the knots $\hat{K}_w$ with $\hat{K}_w \neq K_w$ are 0 in $\mathcal{C}^\#(\Sigma \times I)$. Verifying directly that

$$(z^2 + 4) i_*(K_1^2) = 0 \in \mathcal{C}(\Sigma \times I) \tag{15}$$

gives a good impression of the ideas used in this section. Readers that are mainly interested in the case $\Sigma = P^2$ can concentrate on the verification of Equation (15). Recall that we denote the descending knot K_{s^n} for a fixed choice of a generator $s \in \pi_1(X)$ by K_n. In this section we represent ordered based links in $X \times I$ by drawing only a part of them as shown in Figure 11. We say that we represent an ordered based link in L as the closure of a tangle.

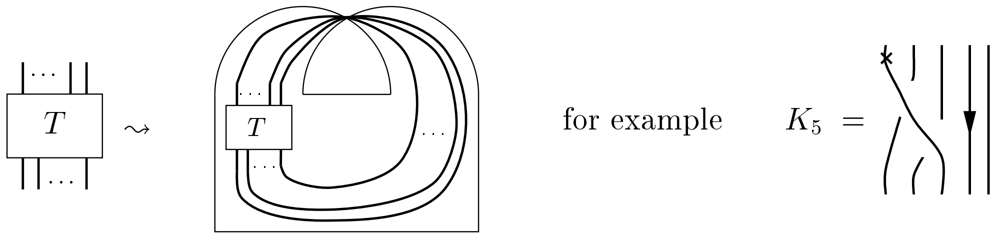


Figure 11. The closure of a tangle

The tangle T must have the same number n of endpoints at the top and at the bottom and the strand at the i-th endpoint at the top (always by counting the endpoints from the left to the right side) is directed downwards if and only if the $(n+1-i)$-th strand at the bottom is directed downwards $(i = 1, \dots, n)$. The ordered set of basepoints of T is in bijection with the components of the closure of T.
The following lemmas are important for determining $\mathcal{C}^\#(X \times I)$. They also provide shortcuts in the computation of the Conway polynomial.

Lemma 17. *For all $k, n, m \in \mathbb{Z}$ the following identities hold in $\mathcal{C}^\#(X \times I)$:*

$$K_{2k} = 0 \qquad \textit{and} \qquad K_{2n+1} K_{2m+1} = -K_{2m+1} K_{2n+1}.$$

As a preparation for the proof of Lemma 17 we first prove the following lemma.

Lemma 18. *Assume that* $K_{2n} = 0 \in \mathcal{C}^{\#}(X \times I)$ *for all* n *with* $-k < n < k$. *Then for* $i \in \mathbb{Z}$ *with* $-k < i < k$ *the following identities hold in* $\mathcal{C}^{\#}(X \times I)$*:*
(1) If a link L *is a closure of a tangle* T *on* $2i$ *strands as in Figure 12 with* $r = i$, *then* $L = 0$.
(2) If a knot K *is a closure of a tangle on* $l \in \{2i+1, 2i+2\}$ *strands as in Figure 12 with* $r = i+1$, K *is descending with respect to its basepoint, the basepoint lies on one of the first* $i+1$ *strands of* K, *and* K *is homotopic to* K_m, *then* $K = K_m = K_m^*$.

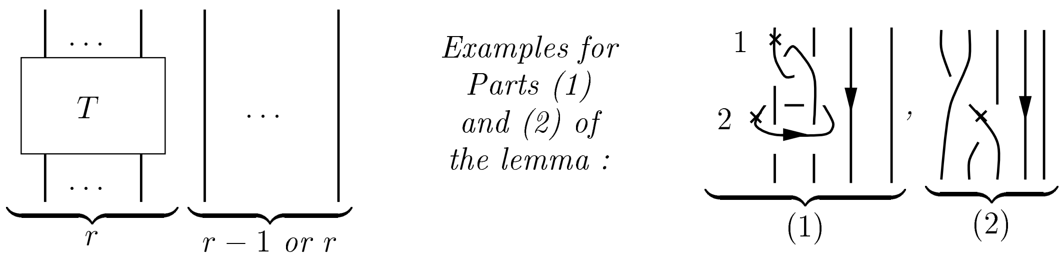


FIGURE 12. Special tangles on an even or odd number of strands

Proof of Lemma 18. (1) Since $2i$ is even and the diagram of L is as shown in Figure 12 we can pass from L to a product $\pm K_{2i_1} \dots K_{2i_r}$ with $-i < i_\nu < i$ by crossing changes and by Relation (Bas). If a crossing of L is spliced, then we obtain again a closure of a tangle as in Figure 12. By induction on the number of crossings of L we can expand L as a linear combination over $\mathbb{Z}[z]$ of monomials $K_{2j_1} \dots K_{2j_m}$ with $-i < j_i < i$. Under the hypotheses of the lemma all these monomials equal 0 in $\mathcal{C}^{\#}(X \times I)$.
(2) Let K be as in Part (2) of the lemma. When we push the basepoint of K along the generator s of $\pi_1(X)$ and l is odd, then Relation (Bas) contributes the sign $(-1)^{1\cdot(l-1)} = 1$. If l is even, then we will see in the following that we will push the basepoint along s an even number of times, because the basepoint is on one of the first $i+1$ strands of K. This will not give a sign contribution either. If the basepoint passes through a crossing, then by changing this crossing the knot will become descending with respect to the new basepoint. If the crossing is spliced, then we obtain a product of two knots that are descending with respect to their basepoints. One of these knots is isotopic to $K_0 = 0$ or satisfies the conditions from Part (1) of the lemma. Therefore we do not change K if we push the basepoint along K and change crossings until the basepoint is close to the left boundary of X in our pictures. This implies $K = K_m$ if K is homotopic to K_m (the number m satisfies $-l \leq m \leq l$ and $l \equiv m \mod 2$). There exists a diagram of K_m as a closure of a tangle as in Figure 12 on m strands having $[(m-1)/2]$ crossings, such that by changing arbitrary crossings and by splicing one arbitrary crossing we obtain a product of two knots (see Figure 11 for $m = 5$). Using this we obtain $K_m = K_m^*$ by similar arguments as above. □

Proof of Lemma 17. We prove the lemma by induction. We have $K_0 = 0$. Assume that the lemma is true for all k, n, m with $|k| \leq \ell$ and $|n| + |m| < \ell$. We have to prove the lemma for $|k|-1=\ell=|n|+|m|$. Since inversion of the orientations of all components of a link induces a $\mathbb{Z}[z]$-linear involution of $\mathcal{C}(M)$, we can assume without loss of generality that $k>0$ and $n\geq 0$. We will prove the following two equations:

$$2K_{2k} = -z\sum_{i=0}^{k-1} K_{2i+1}K_{2(k-i)-1}, \tag{16}$$

$$K_{2n+1}K_{2m+1}+K_{2m+1}K_{2n+1} = zp(z)K_{2k} \quad \text{for some } p(z)\in\mathbb{Z}[z]. \tag{17}$$

Equations (16) and (17) imply that $(2+z^2q(z))K_{2k}=0$ for some $q(z)\in\mathbb{Z}[z]$. Therefore $K_{2k}=0\in\mathcal{C}^{\#}(X\times I)$. This together with Equation (17) implies also $K_{2n+1}K_{2m+1}=-K_{2m+1}K_{2n+1}\in\mathcal{C}^{\#}(X\times I)$.

We prove Equation (16) for $k=3$ in Figure 13 and explain afterwards why this equation holds for arbitrary k.

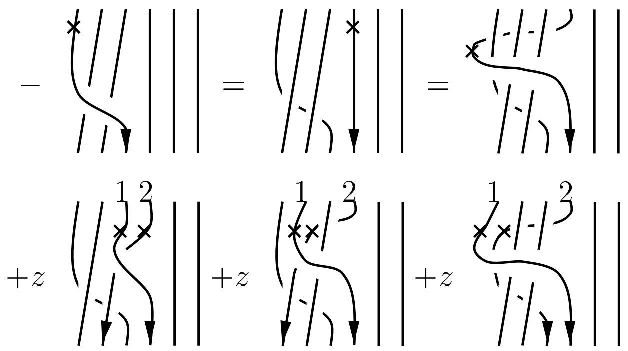


FIGURE 13. A refinement of $2\overline{K}_{2k}=0\in\mathcal{C}_0(\Sigma\times I)$

The first equality in Figure 13 follows in general from Part (2) of Lemma 18 for $l=2k$ and from Relation (Bas). The new knot is descending with respect to its basepoint. The second equality follows by applying the skein relation successively k times to crossings that appear when the new basepoint is pulled to the left. When all crossings are changed, then we obtain again K_{2k}. When one of the k crossings is spliced, then we obtain a product of two descending knots each of which satisfies the conditions of Part (2) of Lemma 18 for odd l. This gives us the k product links K_iK_{2k-i} where i runs successively over the numbers $1,3,\ldots,2k-1$. Looking at signs we obtain Equation (16).

Proof of Equation (17): The link $K_{2n+1}K_{2m+1}$ is the closure of a tangle as shown in Figure 14. In this diagram the crossings between the two components of $K_{2n+1}K_{2m+1}$ are numbered from 1 to $x:=|2m+1|$. The position of the basepoints on $K_{2n+1}K_{2m+1}$ is not important.

Let $D_i(n,m)$ $(i=1,\ldots,x=|2m+1|)$ be the knot obtained by changing the crossings $1,2,\ldots,i-1$ and by splicing the crossing i in Figure 14 (see Figure 15). For later use some crossings of the diagram of $D_i(n,m)$ are marked by the symbol $*$ in Figure 15.

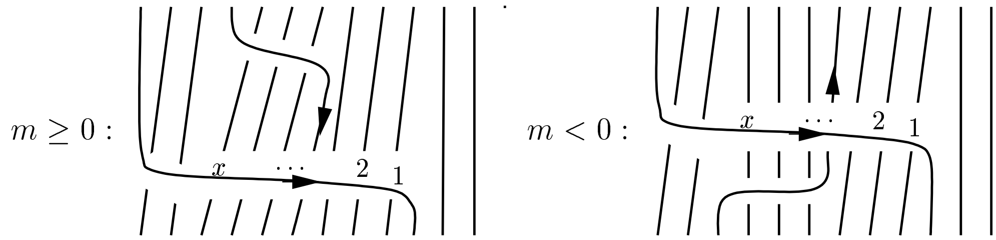


FIGURE 14. The links K_5K_7 and K_5K_{-7}

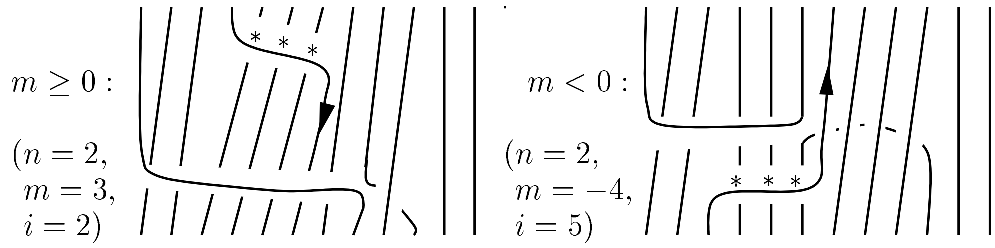


FIGURE 15. The diagram $D_i(n, m)$

Since we can permute the two components of $K_{2n+1}K_{2m+1}$ by changing all crossings between them we obtain by the skein-relation and by Relation (Ord) that

$$K_{2n+1}K_{2m+1} = -K_{2m+1}K_{2n+1} + z \sum_{i=1}^{|2m+1|} \epsilon_i D_i(n, m) \tag{18}$$

for some basepoint on $D_i(n, m)$ and some signs $\epsilon_i \in \{\pm 1\}$. Equation (17) follows from Equation (18), the induction hypothesis and the subsequent lemma. □

Let J_k be the two-sided ideal of $\mathcal{C}(\Sigma \times I)$ generated by knots K_{2n} with $-k < n < k$. Recall the definition of the diagram $D_i(n, m)$ from the proof of Lemma 17 (see Figure 15). Then the following holds.

