Infinite non-simple C^*-algebras: absorbing the Cuntz algebra $\mathcal{O}_\infty$

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Abstract

The first named author has in [14] given a classification theorem for separable, nuclear, C^*-algebras A that absorb the Cuntz algebra $\mathcal{O}_\infty$. (We say that A absorbs $\mathcal{O}_\infty$ if A is isomorphic to $A \otimes \mathcal{O}_\infty$.) Motivated by this theorem, we investigate here if one can intrinsically characterize which C^*-algebras absorb $\mathcal{O}_\infty$. This leads us to three different notions of pure infiniteness of a C^*-algebra, all given in terms of local, algebraic conditions on the C^*-algebra.

The stronger of the three properties, called *strongly purely infinite*, is shown to be equivalent to absorbing $\mathcal{O}_\infty$ for all separable, nuclear C^*-algebras, that either are stable or have an approximate unit consisting of projections. In a previous paper, [16], we studied an intermediate, and perhaps more natural, condition: *purely infinite*, that extends the well-known property for simple C^*-algebras. The weaker condition of the three, called *weakly purely infinite*, is shown to being equivalent to the absence of quasitraces in an ultrapower of the C^*-algebra. The three conditions may be equivalent for all C^*-algebras, and we prove that this is the case in the special cases where the C^*-algebra is either simple, of real rank zero, or approximately divisible.

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Contents

1 Introduction

It is well-known that each von Neumann algebra is the direct sum of two von Neumann algebras: one of which is finite and has a separating family of traces (the type I_n, $n < \infty$, and type II_1 portions), and the other is properly infinite (the type I_∞, II_∞, and III portions). The properly infinite summand is again a direct sum of two von Neumann algebras: one of which has an essential ideal admitting a separating family of (unbounded, densely defined) traces (the type I_∞ and II_∞ portions), and one which is traceless and purely infinite (the type III portion). We investigate here an analogue to C^*-algebras of the type III von Neumann algebras. More generally, we look at the C^*-analogue of the quotient of a general von Neumann algebra by the ideal generated by its finite projections.

Our motivation for studying purely infinite C^*-algebras stems primarily from the possibility of classifying these C^*-algebras along the lines of Elliott's classification program as described in [9]. More specifically, the first named author has recently proved that if A and B are nuclear, separable C^*-algebras both with primitive ideal spectrum homeomorphic to some T_0-space X, then $A \otimes \mathcal{O}_\infty \otimes \mathcal{K}$ is isomorphic to $B \otimes \mathcal{O}_\infty \otimes \mathcal{K}$ if and only if A and B

are KK_X-equivalent, where KK_X is a version of KK-theory that respects the primitive ideal spaces. As a step towards (and also a corollary of) this result, it is shown that two nuclear, separable C^*-algebras A and B satisfy $A \otimes \mathcal{O}_2 \otimes \mathcal{K} \cong B \otimes \mathcal{O}_2 \otimes \mathcal{K}$ if and only if A and B have homeomorphic primitive ideal spaces.

Besides of answering questions, this classification also raises questions: Can one determine when A and B are KK_X-equivalent? Is there an intrinsic characterization of those C^*-algebras that absorb $\mathcal{O}_\infty$? We shall not address the first question here except noting that one should be looking for a version of a universal coefficient theorem (UCT) for KK_X. An example of such a UCT was given in [21], where KK_X-equivalence was determined by isomorphism of 6-term exact sequences in the case where X consists of two points.

The first named author proved (in a paper published in [15]) that for a simple, nuclear, separable C^*-algebra A one has $A \cong A \otimes \mathcal{O}_\infty$ if and only if A is purely infinite (in the sense of J. Cuntz, [6]). The most optimistic generalization of this result to non-simple C^*-algebras would be as follows: For any (separable, nuclear) C^*-algebra A the following three conditions are equivalent:

(i) $A \cong A \otimes \mathcal{O}_\infty$,

(ii) A is purely infinite (cf. Definition 3.4),

(iii) A is traceless, in the sense that no algebraic ideal in A admits a non-zero — possibly unbounded — quasitrace.

We shall here establish an equivalence similar to, but weaker than, this. Some of the technical difficulties are solved by inventing three different notions of being purely infinite. The stronger of the three, strongly purely infinite (defined in Section 5) is in Section 8 shown to be equivalent to (i) above for nuclear, separable C^*-algebras that are either stable or have an approximate unit consisting of projections. In Section 4 we discuss weakly purely infinite C^*-algebras, and it is shown that a C^*-algebra A is weakly purely infinite if and only if its ultrapower A_ω is traceless. The intermediate condition was treated in detail in an earlier paper [16], and a brief survey of the properties of purely infinite C^*-algebras is given in Section 3.

We show in Section 4 that every weakly purely infinite C^*-algebra, that is either simple, approximately divisible, or has real rank zero, is purely infinite. In Section 6 we show that every purely infinite of real rank zero is strongly purely infinite. In particular, each simple purely infinite is strongly purely infinite. We also show that each approximately divisible, purely infinite C^*-algebra is strongly purely infinite.

A more detailed summary of the main results of this paper is given in Section 9. This section also contains a list of open problems related to this article.

The main result on $\mathcal{O}_\infty$-absorbsion is obtained via a local Weyl–von Neumann theorem (Theorem 7.21) which says that an approximately inner completely positive map from a nuclear sub-C^*-algebra of a strongly purely infinite C^*-algebra can always be approximated by 1-step inner completely positive maps. Most of Section 7 is devoted to the proof of that result.

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2 Preliminaries

This section has two subsections containing some background material that will be used frequently throughout this paper.

Cuntz comparison

The various notions of pure infiniteness we shall consider are defined in terms of comparison theory for positive elements in a C^*-algebra. This theory, invented by Cuntz in [5], generalizes the comparison theory for projections in a von Neumann algebra. The reader is referred to [2], [20], and [16] for more reading about Cuntz comparison.

Definition 2.1 (Cuntz comparison) Let A be a C^*-algebra, and let a, b be positive elements in A. Write $a \precsim b$ if there is a sequence $\{x_k\}_{k=1}^\infty$ of elements in A such that $x_k^* b x_k \to a$. Write $a \approx b$ if $a \precsim b$ and $b \precsim a$, and write $a \sim b$ if $a = x^*x$ and $b = xx^*$ for some x in A.

More generally, if a in $M_n(A)$ and b in $M_m(A)$ are positive matrices, then write $a \precsim b$ if $x_k^* b x_k \to a$ for some sequence $\{x_k\}_{k=1}^\infty$ of *rectangular matrices* in $M_{m,n}(A)$, and let $a \approx b$ and $a \sim b$ have similar meanings as above.

With a in $M_n(A)$ and b in $M_m(A)$ one has $a \approx b$ if $a \sim b$ (but not conversely). Let $a \oplus b$ denote the element $\operatorname{diag}(a, b)$ in $M_{n+m}(A)$, and let $a \otimes 1_n$ denote the n-fold direct sum $a \oplus a \oplus \cdots \oplus a$.

To each positive element a in a C^*-algebra A and for each $\varepsilon \geq 0$ define $(a-\varepsilon)_+$ to be the positive part of the self-adjoint element $a - \varepsilon \cdot 1$ in the unitization of A. We remark that $(a-\varepsilon)_+$ actually belongs to A and that $(a-\varepsilon)_+ = h_\varepsilon(a)$, where $h_\varepsilon \colon \mathbb{R}^+ \to \mathbb{R}^+$ is the continuous function given by $h_\varepsilon(t) = \max\{t-\varepsilon, 0\}$. We note the frequently used facts:

$$(a-\varepsilon_1-\varepsilon_2)_+ = \big((a-\varepsilon_1)_+ - \varepsilon_2\big)_+, \qquad \|(a-\varepsilon)_+ - a\| \leq \varepsilon, \tag{2.1}$$

that hold for all a in A^+ and all $\varepsilon, \varepsilon_1, \varepsilon_2 \geq 0$.

Lemma 2.2 *Let A be a C^*-algebra, let a, b be positive elements in A, and let $\varepsilon > \|a-b\|$ be given. Then there is a contraction d in A such that $dbd^* = (a-\varepsilon)_+$.*

Proof: For each $r > 1$ define $g_r \colon \mathbb{R}^+ \to \mathbb{R}^+$ by $g_r(t) = \min\{t, t^r\}$. Observe that $g_r(b) \to b$ as $r \to 1$. Choose $r > 1$ such that $(\varepsilon_1 =)\ \|a - g_r(b)\| < \varepsilon$ and set $b_0 = g_r(b)$. Then $b_0 \leq b$, $b_0 \leq b^r$, and $a - \varepsilon_1 \leq b_0$. Find a positive contraction e in $C^*(a)$ with $e(a-\varepsilon_1)e = (a-\varepsilon)_+$. Then $(a-\varepsilon)_+ \leq eb_0e$. Put $x = b_0^{1/2}e$ and write $x = v|x|$ for some partial isometry v in A^{**}. As $(a-\varepsilon)_+ \leq eb_0e = x^*x$, the element $y = v(a-\varepsilon)_+^{1/2}$ belongs to A, $y^*y = (a-\varepsilon)_+$, and

$$yy^* = v(a-\varepsilon)_+v^* \ \leq\ vx^*xv^* \ =\ xx^* \ =\ b_0^{1/2}e^2b_0^{1/2} \ \leq\ b_0.$$

Now, following the proof of [18, Proposition 1.4.5], put $d_n = y^*\big(\tfrac{1}{n}+b^r\big)^{-1/2}b^{(r-1)/2}$. Because $yy^* \leq b_0 \leq b^r$, [18, Lemma 1.4.4] applies (with $\alpha = 1$ and $\beta = (r-1)/r$) and shows that $\{d_n\}_{n=1}^\infty$ is a Cauchy sequence in A. Let d be the limit of this Cauchy sequence. As in the proof of [18, Proposition 1.4.5], we have $db^{1/2} = y^*$, so that $dbd^* = y^*y = (a-\varepsilon)_+$. Since $yy^* \leq b_0 \leq b$ we get

$$d_n^*d_n \leq b^{(r-1)/2}\big(\tfrac{1}{n}+b^r\big)^{-1/2}b\big(\tfrac{1}{n}+b^r\big)^{-1/2}b^{(r-1)/2} \leq 1.$$

Hence $\|d_n\| \leq 1$ for each n which entails that d is a contraction. □

Lemma 2.3 ([20, Proposition 2.4]) *Let A be a C^*-algebra and let a, b be positive elements in A. The following conditions are equivalent:*

(i) *$a \precsim b$,*

(ii) *$(a-\varepsilon)_+ \precsim b$ for all $\varepsilon > 0$,*

(iii) *for every $\varepsilon > 0$ there is $\delta > 0$ and x in A such that $x^*(b-\delta)_+x = (a-\varepsilon)_+$.*

(iv) *for every $\varepsilon > 0$ there is x in A such that $x^*x = (a-\varepsilon)_+$ and xx^* belongs to $\overline{bAb}$.*

In particular, if a is a positive element in the hereditary sub-C^*-algebra $\overline{bAb}$, then $a \precsim b$.

