# SMALL DATA OSCILLATION IMPLIES THE SATURATION ASSUMPTION

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ABSTRACT. The saturation assumption asserts that the best approximation error in $H_0^1$ with piecewise quadratic finite elements is strictly smaller than that of piecewise linear finite elements. We establish a link between this assumption and the oscillation of $f = -\Delta u$, and prove that small oscillation relative to the best error with piecewise linears implies the saturation assumption. We also show that this condition is necessary, and asymptotically valid provided $f \in L^2$.

## 1. INTRODUCTION

The *saturation assumption* is widely used in a posteriori error analysis of finite element methods [1, Ch. 5], [2], [6]. It asserts, in its simplest form, that the best approximation error $\|\nabla(u-u_2)\|_\Omega$ of a function $u \in H_0^1(\Omega)$ with quadratic finite elements is strictly smaller than that with linear finite elements $\|\nabla(u-u_1)\|_\Omega$, namely,

$$\|\nabla(u-u_2)\|_\Omega \le \alpha \, \|\nabla(u-u_1)\|_\Omega \tag{1.1}$$

for a suitable constant $\alpha \in (0,1)$. Throughout this paper we assume that $\Omega$ is a bounded polyhedral domain in $\mathbb{R}^N$, which coincides with its finite element decomposition; we also use the notation $\|v\|_\omega^2 := \int_\omega |v|^2$ for $\omega \subset \Omega$. If $\mathfrak{T}$ denotes a *graded* shape-regular partition of $\Omega$ of size $h_\mathfrak{T}$, and $\mathfrak{V}_\mathfrak{T}^1, \mathfrak{V}_\mathfrak{T}^2$ stand for the finite element subspaces of $H_0^1(\Omega)$ consisting of piecewise linear and piecewise quadratic functions, respectively, then $u_1, u_2$ are the Ritz projections of $u$ onto such spaces, that is

$$u_k \in \mathfrak{V}_\mathfrak{T}^k : \quad \int_\Omega \nabla(u-u_k)\cdot\nabla\phi = 0, \quad \forall \phi \in \mathfrak{V}_\mathfrak{T}^k. \tag{1.2}$$

The assumption (1.1) becomes true, as $h_\mathfrak{T} \to 0$, for functions $u \in W_p^s(\Omega)$ with $s - 2 > d\max(1/p - 1/2, 0)$ in dimension $d$ as a consequence of standard interpolation theory, provided a non-degeneracy condition holds (see §4). However, since $W_p^s(\Omega) \subset H^{2+\varepsilon}(\Omega)$ with $\varepsilon > 0$, such a regularity is never present for elliptic problems with singularities, and thus requiring adaptive mesh refinement. Even though it is generally believed to be valid asymptotically, (1.1) is not known to hold under any reasonable assumptions on the underlying function $u$ which still allow for singularities. More surprisingly, it is not known whether or not (1.1) may fail to hold for coarse meshes.

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In this paper we discuss the validity of (1.1) and disclose a close relation with data oscillation. Since (1.1) is about approximability in the $H_0^1(\Omega)$-norm, it is natural to consider $f = -\Delta u$ as datum in this discussion. We further assume

$$f \in L_2(\Omega). \tag{1.3}$$

For each interior node $x_i$ of $\mathfrak{T}$, we have a canonical basis function $\phi_i \in \mathfrak{V}_{\mathfrak{T}}^1$ and corresponding star $\omega_i := \mathrm{supp}(\phi_i)$. We denote by $f_i := |\omega_i|^{-1} \int_{\omega_i} f$ the mean value of $f$ in $\omega_i$, and define *data oscillation* to be the quantity

$$\mathrm{osc}(f, \mathfrak{T}) := \Big( \sum_i \|h(f - f_i)\|_{\omega_i}^2 \Big)^{1/2}. \tag{1.4}$$

We prove in §3 the following sufficient condition for the validity of (1.1):

**Theorem 1.** *There exists a constant $0 < \mu < 1$ solely depending on shape regularity of $\mathfrak{T}$, but independent of $u$ and $f$, such that if*

$$\mathrm{osc}(f, \mathfrak{T}) \le \mu \, \|\nabla(u - u_1)\|_\Omega \tag{1.5}$$