Lemma 19. *For $n \in \mathbb{N}$, $m \in \mathbb{Z}$ and $i \leq |2m + 1|$ we have $D_i(n, m) = p(z)K_{2k} + a$, where $p(z) \in \mathbb{Z}[z]$, $k = n + |m| + 1$, and a is some element of J_k.*

Proof. A binary tree with root $L = D_i(n, m)$ is given by the following recursive description:
1. We start at the top left of the diagram of L and follow the orientation of the strand until we come back to the point where we started or until we travel along an undercrossing strand of a crossing where we did not travel along the overcrossing strand of that crossing before.
2.a If we come back to the starting point on L in step 1, then the tree for L consists of a leaf labeled by L.
2.b If we reach a crossing first as an undercrossing in step 1, then we apply Relations (Bas) and (Ord) and then a skein relation to this crossing. We obtain $L = \epsilon_1 L_1 + \epsilon_2 z L_2$ for some $\epsilon_j \in \{\pm 1\}$. The tree for L consists of a root vertex connected to a tree for L_1 and to a tree for L_2.

The knot $D_i(n,m)$ is equal to a weighted sum of the labels L' of the leaves in this tree with weights $\pm z^j$, where j is equal to the number of crossings spliced on the path in the tree leading from the root to L'. The leaves are labeled by links L' that are products KL'', where K is a descending knot. We will further examine the paths in the tree leading from the root to a leaf. There is a unique path in the tree where no crossing is spliced and the label of the leaf at the end of this path is $K_{2n+2m+2}$. If a crossing of $D_i(n,m)$ is spliced, then the first one has to be one of the crossings marked by a $*$ in Figure 15. Let D^s be the resulting link diagram (see Figure 16).

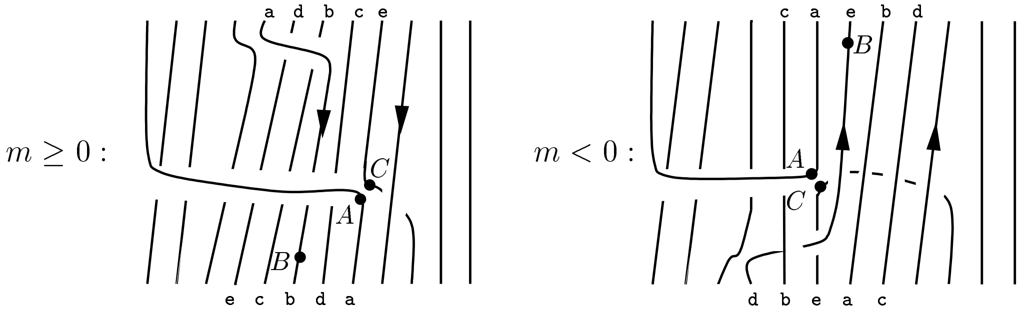


FIGURE 16. The diagram D^s obtained from $D_i(n,m)$

For $m \geq 0$ (resp. $m < 0$) let B be a point on D^s near the $1+n+|m|$-th endpoint at the bottom (resp. at the top) of D^s. Traveling along D^s as in step 1 of the description of the tree for D^s we first pass through a strand of the crossing that was spliced by passing from $K_{2n+1}K_{2m+1}$ to $D_i(n,m)$. We label a point on this strand by A and a point on the other strand of this spliced crossing by C (see Figure 16). When we continue to travel along D^s we will arrive first at the point B and then at the point C on D^s without the need of a further modification of D^s. Forgetting the component K' of D^s containing the points A, B, C, we obtain a knot K''. The upper and lower boundary points of the diagram of K'' consist of a subset of the corresponding boundary points of K_{2m+1} in Figure 14, the $\delta_{1\mathrm{sgn}(m)} + |m|$-th upper and lower boundary point does not belong to K'' because B lies on K', and the strands of K'' to the right of the point B do not intersect. This implies that K'' is the closure of a tangle on $2r$ strands as in Figure 12. The link D^s is not the product of K' and K'', but the point C lies between the r-th and $r+1$-st strand of K'' and when we travel along K' from C back to the starting point we cross K'' only in strands to the right of C. This implies that no matter how D^s is modified in the passage from D^s to a leaf L, the component of L containing the points A, B, C will always be homotopic to K_{2j} for some $j \in \{n-|m|, \ldots, n+|m|+1\} \subseteq \{-k+1, \ldots k\}$. We have $K_{2j} \in J_k$ for $|j| < k$. If the label L of the leaf of the tree is of the form $L = K_{2k}L'$ for some link L', then $L = K_{2k}$. $\square$

Combining Lemma 17 with Lemma 16 we obtain the following proposition.

Proposition 20. *The $\mathbb{Z}[z]$-module $\mathcal{C}^{\#}(\Sigma \times I)$ is generated by the following set of links:*

$$\mathcal{G}_\Sigma = \left\{ \ K_{w_1}\dots K_{w_n} \ \middle| \begin{array}{l} n\in\mathbb{N}, w_i\in\hat{\mathcal{S}}^*, w_i\le w_{i+1}, \\ (w_i=w_{i+1}\Rightarrow \sigma(w_i)=0), (w_i=e\Rightarrow i=n=1) \end{array} \right\}.$$

Proof. The cabled descending knots $\hat{K}_w$ with $\hat{K}_w \neq K_w$ are indexed by $w\in\hat{\mathcal{S}}\setminus(\hat{\mathcal{S}}^*\cup\{e\})$. By definition they are of the form $\hat{K}_w = i_{v*}(K_{2n})$ for some $v\in\pi_1(\Sigma)$ with $\sigma(v)=1$ and $n>0$. By Lemma 17 we have $K_{2n}=0\in\mathcal{C}^\#(X\times I)$ implying $\hat{K}_w = 0\in\mathcal{C}^\#(\Sigma\times I)$. Also by Lemma 17, we have $2K_1^2=0\in\mathcal{C}^\#(X\times I)$ implying $i_*(K_1^2)=0\in\mathcal{C}^\#(\Sigma\times I)$ for an arbitrary embedding $i: X\times I\longrightarrow\Sigma\times I$. This shows that the ideal J of Lemma 16 is contained in $\mathrm{Tor}_{\mathbb{Z}[z]}(\mathcal{C}(\Sigma\times I))$. It is easy to see that for all 3-manifolds M we have $zK_eK_w=0\in\mathcal{C}(M)$. For non-orientable M we have $K_e=0$ and $e\notin\hat{\mathcal{S}}^*$. The three arguments from above together with Lemma 16 imply the proposition. □

6. The weight system of the Conway polynomial

We used ordered based links to define the Conway polynomial ∇ taking values in $\mathbb{Z}[z]\otimes R(\Sigma\times I)$. In this section we will construct a map $W^{\mathrm{ob}}_{\mathfrak{gl}_0}$ from a space of ordered based chord diagrams to $R(\Sigma\times I)\otimes\mathbb{Q}$. In the next section we will use the map $W^{\mathrm{ob}}_{\mathfrak{gl}_0}$ together with a universal Vassiliev invariant $Z^{\mathrm{ob}}_{\Sigma\times I}$ to show that ∇ is well-defined.

Let G be a group. Let $\Gamma = S^{1^{\amalg l}}$ be a disjoint union of l oriented circles. A *G-labeled ordered based chord diagram* D with support Γ consists of a finite set $S=A\amalg B\amalg C$ of mutually distinct points on Γ, such that

- on each circle lies exactly one element of B called *basepoint*, the set B is linearly ordered,
- C is partitioned into subsets of cardinality two called *chords*, and
- to each point p in A there is assigned an element of the group G called *label* of p.

Usually we call G-labeled ordered based chord diagrams simply chord diagrams. We consider two chord diagrams D and D' with support Γ and Γ' respectively as being equal, if there exists a homeomorphism between Γ and Γ' that preserves all additional data. Define the *degree* $\deg(D)$ of a chord diagram D as the number of its chords.

We represent a chord diagram graphically as follows: the circles of the 1-manifold Γ are oriented counterclockwise in the pictures. We connect the two endpoints of a chord by a thin line. The labels of a chord diagram are represented by marking the points of A by a dot on a circle and by writing the labels close to these marked points. The basepoints are marked by the symbol $\times$ on the circle. We draw the basepoints from the left to the right in increasing order or we label the basepoints by elements of an ordered set. An example of a picture of a $\mathbb{Z}$-labeled chord diagram D of degree 5 is shown in Figure 17.

In the following definitions we define vector spaces by generators and graphical relations. We use the convention that all diagrams in a graphical relation coincide everywhere except for the parts we show, and that all configurations of the hidden parts are possible.

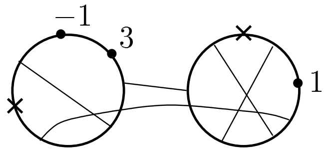
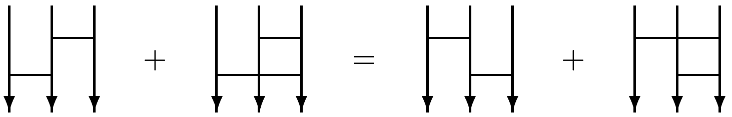
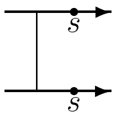


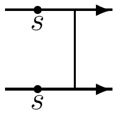
FIGURE 17. A picture of a $\mathbb{Z}$-labeled chord diagram

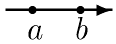
Definition 21. *Let G be a group, and $\sigma : G \longrightarrow \mathbb{Z}/2$ a homomorphism of groups. Define $\mathcal{A}^{\mathrm{ob}}(G,\sigma)$ to be the graded $\mathbb{Q}$-vector space generated by G-labeled ordered based chord diagrams D modulo the Relations (4T), (σ-Nat), (Rep), (Bas), and (Ord) below.*

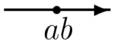
The four-term relation (4T):

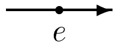


The Relation (σ-Nat): 

$= (-1)^{\sigma(s)}$ 

The Relations (Rep): 

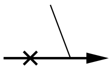
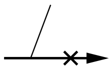
$=$ 

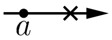
, 

$=$,

where e is the neutral element of G and ab denotes the product of a and b in the group G.