Lemma 2.4 *Let a, b be positive elements in a C^*-algebra A and let $\delta > 0$.*

(i) *If $a = x^*(b-\delta)_+x$ for some x in A, then $a = y^*by$ for some y in A with $\|y\| \le \delta^{-1/2}\|a\|^{1/2}$.*

(ii) *If $a \precsim (b-\delta)_+$, then for each $r > 1$ there exists y in A with $a = y^*by$ and with $\|y\| \le r\delta^{-1/2}\|a\|^{1/2}$.*

(iii) *If $a \le b$, then for each $\varepsilon > 0$ there is a contraction d in A with $d^*bd = (a-\varepsilon)_+$.*

Proof: We shall need some functions in the proof. For $\delta > 0$ and for α in the interval $[0, 1/2]$ define $f_{\delta,\alpha}, g_\delta \colon \mathbb{R}^+ \to \mathbb{R}^+$ by

$$f_{\delta,\alpha}(t) = \begin{cases} \sqrt{\frac{(t-\delta)^{2\alpha}}{t}}, & t \ge \delta \\ 0, & t < \delta \end{cases}, \qquad g_\delta(t) = \begin{cases} 1/t, & t \ge \delta \\ \delta^{-2}t, & t < \delta \end{cases}, \tag{2.2}$$

and put $f_\delta = f_{\delta,1/2}$. Then

$$tf_{\delta,\alpha}(t)^2 = (t-\delta)_+^{2\alpha}, \qquad f_\delta(t)^2 = (t-\delta)_+ g_\delta(t).$$

(i). Put $y = f_\delta(b)x$. Then

$$y^*by = x^*bf_\delta(b)^2x = x^*(b-\delta)_+x = a, \qquad y^*y = x^*f_\delta(b)^2x \le \|g_\delta(b)\|x^*(b-\delta)_+x \le \delta^{-1}a.$$

The latter inequality yields the desired norm estimate for y.

(ii). Choose $\delta_0 > 0$ such that $\delta_0 < \delta$ and $\delta_0^{-1/2} \le r\delta^{1/2}$. If $a \precsim (b-\delta)_+$, then $a = x^*(b-\delta_0)_+x$ for some x in A by Lemma 2.3 (iii) and (2.1). Hence $a = y^*by$ for some y in A with $\|y\| \le \delta_0^{-1/2}\|a\|^{1/2} \le r\delta^{-1/2}\|a\|^{1/2}$ by (i).

(iii). The system $\{f_\delta(b)\}_{\delta>0}$ is an approximate unit for $\overline{bAb}$, and so we can choose $\delta > 0$ with $\|f_\delta(b)af_\delta(b) - a\| < \varepsilon$. Observe that

$$\lim_{\alpha \to 1/2^-} \|f_{\delta,\alpha}(b)\| = \|f_\delta(b)\| = \big((\|b\| - \delta)/\|b\|\big)^{1/2} < 1.$$

We can therefore choose α in the interval $[0, 1/2)$ such that $\|a\|^{1/2-\alpha}\|f_{\delta,\alpha}(b)\| \le 1$. Put $y = f_{\delta,\alpha}(b)$, so that $y^*by = (b-\delta)_+^{2\alpha}$. By Lemma 2.2 there is a contraction t in A such that $(a-\varepsilon)_+ = t^*f_\delta(b)af_\delta(b)t$. We have

$$f_\delta(b)af_\delta(b) \le f_\delta(b)bf_\delta(b) = (b-\delta)_+.$$

Use [18, Proposition 1.4.5] to find u in A with $u^*(b-\delta)_+^{2\alpha}u = f_\delta(b)af_\delta(b)$ and $\|u\| \le \|a\|^{1/2-\alpha}$. Put $d = yut$. Then $d^*bd = (a-\varepsilon)_+$ and $\|d\| \le \|y\|\|u\|\|t\| \le 1$. $\square$

The proof of Lemma 2.4 (iii) actually yields an element d in A of norm slightly less than 1 with $d^*bd = (a-\varepsilon)_+$.

Limit algebras

A filter on a set Ω is an upper directed collection of subsets of Ω that is closed under finite intersections. To each filter ω on $\mathbb{N}$ and to each C^*-algebra A one defines the C^*-algebra A_ω to be the quotient $\ell^\infty(A)/c_\omega(A)$, where $c_\omega(A)$ is the closed two-sided ideal in $\ell^\infty(A)$ consisting of those sequences $\{a_n\}_{n=1}^\infty$ for which $\lim_\omega \|a_n\| = 0$. Recall that $\lim_\omega \|\alpha_n\| = \alpha$ if for each $\varepsilon > 0$ there is a subset X in ω such that $|\alpha - \alpha_n| < \varepsilon$ for all n in X. We shall denote the quotient mapping $\ell^\infty(A) \to A_\omega$ by π_ω.

There is a canonical embedding of A into A_ω given by $a \mapsto \pi_\omega(a, a, \dots,)$. We shall often view A as a sub-C^*-algebra of A_ω using this embedding implicitly.

A filter ω on $\mathbb{N}$ is called *free* if it contains all co-finite subsets of $\mathbb{N}$, and ω is called an *ultrafilter* if it is a maximal filter. Each filter is contained in an ultrafilter. The set of all cofinite subsets of $\mathbb{N}$ is a free filter, and any ultrafilter containing this filter is a free ultra filter. If ω contains a finite set, then it is not free and there is a finite subset $X_0 = \{n_1, \dots, n_k\}$ of $\mathbb{N}$ such that ω is the collection of all subsets of $\mathbb{N}$ containing X_0. In this case, $A_\omega = A \oplus A \oplus \cdots \oplus A$ (with k summands), and $\pi_\omega(a_1, a_2, \dots) = (a_{n_1}, \dots, a_{n_k})$. There are filters on $\mathbb{N}$ that are neither free nor contain a finite set.

Lemma 2.5 *Let A be a C^*-algebra and let ω be a free filter on $\mathbb{N}$. Let $\{p_i\}_{i\in\mathbb{I}}$ be a finite or countably infinite family of polynomials over A_ω in two non-commuting variables (cf. the examples below). Suppose for some finite constant C there is a sequence $\{d_n\}_{n=1}^\infty$ of elements in A_ω such that $\|d_n\| \le C$ for all n in $\mathbb{N}$ and*

$$\lim_{n\to\infty} \|p_i(d_n, d_n^*)\| = 0$$

for all $i \in \mathbb{I}$. Then there is d in A_ω such that $\|d\| \le C$ and $p_i(d, d^*) = 0$ for all $i \in \mathbb{I}$.

We shall typically apply the lemma in situations where the polynomials p_i are of the form:

$$p_1(d,d^*) = d^*ad - b, \qquad p_2(d,d^*) = [d^*ad, b], \qquad p_3(d,d^*) = [d^*ad, d^*bd]$$

for some a, b in A_ω.

Proof: Each coefficient of each p_i is an element in A_ω. Upon lifting each such coefficient to an element (of the same norm) in $\prod_{n=1}^\infty A$ we obtain a sequence $\{p_{i,n}\}_{n=1}^\infty$ of polynomials over A in two non-commuting variables such that $\{p_{i,n}(e_n, e_n^*)\}_{n=1}^\infty$ is a bounded sequence for each bounded sequence $\{e_n\}_{n=1}^\infty$ in A, and such that

$$p_i(e, e^*) = \pi_\omega(p_{i,1}(e_1, e_1^*), p_{i,2}(e_2, e_2^*), p_{i,3}(e_3, e_3^*), \dots) \quad \text{when} \quad e = \pi_\omega(e_1, e_2, e_3, \dots).$$

Recall that

$$\|p_i(e, e^*)\| = \inf_{X \in \omega} \sup_{n \in X} \|p_{i,n}(e_n, e_n^*)\|.$$

Write I as an increasing union of finite subsets $\{I_k\}_{k=1}^\infty$ of I. For each k find d_k in A_ω such that $\|d_k\| \le C$ and $\|p_i(d_k, d_k^*)\| < 1/k$ for all i in I_k. Write $d_k = \pi_\omega(d_{k,1}, d_{k,2}, d_{k,3}, \dots)$, where each $d_{k,n}$ is an element in A with $\|d_{k,n}\| \le C$.