*holds, then (1.1) is valid with $\alpha := 1 - \mu^2$.*

For simplicity, we prove the Main Result in two dimensions, but comment on Remark 3.5 about the minor modifications for higher dimensions. We note that (1.5) is asymptotically valid as $h_{\mathfrak{T}} \downarrow 0$, and we explore this matter in detail in §4. Alternatively, (1.5) can be replaced by the more practical condition (3.9) which does not involve $u$ directly (see Remark 3.4). In §2 we exhibit an elementary example showing that (1.5) is a necessary condition for (1.1). This also demonstrates that (1.1) may not in general be valid in the preasymptotic regime, when the oscillations of $f$ are not yet well resolved by $\mathfrak{T}$.

## 2. Counterexample

The purpose of this section is twofold. We give an explicit example violating (1.1) and the same time argue about its connection with data oscillation. Let the domain be the square $\Omega = (0,1)^2$, the partition $\mathfrak{T}$ have one interior node, and the forcing function $f$ be piecewise constant (see Figure 1). By symmetry, it easily follows that

$$\int_\Omega f\phi = 0 \quad \forall \phi \in \mathfrak{V}_{\mathfrak{T}}^1, \mathfrak{V}_{\mathfrak{T}}^2 \qquad \Longrightarrow \qquad u_1 = u_2 = 0,$$

which violates (1.1). We realize that the oscillations of $f$ at the star level are responsible for this outcome. Since the counterexample is rather elementary, we conclude that we could not expect in general the saturation assumption to be valid in the *preasymptotic regime*, that is whenever data oscillation has not yet been resolved by the mesh $\mathfrak{T}$. But we might still expect that once $\mathfrak{T}$ becomes fine enough to detect the structure of $f$, then (1.1) should hold.

In §3 we establish this conjecture upon quantifying the size of data oscillation. In §4 we prove that (1.1) is always valid in the *asymptotic regime* provided $f \in L_2(\Omega)$ and $u$ satisfies a non-degeneracy condition.

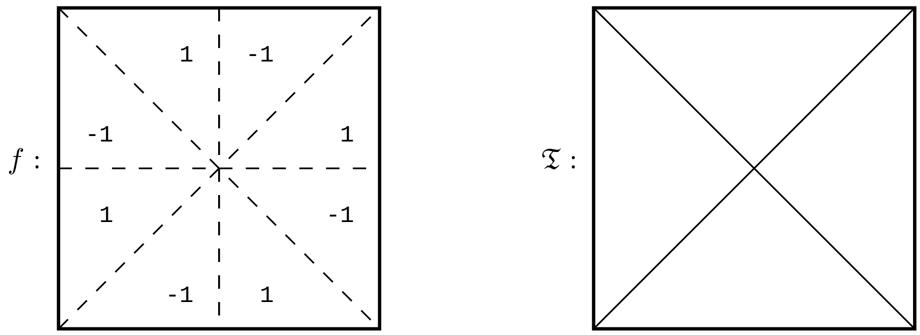


FIGURE 1. Function $f$ and mesh $\mathfrak{T}$ of counterexample.

## 3. Validity of the Saturation Assumption

In this section we prove the Main Result of this paper, namely that *small data oscillation implies the saturation assumption.* For simplicity, the result is derived in two space dimensions but it is valid in any dimension.

Even though the saturation assumption is a basic issue in approximation theory, the technique used here comes from a posteriori error analysis. In fact, it consists of exploiting orthogonality to relate the errors $\|\nabla(u-u_2)\|_\Omega$ and $\|\nabla(u-u_1)\|_\Omega$ via $\|\nabla(u_2-u_1)\|_\Omega$, and then showing a lower bound for the latter in terms of $\|\nabla(u-u_1)\|_\Omega$. This entails deriving precise expressions for interior and jump residuals on *stars.* We split the argument into several steps.