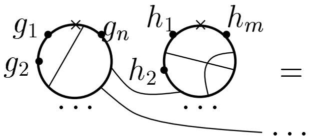
The Relations (Bas):

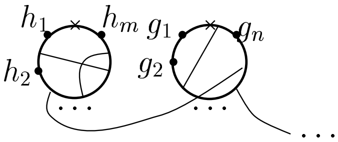
 $=$  , 

$= (-1)^{\sigma(a)\sigma(b)}$ 

where the element $b \in G$ is defined as the product of the labels on the hidden part of the shown component (the order of the multiplication is not important).

The Relation (Ord):



$= (-1)^{\sigma(g_1 \dots g_n)\sigma(h_1 \dots h_m)}$ 

meaning that the change of the order of two neighbored circles gives the sign contribution $(-1)^{\sigma(a)\sigma(b)}$, where a and b are the products of the labels on the two circles.

Definition 22. *Let $\bar{\mathcal{A}}^{\mathrm{ob}}(G,\sigma)$ be the quotient of $\mathcal{A}^{\mathrm{ob}}(G,\sigma)$ by the framing independence relation (FI):*  $= \quad 0.$

Extending the order of the basepoints of chord diagrams D_1, D_2 to the disjoint union $D_1 \amalg D_2$ by requiring that all basepoints of D_1 are smaller than those of D_2 induces a multiplication on $\mathcal{A}(G,\sigma)$ turning it into a graded ring. If $G = \pi_1(M)$ and $\sigma : \pi_1(M) \longrightarrow \mathbb{Z}/2$ is the first Stiefel-Whitney class of M, then one has an isomorphism of rings $\mathcal{C}_0(M) \otimes \mathbb{Q} \cong \mathcal{A}_0(G,\sigma)$, where $\mathcal{A}_0(G,\sigma)$ is the degree-0 part of $\mathcal{A}(G,\sigma)$.

Let D be a picture of a chord diagram such that the basepoint is the highest point on each circle and all labels lie on a horizontal line. An example is shown in Figure 18.

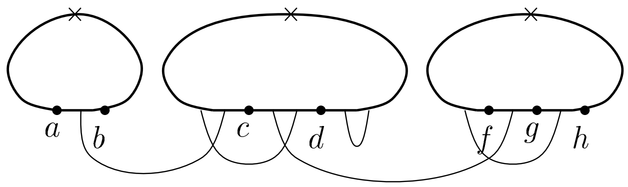


FIGURE 18. A special picture of a chord diagram

We used the order of the oriented circles and the basepoints to draw Figure 18. Now we forget this data. We replace each chord as shown in Figure 19.

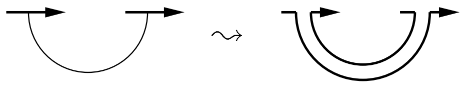


FIGURE 19. Replacing a chord

The result is an immersion of labeled circles into the plane. By a homotopy this immersion can be rearranged to a *standard embedding*, meaning that all labels lie on a horizontal line, no parts of the circles lie below this line, and the projections of the circles to the horizontal line are disjoint. As an example, after having replaced the chords of the diagram in Figure 18 we can rearrange the immersion to a standard embedding as shown in Figure 20.

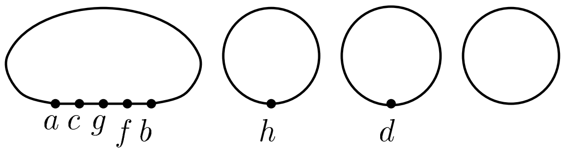


FIGURE 20. A standard embedding of labeled circles

During the homotopy we count the number m of times the order of the projections of two labels s, s' with $\sigma(s) = \sigma(s') = 1$ to the horizontal line changes. For the example in Figures 18 and 20 we have $(-1)^m = (-1)^{\sigma(b)\sigma(cfg)+\sigma(d)\sigma(fgh)+\sigma(f)\sigma(g)}$. The chosen order of the circles and the chosen highest points of the circles in a standard embedding of labeled circles determine an ordered based chord diagram (without chords) that we call D' for a moment.

Proposition 23. *With the notation from above, the linear map*

$$W_{\mathfrak{gl}}^{\mathrm{ob}} : \mathcal{A}^{\mathrm{ob}}(G,\sigma) \longrightarrow \mathcal{A}_0^{\mathrm{ob}}(G,\sigma)$$

that sends a G-labeled ordered based chord diagram D to $(-1)^m D'$ is well-defined.

Proof. Changing the order of two circles or pushing a label around a circle in a standard embedding of labeled circles as in Figure 20 gives the same sign contribution to $(-1)^m$ and to D'. So $W_{\mathfrak{gl}}^{\mathrm{ob}}$ is well-defined for chord diagrams. We have to verify that $W_{\mathfrak{gl}}^{\mathrm{ob}}$ respects the defining relations of $\mathcal{A}^{\mathrm{ob}}(G,\sigma)$:
The map $W_{\mathfrak{gl}}^{\mathrm{ob}}$ respects the relation shown in Figure 21, where any order of the shown parts of the diagram is possible, because one can slide one thickened chord as in Figure 19 along the other one (see [BNG], Section 3). This relation implies the Relation (4T).

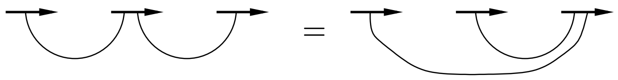


FIGURE 21. A two term relation

We can easily verify the compatibility of $W_{\mathfrak{gl}}^{\mathrm{ob}}$ with the Relations (Rep), (Bas), and (Ord). The compatibility of $W_{\mathfrak{gl}}^{\mathrm{ob}}$ with the Relation (σ-Nat) follows from Figure 22 which shows an equation between elements of $\mathcal{A}_0^{\mathrm{ob}}(G,\sigma)$ defined by immersed labeled circles as in the definition of $W_{\mathfrak{gl}}^{\mathrm{ob}}$.

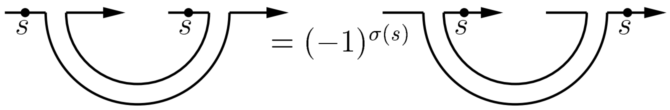


FIGURE 22. The compatibility of $W_{\mathfrak{gl}}^{\mathrm{ob}}$ with the Relation (σ-Nat)

The formula in Figure 22 holds because when we push the labels s to their new positions, then in the projection to the horizontal line they are commuted with the same labels from the remaining part of the diagram and the commutation between the s on the right side with the s on the left side gives the sign contribution $(-1)^{\sigma(s)}$. □

Let $p : \mathcal{A}_0(\pi_1(\Sigma),\sigma) \longrightarrow R(\Sigma\times I)\otimes\mathbb{Q}$ be the homomorphism of rings induced by mapping a circle with a single mark g to $\overline{K}_g$. Replacing the chords in a Relation (FI) as in Figure 19 produces a circle without labels together with at least one additional component. This implies the following corollary.

Corollary 24. *For non-orientable surfaces Σ the map $p\circ W_{\mathfrak{gl}}^{\mathrm{ob}}$ descends to a map*

$$W_{\mathfrak{gl}_0}^{\mathrm{ob}} : \bar{\mathcal{A}}(\pi_1(\Sigma),\sigma) \longrightarrow R(\Sigma\times I)\otimes\mathbb{Q}.$$

The reason for the notation $W_{\mathfrak{gl}}^{\mathrm{ob}}$ and $W_{\mathfrak{gl}_0}^{\mathrm{ob}}$ is given in Section 1.17 of [Lie], where $W_{\mathfrak{gl}}^{\mathrm{ob}}$ and $W_{\mathfrak{gl}_0}^{\mathrm{ob}}$ are related to weight systems associated with the Lie superalgebra $\mathfrak{gl}(V)$ of endomorphisms of a $\mathbb{Z}/2$-graded vector space V. In the case of $W_{\mathfrak{gl}_0}^{\mathrm{ob}}$ the superdimension of V is 0. We will make use of this relation in Section 8.
The map $W_{\mathfrak{gl}}^{\mathrm{ob}}$ can be described recursively as follows. For chord diagrams D of degree 0 we have $W_{\mathfrak{gl}}^{\mathrm{ob}}(D) = D$. If D is a chord diagram with at least one chord c,

then one can use Relations (Bas) and (Ord) such that D looks like one of the two diagrams to the left of the symbol $\leadsto$ in Figure 23.

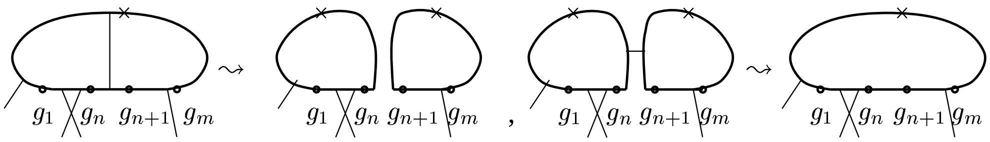


FIGURE 23. A recursive description of $W_{\mathfrak{gl}}^{\mathrm{ob}}$

In Figure 23 only the chord c of D is replaced as shown in Figure 19. We obtain a chord diagram $D(c)$ with $\deg(D(c)) = \deg(D) - 1$ and with

$$W_{\mathfrak{gl}}^{\mathrm{ob}}(D) = W_{\mathfrak{gl}}^{\mathrm{ob}}(D(c)). \tag{19}$$

We can express the recursion formula for $W_{\mathfrak{gl}}^{\mathrm{ob}}$ from Figure 23 without using embeddings by the local replacement rules involving basepoints as shown in Figure 24.

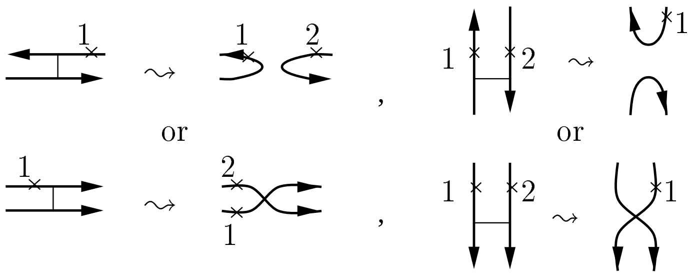


FIGURE 24. Descriptions of $W_{\mathfrak{gl}}^{\mathrm{ob}}$ and $W_{\mathfrak{gl}_0}^{\mathrm{ob}}$ by local replacement rules

7. Proof of Theorems 10 and 11

Let $\Sigma = \bigcup_{i=0}^{k} B_i$ be a decomposed surface or let $\Sigma = P^2$. Let L be a link in $\Sigma \times I$. Recall the definition of the universal Vassiliev invariant $Z_{\Sigma\times I}(L)$ from [Lie]. For a decomposed surface Σ this definition is roughly as follows: after applying an isotopy we can assume that $L \cap B_i \times I$ is in a standard position for all $i > 0$. The tangle $L \cap B_0 \times I$ is turned into a non-associative tangle T by choosing a bracketing on the ordered set $L \cap (\partial B_0) \times I$ satisfying some conditions. Then $Z_{\Sigma\times I}(L)$ is obtained from the universal Vassiliev invariant $Z(T)$ of the non-associative tangle T (see [LM2], [BN2]) by gluing labeled intervals to those pairs of boundary points of diagrams in $Z(T)$ whose corresponding boundary points of T are connected by an interval in $L \cap (\Sigma \setminus B_0) \times I$.