For each k find Y_k in ω such that

$$\|p_{i,n}(d_{k,n}, d_{k,n}^*)\| < 1/k, \qquad n \in Y_k, \quad i \in I_k. \tag{2.3}$$

Define $X_k \in \omega$ inductively by setting $X_1 = Y_1$, and

$$X_k = Y_k \cap X_{k-1} \cap (\mathbb{N} \setminus \{1, 2, \dots, k\})$$

for $k \ge 2$. (The set $\mathbb{N} \setminus \{1, 2, \dots, k\}$ belongs to ω by the assumption that ω is free.) Then (2.3) holds for all n in X_k and for all i in I_k, the sequence $\{X_k\}_{k=1}^\infty$ is decreasing and $\cap_{k=1}^\infty X_k = \emptyset$. The last assumption implies that

$$\mathbb{N} = (\mathbb{N} \setminus X_1) \cup (X_1 \setminus X_2) \cup (X_2 \setminus X_3) \cup \cdots.$$

Let $\{e_n\}_{n=1}^\infty$ in $\ell^\infty(A)$ be given by

$$e_n = \begin{cases} 0, & \text{if } n \in \mathbb{N} \setminus X_1, \\ d_{1,n}, & \text{if } n \in X_1 \setminus X_2, \\ d_{2,n}, & \text{if } n \in X_2 \setminus X_3, \\ \vdots & \vdots \end{cases}$$

and put $d = \pi_\omega(e_1, e_2, e_3, \dots)$ in A_ω. Then $\|p_{i,n}(e_n, e_n^*)\| \le 1/k$ for all n in X_k and for all i in I_k. Hence $\|p_i(d, d^*)\| \le 1/k$ for all i in I_k. This holds for all k, and so $p_i(d, d^*) = 0$ for all i in I as desired. $\square$

3 Purely infinite C^*-algebras

We give here a brief review of some of the results on purely infinite C^*-algebras from [16].

Definition 3.1 (Properly infinite elements) A positive element a in a C^*-algebra A is said to be *properly infinite* if a is non-zero and $a \oplus a \precsim a$.

The condition $a \oplus a \precsim a$ means by definition that there is a sequence $\{d_n\}_{n=1}^{\infty}$ of elements in $M_2(A)$ such that

$$\Big\| d_n^* \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} d_n - \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \Big\| \to 0. \tag{3.1}$$

We note as a side remark that if there is a sequence $\{d_n\}$ such that (3.1) holds for *all* a in A, and if A is separable, then A is isomorphic to $A \otimes \mathcal{O}_\infty$ by Proposition 8.4. (See also Proposition 7.8.)

Some properties of properly infinite elements, established in [16, Proposition 3.3], include:

Lemma 3.2 *The following conditions are equivalent when a is a positive element in a C^*-algebra A:*

(i) *a is properly infinite.*

(ii) *For each $\varepsilon > 0$ there are positive elements a_1, a_2 in $\overline{aAa}$ such that $a_1 a_2 = 0$ and $(a-\varepsilon)_+ \precsim a_j$ for $j = 1, 2$.*

(iii) *For each natural number n and for each $\varepsilon > 0$ there are elements $d_1, \dots, d_n$ in $\overline{aAa}$ such that $d_j^* a d_i = \delta_{ij}(a-\varepsilon)_+$.*

The property that an element in a C^*-algebra is properly infinite depends on the C^*-algebra to which the element belongs. However, as follows readily from Lemma 3.2 (iii), if a is a positive element in a hereditary sub-C^*-algebra B of a C^*-algebra A, and if a is properly infinite relatively to A, then a is also properly infinite relatively to B.

In the spirit of comparison theory for projections, a positive element a is called *infinite* if $a \oplus b \precsim a$ for some non-zero positive element b in A, cf. Definition 2.1.

If $\varphi \colon A \to B$ is a *-homomorphism between C^*-algebras A and B, and if a is a properly infinite element in A, then $\varphi(a)$ is properly infinite if non-zero. Moreover, a positive element a is properly infinite if and only if $\varphi(a)$ is either infinite or zero for every *-homomorphism φ on A, cf. [16, Proposition 3.14].

If a is a properly infinite element in A, then $b \precsim a$ for each positive element b in the closed two-sided ideal, $\overline{AaA}$, generated by a, cf. [16, Proposition 3.5].

The set of properly infinite positive elements in a C^*-algebra is not dense. For example, no finite rank projections on a Hilbert space H is properly infinite, but they can be approximated in norm by properly infinite positive elements in $B(H)$. We have the following (weaker) approximation lemma:

Lemma 3.3 *Let a be a positive element in a C^*-algebra A and suppose that for each $\varepsilon > 0$ there is a properly infinite element b in A with $b \precsim a$ and $\|a - b\| < \varepsilon$. Then a is properly infinite.*

Proof: By Lemma 2.2 we have

$$(a - \varepsilon)_+ \oplus (a - \varepsilon)_+ \precsim b \otimes b \precsim b \precsim a,$$

and if $(a - \varepsilon)_+ \oplus (a - \varepsilon)_+ \precsim a$ for all $\varepsilon > 0$, then a is properly infinite, cf. Lemma 2.3. $\square$

Definition 3.4 (Purely infinite C^*-algebras) A C^*-algebra A is said to be *purely infinite* if A has no Abelian quotients and if for each pair of positive elements a, b in A such that a belongs to $\overline{AbA}$, the closed two-sided ideal generated by b, we have $a \precsim b$

It is shown in [16, Theorem 4.16] that A is purely infinite if and only if each non-zero positive element in A is properly infinite. Other facts about purely infinite C^*-algebras, proved in [16], include:

Proposition 3.5 (Permanence properties)

(i) *For each short exact sequence $0 \to I \to A \to B \to 0$ one has that A is purely infinite if and only if I and B are purely infinite.*

(ii) *If A and B are stably isomorphic and if A is purely infinite, then so is B.*

(iii) *Each hereditary sub-C^*-algebra of a purely infinite C^*-algebra is purely infinite.*

(iv) *If A is a purely infinite C^*-algebra and if ω is a free filter on $\mathbb{N}$, then A_ω is purely infinite.*

(v) *Any inductive limit of a system of purely infinite C^*-algebras is purely infinite.*

(vi) *$A \otimes \mathcal{O}_\infty$ is purely infinite for every C^*-algebra A.*

A simple C^*-algebra is purely infinite if and only if each of its non-zero hereditary sub-C^*-algebras contain an infinite projection, in agreement with Cuntz' original definition in [6].

4 Weakly purely infinite C^*-algebras

One motivation for introducing the notion of weakly purely infinite C^*-algebras is found in [16, Theorem 5.9] which says that an approximate divisible C^*-algebra is purely infinite if and only if it is traceless. (The notions of being approximate divisible and traceless are defined below.) We shall here characterize tracelessness in terms of being weakly purely infinite (without assuming approximate divisibility).

Definition 4.1 (Approximate divisibility) A C^*-algebra A is said to be *approximately divisible* if for every natural number n, for every finite subset F of A, and for every $\varepsilon > 0$ there is a unital *-homomorphism $\varphi\colon M_n(\mathbb{C}) \oplus M_{n+1}(\mathbb{C}) \to \mathcal{M}(A)$, where $\mathcal{M}(A)$ denotes the multiplier algebra of A, such that

$$\|\varphi(x)a - a\varphi(x)\| \leq \varepsilon\|x\|$$

for all a in F and all x in $M_n(\mathbb{C}) \oplus M_{n+1}(\mathbb{C})$.

As remarked in [16, Lemma 5.6], if A is approximately divisible and if B is any C^*-algebra, then $A \otimes B$ is approximately divisible (where $\otimes$ is any tensor product). As $\mathcal{O}_\infty$ is approximately divisible (a consequence of [15], see also Corollary 8.3) we find that $A \otimes \mathcal{O}_\infty$ is approximately divisible for every C^*-algebra A.

Definition 4.2 (Traceless C^*-algebras) A C^*-algebra A will be called *traceless* if no algebraic ideal of A admits a non-zero quasitrace [1].

There is a one-to-one correspondence between quasitraces and lower semi-continuous dimension functions (established by Blackadar and Handelman in [3]), and so being traceless is the same as being absent of non-zero lower semi-continuous dimension functions, cf. [16, Theorem 5.9].

[1]By a quasitrace we shall always mean a lower semi-continuous, positive 2-quasitrace. A 2-quasitrace on a C^*-algebra A is a quasitrace that extends (not necessarily in the obvious way) to a quasitrace on $M_2(A)$. Haagerup proved in [10] that each quasitrace on a unital, exact C^*-algebra extends to a trace (and the first named author has extended this result to non-unital, exact C^*-algebras in [13]). Thus an exact C^*-algebra A is traceless if and only if no algebraic ideal of A admits a non-zero trace.

The C^*-algebra $B(H)$ of all bounded operators on an infinite dimensional Hilbert space H is not traceless, since it has a non-zero trace defined on the trace class operators on H. We do not require our traces (or quasitraces) to be bounded.

Definition 4.3 (Weakly purely infinite C^*-algebras) A C^*-algebra A will be said to have property *pi-n* if the n-fold direct sum $a \oplus a \oplus \cdots \oplus a = a \otimes 1_n$ is properly infinite (cf. Definition 3.1) for every non-zero positive element a in A. If A is pi-n for some n, then we shall call A *weakly purely infinite.*

By [16, Theorem 4.16], a C^*-algebra is pi-1 if and only if it is purely infinite. By definition, $a \otimes 1_n$ is properly infinite if and only if $a \otimes 1_{2n} \precsim a \otimes 1_n$.