*Orthogonality.* Since $u_2$ is the orthogonal projection of $u$ onto $\mathfrak{V}^2_{\mathfrak{T}}$ with the scalar product of $H^1_0(\Omega)$, we deduce that $\int_\Omega \nabla(u-u_2)\cdot\nabla v = 0$ for all $v\in\mathfrak{V}^2_{\mathfrak{T}}$. Therefore $u_2-u_1\in\mathfrak{V}^2_{\mathfrak{T}}$ is perpendicular to $u-u_2$, and $u-u_1=(u-u_2)+(u_2-u_1)$ satisfies the orthogonality (Pythagoras) relation

$$\|\nabla(u-u_1)\|^2_\Omega = \|\nabla(u-u_2)\|^2_\Omega + \|\nabla(u_2-u_1)\|^2_\Omega. \tag{3.1}$$

We conclude that to prove the saturation assumption (1.1), it suffices to establish a lower bound of $\|\nabla(u_2-u_1)\|_\Omega$ in terms of $\|\nabla(u-u_1)\|_\Omega$. This is possible at the expense of an additional term involving the oscillation of $f$, and is shown below. But before we prove an elementary upper bound of $\|\nabla(u-u_1)\|_\Omega$ in terms of residual estimators.

*Upper A Posteriori Bound.* We intend to express the usual upper a posteriori bound with the interior residual accumulated by stars instead of by elements. The following estimate is well-known [1, Ch. 2]

$$\|\nabla(u-u_1)\|^2_\Omega \le C_1\Big(\sum_{S\in\mathfrak{S}}\|h^{1/2}J\|^2_S + \sum_{T\in\mathfrak{T}}\|hf\|^2_T\Big).$$

Here, $\mathfrak{S}$ denotes the set of all interior sides. We recall that $\omega_i$ denotes an interior star and $f_i$ indicates the mean value of $f$ in $\omega_i$. Since any element of $\mathfrak{T}$ belongs at most to 3 stars in $\mathbb{R}^2$, we can replace the interior residual by

$$\sum_{T\in\mathfrak{T}} \|hf\|_T^2 \le C\sum_i \|hf\|_{\omega_i}^2 \le C\Big(\sum_i \|hf_i\|_{\omega_i}^2 + \sum_i \|h(f-f_i)\|_{\omega_i}^2\Big).$$

The above two estimates together give rise to the modified upper bound

$$\|\nabla(u-u_1)\|_\Omega^2 \le C_2\Big(\sum_{S\in\mathfrak{S}} \|h^{1/2}J\|_S^2 + \sum_i \|hf_i\|_{\omega_i}^2 + \sum_i \|h(f-f_i)\|_{\omega_i}^2\Big). \tag{3.2}$$

We conclude that to prove (1.1) we need to bound the jump and interior residuals, the first two terms on the right hand side of (3.2), by $\|\nabla(u_2-u_1)\|_\Omega$.

*Local Estimate of Jump Residuals.* For any side $S$ let $\omega_S$ be the union of the elements that meet at $S$ and let $\phi_S \in \mathfrak{V}_{\mathfrak{T}}^2$ be the canonical functions with degree of freedom at the midpoint of $S$. Using Simpson's rule, integration by parts, and the equation for $u_2$ (in this order), we obtain for any interior side $S$

$$\begin{aligned} \frac{2}{3}h_S J_S &= \int_S J_S\phi_S = -\int_{\omega_S} \nabla u_1\cdot\nabla\phi_S \\ &= \int_{\omega_S}\Big(\nabla(u_2-u_1)\cdot\nabla\phi_S - f_i\phi_S + (f_i-f)\phi_S\Big); \end{aligned} \tag{3.3}$$

here it is crucial that $J_S$ is constant. Consequently, squaring and recalling that

$$\|\nabla\phi_S\|_{\omega_S} \le C, \qquad \|\phi_S\|_{\omega_S},\ \|\phi_i\|_{\omega_i} \le Ch_i, \tag{3.4}$$

as well as $h_S\|J_S\|_S^2 = \|h^{1/2}J\|_S^2$, we end up with the local estimate

$$\|h^{1/2}J\|_S^2 \le C_3\Big(\|\nabla(u_2-u_1)\|_{\omega_S}^2 + \|hf_i\|_{\omega_S}^2 + \|h(f-f_i)\|_{\omega_S}^2\Big). \tag{3.5}$$

*Local Estimate of Interior Residuals.* This is the key estimate which relates $\|hf_i\|_{\omega_i}$ with $\|\nabla(u_2-u_1)\|_{\omega_i}$. The proof entails deriving a sharp relation for the jump residuals in terms of piecewise linear and piecewise quadratic functions. To this end, it is essential to work on stars because on these sets there is a natural relation between interior and jump residuals for piecewise linear finite elements. A similar idea was used in [8] to remove the saturation assumption.