Let $\mathcal{L}^{\mathrm{ob}}(M)$ be the set of isotopy classes of ordered based links in M. Let $\widehat{\mathcal{A}}^{\mathrm{ob}}(M)$ be the graded completion of $\bar{\mathcal{A}}^{\mathrm{ob}}(M) = \bar{\mathcal{A}}^{\mathrm{ob}}(\pi_1(M), \sigma)$. Define a map

$$Z_{\Sigma\times I}^{\mathrm{ob}} : \mathcal{L}^{\mathrm{ob}}(\Sigma \times I) \longrightarrow \widehat{\mathcal{A}}^{\mathrm{ob}}(\Sigma \times I) \tag{20}$$

by equipping the chord diagrams in the series $Z_{\Sigma\times I}(L)$ with an order and with basepoints according to the order and the basepoints of L. The map $Z^{\mathrm{ob}}_{\Sigma\times I}$ is well-defined because chord endpoints commute with the basepoints by the Relation (Bas) (see [Li2]). Let $\widehat{R}(M) := (R(M)\otimes\mathbb{Q})[[h]]$. Define

$$\widehat{W}^{\mathrm{ob}}_{\mathfrak{gl}_0} : \widehat{\mathcal{A}}^{\mathrm{ob}}(G,\sigma) \longrightarrow \widehat{R}(\Sigma\times I) \tag{21}$$

by extending $\widehat{W}^{\mathrm{ob}}_{\mathfrak{gl}_0}(D) = W^{\mathrm{ob}}_{\mathfrak{gl}_0}(D)h^{\deg D}$ for any chord diagram D to formal power series. Finally, define the invariant $\widehat{\nabla}$ of ordered based links by

$$\widehat{\nabla} = \widehat{W}^{\mathrm{ob}}_{\mathfrak{gl}_0}\circ Z^{\mathrm{ob}}_{\Sigma\times I}.$$

Turn $\widehat{R}(M)$ into a $\mathbb{Z}[z]$-module by $z\cdot a := (e^{h/2}-e^{-h/2})a$ for $a\in\widehat{R}(M)$.

Lemma 25. *The invariant $\widehat{\nabla}$ of ordered based links induces a $\mathbb{Z}[z]$-linear map*

$$\begin{aligned} &\widehat{\nabla} : \mathcal{C}^{\#}(\Sigma\times I) \longrightarrow \widehat{R}(\Sigma\times I) \ \textit{satisfying} \\ &\widehat{\nabla}(L) \equiv \overline{L} \bmod h \quad \textit{for all ordered based links } L. \end{aligned} \tag{22}$$

Proof. Let $(L_+, L_-, L_{||})$ be a skein triple of ordered based links. Let $B_1 = \quad 1$ and $B_2 = \ 1\ \ 2$. Define the composition of diagrams D_1, D_2 by placing D_1 onto the top of D_2. In the recursive description of $W^{\mathrm{ob}}_{\mathfrak{gl}}$ (see Figure 24) a chord $B_i\circ$ between two parallel parts of the oriented circles is replaced by $B_{3-i}\circ$. By the explicit description of $Z^{\mathrm{ob}}_{\Sigma\times I}$ and by Equation (19) we have for some $i\in\{1,2\}$:

$$\begin{aligned} &\widehat{\nabla}(L_+) - \widehat{\nabla}(L_-) \\ = \ &\widehat{W}^{\mathrm{ob}}_{\mathfrak{gl}_0}\left(B_i\circ\exp\left(\ /2\right)\circ\ \right) - \widehat{W}^{\mathrm{ob}}_{\mathfrak{gl}_0}\left(B_i\circ\exp\left(-\ /2\right)\circ\ \right) \\ = \ &\sum_{n=0}^{\infty}\frac{1}{4^n(2n+1)!}\widehat{W}^{\mathrm{ob}}_{\mathfrak{gl}_0}\left(B_i\circ\ ^{2n+1}\circ\ \right) \\ = \ &\sum_{n=0}^{\infty}\frac{h^{2n+1}}{4^n(2n+1)!}\widehat{W}^{\mathrm{ob}}_{\mathfrak{gl}_0}(B_{3-i}) = \left(e^{h/2}-e^{-h/2}\right)\widehat{\nabla}(L_{||}) = z\cdot\widehat{\nabla}(L_{||}). \end{aligned}$$

It is obvious by Relations (Ord) and (Bas) for ordered based links and chord diagrams that $\widehat{\nabla}$ is well-defined on $\mathcal{C}(\Sigma\times I)$. Since $\widehat{R}(\Sigma\times I)$ has no non-trivial torsion elements as a $\mathbb{Z}[z]$-module, $\widehat{\nabla}$ descends to $\mathcal{C}^{\#}(\Sigma\times I)$. Equation (22) follows because for ordered based links L whose i-th component is homotopic to $w_i\in\pi_1(\Sigma\times I)$ we have $Z^{\mathrm{ob}}_{\Sigma\times I}(L)\equiv D_L \bmod h$ where $D_L\in\bar{\mathcal{A}}(\Sigma\times I)_0$ is a product of r oriented circles with a single label w_i and $W^{\mathrm{ob}}_{\mathfrak{gl}_0}(D_L) = \pi(D_L) = \overline{L}\in R(\Sigma\times I)\otimes\mathbb{Q}$. ☐

It remains to make a change of parameters to show that the Conway polynomial is well-defined.

Proof of Theorem 10. (1) The uniqueness of ∇ follows because the $\mathbb{Z}[z]$-module $\mathbb{Z}[z]\otimes R(\Sigma\times I)$ has a trivial torsion submodule and the condition $\nabla(L)=\overline{L}$ for descending links L with $\theta(L)\neq 0$ prescribes the image of the generators $\mathcal{G}_\Sigma$ of $\mathcal{C}^\#(\Sigma\times I)$ (see Proposition 20). Let us prove the existence of ∇. Proposition 5 implies that the set $\overline{\mathcal{G}}_\Sigma=\{\overline{L}\mid L\in\mathcal{G}_\Sigma\}$ is a basis of the $\mathbb{Z}$-module $R(\Sigma\times I)$. Define a $\mathbb{Z}[z]$-linear map $\kappa:\mathbb{Z}[z]\otimes_{\mathbb{Z}} R(\Sigma\times I)\longrightarrow\mathcal{C}^\#(\Sigma\times I)$ on basis elements by

$$\kappa(1\otimes\overline{K}_{w_1}\ldots\overline{K}_{w_n}):=K_{w_1}\ldots K_{w_n}.$$

The map κ is surjective. Consider the composition

$$\mathbb{Z}[z]\otimes_{\mathbb{Z}} R(\Sigma\times I)\ \xrightarrow{\ \kappa\ }\ \mathcal{C}^\#(\Sigma\times I)\ \xrightarrow{\ \widehat{\nabla}\ }\ \widehat{R}(\Sigma\times I).$$

By Lemma 25 we have $\widehat{\nabla}(\kappa(\overline{L}))\equiv\overline{L}\bmod h$ for all ordered based links L. This implies that $\overline{\mathcal{G}}_\Sigma$ is mapped injectively to the basis $\widehat{\nabla}(\kappa(\overline{\mathcal{G}}_\Sigma))\bmod h$ of the $\mathbb{Q}$-vector space $\widehat{R}(\Sigma\times I)/(h)$. Considering inductively the sets $\widehat{\nabla}(\kappa(\overline{\mathcal{G}}_\Sigma))\bmod h^n$ we see that $\widehat{\nabla}(\kappa(\overline{\mathcal{G}}_\Sigma))$ is linearly independent over $\mathbb{Q}[z]$ (where $z\cdot a=(e^{h/2}-e^{-h/2})a$). This implies that $\widehat{\nabla}\circ\kappa:\mathbb{Z}[z]\otimes R(\Sigma\times I)\longrightarrow\widehat{R}(\Sigma\times I)$ is injective. Hence κ is an isomorphism and the map $\kappa^{-1}:\mathcal{C}^\#(\Sigma\times I)\longrightarrow\mathbb{Z}[z]\otimes R(\Sigma\times I)$ has the property $\kappa^{-1}(L)=\overline{L}$ for all $\hat{\mathcal{S}}$-descending links L with $\theta(\overline{L})\neq 0$. We obtain ∇ as the composition of the canonical projection $\mathcal{C}(\Sigma\times I)\longrightarrow\mathcal{C}^\#(\Sigma\times I)$ with κ^{-1}. $\square$

Proof of Theorem 11. (1) Let Σ be the Möbius strip X. Let $L\subset X\times I$ be a $\hat{\mathcal{S}}$-descending link. If $\theta\left(\overline{L}\right)=0$, then $L=0\in\mathcal{C}^\#(X\times I)$ by Lemma 17. Theorem 10 implies $\nabla(L)=0$ because ∇ factors through $\mathcal{C}^\#(X\times I)$.
It follows easily from Lemma 17 and Theorem 10 that $\nabla(L_1L_2)=\nabla(L_1)\nabla(L_2)$ for all $\hat{\mathcal{S}}$-descending links $L_1,L_2\subset X\times I$. Lemma 14 implies this formula for all ordered based links $L_1,L_2\subset X\times I$.
The case $\Sigma=P^2$ of the theorem can be deduced from the case $\Sigma=X$. For $\Sigma=I^2$ and $\Sigma=S^1\times I$ Part (1) of the theorem is trivial.
(2) It is enough to show $L=(-1)^{|L|_0}L^*\in\mathcal{C}^\#(\Sigma\times I)$ for $\Sigma=X$. Let L be a diagram of an ordered based link in $X\times I$. Let $k_1(L)$ be the number of crossings of L and let $k_2(L)$ be the minimal number of crossing changes needed to pass from L to a diagram of an $\hat{\mathcal{S}}$-descending link. We prove the theorem by induction on the lexicographical order on $(k_1(L),k_2(L))$.
Assume that $k_2(L)=0$. Then $L=K_{i_1}\ldots K_{i_r}$ is a product of descending knots. By Lemma 17 we have $L=0=L^*$ if $|L|_0>0$. Assume that $|L|_0=0$. Then Lemma 17 and Lemma 18 imply

$$\begin{aligned}K_{i_1}\ldots K_{i_r}&=(-1)^{r(r-1)/2}K_{i_r}\ldots K_{i_1}=(-1)^{r(r-1)/2}K^*_{i_r}\ldots K^*_{i_1}\\&=(K_{i_1}\ldots K_{i_r})^*=(-1)^{|L|_0}\left(K_{i_1}\ldots K_{i_r}\right)^*.\end{aligned}$$

If $k_1(L)=0$, then we also have $k_2(L)=0$ and we are back in the first case. Now let $k_1(L)>0$ and $k_2(L)>0$. Choose a crossing of L such that $k_2(L_1)=k_2(L)-1$, where L_1 is obtained from L by changing this crossing and let L_2 be obtained by splicing the crossing. Then for an $\epsilon \in \{\pm 1\}$ we have by induction

$$\begin{aligned} L &= L_1 + \epsilon z L_2 = (-1)^{|L_1|_0} L_1^* + \epsilon z (-1)^{|L_2|_0} L_2^* \\ &= (-1)^{|L|_0} (L_1^* - \epsilon z L_2^*) = (-1)^{|L|_0} L^*. \end{aligned}$$

In this computation we used that for a skein triple $(L_+, L_-, L_{||})$ of links we have $|L_+|_0 = |L_-|_0 = |L_{||}|_0 \pm 1$ and $(L_-^*, L_+^*, L_{||}^*)$ is also a skein triple.