Lemma 4.4 *Let a be a positive element in a C^*-algebra A and let n be a natural number. The following conditions are equivalent:*

(i) $a \otimes 1_n$ *is properly infinite,*

(ii) $a \otimes 1_m \precsim a \otimes 1_n$ *for all natural numbers m,*

(iii) $a \otimes 1_m \precsim a \otimes 1_r$ *for all natural numbers r, m with $r \geq n$,*

(iv) $a \otimes 1_{n+1} \precsim a \otimes 1_n$,

(v) *for each $\varepsilon > 0$ and for each m in $\mathbb{N}$ there is x in $M_{m,n}(A)$ such that x^*x belongs to $M_n(\overline{aAa})$ and $xx^* = (a-\varepsilon)_+ \otimes 1_m$.*

Proof: The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iv) are trivial, and (ii) and (v) are equivalent by Lemma 2.3. Assume that (iv) holds. Then

$$a \otimes 1_{k+1} = (a \otimes 1_{n+1}) \oplus (a \otimes 1_{k-n}) \precsim (a \otimes 1_n) \oplus (a \otimes 1_{k-n}) = a \otimes 1_k$$

for all $k \geq n$. By transitivity of the relation $\precsim$ and by induction we obtain (iii). $\square$

Proposition 4.5 (Permanence properties)

(i) *If A is pi-n, then A is pi-m for every $m \geq n$.*

(ii) *Every hereditary sub-C^*-algebra of a pi-n C^*-algebra is pi-n.*

(iii) *Every non-zero quotient of a pi-n C^*-algebra is again pi-n.*

(iv) *If $0 \to I \to A \to B \to 0$ is a short exact sequence, if I is pi-n, and B is pi-m, then A is pi-$(n+m)$.*

(v) *If A is an inductive limit of a system of C^*-algebras, each of which is pi-n for some n, then A is pi-n.*

(vi) *If A and B are stably isomorphic C^*-algebras, and if A is pi-n, then B is pi-n^2.*

By (iii), an extension of two weakly purely infinite C^*-algebras is again weakly purely infinite. The given estimate on the degree of pure infiniteness is not optimal (cf. Proposition 3.5 (i)), but it suffices for our purposes.

Proof: (i) follows from Lemma 4.4.

(ii). This follows from the remark below Lemma 3.2 that being properly infinite is preserved when passing to hereditary sub-C^*-algebras.

(iii). This follows from the fact, remarked below Lemma 3.2, that a non-zero image under a *-homomorphism of a properly infinite element is again properly infinite.

(iv). Let a be a non-zero positive element in A and let $\varepsilon > 0$ be given. View I as an ideal of A and let π denote the quotient mapping $A \to B$. If a belongs to I, then $a \otimes 1_n$ is properly infinite because I is pi-n.

Assume now that $\pi(a)$ is non-zero. Then $\pi(a) \otimes 1_m$ is properly infinite in $M_m(B)$. By Lemma 3.2 we can find positive elements b_1, b_2 in $M_m(\overline{\pi(a)B\pi(a)})$ such that $b_1 b_2 = 0$ and $\pi((a-\varepsilon/3)_+) \otimes 1_m \precsim b_j$ for $j = 1,2$. Since $\pi((a-\varepsilon/3)_+) \otimes 1_m$ is properly infinite we also have

$$\pi((a-\varepsilon/3)_+) \otimes 1_{m+n} \precsim b_j, \qquad j = 1,2.$$

Lift b_1, b_2 to mutually orthogonal, positive elements a_1, a_2 in $M_m(\overline{aAa})$. There are positive elements c_1', c_2' in $M_{m+n}(\overline{aIa})$ such that $(a-\varepsilon/2)_+ \otimes 1_{m+n} \precsim a_j \oplus c_j'$ for $j = 1,2$, cf. [16, Lemma 4.12]. There is $\delta > 0$ such that

$$(a-\varepsilon)_+ \otimes 1_{m+n} \precsim a_j \oplus (c_j' - \delta)_+, \quad j = 1,2, \tag{4.1}$$

cf. Lemma 2.3. We show next that there are positive elements c_1, c_2 in $M_n(\overline{aIa})$ such that $c_1 c_2 = 0$ and $(c_j' - \delta)_+ \precsim c_j$. Find a positive element c_0 in $\overline{aIa}$ and $\eta > 0$ such that $(c_1' - \delta)_+$ and $(c_2' - \delta)_+$ belong to $M_{m+n}(\overline{I(c_0 - \eta)_+ I})$. Since I is pi-n, $(c_0 - \eta)_+ \otimes 1_n$ is properly infinite, and so $(c_j' - \delta)_+ \precsim (c_0 - \eta)_+ \otimes 1_n$. Because $c_0 \otimes 1_n$ also is properly infinite we can use Lemma 3.2 to find c_1, c_2 in the hereditary sub-C^*-algebra of $M_n(I)$ generated by $c_0 \otimes 1_n$ such that $c_1 c_2 = 0$ and $(c_0 - \eta)_+ \otimes 1_n \precsim c_j$ for $j = 1,2$. Notice that c_1, c_2 belong to $M_n(\overline{aAa})$. Hence $d_j = \mathrm{diag}(a_j, c_j)$ belongs to $M_{m+n}(\overline{aAa})$, $d_1 d_2 = 0$, and

$$(a-\varepsilon)_+ \otimes 1_{m+n} \precsim a_j \oplus (c_j' - \delta)_+ \precsim a_j \oplus c_j \precsim d_j.$$

It follows from Lemma 3.2 that $a \otimes 1_{m+n}$ is properly infinite, and this proves (iv).

(v). Assume that A is the inductive limit of a system $\{A_i\}_{i\in\mathbb{I}}$ of C^*-algebras each of which is pi-n. By (iii) we can assume that each map $A_i \to A$ is an inclusion mapping (so that A is the closure of the directed union of the algebras A_i). Let a be a non-zero positive element in A and let $\varepsilon > 0$ be given. Use [16, Lemma 2.5] to find i in $\mathbb{I}$ and a non-zero positive element b in A_i such that $(a-\varepsilon)_+ \precsim b \precsim a$. Then

$$(a-\varepsilon)_+ \otimes 1_{n+1} \precsim b \otimes 1_{n+1} \precsim b \otimes 1_n \precsim a \otimes 1_n.$$

Since this holds for all $\varepsilon > 0$, we get $a \otimes 1_{n+1} \precsim a \otimes 1_n$, and so A is pi-n.

(vi). By (ii) and (v) it suffices to show that if A is pi-n, then $M_k(A)$ is pi-n^2 for all natural numbers k.

We show first that $b \otimes 1_n$ is properly infinite for every non-zero positive element b in $M_n(A)$. Indeed, let b_{ij} be the matrix entries of b. Then $b_{ii} \precsim b \precsim b_{11} \oplus \cdots \oplus b_{nn}$ for $i = 1, \dots, n$ (cf. Lemma 5.3 below) and by Lemma 4.4 this implies:

$$b \otimes 1_{n^2+1} \precsim (b_{11} \oplus \cdots \oplus b_{nn}) \otimes 1_{n^2+1} \precsim (b_{11} \oplus \cdots \oplus b_{nn}) \otimes 1_n \precsim b \otimes 1_{n^2}.$$

Take now a non-zero positive matrix a in $M_k(A)$ and let $\varepsilon > 0$ be given. Choose $\varepsilon_1 > 0$ such that if $\|a - c\| < \varepsilon_1$, then $\|a - c\| < \varepsilon/2$ and $\|(a-\varepsilon/2)_+ - (c-\varepsilon/2)_+\| < \varepsilon/2$. By Lemma 2.2, this will entail that $(a-\varepsilon)_+ \precsim (c-\varepsilon/2)_+ \precsim a$. Find a positive contraction e in A and $\delta > 0$ such that

$$\|a - a^{1/2}\big((e-\delta)_+ \otimes 1_k\big)a^{1/2}\| < \varepsilon_1.$$

Since $e \otimes 1_n$ is properly infinite there is an element d in $M_{n,k}(A)$ such that $d^*(e \otimes 1_n)d = (e-\delta)_+ \otimes 1_k$. With $t = (e^{1/2} \otimes 1_n)da^{1/2}$ in $M_{n,k}(A)$ we have $\|t^*t - a\| < \varepsilon_1$ and tt^* $(= b)$ is a positive element in $M_n(A)$. Now $(tt^* - \varepsilon/2)_+ \otimes 1_{n^2}$, and hence $(t^*t - \varepsilon/2)_+ \otimes 1_{n^2}$, are properly infinite by the first part of the proof. By the choice of ε_1 we therefore get

$$(a-\varepsilon)_+ \otimes 1_{n^2+1} \precsim (t^*t - \varepsilon/2)_+ \otimes 1_{n^2+1} \precsim (t^*t - \varepsilon/2)_+ \otimes 1_{n^2} \precsim a \otimes 1_{n^2}.$$

As $\varepsilon > 0$ was arbitrary, this proves that $a \otimes 1_{n^2}$ is properly infinite. $\square$

Proposition 4.6 *The following conditions are equivalent for every C^*-algebra A:*

(i) *A is pi-n.*

(ii) *$\ell^\infty(A)$ is pi-n.*

(iii) *A_ω is pi-n for every filter ω on $\mathbb{N}$.*

(iv) *A_ω is pi-n for some filter ω on $\mathbb{N}$.*

Proof: (i) $\Rightarrow$ (ii). Let $a = (a_1, a_2, \dots)$ be a non-zero positive element in $\ell^\infty(A)$ and let $\varepsilon > 0$ be given. We show that $a \otimes 1_n$ is properly infinite. It is no loss of generality to assume that $\|a\| = 1$. Since A is pi-n, $(a_k - \varepsilon)_+ \otimes 1_{n+1} \precsim (a_k - \varepsilon)_+ \otimes 1_n$, and hence $(a_k - \varepsilon)_+ \otimes 1_{n+1} = x_k^*(a_k \otimes 1_n)x_k$ for some x_k in $M_{n,n+1}(A)$ with $\|x_k\| \le 2\varepsilon^{-1/2}$, cf. Lemma 2.4 (ii). It follows that $x = (x_1, x_2, \dots)$ belongs to $\ell^\infty(A)$ and that $(a - \varepsilon)_+ \otimes 1_{n+1} = x^*(a \otimes 1_n)x$. Since this holds for all $\varepsilon > 0$ we get $a \otimes 1_{n+1} \precsim a \otimes 1_n$. Hence $a \otimes 1_n$ is properly infinite, and (ii) must hold.