Let $x_i$ be an interior node, $\phi_i \in \mathfrak{V}_{\mathfrak{T}}^1$ be the corresponding nodal basis function and $\omega_i$ be its star. Denote by $\mathfrak{S}_i$ the set of interior sides $S$ in $\omega_i$. Since $\phi_i$ is piecewise linear, the trapezoidal rule combined with the equation for $u_1$ implies

$$\begin{aligned} \frac{1}{2}\sum_{S\in\mathfrak{S}_i} h_S J_S &= \sum_{S\in\mathfrak{S}_i}\int_S J_S\phi_i = -\int_{\omega_i}\nabla u_1\cdot\nabla\phi_i \\ &= -\int_{\omega_i} f\phi_i = -\frac{1}{3}f_i|\omega_i| + \int_{\omega_i}(f_i-f)\phi_i. \end{aligned} \tag{3.6}$$

We now repeat this calculation for quadratics, namely $\phi_S \in \mathfrak{V}_{\mathfrak{T}}^2$. Adding (3.3) for all sides $S$ in $\mathfrak{S}_i$, we obtain

$$\frac{2}{3}\sum_{S\in\mathfrak{S}_i} h_S J_S = -f_i\sum_{S\in\mathfrak{S}_i}\int_{\omega_S}\phi_S + \sum_{S\in\mathfrak{S}_i}\int_{\omega_S}\Big(\nabla(u_2-u_1)\cdot\nabla\phi_S + (f_i-f)\phi_S\Big).$$

Since

$$\sum_{S\in\mathfrak{S}_i}\int_{\omega_S}\phi_S = \frac{1}{3}\sum_{S\in\mathfrak{S}_i}|\omega_S| = \frac{2}{3}|\omega_i|$$

we readily deduce

$$\frac{2}{3}\sum_{S\in\mathfrak{S}_i} h_S J_S = -\frac{2}{3}f_i|\omega_i| - \sum_{S\in\mathfrak{S}_i}\int_{\omega_S}\Big(\nabla(u_1-u_2)\cdot\nabla\phi_S + (f-f_i)\phi_S\Big). \tag{3.7}$$

Examining (3.6) and (3.7) reveals the main idea of the proof: the jump residual can be eliminated, thereby giving an expression for the interior residual in terms of $\nabla(u_2-u_1)$ and data oscillation. In fact, we get

$$f_i|\omega_i| = \frac{9}{2}\sum_{S\in\mathfrak{S}_i}\int_{\omega_S}\Big(\nabla(u_2-u_1)\cdot\nabla\phi_S + (f_i-f)\phi_S\Big) + 6\int_{\omega_i}(f-f_i)\phi_i.$$

Since each $\omega_S$ is only counted twice in the above sum, in light of (3.4) we obtain the crucial local upper bound

$$\|hf_i\|_{\omega_i} \le C|f_i||\omega_i| \le C_5\Big(\|\nabla(u_2-u_1)\|_{\omega_i} + \|h(f-f_i)\|_{\omega_i}\Big). \tag{3.8}$$

*Proof of Theorem 1.* We now collect the above estimates. We first note that combining (3.5) with (3.8) yields the local estimate

$$\sum_{S\in\mathfrak{S}_i}\|h^{1/2}J\|_S^2 + \|hf_i\|_{\omega_i}^2 \le C\Big(\|\nabla(u_2-u_1)\|_{\omega_i}^2 + \|h(f-f_i)\|_{\omega_i}^2\Big).$$