□

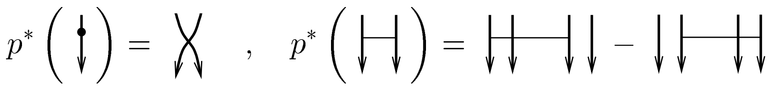
8. Coverings and the Conway polynomial

Let $\bar{\mathcal{A}}(G, \sigma)$ be the vector space generated by G-labeled chord diagrams (without order and basepoints) modulo the Relations (4T), (σ-Nat), (Rep) and (FI) (see Section 6). For a 3-manifold M we define $\bar{\mathcal{A}}(M) := \bar{\mathcal{A}}(\pi_1(M), \sigma)$ where σ is the orientation character of M. The universal Vassiliev invariant $Z_{\Sigma\times I}$ (see [Lie]) takes values in a completion of $\bar{\mathcal{A}}(\Sigma \times I)$. For a labeled diagram D define

$$\sigma(D) = \sum \sigma(g) \in \mathbb{Z}/2$$

where the sum runs over all labels g on the skeleton of D.

Let us recall a result of [Li2] for the special case of the 2-fold covering $p : S^2 \times I \longrightarrow P^2 \times I$. For a chord diagram D labeled by elements of $\pi_1(P^2, *) \cong \mathbb{Z}/2$ we omit the points labeled by the neutral element in our pictures. The points of D labeled by the non-trivial element are simply represented by a dot (without label). There exists a map $p^* : \bar{\mathcal{A}}(P^2 \times I) \longrightarrow \bar{\mathcal{A}}(S^2 \times I)$ defined by the replacement rules shown in Equation (23).

(23) 

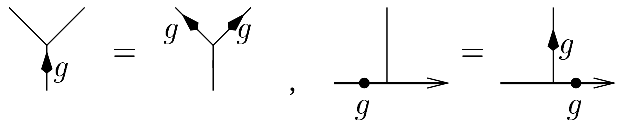
For a link $L \subset P^2 \times I$ we have

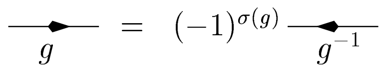
$$p^* \circ Z_{P^2\times I}(L) = Z_{S^2\times I}(p^{-1}(L)) \tag{24}$$

(see [Li2], Theorem 4). Links in $S^2 \times I$ are in a natural bijection with links in S^3 or $\mathbb{R}^3$ and $Z_{S^2\times I}$ corresponds to the usual Kontsevich integral under this bijection.

Comparing $\bar{\mathcal{A}}(P^2 \times I)$ with $\bar{\mathcal{A}}^{\mathrm{ob}}(P^2 \times I)$ we remark huge differences: elements of odd degree in $\bar{\mathcal{A}}(P^2 \times I)$ are 0 (see [Lie], proof of Proposition 1.2) whereas this is not true for $\bar{\mathcal{A}}^{\mathrm{ob}}(P^2 \times I)$ (otherwise the recursive description in Figure 24 of the non-trivial map $W_{\mathfrak{gl}_0}^{\mathrm{ob}}$ could not exist). However, for ordered based diagrams D on a single circle with $\sigma(D) = 1$, order and basepoints are superfluous and the difference between $\bar{\mathcal{A}}(P^2 \times I)$ and $\bar{\mathcal{A}}^{\mathrm{ob}}(P^2 \times I)$ vanishes for these special diagrams.

As for usual chord diagrams there are many descriptions of spaces isomorphic to $\bar{\mathcal{A}}(G, \sigma)$. By G-labeled trivalent diagrams we shall mean usual trivalent diagrams D (see [BN1]) together with a distinguished subset of points on edges of D that are equipped with local orientations and labeled by elements of G. For points on the skeleton of D we assume that the local orientation coincides with the orientation of the skeleton. In particular, labeled chord diagrams are labeled trivalent diagrams. The space $\bar{\mathcal{A}}(G,\sigma)$ is isomorphic to the $\mathbb{Q}$-vector space generated by trivalent diagrams modulo Relations (FI), (Rep) (see Section 6), (STU), (IHX), (AS) (see [BN1]) and modulo the Relations (Comult) and (Inv1) shown below.

(Comult) 

(Inv1) 

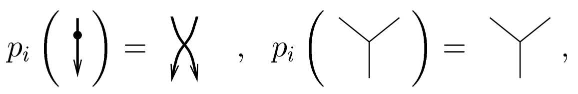
This can be proven along the same lines as in [BN1] (with a slight difference in the case where an internal trivalent vertex connects to the skeleton of the diagram via two edges). A similar description by generators and relations exists for $\bar{\mathcal{A}}^{\mathrm{ob}}(G,\sigma)$. The space $\bar{\mathcal{A}}(G,\sigma)$ is a coalgebra with comultiplication Δ given by

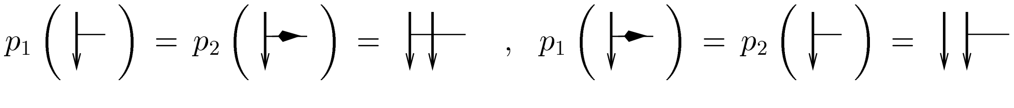
$$\Delta(D) = \sum_{D=D_1\cup D_2} D_1 \otimes D_2, \tag{25}$$

where the sum runs over all trivalent diagrams $D_1, D_2 \subset D$, such that $D\setminus\Gamma = (D_1\setminus\Gamma)\amalg(D_2\setminus\Gamma)$. Besides the comultiplication Δ, there exists a map

$$\Delta^{\mathrm{ob}} : \bar{\mathcal{A}}(G,\sigma) \longrightarrow \bar{\mathcal{A}}^{\mathrm{ob}}(G,\sigma)\otimes\bar{\mathcal{A}}^{\mathrm{ob}}(G,\sigma) \tag{26}$$

defined by the same formula as in Equation (25), where this time D_1 and D_2 are equipped with order and basepoints that are arbitrarily chosen for the first diagram and copied to the second.
Now consider $\mathbb{Z}/2$-labeled trivalent diagrams D with skeleton Γ where $D\setminus\Gamma$ is connected and all labeled points lie close to univalent vertices of $D\setminus\Gamma$. Define diagrams $p_1(D)$ and $p_2(D)$ by the local replacement rules shown in Equations (27) and (28).

(27) 

(28) 

It is easy to see that the map p^* satisfies Equation (29) for diagrams D as above.

$$p^*(D) = p_1(D) + (-1)^{\deg D} p_2(D). \tag{29}$$

For diagrams D where each of the $c > 1$ connected components of $D \setminus \Gamma$ satisfies the conditions above, $p^*(D)$ can be expressed as a sum with 2^c terms by extending Equation (29) multilinearly over these connected components. Using Relations (Comult) and (Inv1), we can apply this formula whenever $D \setminus \Gamma$ is a forest.
We call a connected component of a diagram $D \setminus \Gamma$ a wheel with (resp. without) a dot, if it looks like shown on the left (resp. right) side of Figure 25.

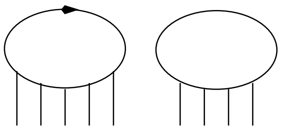


FIGURE 25. A wheel with and without a dot

As a consequence of Equation (29) we can construct many elements in $\mathrm{Ker}(p^*)$. An example is given in the following lemma.

Lemma 26. *Let D be a trivalent diagram such that one component of $D \setminus \Gamma$ is a wheel with a dot. Then $p^*(D) = 0$.*

Proof. Applying the (STU)-relation to the wheel with a dot we obtain the two diagrams D_1 and D_2 shown in Figure 26.

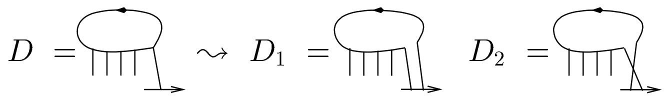


FIGURE 26. $D = D_1 - D_2$ by the (STU)-relation

By Equation (29) we then have

$$\begin{aligned} & p^*(D) = p^*(D_1 - D_2) \\ = \; & p_1(D_1) + (-1)^{\deg(D_1)} p_2(D_1) - p_1(D_2) - (-1)^{\deg(D_2)} p_2(D_2) = 0 \end{aligned}$$

because $p_i(D_1) = p_i(D_2)$. $\square$

Consider an interval J on the skeleton $\Gamma = S^{1^{\amalg \ell}}$ of a diagram D. Cutting D at the endpoints of J we obtain a diagram D' with skeleton $S^{1^{\amalg \ell - 1}} \amalg J \amalg (S^1 \setminus J)$. When D' is the union of diagrams D_1 and D_2 with skeletons J and $S^{1^{\amalg \ell - 1}} \amalg (S^1 \setminus J)$, respectively, then we say that D_1 is an *isolated part* of D. The map $W^{\mathrm{ob}}_{\mathfrak{gl}_0} : \bar{\mathcal{A}}^{\mathrm{ob}}(P^2 \times I) \longrightarrow R(P^2 \times I) \otimes \mathbb{Q}$ has the following special property.

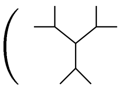
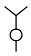
Lemma 27. *Let D be an ordered based $\mathbb{Z}/2$-labeled diagram such that $\sigma(D) = 0$, or the number of circles of the skeleton of D plus the degree of D is even, or D contains an isolated part of odd degree. Then $W^{\mathrm{ob}}_{\mathfrak{gl}_0}(D) = 0$.*

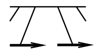
The proof of the lemma follows easily by expanding $W^{\mathrm{ob}}_{\mathfrak{gl}_0}(D)$ as a linear combination of multiples of $\overline{K}_e = 0$ and $\overline{K}_s^2 = 0 \in R(P^2 \times I) \otimes \mathbb{Q}$.