(ii) $\Rightarrow$ (iii) follows from Proposition 4.5 (iii); and (iii) $\Rightarrow$ (iv) is trivial.

(iv) $\Rightarrow$ (i). Let a be a non-zero positive element in A and let $\varepsilon > 0$. Since A_ω is pi-n there is $x = \pi_\omega(x_1, x_2, \dots)$ in $M_{n,n+1}(A_\omega)$ such that $x^*(a \otimes 1_n)x = (a - \varepsilon/2)_+ \otimes 1_{n+1}$. Each x_k belongs to $M_{n,n+1}(A)$ and

$$\lim_\omega \|x_k^*(a \otimes 1_n)x_k - (a - \varepsilon/2)_+ \otimes 1_{n+1}\| = 0.$$

Hence $\|x_k^*(a \otimes 1_n)x_k - (a - \varepsilon/2)_+ \otimes 1_{n+1}\| < \varepsilon/2$ for some k. By Lemma 2.2 this entails that $(a - \varepsilon)_+ \otimes 1_{n+1} \precsim a \otimes 1_n$. Since $\varepsilon > 0$ was arbitrary we conclude that $a \otimes 1_{n+1} \precsim a \otimes 1_n$, and so A is pi-n. $\square$

Lemma 4.7 *Let A be a C^*-algebra which is pi-n. Then for each pair of positive elements a, b in A, where a belongs to $\overline{AbA}$, and for each $\varepsilon > 0$ there are elements $x_1, \dots, x_n$ in A such that*

$$\sum_{j=1}^{n} x_j^* b x_j = (a - \varepsilon)_+.$$

Proof: Let $\varepsilon > 0$ be given. By [16, Proposition 2.7 (v)] there is a natural number k such that $(a - \varepsilon/2)_+ \precsim b \otimes 1_k$. By Lemma 4.4, $b \otimes 1_k \precsim b \otimes 1_n$, and so $(a - \varepsilon/2)_+ \precsim b \otimes 1_n$. It follows that $(a - \varepsilon)_+ = x^*(b \otimes 1_n)x$ for some x in $M_{n,1}(A)$. With $x_1, \dots, x_n$ the entries of x, we obtain the desired identity. $\square$

Theorem 4.8 *Let A be a C^*-algebra.*

(i) *For each free filter ω on $\mathbb{N}$ the following three conditions are equivalent:*

 (a) *A_ω is traceless,*

 (b) *A_ω is weakly purely infinite,*

 (c) *A is weakly purely infinite.*

(ii) *If A is weakly purely infinite, then A is traceless.*

Proof: (i). (c) $\Rightarrow$ (b) follows from Proposition 4.6, and (b) $\Rightarrow$ (a) follows from (i). We proceed to prove (a) $\Rightarrow$ (c). We do so indirectly by assuming that A is not weakly purely infinite. We construct below a positive element a in A_ω which satisfies

$$\big((a-1/4)_+ \oplus (a-1/4)_+\big) \otimes 1_k \not\precsim a \otimes 1_k \tag{4.2}$$

for all natural numbers k. It then follows from [16, Proposition 5.7] that A_ω admits a non-zero lower semi-continuous dimension functions, and hence that A is not traceless.

For each natural number k there is a positive contraction a_k in A such that

$$\big((a_k-1/2)_+ \oplus (a_k-1/2)_+\big) \otimes 1_m \not\precsim a_k \otimes 1_m, \qquad m = 1,2,\dots,k. \tag{4.3}$$

To see this, find for each k a non-zero positive element b_k in A such that $b_k \otimes 1_k$ is not properly infinite. Then $b_k \otimes 1_m$ is not properly infinite for $m = 1,2,\dots,k$, cf. Lemma 4.4. Hence $b_k \otimes 1_{2m} \not\precsim b_k \otimes 1_m$ for $m \le k$. By Lemma 2.3, $(b_k - \varepsilon_m)_+ \otimes 1_{2m} \not\precsim b_k \otimes 1_m$ for some $\varepsilon_m > 0$. Taking ε to be the minimum of $\varepsilon_1,\dots,\varepsilon_k$ we can assume that $\varepsilon_m = \varepsilon$ for each m. Define $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ to be $h(t) = \min\{t/(2\varepsilon), 1\}$, and set $a_k = h(b_k)$. Then a_k is a positive contraction in A, $(b_k-\varepsilon)_+ \approx (a_k-1/2)_+$, and $a_k \approx b_k$. This shows that (4.3) holds.

With $\pi_\omega\colon \ell^\infty(A) \to A_\omega$ denoting the quotient mapping, set $a = \pi_\omega(a_1,a_2,\dots)$. We proceed to show that (4.2) holds. Assume the contrary. Then $(a-1/4)_+ \otimes 1_{2k} \precsim a \otimes 1_k$ for some natural number k, and so $\|x^*(a\otimes 1_k)x - (a-1/4)_+ \otimes 1_{2k}\| < 1/4$ for some x in $M_{k,2k}(A_\omega)$. Write $x = \pi_\omega(x_1,x_2,\dots)$. Then

$$\lim_{n\to\omega} \|x_n^*(a_n \otimes 1_k)x_n - (a_n-1/4)_+ \otimes 1_{2k}\| < 1/4.$$

Because ω is a free filter there is an infinite subset X of $\mathbb{N}$ such that

$$\|x_n^*(a_n \otimes 1_k)x_n - (a_n-1/4)_+ \otimes 1_{2k}\| < 1/4, \qquad n \in X.$$

Use Lemma 2.2 to deduce that $(a_n - 1/2)_+ \otimes 1_{2k} \precsim a_n \otimes 1_k$ for all n in X. Because X is infinite it contains an element $n \geq k$. But this is in contradiction with (4.3).

(ii). If A is weakly purely infinite, then A is pi-n for some n. Arguing as in the proof of [16, Proposition 5.1] we see that A admits no non-zero dimension function, and hence no quasitrace: Indeed, if d were a dimension function with domain I (an algebraic ideal of A) and if a is a non-zero positive element in I, then $a \otimes 1_n$ is properly infinite, and hence $a \otimes 1_{n+1} \precsim a \otimes 1_n$. This implies $nd(a) \leq (n+1)d(a)$, and so $d(a) = 0$. □

It is not known if the sum of two properly infinite elements is again properly infinite. We do however have the following weaker result.

Lemma 4.9 *Let $a_1, \dots, a_n$ be properly infinite elements in a C^*-algebra A, and put $a = a_1 + a_2 + \cdots + a_n$. Then:*

(i) *$a \otimes 1_n$ is properly infinite,*

(ii) *and if $a_1, \dots, a_n$ are mutually orthogonal, then a is properly infinite.*

Proof: (i) follows from the relations

$$a \otimes 1_{2n} \precsim (a_1 \oplus a_2 \oplus \cdots \oplus a_n) \otimes 1_{2n} \precsim (a_1 \oplus a_2 \oplus \cdots \oplus a_n) \precsim a \otimes 1_n.$$

(ii) follows from [16, Lemma 3.9]. □

The next lemma is used for our proof of Proposition 4.11 saying that the multiplier algebra of a weakly purely infinite C^*-algebra is weakly purely infinite. Its proof uses a technique of Elliott's from [8].

Lemma 4.10 *Let A be a σ-unital weakly purely infinite C^*-algebra, let T be a positive element in $\mathcal{M}(A)$, and let $\varepsilon > 0$. Then there is an increasing, countable, approximate unit $\{e_n\}_{n=1}^{\infty}$ for A consisting of positive contractions satisfying $e_{n+1}e_n = e_n$ and $e_0 = 0$, such that*

$$T = a + \sum_{n=1}^{\infty} f_{2n-1}^{1/2} T f_{2n-1}^{1/2} + \sum_{n=1}^{\infty} f_{2n}^{1/2} T f_{2n}^{1/2}, \qquad f_n = e_n - e_{n-1}, \tag{4.4}$$

where a is a (not necessarily positive) element in A with $\|a\| \leq \varepsilon$.

We have $f_n \perp f_m$ whenever $|n - m| \geq 2$, so that the summands in the two (strictly convergent) sums in (4.4) are mutually orthogonal elements in A.

Proof: Take a countable approximate unit $\{e_n\}_{n=1}^\infty$ for A consisting of positive contractions satisfying $e_{n+1}e_n = e_n$ and $\|e_nT - Te_n\| \to 0$, cf. [18, Theorem 3.12.14]. Upon passing to a subsequence of $\{e_n\}_{n=1}^\infty$ we may assume that

$$\sum_{n=1}^{\infty} \|(e_n - e_{n-1})^{1/2}T - T(e_n - e_{n-1})^{1/2}\| < \varepsilon, \tag{4.5}$$

(where $e_0 = 0$.) Now, $1 = \sum_{n=1}^\infty f_n$ (the sum is strictly convergent), and so

$$\begin{aligned} a &= T - \sum_{n=1}^{\infty} f_{2n-1}^{1/2} T f_{2n-1}^{1/2} + \sum_{n=1}^{\infty} f_{2n}^{1/2} T f_{2n}^{1/2} \\ &= \sum_{n=1}^{\infty} T f_n - \sum_{n=1}^{\infty} f_n^{1/2} T f_n^{1/2} \;=\; \sum_{n=1}^{\infty} (T f_n^{1/2} - f_n^{1/2} T) f_n^{1/2}. \end{aligned}$$

By (4.5) we have $\|a\| \le \sum_{n=1}^\infty \|Tf_n^{1/2} - f_n^{1/2}T\| < \varepsilon$, and because this sum is norm convergent, we can also conclude that a belongs to A. □

Proposition 4.11 *Let A be a σ-unital weakly purely infinite C^*-algebra. Then its multiplier algebra $\mathcal{M}(A)$ is weakly purely infinite.*

Proof: Assume that A is pi-r.