Inserting this bound into (3.2), and using the finite overlapping property of stars, gives

$$\|\nabla(u-u_1)\|_\Omega^2 \le C_6\Big(\|\nabla(u_2-u_1)\|_\Omega^2 + \operatorname{osc}(f,\mathfrak{T})^2\Big).$$

In view of (3.1) and the small oscillation assumption $\operatorname{osc}(f,\mathfrak{T}) \le \mu\|\nabla(u-u_1)\|_\Omega$, we thus end up with

$$\begin{aligned}\|\nabla(u-u_2)\|_\Omega^2 &\le (1-C_6^{-1})\,\|\nabla(u-u_1)\|_\Omega^2 + \operatorname{osc}(f,\mathfrak{T})^2\\ &\le (1-C_6^{-1}+\mu^2)\,\|\nabla(u-u_1)\|_\Omega^2.\end{aligned}$$

The asserted estimate (1.1) follows from choosing $\mu^2 = (2C_6)^{-1}$ and $\alpha = 1-\mu^2$. Note that both $\mu$ and $\alpha$ are quantities which solely depend on shape regularity of $\mathfrak{T}$; the argument carries over even for highly refined meshes though.

**Remark 3.1.** It is worth stressing once more that the chief idea of the proof is to work on stars, which are viewed as basic cells for piecewise linear approximation. This gives rise to the link between interior and jump residual of (3.6), which would not be possible otherwise.

**Remark 3.2.** Suppose that quadratics over a mesh $\mathfrak{T}_H$ with mesh-size $H$ are replaced by linears over a uniformly refined mesh $\mathfrak{T}_{H/2}$ obtained from $\mathfrak{T}_H$ via two bisections. The number of degrees of freedom and their location is the same for both spaces. The question thus arises whether or not $\mathfrak{V}_H^2$ could be replaced by $\mathfrak{V}_{H/2}^1$ in the above construction and argument. The example of Figure 2 with $f = 1$ and a couple of uniform meshes $\mathfrak{T}_H$ and $\mathfrak{T}_{H/2}$, introduced in [7], shows that the answer is in general negative.

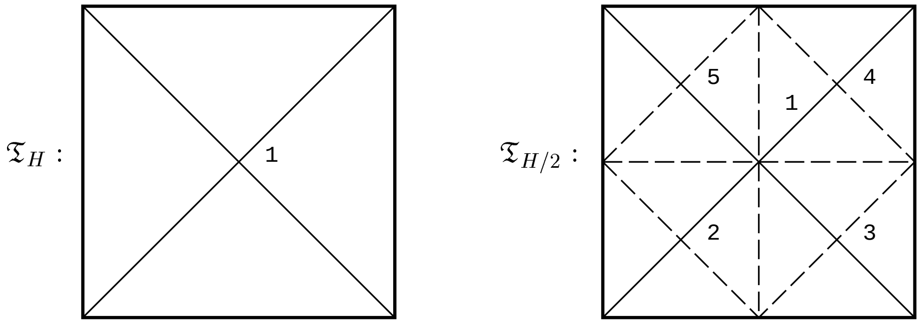


FIGURE 2. Example with $f = 1$ and $u_H = u_{H/2}$.

Let $\phi_1 \in \mathfrak{V}_H^1$ be the canonical basis function over $\mathfrak{T}_H$. It is easy to see that

$$u_H = u_{H/2} = \phi_1/12,$$

whence

$$\|\nabla(u - u_H)\|_\Omega = \|\nabla(u - u_{H/2})\|_\Omega$$

which violates (1.1). We may thus wonder what goes wrong in the above argument which seems to extend to this situation as well. What happens is that in trying to eliminate the jump residuals from (3.6) and (3.7), the interior residuals also cancel out, thereby providing no useful information. *Quadratics do indeed encode finer information than refined linears.*