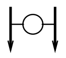
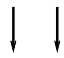
Some properties of $W^{\mathrm{ob}}_{\mathfrak{gl}_0}$ for ordered based $\mathbb{Z}/2$-labeled diagrams can more conveniently be proven using the Lie superalgebra $\mathfrak{gl}(1|1)$. We identify $\mathfrak{gl}(1|1)$ with 2×2-matrices by using a homogeneous basis of the defining $(1|1)$-dimensional representation of $\mathfrak{gl}(1|1)$. The rough idea, how the usual description of the weight system associated to $\mathfrak{gl}(1|1)$ (see [Vai]) extends to $\mathbb{Z}/2$-labeled diagrams is by mapping a dot on the skeleton to the morphism τ in Equation (30).

(30) $$\mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \tau$$

(compare [Lie], Section 1.17; recall from there how labels can influence signs). Given an ordered based $\mathbb{Z}/2$-labeled diagram D with $\sigma(D)=1$ we cut the first circle of the skeleton of D at its basepoint. Then the construction of weight systems maps the resulting diagram to a morphism of the form $\lambda\cdot\tau$. It is easy to show that $W^{\mathrm{ob}}_{\mathfrak{gl}_0}(D) = \lambda\overline{K}_s$. In particular, the number $\lambda =: W^s(D)$ depends only on $D\in\bar{\mathcal{A}}^{\mathrm{ob}}(P^2\times I)$.
Define $W^e : \bar{\mathcal{A}}(S^2\times I) \longrightarrow \mathbb{Q}$ by $W^e(D)=\lambda$ whenever $W^{\mathrm{ob}}_{\mathfrak{gl}_0}(D) = \lambda\overline{K}_e$. The following properties of $W=W^e$ for diagrams without labels are proven in [Vai] [1]. They extend to $W=W^s$ in a straightforward way.

(31) $W\Big($  $\Big) = W\Big($  $\Big) = 0$

(32) $W\Big($  $\Big) = W\Big($  $\Big)$

(33) $W\Big($  $\Big) = -2W\Big($  $\Big)$

Equations (31)–(33) for $W=W^e, W^s$, Equations (25), (26), (29), and the (STU)-relation imply the following lemma.

Lemma 28. *Equations (31)–(33) also hold for*

$$W = W^e\circ p^* \quad \textit{and} \quad W = (W^s\otimes W^s)\circ\Delta^{\mathrm{ob}}.$$

Now we are ready to prove the following important lemma.

Lemma 29. *Let D be a $\mathbb{Z}/2$-labeled diagram with skeleton S^1 and $\sigma(D)=1$. Then*

$$(W^s\otimes W^s)\circ\Delta^{\mathrm{ob}}(D) = W^e(p^*(D)).$$

Proof. The proof is devided into four steps.
(1) Assume that a component of $D\setminus\Gamma$ is a wheel with a dot. Then $W^s(D)=0$ (by Equation (32) and Lemma 27 it is sufficient to verify this for a degree-2 wheel with a dot. This case follows from a straightforward compuation).

[1] The diagram on the right side of Equation (32) is a trivalent diagram in the sense of this paper (that means it must not contain a connected component that does not connect to the skeleton of the diagram).

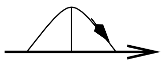
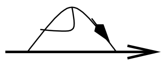
(2) We show now that when the lemma holds for D, then it also holds for all diagrams D' with $D \subset D'$ where $C = D' \setminus D$ is connected and C is not a tree. If C is a wheel with a dot then we have

$$(W^s \otimes W^s) \circ \Delta^{\mathrm{ob}}(D') = 0 \quad \text{and} \quad W^e(p^*(D')) = 0 \tag{34}$$

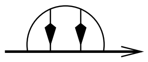
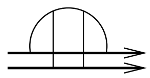
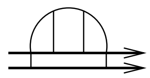
by Part (1) of this proof and Lemma 26. If C is not a wheel, then Equation (34) follows from Lemma 29 and Equation (31). If C is a wheel of odd degree, then Equation (34) is implied by the (STU)-relation, the previous case (where C was not a wheel), and Lemma 27. Finally, if C is a wheel of even degree without a dot, then we have $W^{\mathrm{ob}}_{\mathfrak{gl}_0}(D') = -2W^{\mathrm{ob}}_{\mathfrak{gl}_0}(D)$ by Equations (32) and (33) which implies

$$(W^s \otimes W^s) \circ \Delta^{\mathrm{ob}}(D') = -4(W^s \otimes W^s) \circ \Delta^{\mathrm{ob}}(D) = -4W^e(p^*(D)) = W^e(p^*(D')).$$

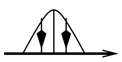
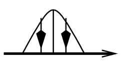
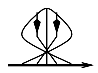
(3) Now assume that the lemma holds for a diagram D and consider $D \subset D'$ such that $C = D' \setminus D$ is a tree. We call the tree C a comb if we cannot apply Equation (31) to the part C of the diagram D. By Lemma 28 and Equation (32) we only need to consider combs of degrees 1, 2, 3, and 4. By the (STU)-relation and Part (2) of this proof, we may arrange the univalent vertices of C on the skeleton S^1 in any order we want to. We then apply Relation (Comult) to reduce the configuration of labels we need to consider. In the following we investigate the remaining possibilities.
deg $C = 1$: By Relation (FI) and Lemma 1.9 of [Lie] there is nothing to prove in this case (alternatively, there is a direct proof similar to the case deg $C = 3$).
deg $C = 2$: We only need to consider the case shown in Equation (35), where we apply Relations (STU) and (AS) to reduce this case to Part (2) of this proof.

(35)  $=\ \frac{1}{2}$ 

deg $C = 3$: All possible configurations of $C \subset D'$ can be reduced the case shown in Equation (36), where we prove that $p^*(D') = 0$.

(36) $p^*\Big($  $\Big) =$  $-$  $= 0$

By Lemma 27 we have $(W^s \otimes W^s) \circ \Delta^{\mathrm{ob}}(D') = 0$ because $\Delta^{\mathrm{ob}}(D')$ is a linear combination of elements $D_1 \otimes D_2$ such that D_1 or D_2 has an isolated part of odd degree.
deg $C = 4$: It is sufficient to consider a comb as shown on the left side of Equation (37).

(37)  $\rightsquigarrow \frac{1}{2}$  $+\frac{1}{2}$  $= 0$

By the (STU)-relation and Part (2) of this proof, we can equivalently consider the linear combination of two diagrams shown on the right side of Equation (37). But by Relation (AS) this element equals 0.

(4) Conclusion: For diagrams of degree 0 the lemma is obvious. Using Parts (2) and (3) of this proof the lemma follows by induction. □

It would be interesting to know to what extent Theorem 12 can be generalized to links. A counterexample to Lemma 29 for chord diagrams on more than one circle is shown in Figure 27.

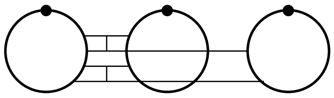


FIGURE 27. A diagram D with $(W^s \otimes W^s) \circ \Delta^{\mathrm{ob}}(D) = 0$ and $W^e(p^*(D)) \neq 0$

Lemma 29 is the main ingredient in the following proof of Theorem 12.

Proof of Theorem 12. Define $\mathbb{Q}[[h]]$-valued maps $\widehat{W}^e$, $\widehat{W}^s$ by $\widehat{W}^e(D) = W^e(D)h^{\deg D}$ and $\widehat{W}^s(D) = W^s(D)h^{\deg D}$. Let $K^{\mathrm{ob}} \subset P^2 \times I$ be a based knot with $\sigma(K) = 1$ and let K be K^{ob} without its basepoint. Then we have

$$\widehat{W}^e \circ Z_{S^2\times I}(p^{-1}(K)) = \widehat{W}^e \circ p^*(Z_{P^2\times I}(K)) \tag{38}$$

$$= \left(\widehat{W}^s \otimes \widehat{W}^s\right) \circ \Delta^{\mathrm{ob}}(Z_{P^2\times I}(K)) = \left(\widehat{W}^s \circ Z^{\mathrm{ob}}_{P^2\times I}(K^{\mathrm{ob}})\right)^2, \tag{39}$$

where the first equality follows from Equation (24), the second equality follows from Lemma 29, and the last one follows because $Z_{P^2\times I}(K)$ is group-like.
It remains to verify that Equations (38), (39) are compatible with the replacement of parameters in the proof of Theorem 10: by Proposition 20 there exist $f, g \in \mathbb{Z}[z]$ such that

$$\nabla(p^{-1}(K)) = g(z)\overline{K}_e \quad \text{and} \quad \nabla(K^{\mathrm{ob}}) = f(z)\overline{K}_s. \tag{40}$$

By Lemma 25 we have

$$\widehat{W}^e \circ Z_{S^2\times I}(p^{-1}(K)) = g(e^{h/2} - e^{-h/2})\widehat{W}^e \circ Z_{S^2\times I}(K_e) \tag{41}$$

$$\text{and} \quad \widehat{W}^s \circ Z_{P^2\times I}(K^{\mathrm{ob}}) = f(e^{h/2} - e^{-h/2})\widehat{W}^s \circ Z_{P^2\times I}(K_s). \tag{42}$$

As a simple consequence of the construction of $Z_{P^2\times I}$ we have

$$\left(\widehat{W}^s \circ Z_{P^2\times I}(K_s)\right)^2 = \widehat{W}^e \circ Z_{S^2\times I}(K_e). \tag{43}$$

Equations (38)–(43) imply $g(z) = f(z)^2$ what we wanted to prove. □

Lemma 29 can be generalized from P^2 to the Möbius strip X using the weight system $W^{\mathrm{ob}}_{\mathfrak{gl}_0} \bmod (\overline{K}_i\overline{K}_j \mid i, j \in \mathbb{Z})$. A generalization of Theorem 12 to this case is not straightforward because Equation (40) becomes more complicated and for ordered based links L in the ideal $(K_iK_j \mid i, j \in \mathbb{Z}) \subset \mathcal{C}(X \times I)$ we may have

$$\widehat{W}^{\mathrm{ob}}_{\mathfrak{gl}} \circ Z_{X\times I}(L) \not\equiv 0 \bmod (\overline{K}_i\overline{K}_j \mid i,j\in\mathbb{Z}) \tag{44}$$

(for example for $L = K_{-1}K_1K_3$).

Appendix

In the following we apply the methods of this paper to determine the Homfly and Kauffman skein modules of cylinders over oriented surfaces with boundary. For the Homfly skein module we rediscover the main result of [Pr1]. Our choice of representatives of conjugacy classes in $\pi_1(\Sigma)$ is different to the choice made in [Pr1]. Our choice has the advantage that it allows us to determine the structure of the 2-variable Kauffman skein module with similar methods. This result seems to be new.