We show first that if $\{e_n\}_{n=1}^\infty$ is an approximate unit as in Lemma 4.10 and if R is non-zero and given by a (strictly convergent) sum $R = \sum_{k=1}^\infty a_k$, where $a_1, a_2, \dots$ is a bounded sequence of positive, mutually orthogonal elements of A such that $a_k e_{k-1} = 0$ for all k, then $R \otimes 1_r$ is properly infinite. Let $\varepsilon > 0$ be given. Since A is pi-r there are elements x_k in $M_{r,2r}(A)$ such that $x_k^* x_k = (a_k - \varepsilon)_+ \otimes 1_{2r}$ and $x_k x_k^* \in M_r(\overline{a_k A a_k})$, cf. Lemma 4.4. Now, $\|x_k\| = \|(a_k - \varepsilon)_+\|^{1/2}$ which shows that the sequence $\{x_k\}_{k=1}^\infty$ is norm-bounded. The (i,j)th entry, $x_k(i,j)$, of x_k belongs to $\overline{a_k A a_k}$, and so $x_k(i,j)e_{k-1} = 0$. It follows that $X(i,j) = \sum_{k=1}^\infty x_k(i,j)$ is strictly convergent in $\mathcal{M}(A)$. The resulting matrix X in $M_{r,2r}(\mathcal{M}(A))$, whose (i,j)th entry is $X(i,j)$, satisfies $X^*X = (R-\varepsilon)_+ \otimes 1_{2r}$ and $XX^* \in M_r(\overline{R\mathcal{M}(A)R})$. Since $\varepsilon > 0$ was arbitrary, we conclude that $R \otimes 1_{2r} \precsim R \otimes 1_r$, cf. Lemma 2.3.

The argument above and Lemma 4.10 show that each positive element T in $\mathcal{M}(A)$ can be written as $T = a + T_1 + T_2$, where T_1, T_2 are positive elements in $\mathcal{M}(A)$ for which $T_j \otimes 1_r$ is properly infinite and a belongs to A. With $\pi\colon \mathcal{M}(A) \to \mathcal{M}(A)/A$ denoting the quotient mapping we get $\pi(T) = \pi(T_1) + \pi(T_2)$, and $\pi(T_j) \otimes 1_r$ is properly infinite (cf. the remarks below Lemma 3.2). Hence $\pi(T) \otimes 1_{2r}$ is properly infinite by Lemma 4.9 (i). This proves that $\mathcal{M}(A)/A$ is pi-$2r$. By Proposition 4.5 we conclude that $\mathcal{M}(A)$ is pi-$3r$ and hence weakly purely infinite. $\square$

We do not know if the multiplier algebra of a purely infinite C^*-algebra is again purely infinite.

We end this section by discussing a sufficient condition under which one can deduce pure infiniteness from weak pure infiniteness.

It is a consequence of a lemma by Glimm that if n is a natural number and if A is a C^*-algebra that admits an irreducible representation on a Hilbert space of dimension at least n, then there is a non-zero *-homomorphism from $M_n\big(C_0((0,1])\big)$ into A, cf. [16, Proposition 4.10].

Definition 4.12 (The global Glimm property) A C^*-algebra A is said to have the *global Glimm property* if for each natural number n, for each positive element a in A, and for each $\varepsilon > 0$ there is a *-homomorphism $\varphi\colon M_n\big(C_0((0,1])\big) \to \overline{aAa}$ such that $(a-\varepsilon)_+$ belongs to the closed two-sided ideal of A generated by the image of φ.

Equivalently, A has the global Glimm property if for each positive element a in A, for each $\varepsilon > 0$, and for each natural number n there are mutually orthogonal elements $t_1, \dots, t_n$ in $\overline{aAa}$ such that $t_1 \sim t_2 \sim \cdots \sim t_n$, cf. Definition 2.1, and such that $(a-\varepsilon)_+$ belongs to the closed two-sided ideal generated by $t = t_1 + \cdots + t_n$.

The latter condition is illustrated in our next result:

Lemma 4.13 *Let a be a positive element in a C^*-algebra A and suppose that there is a full [2] *-homomorphism $\varphi\colon M_n(\mathbb{C}) \to \mathcal{M}(\overline{aAa})$. Then, for each $\varepsilon > 0$, there are mutually orthogonal and mutually equivalent positive elements $t_1, \dots, t_n$ in $\overline{aAa}$ such that $(a-\varepsilon)_+$ belongs to the closed two-sided ideal generated by $t = t_1 + \cdots + t_n$.*

Proof: Observe first that if p is a full projection in $\mathcal{M}(\overline{aAa})$, then $p\overline{aAa}p$ is a full hereditary sub-C^*-algebra of $\overline{aAa}$. Let $\{e_{ij}\}$ denote the matrix units of $M_n(\mathbb{C})$. Take a positive contraction h in $\varphi(e_{11})\overline{aAa}\varphi(e_{11})$ such that $(a-\varepsilon)_+$ belongs to $\overline{AhA}$. Put $t_j = \varphi(e_{j1})h\varphi(e_{1j})$. Then $t_1, \dots, t_n$ are mutually orthogonal and mutually equivalent positive elements in $\overline{aAa}$ and $(a-\varepsilon)_+$ belongs to the ideal generated by $t = t_1 + \cdots + t_n$. $\square$

Lemma 4.14 *Let A be a C^*-algebra such that no non-zero hereditary sub-C^*-algebra of A a finite dimensional representation. Then A has the global Glimm property if A is either simple, approximately divisible, or purely infinite.*

[2] A *-homomorphism $A \to B$ is called *full* if its image is not contained in a proper closed two-sided ideal in B.

Proof: The local version of Glimm's lemma (see [16, Proposition 4.10]) implies the global Glimm property when A is simple.

Assume next that A is approximately divisible. Let a be a positive element in A and let $\varepsilon > 0$ be given. Let B denote the algebra $M_n(\mathbb{C}) \oplus M_{n+1}(\mathbb{C})$ and find an asymptotically central sequence of unital *-homomorphisms $\varphi_k \colon B \to \mathcal{M}(A)$. Let μ denote the Haar measure on the compact unitary group of B, and define a conditional expectation $E_k \colon A \to A \cap \varphi_k(B)'$ by

$$E_k(a) = \int_{U(B)} \varphi_k(u) a \varphi_k(u)^* \, d\mu(u).$$

Then $\|a - E_k(a)\| \to 0$. Choose k large enough so that $\|a - E_k(a)\| < \varepsilon/2$, and put $a_0 = (E_k(a) - \varepsilon/2)_+$. Then a_0 is a positive element in $A \cap \varphi_k(B)'$ and $\|a - a_0\| < \varepsilon$. By Lemma 2.2 there are elements d, e in A such that $a_0 = d^*ad$ and $(a-\varepsilon)_+ = e^* a_0 e$. Put $a_1 = a^{1/2} d d^* a^{1/2}$. There is an isomorphism

$$\alpha \colon \overline{a_0 A a_0} \to \overline{a_1 A a_1} \subseteq \overline{aAa}$$

such that $\alpha(a_0) = a_1$, cf. [16, Lemma 2.4]. Being equivalent, a_0 and a_1 generate the same closed two-sided ideal of A, and $(a-\varepsilon)_+$ belongs to this ideal.

The canonical isomorphism $\beta \colon B \otimes C(\mathrm{sp}(a_0)) \to C^*(\varphi_k(B), a_0)$ restricts to a *-homomorphism

$$\beta_0 \colon B \otimes C_0(\mathrm{sp}(a_0) \setminus \{0\}) \to \overline{a_0 A a_0},$$

and $a_0 = \beta_0(1 \otimes \iota)$, where ι in $C_0(\mathrm{sp}(a_0) \setminus \{0\})$ is given by $\iota(t) = t$. Take a full *-homomorphism $M_n(\mathbb{C}) \to B$, a surjective *-homomorphism $C_0((0,1]) \to C_0(\mathrm{sp}(a_0) \setminus \{0\})$, and obtain in this way a *-homomorphism

$$M_n\big(C_0((0,1])\big) = M_n(\mathbb{C}) \otimes C_0((0,1]) \to B \otimes C_0(\mathrm{sp}(a_0) \setminus \{0\}) \to \overline{a_0 A a_0},$$

whose image generates an ideal of A which contains $(a-\varepsilon)_+$.

Suppose finally that A is purely infinite. Let a be a positive element in A and let $\varepsilon > 0$ and n in $\mathbb{N}$ be given. By Lemma 2.3 there is d in $M_{1,n}(A)$ such that $d^*ad = (a-\varepsilon)_+ \otimes 1_n$. Write $d = (d_1, d_2, \dots, d_n)$. Then $d_i^* a d_j = \delta_{ij}(a-\varepsilon)_+$ for all i, j. Put $t_j = a^{1/2} d_j d_j^* a^{1/2}$. Then $t_1, \dots, t_n$ are mutually orthogonal elements in $\overline{aAa}$ each equivalent to $(a-\varepsilon)_+$, and this shows that A has the global Glimm property. $\square$

Proposition 4.15 *A weakly purely infinite is purely infinite if and only if it has the global Glimm property.*

Proof: The "only if" part follows from Proposition 4.14. Assume next that A is weakly purely infinite, say pi-n, with the global Glimm property. Let a be a non-zero positive element in A and let $\varepsilon > 0$ be given. Then $\overline{aAa}$ contains pairwise orthogonal and equivalent positive elements $t_1, t_2, \ldots, t_n$ such that $(a-\varepsilon)_+$ belongs to the closed two-sided ideal generated by $t = t_1 + t_2 + \cdots + t_n$. Now,

$$t_1 \otimes 1_n \sim t_1 \oplus t_2 \oplus \cdots \oplus t_n \sim t_1 + t_2 + \cdots + t_n = t.$$

Hence t is properly infinite. Since $(a-\varepsilon)_+$ belongs to the ideal generated by t we conclude that $(a-\varepsilon)_+ \precsim t$. Conversely, $t \precsim a$ because t belongs to $\overline{aAa}$, cf. [16, Proposition 2.7 (i)]. Using again that t is properly infinite, we get

$$(a-\varepsilon)_+ \oplus (a-\varepsilon)_+ \precsim t \oplus t \precsim t \precsim a.$$

Since this holds for all $\varepsilon > 0$ we conclude that a is properly infinite. This shows that A is purely infinite because a was arbitrary. $\square$

Every hereditary sub-C^*-algebra and every quotient of a weakly purely infinite is again weakly purely infinite, and no weakly purely infinite C^*-algebra can be finite dimensional (e.g. by Theorem 4.8). Hence no hereditary sub-C^*-algebra of a weakly purely infinite C^*-algebra admits a finite dimensional representation. Together with Proposition 4.14 and Lemma 4.15 this proves:

Corollary 4.16 *Any weakly purely infinite C^*-algebra, which is either simple or approximately divisible, is purely infinite.*

A C^*-algebra A is said to have property (SP) ("small projections") if every non-zero hereditary sub-C^*-algebra of A contains a non-zero projection.