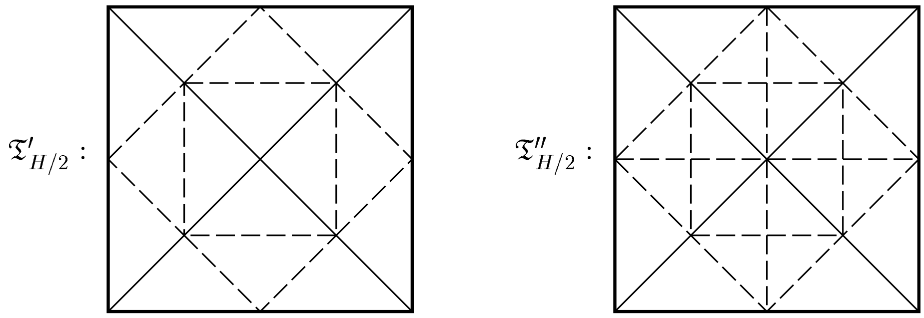


FIGURE 3. Meshes produced by red refinement of $\mathfrak{T}_H$ and three bisections of $\mathfrak{T}_H$.

**Remark 3.3.** Suppose that $\mathfrak{T}_{H/2}$, obtained from $\mathfrak{T}_H$ by two bisections, is replaced by a *red refinement* $\mathfrak{T}'_{H/2}$ of all triangles around $x_1$ [9, Ch. 4] (see left picture in Figure 3). Then the inequality (1.1) can be established for $u_1 = u_H$ and $u_2 = u'_{H/2}$ with the given technique. The same happens for three consecutive bisections $\mathfrak{T}''_{H/2}$ of $\mathfrak{T}_H$ because the basis functions of $\mathfrak{T}'_{H/2}$ are contained in the resulting (and reacher) finite element space of $\mathfrak{T}''_{H/2}$ (see right picture in Figure 3).

**Remark 3.4.** It is perhaps useful in the context of a posteriori error estimation, to express data oscillation fineness (1.5) in a computable fashion, that is without referring to the unknown function $u$. We thus propose the following computable alternative to (1.5):

$$\operatorname{osc}(f, \mathfrak{T}) \leq \hat{\mu}\, \|h^{1/2} J\|_{\Omega}. \tag{3.9}$$

To see that this implies (1.5), we resort to the lower bound for the error

$$\|h^{1/2} J\|_{\Omega} \leq \hat{C}_1\, \|\nabla(u - u_1)\|_{\Omega} + \hat{C}_2 \operatorname{osc}(f, \mathfrak{T}),$$

which simply arises from dropping the interior residual in the usual lower bound [9]. Hence

$$\operatorname{osc}(f, \mathfrak{T}) \leq \frac{\hat{\mu}\hat{C}_1}{(1 - \hat{\mu}\hat{C}_2)}\, \|\nabla(u - u_1)\|_{\Omega} = \mu\, \|\nabla(u - u_1)\|_{\Omega},$$

for a suitably small value of $\hat{\mu}$, still solely depending on mesh geometry.

**Remark 3.5.** To extend the proof of the Main Result to dimension $d > 2$, a few minor modifications are necessary: we must check how (3.6) and (3.7) change. Let $\omega_i$ be the star corresponding to an interior node $x_i$, $T \in \mathfrak{T}$ a $d$-simplex contained in $\omega_i$, and $S$ be a (closed) side of $T$ containing the node $x_i$. Instead of (3.6), we now have

$$\frac{1}{d} \sum_{S \in \mathfrak{S}_i} |S| J_S = -\frac{1}{d+1} f_i |\omega_i| + \int_{\omega_i} (f_i - f)\phi_i. \tag{3.10}$$

On the other hand, to deal with quadratics we first recall the quadrature rule over $T$ which uses the $d+1$ vertices $v_i$ of $T$ and the $d(d+1)/2$ midpoints of edges $e_i$ of $T$ as quadrature points:

$$\int_T \phi \approx \frac{|T|}{(d+1)(d+2)} \Big( \sum_{i=1}^{d+1} (2-d)\phi(v_i) + \sum_{i=1}^{d(d+1)/2} 4\phi(e_i) \Big).$$