The Homfly skein module. The *Homfly skein module* of an oriented 3-manifold is generated over the ring $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ by isotopy classes of oriented links modulo the skein relation of the Homfly polynomial

$$x \quad - x^{-1} \quad = y \quad, \tag{45}$$

where the links (, ,) differ only locally as shown in the diagrams. For technical reasons we include the empty link $\emptyset$ in the definition of the Homfly skein module and relate it to the trivial knot O by the equation $(x-x^{-1})\emptyset = yO$.
In difference to the Conway skein module, the 3-manifold must be oriented for the definition of the Homfly skein module, and we need no order or basepoints on L. Let Σ be an oriented decomposed surface in the sense of Section 2. Define the ordered set $\hat{\mathcal{S}}$ of representatives of conjugacy classes of elements in $\pi_1(\Sigma)$ as in Section 2. Then $\hat{\mathcal{S}}$-descending links are defined by forgetting the order and basepoints of the links in Definition 9. Define $\hat{\mathcal{S}}^\dagger = \hat{\mathcal{S}} \setminus \{e\}$, where $e \in \pi_1(\Sigma)$ is the neutral element. For a link $L \subset \Sigma \times I$ we define $L^\dagger$ as L without the components that are homotopic to e. Let $n(L)$ be the number of components of $L \setminus L^\dagger$. Let $\mathbb{Z}[t_{\hat{\mathcal{S}}^\dagger}]$ (resp. $\mathbb{Z}[t_{\hat{\mathcal{S}}}]$) be a polynomial ring with indeterminates t_w in one-to-one correspondence with elements $w \in \hat{\mathcal{S}}^\dagger$ (resp. $w \in \hat{\mathcal{S}}$). For a knot $K \subset \Sigma \times I$ the element $t_K \in \mathbb{Z}[t_{\hat{\mathcal{S}}}]$ is defined as $t_K = t_w$, where K is homotopic to w. For a link L the element t_L is defined as the product of elements t_K where K runs over all components of L. By $t_\emptyset$ we mean $1 \in \mathbb{Z}[t_{\hat{\mathcal{S}}}]$. Then we have the following theorem.

Theorem 30. *There exists a unique invariant*

$$H(L) \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}] \otimes_{\mathbb{Z}} \mathbb{Z}[t_{\hat{\mathcal{S}}^\dagger}]$$

of links L in $\Sigma \times I$ that depends only on the class of L in the Homfly skein module and satisfies

$$H(L) = t_{L^\dagger}\left(\frac{x-x^{-1}}{y}\right)^{n(L)} \quad \textit{whenever } L \textit{ is } \hat{\mathcal{S}}\textit{-descending.} \tag{46}$$

The link invariant H is called the Homfly polynomial. As in the proof of Lemma 14 we see that Equations (45) and (46) are sufficient to calculate $H(L)$ for every link[2]. This implies the uniqueness stated in Theorem 30. Under the condition that H factors through the Homfly skein module it can easily be shown that Equation (46) is implied by the same equation for links L with $n(L) = 0$. This implies that the Homfly skein module of $\Sigma \times I$ is a quotient of $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}] \otimes_{\mathbb{Z}} \mathbb{Z}[t_{\hat{\mathcal{S}}^\dagger}]$. We give a sketch of the proof of the existence of H in the rest of this section. This will imply that the Homfly skein module of $\Sigma \times I$ is isomorphic to $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}] \otimes_{\mathbb{Z}} \mathbb{Z}[t_{\hat{\mathcal{S}}^\dagger}]$.
For a group G define $\mathcal{A}(G) = \mathcal{A}^{\mathrm{ob}}(G, 0)$ and $\bar{\mathcal{A}}(G) = \bar{\mathcal{A}}^{\mathrm{ob}}(G, 0)$ (see Definitions 21 and 22). In these definitions the order and the basepoints on chord diagrams and relations (Ord) and (Bas) are superfluous because $\sigma = 0$. We identify the degree-0 part $\mathcal{A}_0(G)$ of $\mathcal{A}(G)$ with the isomorphic polynomial ring $\mathbb{Q}[t_{\hat{\mathcal{S}}}]$. The analogue of Corollary 24 is not true for $\sigma = 0$ because $t_e t_w \neq 0$ for $w \in G$. There are two equivalent ways of solving this problem. One way is used in the following, and the second way is used in Section 8 for the Kauffman polynomial. For a G-labeled chord diagram D, we define the element $\iota(D)$ by replacing each chord as shown in Figure 28.

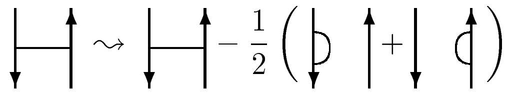


FIGURE 28. The deframing map ι

The definition determines a linear map $\iota : \bar{\mathcal{A}}(G) \longrightarrow \mathcal{A}(G)$, such that $p \circ \iota = \mathrm{id}$ where $p : \mathcal{A}(G) \longrightarrow \bar{\mathcal{A}}(G)$ denotes the canonical projection (compare [BN1], Exercise 3.16). The map ι induces a map $\iota^* : \mathcal{A}(G)^* \longrightarrow \bar{\mathcal{A}}(G)^*$ called deframing projection. For $\sigma = 0$ we denote the map $W_{\mathfrak{gl}}^{\mathrm{ob}}$ (see Proposition 23) also by $W_{\mathfrak{gl}}$. Define

$$\overline{W}_{\mathfrak{gl}} : \bar{\mathcal{A}}(G) \longrightarrow \mathbb{Q}[t_{\hat{\mathcal{S}}}] \;, \quad \overline{W}_{\mathfrak{gl}} = W_{\mathfrak{gl}} \circ \iota. \tag{47}$$

The map $\overline{W}_{\mathfrak{gl}}$ is called the weight system of the Homfly polynomial. Proceeding as in Section 7, we extend $\overline{W}_{\mathfrak{gl}}$ to the completion $\widehat{\mathcal{A}}(G)$ of $\bar{\mathcal{A}}(G)$ by

$$\widehat{W}_{\mathfrak{gl}}(D) = \overline{W}_{\mathfrak{gl}}(D) h^{\deg D} \in \mathbb{Q}[t_{\hat{\mathcal{S}}}][[h]]. \tag{48}$$

For a link $L \subset \Sigma \times I$ let $\widehat{H}(L) \in \mathbb{Q}[t_{\hat{\mathcal{S}}}][[h]]$ be given by $\widehat{H}(L) = \widehat{W}_{\mathfrak{gl}} \circ Z_{\Sigma \times I}(L)$ where $Z_{\Sigma \times I}$ denotes the universal Vassiliev invariant of links in $\Sigma \times I$. Denote the inclusion map $\mathbb{Q}[t_{\hat{\mathcal{S}}}][[h]] \longrightarrow \mathbb{Q}[t_{\hat{\mathcal{S}}}][[h, h^{-1}]]$ by i. We turn $\mathbb{Q}[t_{\hat{\mathcal{S}}}][[h, h^{-1}]]$ into a $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$-module by

$$x \cdot a = e^{t_e h/2} a \quad \text{and} \quad y \cdot a = (e^{h/2} - e^{-h/2}) a. \tag{49}$$

With the notation from above we have the following lemma.

[2]This argument is much simpler than the corresponding proof for the Conway polynomial in Sections 4 and 5.

Lemma 31. *The link invariant $i\circ\widehat{H}$ induces a $\mathbb{Z}[x^{\pm1},y^{\pm1}]$-linear map from the Homfly skein module to $\mathbb{Q}[t_{\hat{\mathcal{S}}}][[h,h^{-1}]]$ and satisfies*

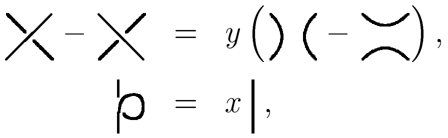
$$\widehat{H}(L)\equiv t_L \operatorname{mod} h \textit{ for every link } L.$$

The proof of Lemma 31 and the completion of the proof of Theorem 30 are similar to the proofs in Section 7.

The Kauffman skein module. The *Kauffman skein module* of an oriented 3-manifold is generated over the ring $\mathbb{Z}[x^{\pm1},y^{\pm1}]$ by isotopy classes of framed oriented links modulo the relation that reversing the orientation of a component induces the identity map of the Kauffman skein module, and modulo the skein relations of the Kauffman polynomial

(50)

(51)



where the framed links (, , ,) and (,) differ only locally as shown by the diagrams. In diagrams of framed links the framing is assumed to be the so-called blackboard framing (the framing pointing to the reader). For technical reasons we include the empty link $\emptyset$ in the definition of the Kauffman skein module and relate it to the trivial knot with 0-framing O by the equation $(x-x^{-1}+1)\emptyset=yO$.
Let Σ be an oriented decomposed surface in the sense of Section 2. Define the ordered set $\hat{\mathcal{S}}$ of representatives of conjugacy classes of elements in $\pi_1(\Sigma)$ as in Section 2. Let $\hat{\mathcal{S}}_\pm=\{\min\{a,a^{-1}\}\,|\,a\in\hat{\mathcal{S}}\}$. The set $\hat{\mathcal{S}}_\pm$ is in one-to-one correspondence with homotopy classes of non-oriented knots. In this section, $\hat{\mathcal{S}}$-descending links are defined by forgetting the order and basepoints of the links in Definition 9 and equipping them with arbitrary framing.
For a diagram L of a framed oriented link in $\Sigma\times I$, define the *writhe* of L as $\mathrm{w}(L)=k_+-k_-$, where k_+ is the number of positive crossings and k_- is the number of negative crossings in the diagram. The writhe is an isotopy invariant of framed oriented links and of framed non-oriented knots.
Define $\hat{\mathcal{S}}_\pm^\dagger=\hat{\mathcal{S}}_\pm\setminus\{e\}$, where $e\in\pi_1(\Sigma)$ is the neutral element. For a framed link L we define $L^\dagger$, $n(L)$ and $t_L\in\mathbb{Z}[t_{\hat{\mathcal{S}}_\pm}]$ in the same way as in Section 8. Then we have the following theorem.

Theorem 32. *There exists a unique invariant*

$$F(L)\in\mathbb{Z}[x^{\pm1},y^{\pm1}]\otimes_{\mathbb{Z}}\mathbb{Z}[t_{\hat{\mathcal{S}}_\pm^\dagger}]$$

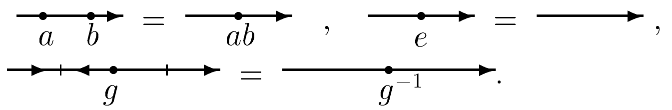
of framed links L in $\Sigma\times I$ that depends only on the class of L in the Kauffman skein module and satisfies

$$F(L)=t_{L^\dagger}\left(\frac{x-x^{-1}}{y}+1\right)^{n(L)}x^{\mathrm{w}(L)} \textit{ whenever } L \textit{ is } \hat{\mathcal{S}}_\pm\textit{-descending.}$$

The link invariant F is called the Kauffman polynomial. The polynomial $x^{-w(L)}F(L)$ is defined for each framed oriented link L and does not depend on the framing of L, hence is an isotopy invariant of oriented links. As in the proof of Lemma 14 we see that the conditions in Theorem 32 are sufficient to calculate $F(L)$ for every link. This implies the uniqueness stated in Theorem 32. We give a sketch of the proof of the existence of F in the rest of this section. This will also imply that the Kauffman skein module of $\Sigma \times I$ is isomorphic to $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}] \otimes_{\mathbb{Z}} \mathbb{Z}[t_{\hat{\mathcal{S}}_\pm^\dagger}]$.