Proposition 4.17 *Each non-zero projection in a weakly purely infinite C^*-algebra with property (SP) is infinite.*

Proof: Let A be a weakly purely infinite C^*-algebra with property (SP), and let p be a non-zero projection in A. Upon replacing A with pAp (which again is weakly purely infinite by Proposition 4.5 (ii) and has property (SP)), it suffices to show that the unit is infinite in a unital weakly purely infinite C^*-algebra A with property (SP).

Since A is weakly purely infinite it admits no finite dimensional representations. Hence Glimm's lemma applies (cf. [16, Proposition 4.10]), and there is a non-zero *-homomorphism from $M_n\big(C_0((0,1])\big)$ into A. It follows that A contains mutually orthogonal positive elements $t_1, t_2, \dots, t_n$ and elements $x_2, \dots, x_n$ such that $x_j x_j^* = t_j$ and $x_j^* x_j = t_1$. Find a non-zero projection p_1 in $\overline{t_1 A t_1}$. Write $x_j = v_j |x_j|$ for some partial isometry v_j in A^{**}. Then $u_j = v_j p_1$ belongs to A, $u_j^* u_j = p_1$, and $p_j = u_j u_j^*$ belongs to $\overline{t_j A t_j}$. In particular, $p_1, p_2, \dots, p_n$ are mutually orthogonal and mutually equivalent projections. Consequently,

$$p = p_1 + p_2 + \cdots + p_n \sim p_1 \oplus p_2 \oplus \cdots \oplus p_n \sim p_1 \otimes 1_n$$

is properly infinite. As A contains a non-zero properly infinite projection, the unit of A must be infinite. □

Proposition 4.18 *Every weakly purely infinite C^*-algebra of real rank zero is purely infinite.*

Proof: We check that if B is a non-zero hereditary sub-C^*-algebra of a quotient of a weakly purely infinite C^*-algebra A, then B contains an infinite projection. By [16, Proposition 4.7], this will ensure that A is purely infinite. By Proposition 4.5 (ii), B is weakly purely infinite. Any C^*-algebra of real rank zero has property (SP). Proposition 4.17 therefore yields that every non-zero projection in B is infinite. □

One can relax the real rank zero condition in Proposition 4.18 to the weaker condition that every quotient of the C^*-algebra has property (SP).

We conclude this section with further reformulations of the open problem, if weak pure infiniteness implies pure infiniteness:

Proposition 4.19 *The following four conditions are equivalent:*

(i) *all weakly purely infinite C^*-algebras are purely infinite,*

(ii) *the (unit of the) multiplier algebra of any σ-unital weakly purely infinite C^*-algebra is properly infinite,*

(iii) *all non-zero projections in any weakly purely infinite C^*-algebra are properly infinite,*

(iv) *all non-zero projections in any weakly purely infinite C^*-algebra are infinite.*

Proof: All non-zero projections in a purely infinite C^*-algebra are properly infinite by [16, Theorem 4.16], and therefore (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii).

(iv) $\Rightarrow$ (iii). Suppose that p is a non-zero, not properly infinite projection in a weakly purely infinite A. Then there is a closed two-sided ideal I of A such that $p+I$ is a non-zero finite projection in A/I, cf. [16, Corollary 3.15]. But A/I is weakly purely infinite by Lemma 4.5 (iii), thus contradicting (iv).

(ii) $\Rightarrow$ (i). If $\mathcal{M}(\overline{aAa})$ is properly infinite, then there is a full (possibly non-unital) embedding of $M_n(\mathbb{C})$ into $\mathcal{M}(\overline{aAa})$ for every natural number n. Lemma 4.13 therefore implies that A has the global Glimm property, and by Proposition 4.15 we conclude that A is purely infinite if A is weakly purely infinite. $\square$

5 Strongly purely infinite C^*-algebras

Our third notion of pure infiniteness is perhaps not very intuitive, but the definition is still local and algebraic like the definitions of being purely infinite and weakly purely infinite. It turns out that this notion is precisely what is required to obtain an $\mathcal{O}_\infty$-absorbsion theorem (Theorem 8.6) and a Weyl–von Neumann type result such as Theorem 7.21.

Definition 5.1 A C^*-algebra A is said to have be *strongly purely infinite* if for every

$$\begin{pmatrix} a & x^* \\ x & b \end{pmatrix} \in M_2(A)^+$$

and for every $\varepsilon > 0$ there exist d_1, d_2 in A such that

$$\left\| \begin{pmatrix} d_1^* & 0 \\ 0 & d_2^* \end{pmatrix} \begin{pmatrix} a & x^* \\ x & b \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\| \leq \varepsilon.$$

The matrix diagonalization appearing in this definition can be rephrased in a number of ways (see also Remark 5.10 below):

Lemma 5.2 *For each element* $\left(\begin{smallmatrix} a & x^* \\ x & b \end{smallmatrix}\right)$ *in* $M_2(A)^+$*, the following conditions are equivalent:*

(i) *For each* $\varepsilon > 0$ *there exist* d_1, d_2 *in* $\mathcal{M}(A)$ *such that* $\|d_1^* a d_1 - a\| \leq \varepsilon$, $\|d_2^* b d_2 - b\| \leq \varepsilon$, *and* $\|d_2^* x d_1\| \leq \varepsilon$.

(ii) *For each* $\varepsilon > 0$ *there exist* d_1, d_2 *in* A *such that*

$$\left\| \begin{pmatrix} d_1^* & 0 \\ 0 & d_2^* \end{pmatrix} \begin{pmatrix} a & x^* \\ x & b \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\| \leq \varepsilon.$$

(iii) *For each* $\varepsilon > 0$ *and* $\delta > 0$ *there exist* d_1 *in* $\overline{aAa}$ *and* d_2 *in* $\overline{bAb}$ *such that* $d_1^* a d_1 = (a-\varepsilon)_+$, $d_2^* b d_2 = (b-\varepsilon)_+$, *and* $\|d_2^* x d_1\| \leq \delta$.

Proof: The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are obvious. Assume that (i) holds. Let $\varepsilon > 0$ and $\delta > 0$ be given, and find e_1, e_2 in $\mathcal{M}(A)$ such that $\|e_1^* a e_1 - a\| \le \varepsilon/2$, $\|e_2^* b e_2 - b\| \le \varepsilon/2$, and $\|e_2^* x e_1\| \le \delta/2$. Set $f_j = g_j^{1/2} e_j g_j^{1/2}$, $j = 1, 2$, where g_1 and g_2 are positive contractions in $\overline{aAa}$, respectively $\overline{bAb}$, chosen such that $\|f_1^* a f_1 - a\| < \varepsilon$, $\|f_2^* b f_2 - b\| < \varepsilon$, and $\|f_2^* x f_1\| \le \delta$. By Lemma 2.2 there are contractions h_1 and h_2 in $\overline{aAa}$, respectively $\overline{bAb}$, such that $h_1^* f_1^* a f_1 h_1 = (a - \varepsilon)_+$ and $h_2^* f_2^* b f_2 h_2 = (b - \varepsilon)_+$. We can therefore take d_j to be $f_j h_j$, $j = 1, 2$. $\square$

In a strongly purely infinite (or in a purely infinite) C^*-algebra A, if $a = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)$ is a positive element in $M_2(A)$, then $\operatorname{diag}(a_{11}, a_{22}) \precsim a$. The converse holds in any C^*-algebra:

Lemma 5.3 *Let A be any C^*-algebra, let n be a positive integer, and let $a = (a_{ij})$ be a positive element in $M_n(A)$. Then $a \precsim \operatorname{diag}(a_{11}, \dots, a_{nn})$.*

Proof: This follows from the fact that a belongs to the hereditary sub-C^*-algebra generated by $\operatorname{diag}(a_{11}, \dots, a_{nn})$. $\square$

Proposition 5.4 *Every strongly purely infinite C^*-algebra is purely infinite.*

Proof: If A is strongly purely infinite and if a is a positive element in A, then

$$a \oplus a = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \precsim \begin{pmatrix} a & a \\ a & a \end{pmatrix} = rr^* \sim r^*r = 2a \approx a,$$

cf. Definition 2.1, where r is the column matrix in $M_{2,1}(A)$ with both entries equal to $a^{1/2}$. $\square$

The following definition is convenient for the formulation of some of our next lemmas.