This formula is exact for quadratics. Let $\phi_S$ be the quadratic function which is 1 at the $d-1$ midpoints of (closed) edges of $S$ containing $x_i$, and vanishes at the vertices of $T$ as well as the remaining midpoints of edges of $T$. We note that $\phi_S$ vanishes on $\partial\omega_i$ and thus $\phi_S \in \mathfrak{V}^2_{\mathfrak{T}}$. The quadrature rule gives rise to the following substitute for (3.7)

$$\begin{aligned} \frac{4(d-1)}{d(d+1)} \sum_{S \in \mathfrak{S}_i} |S| J_S = &-\frac{8(d-1)}{(d+1)(d+2)} f_i |\omega_i| \\ &+ \sum_{S \in \mathfrak{S}_i} \int_{\omega_S} \Big( \nabla(u_2 - u_1) \cdot \nabla\phi_S + (f_i - f)\phi_S \Big). \end{aligned} \tag{3.11}$$

Since we can still eliminate the jump residual between (3.10) and (3.11) and obtain a representation formula for the interior residual, for all $d > 2$, then the proof continues as above.

## 4. Asymptotics

We finally consider the generic situation in which both $u$ and a sequence of meshes $\mathfrak{T}$ satisfy the *non-degeneracy* property: *there exists a constant* $A > 0$ *independent of* $\mathfrak{T}$ *such that*

$$\|\nabla(u - u^1_{\mathfrak{T}})\|_\Omega \geq A h_{\mathfrak{T}}, \tag{4.1}$$

where $u^1_{\mathfrak{T}}$ is the Ritz projection onto $\mathfrak{V}^1_{\mathfrak{T}}$ and $h_{\mathfrak{T}}$ is the largest mesh-size of $\mathfrak{T}$. This is guaranteed, for instance, if $|D^2u(x)| \geq C > 0$ for all $x$ in a fixed region $\omega$ of $\Omega$, where the local mesh-size is of order $h_{\mathfrak{T}}$, namely $h_T \geq C h_{\mathfrak{T}}$; in particular, this is valid provided $\pm f(x) \geq C > 0$ for all $x \in \omega$. Therefore, (4.1) is not a very restrictive condition in practice.

We first show that, as asserted in the introduction, the saturation assumption (1.1) is valid as $h_{\mathfrak{T}} \downarrow 0$ provided $u \in W^s_p(\Omega)$ with $s - 2 > d\max(1/p - 1/2, 0)$. Since

$$t := \min\big(s - 1 - d\max(1/p - 1/2, 0),\ 2\big) > 1,$$

standard approximation theory in Sobolev spaces [3, Theorem 16.2], together with (4.1), yields

$$\|\nabla(u - u_2)\|_{L_2(\Omega)} \leq C\|h^t D^s u\|_{L_p(\Omega)} \leq C h^t_{\mathfrak{T}} \leq C h^{t-1}_{\mathfrak{T}} \|\nabla(u - u_1)\|_{L_2(\Omega)}.$$

We next prove the asymptotic validity of (1.5). To this end, we use a simple density argument for $f \in L_2(\Omega)$. Given $\varepsilon > 0$, let $\phi$ be a smooth approximation of $f$ satisfying $\|f - \phi\|_\Omega \leq \varepsilon$. Since the mean value $f_i$ satisfies $\|f_i\|_{\omega_i} \leq \|f\|_{\omega_i}$ for all interior stars $\omega_i$, we have

$$\begin{aligned}\|f - f_i\|_{\omega_i} &\leq \|f - \phi\|_{\omega_i} + \|\phi - \phi_i\|_{\omega_i} + \|(\phi - f)_i\|_{\omega_i} \\ &\leq 2\|f - \phi\|_{\omega_i} + C h_i \|\nabla\phi\|_{\omega_i},\end{aligned}$$

whence, making use of the finite overlapping property of stars,

$$\operatorname{osc}(f, \mathfrak{T})^2 \leq h^2_{\mathfrak{T}} \sum_i \|f - f_i\|^2_{\omega_i} \leq C h^2_{\mathfrak{T}}(\varepsilon + h_{\mathfrak{T}}) = o(h^2_{\mathfrak{T}}). \tag{4.2}$$

Combining (4.1) with (4.2), we deduce that (1.5) is valid for all $f \in L_2(\Omega)$ provided the mesh-size $h_{\mathfrak{T}} \leq h_*$ is sufficiently small. The size of the threshold $h_*$ depends on both $u$ and $f$. This implies the saturation assumption (1.1).

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