We say that a circle is locally oriented if it is decomposed into a finite number of oriented intervals. Let $C_{\mathrm{loc}}(G)$ be the $\mathbb{Q}$-vector space generated by disjoint unions of locally oriented circles with a finite number of distinct points on oriented parts of the circles labeled by elements of G modulo homeomorphisms of these diagrams and the following relations:

(Rep):
(Ori):



Here and in Figure 29 we represent elements of $C_{\mathrm{loc}}(G)$ graphically by formal linear combinations of parts of pictures of immersions of labeled, locally oriented circles. The points where the local orientation changes are marked by the symbol $|$.

Recall the definition of the vector space $\mathcal{A}(G) = \mathcal{A}^{\mathrm{ob}}(G, 0)$ (see Definition 21). Define a map $\beta_1 : \mathcal{A}(G) \longrightarrow C_{\mathrm{loc}}(G)$ by replacing each chord as shown in Figure 29 (it is easy to see that β_1 is well-defined).

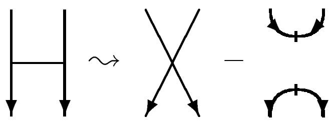


FIGURE 29. The map β_1

Using Relations (*Rep*) and (*Ori*) we can replace each element of $C_{\mathrm{loc}}(G)$ by a linear combination of disjoint unions of oriented circles labeled by a single element of G. Again by Relations (*Rep*) and (*Ori*), oriented circles with labels g, hgh^{-1}, and g^{-1} define the same element of $C_{\mathrm{loc}}(G)$. A map $\beta_2 : C_{\mathrm{loc}}(G) \longrightarrow \mathbb{Q}[t_{\hat{\mathcal{S}}_\pm}]$ is defined by replacing oriented circles with a label $g \in \hat{\mathcal{S}}_\pm$ by t_g. It is easy to see that β_2 is well-defined. We define the linear map

$$W_{\mathfrak{osp}} : \mathcal{A}(G) \longrightarrow \mathbb{Q}[t_{\hat{\mathcal{S}}_\pm}], \quad W_{\mathfrak{osp}} = \beta_2 \circ \beta_1. \tag{52}$$

The map $W_{\mathfrak{osp}}$ is called the weight system of the Kauffman polynomial. Proceeding as in Section 7, we extend $W_{\mathfrak{osp}}$ to the completion $\widetilde{\mathcal{A}}(G)$ of $\mathcal{A}(G)$ by

$$\widetilde{W}_{\mathfrak{osp}}(D) = W_{\mathfrak{osp}}(D) h^{\deg D} \in \mathbb{Q}[t_{\hat{\mathcal{S}}_\pm}][[h]].$$

For a framed oriented link $L \subset \Sigma \times I$ let $\widetilde{F}(L) \in \mathbb{Q}[t_{\hat{\mathcal{S}}}][[h]]$ be given by

$$\widetilde{F}(L) = \widetilde{W}_{\mathfrak{osp}} \circ Z^f_{\Sigma\times I}(L)$$

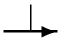
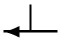
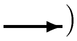
where $Z^f_{\Sigma\times I}$ denotes the universal Vassiliev invariant of framed oriented links in $\Sigma\times I$. [3] As in Section 8 we denote the inclusion map $\mathbb{Q}[t_{\hat{\mathcal{S}}_\pm}][[h]] \longrightarrow \mathbb{Q}[t_{\hat{\mathcal{S}}_\pm}][[h,h^{-1}]]$ by i. This time, we turn $\mathbb{Q}[t_{\hat{\mathcal{S}}_\pm}][[h,h^{-1}]]$ into a $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$-module by

$$x\cdot a = e^{(t_e-1)h/2}a \quad \text{and} \quad y\cdot a = (e^{h/2}-e^{-h/2})a. \tag{53}$$

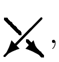
With the notation from above we have the following lemma (compare [LM1]).

Lemma 33. *The invariant $i\circ\widetilde{F}$ of framed oriented links induces a $\mathbb{Z}[x^{\pm1},y^{\pm1}]$-linear map from the Kauffman skein module to $\mathbb{Q}[t_{\hat{\mathcal{S}}_\pm}][[h,h^{-1}]]$ and satisfies*

$$\widetilde{F}(L) \equiv t_L \bmod h \text{ for every link } L.$$

Sketch of proof: Using $W_{\mathfrak{osp}}($$) = -W_{\mathfrak{osp}}($$)$ and the analogous dependence of $Z^f_{\Sigma\times I}$ on orientations we see that $\widetilde{F}(L)$ does not depend on the orientation of L. Using $W_{\mathfrak{osp}}($$) = (t_e-1)W_{\mathfrak{osp}}($$)$ and the explicit description of $Z^f_{\Sigma\times I}$ it is easy to see that

(54) $\widetilde{F}\Big($$\Big) = e^{(t_e-1)h/2}\widetilde{F}\Big($$\Big).$

The property $\widetilde{F}(L)\equiv t_L \bmod h$ follows directly from the analogous property of the degre-0 part of $\tilde{Z}^f_{\Sigma\times I}$ and the definition of $W_{\mathfrak{osp}}$. Therefore it remains only to show that Equation (55) holds with $\kappa = 1$, $\tilde{y} = e^{h/2}-e^{-h/2}$, and $(L_+, L_-, L_{||}, L_=) = \Big($, , , $\Big)$, where the link $L_=$ has arbitrary orientation.

$$\widetilde{F}(L_+) - \widetilde{F}(L_-) = \tilde{y}\left(\widetilde{F}(L_{||}) - \kappa\widetilde{F}(L_=)\right) \tag{55}$$

For the following computation we extend $\widetilde{W}_{\mathfrak{osp}}$ to locally oriented labeled chord diagrams in the unique way that respects the Relation (Ori). We do not indicate the local orientation in the picture if it is not of importance.

[3]The invariant $Z^f_{\Sigma\times I}$ is defined by the same formulas as $Z_{\Sigma\times I}$ (see [Lie], [LM2], [BN2]).

$$
\begin{aligned}
&\widetilde{F}(L_+) - \widetilde{F}(L_-) = \widetilde{W}_{\mathfrak{osp}}\left(\exp\left(\downarrow\!\!-\!\!\downarrow/2\right)\circ \times\right) - \widetilde{W}_{\mathfrak{osp}}\left(\exp\left(-\downarrow\!\!-\!\!\downarrow/2\right)\circ \times\right) = \\
&\sum_{n=0}^{\infty} \widetilde{W}_{\mathfrak{osp}}\left(((\times - \asymp)^n - (\asymp - \times)^n)\circ \times\right)\frac{h^n}{2^n n!} = \\
&\sum_{n=0}^{\infty} \widetilde{W}_{\mathfrak{osp}}\left((\times^n -)(+ ()(- \asymp)^n - (-\times)^n + (-)()^n - (\asymp -)()^n)\circ \times\right)\frac{h^n}{2^n n!} = \\
&\widetilde{y}\widetilde{F}(L_{||}) + \sum_{n=0}^{\infty} \widetilde{W}_{\mathfrak{osp}}\left((-)(+ ()(- \asymp)^n + (-)()^n - (\asymp -)()^n)\circ \times\right)\frac{h^n}{2^n n!} = \\
&\widetilde{y}\widetilde{F}(L_{||}) + \frac{1}{t_e}\sum_{n=0}^{\infty} \widetilde{W}_{\mathfrak{osp}}\left((-)(+ ()(- \asymp)^n + (-)()^n - (\asymp -)()^n)\circ \asymp\right)\frac{h^n}{2^n n!} = \\
&\widetilde{y}\widetilde{F}(L_{||}) + \frac{-e^{h/2} + e^{(1-t_e)h/2} + e^{-h/2} - e^{(t_e-1)h/2}}{t_e}\widetilde{W}_{\mathfrak{osp}}(\asymp) = \\
&\widetilde{y}\left(\widetilde{F}(L_{||}) - \frac{[t_e-1]_{e^{h/2}}+1}{t_e}\widetilde{W}_{\mathfrak{osp}}(\asymp)\right),
\end{aligned}
$$

where in the last equation we used the notation

$$[t_e - 1]_{e^{h/2}} = \frac{e^{(t_e-1)h/2} - e^{-(t_e-1)h/2}}{e^{h/2} - e^{-h/2}}.$$

Let O be the trivial knot with 0-framing. It is easy to see that we have

$$\widetilde{W}_{\mathfrak{osp}}\left(\begin{matrix}\boxed{\nu^{1/2}}\\ \boxed{\nu^{1/2}}\end{matrix}\right) = \frac{\widetilde{F}(O)}{t_e}\widetilde{W}_{\mathfrak{osp}}(\asymp),$$

where $\nu = Z_f(O)$. We have $\widetilde{F}(O) \equiv t_e \bmod h$ which implies that $\widetilde{F}(O)$ is invertible in $\mathbb{Q}[t_{\hat{\mathcal{S}}_\pm}][t_e^{-1}][[h]]$. The computations above show that Equation (55) holds with

$$\kappa = ([t_e-1]_{e^{h/2}} + 1)/\widetilde{F}(O) \in \mathbb{Q}[t_{\hat{\mathcal{S}}_\pm}][t_e^{-1}][[h]]. \tag{56}$$

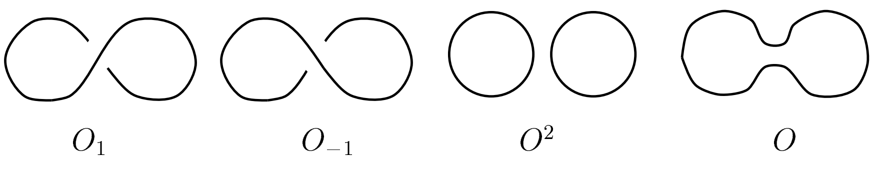


FIGURE 30. Link diagrams used to determine $\widetilde{F}(O)$

Applying Equation (55) to the link diagrams of Figure 30 and using Equation (54) we obtain

$$\widetilde{F}(O^2) = ([t_e - 1]_{e^{h/2}} + \kappa)\widetilde{F}(O). \tag{57}$$

It is easy to see that $\widetilde{F}(O^2) = \widetilde{F}(O)^2$. With κ as in Equation (56) Equation (57) has a unique solution satisfying $\widetilde{F}(O) \equiv t_e \bmod h$, namely $\widetilde{F}(O) = [t_e - 1]_{e^{h/2}} + 1$. This implies $\kappa = 1$ which completes the proof. □

Theorem 32 follows from Lemma 33 in a similar way as Theorem 10 follows from Lemma 25 (see Section 7).

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