Definition 5.5 (The matrix diagonalization property) An n-tuple $(a_1, \dots, a_n)$ in a C^*-algebra A is said to have the *matrix diagonalization property* if for every positive matrix $a = (a_{ij})$ in $M_n(A)$ with $a_{jj} = a_j$, and for every $\varepsilon_1 > 0, \dots, \varepsilon_n > 0$ and every $\delta > 0$ there are elements $d_1, \dots, d_n$ in A with

$$d_j^* a_{jj} d_j = (a_{jj} - \varepsilon_j)_+, \qquad \|d_i^* a_{ij} d_j\| \le \delta \quad \text{for} \quad i \ne j. \tag{5.1}$$

Lemma 5.6 *Let $a_1, \dots, a_n$ be positive elements in a C^*-algebra A, let $\varepsilon_1 > 0, \dots, \varepsilon_n > 0$, and suppose that the n-tuple $\big((a_1 - r\varepsilon_1)_+, \dots, (a_n - r\varepsilon_n)_+\big)$ has the matrix diagonalization property for some r in $(1/2, 1)$. Then for each positive matrix $a = (a_{ij})$ in $M_n(A)$, with*

$a_{jj} = a_j$, *and for each* $\delta > 0$ *there are elements* $d_1, \dots, d_n$ *in* A *such that* (5.1) *holds and such that* $\|d_j\|^2 \le 2\varepsilon_j^{-1}\|a_j\|$.

Proof: Let $f_\varepsilon \colon \mathbb{R}^+ \to [0,1]$ be a continuous function satisfying $f_\varepsilon(t)^2 = (t - r\varepsilon)/t$ for $t \ge r\varepsilon$, and $f_\varepsilon(t) = 0$ for $t \le \varepsilon/2$. Then

$$(a - r\varepsilon)_+ = f_\varepsilon(a)^2 a \ge \frac{\varepsilon}{2} f_\varepsilon(a)^2$$

for all positive a in A. Put $b_{ij} = f_{\varepsilon_i}(a_i) a_{ij} f_{\varepsilon_j}(a_j)$. Then $b = (b_{ij})$ is a positive matrix in $M_n(A)$ and $b_{jj} = (a_j - r\varepsilon_j)_+$. By assumption, and because $r\varepsilon_j < \varepsilon_j$, there are elements $e_1, \dots, e_n$ in A such that $e_j^* b_{jj} e_j = (a_j - \varepsilon_j)_+$ and $\|e_i^* c_{ij} e_j\| \le \delta$ for $i \ne j$. Put $d_j = f_{\varepsilon_j}(a_j) e_j$, so that $d_i^* a_{ij} d_j = e_i^* b_{ij} e_j$. Then (5.1) holds, and

$$\|d_j\|^2 = \|e_j^* f_{\varepsilon_j}(a_j)^2 e_j\| \le \frac{2}{\varepsilon_j}\|e_j^* f_{\varepsilon_j}(a_j) a_j f_{\varepsilon_j}(a_j) e_j\| = \frac{2}{\varepsilon_j}\|(a_j - \varepsilon_j)_+\| \le \frac{2}{\varepsilon_j}\|a_j\|$$

as desired. □

The norm estimate on the elements d_j in the next lemma will be improved later in Corollary 7.22, where it is shown that we can take $d_1, \dots, d_n$ to be contractions.

Lemma 5.7 *Let* $a_1, \dots, a_n$ *be positive elements in a* C^**-algebras* A*, and suppose that each pair* $\big((a_i - \eta_i)_+, (a_j - \eta_j)_+\big)$, $i \ne j$, *has the matrix diagonalization property for every choice of* $\eta_k \ge 0$*. Then the* n*-tuple* $(a_1, \dots, a_n)$ *has the matrix diagonalization property.*

Proof: The lemma is proved by induction on n. For $n = 2$ there is nothing to prove. Suppose that $n \ge 3$ and that the lemma has been verified for $n - 1$. Let $a = (a_{ij})$ be a positive matrix in $M_n(A)$ with $a_{jj} = a_j$, and let $\varepsilon_j > 0$ and $\delta > 0$ be given. We find $d_1, \dots, d_n$ in A with $d_j^* a_{jj} d_j = (a_{jj} - \varepsilon_j)_+$ and $\|d_i^* a_{ij} d_i\| \le \delta$ for $i \ne j$. The proof has three steps: First we diagonalize the lower $n-1$ by $n-1$ sub-matrix of a, then we diagonalize the resulting upper $n-1$ by $n-1$ sub-matrix, and at the end we take care of the entries at the positions $(1, n)$ and $(n, 1)$.

Take $\delta_0 > 0$ and $\delta_1 > 0$ (to be determined later). By the induction hypothesis there are elements $f_2, \dots, f_n$ in A such that $f_j^* a_{jj} f_j = (a_{jj} - \varepsilon_j/2)_+$ and $\|f_i^* a_{ij} f_j\| \le \delta_0$ for $i \ne j$. Set $f_1 = 1$ and put $b_{ij} = f_i^* a_{ij} f_j$, making $b = (b_{ij})$ a positive matrix in $M_n(A)$. Use the induction hypothesis and Lemma 5.6 to find elements $g_1, \dots, g_{n-1}$ in A such that

$$\|g_j\| \le 2\varepsilon_j^{-1}\|a_{jj}\|, \qquad g_j^* b_{jj} g_j = (b_{jj} - \varepsilon_j/2)_+, \qquad \|g_i^* b_{ij} g_j\| \le \delta_1 \quad \text{for } i \ne j.$$

Set $g_n = 1$ and set $c_{ij} = g_i^* b_{ij} g_j$. Then

$$c_{jj} = \begin{cases} (a_{jj} - \varepsilon_j/2)_+, & j = 1,\ j = n, \\ (a_{jj} - \varepsilon_j)_+, & 2 \le j \le n-1, \end{cases} \qquad \|c_{ij}\| \le \begin{cases} \delta_1, & i \ne j,\ 1 \le i,j \le n-1, \\ \|g_j\|\delta_0, & i = n,\ 2 \le j \le n-1 \\ \|g_i\|\delta_0, & j = n,\ 2 \le i \le n-1 \end{cases}$$

Use finally the assumption (and Lemma 5.6) to find h_1 and h_n in A with $\|h_j\| \le 2\varepsilon_j^{-1}\|a_{jj}\|$ and

$$\|h_1^* c_{1n} h_n\| = \|h_n^* c_{n1} h_1\| \le \delta, \qquad h_j^* c_{jj} h_j = (c_{jj} - \varepsilon_j/2)_+ = (a_{jj} - \varepsilon_j)_+ \quad \text{for } j = 1, n,$$

and put $h_j = 1$ for $j = 2, \dots, n$. Then $d_j = f_j g_j h_j$ satisfies $d_j^* a_{jj} d_j = (a_{jj} - \varepsilon_j)_+$, and if δ_0 and δ_1 have been chosen small enough, then also $\|d_i^* a_{ij} d_j\| \le \delta$ whenever $i \ne j$. □

As each pair of positive elements in a strongly purely infinite has the matrix diagonalization property, Lemma 5.7 and Lemma 5.6 imply:

Lemma 5.8 *Let A be a strongly purely infinite C^*-algebra. Then for each positive matrix $a = (a_{ij})$ in $M_n(A)$, for each choice of $\varepsilon_j > 0$, $j = 1, \dots, n$, and for each $\delta > 0$ there are elements $d_1, \dots, d_n$ in A such that*

$$d_j^* a_{jj} d_j = (a_{jj} - \varepsilon_j)_+, \qquad \|d_i^* a_{ij} d_j\| \le \delta \quad \textit{for } i \ne j, \qquad \|d_j\|^2 \le 2\varepsilon_j^{-1}\|a_{jj}\|. \tag{5.2}$$

Lemma 5.9 *Let A be a C^*-algebra, and let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_m$ be two families of positive elements in A such that the $(n+m)$-tuple $(a_1, \dots, a_n, b_1, \dots, b_m)$ has the matrix diagonalization property. Then the pair $\big(\sum_{i=1}^n a_i, \sum_{j=1}^m b_j\big)$ has the matrix diagonalization property.*

Proof: We may assume that $m = n$ (otherwise take for example $b_{m+1} = \dots = b_n = 0$). Put $a = \sum_{i=1}^n a_i$ and $b = \sum_{i=1}^n b_i$, and let x in A be such that $\left(\begin{smallmatrix} a & x^* \\ x & b \end{smallmatrix}\right)$ is a positive matrix in $M_2(A)$. Let $\varepsilon > 0$ be given. Put

$$s = \begin{pmatrix} a_1^{1/2} \\ \vdots \\ a_n^{1/2} \end{pmatrix}, \qquad t = \begin{pmatrix} b_1^{1/2} \\ \vdots \\ b_n^{1/2} \end{pmatrix}.$$

Then $s^*s = a$ and $t^*t = b$. Write $s = ua^{1/2}$ and $t = vb^{1/2}$ for some partial isometries u, v in $M_{n,1}(A^{**})$. Observe that uc, respectively, vc, belongs to $M_{n,1}(A)$ for every c in $\overline{aAa}$, respectively, in $\overline{bAb}$. Also,

$$uau^* = ss^* = \begin{pmatrix} a_1 & \cdots & a_1^{1/2}a_n^{1/2} \\ \vdots & & \vdots \\ a_n^{1/2}a_1^{1/2} & \cdots & a_n \end{pmatrix}, \qquad vbv^* = tt^* = \begin{pmatrix} b_1 & \cdots & b_1^{1/2}b_n^{1/2} \\ \vdots & & \vdots \\ b_n^{1/2}b_1^{1/2} & \cdots & b_n \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} uau^* & ux^*v^* \\ vxu^* & vbv^* \end{pmatrix} \in M_{2n}(A)^+,$$

and that the diagonal entries of this matrix are $(a_1, \dots, a_n, b_1, \dots, b_n)$, and this $2n$-tuple has the matrix diagonalization property by assumption. Arguing as in the proof of Proposition 5.11 (iv) we find matrices f_1, f_2 in $M_n(A)$ such that

$$\|f_1^*uau^*f_1 - uau^*\| < \varepsilon, \qquad \|f_2^*vbv^*f_2 - vbv^*\| < \varepsilon, \qquad \|f_2^*vxu^*f_1\| < \varepsilon.$$

Replacing f_j by $e_jf_je_j$ where e_1 and e_2 are suitable (approximate units) in hereditary sub-C^*-algebras generated by uau^*, respectively, vbv^*, we can assume that f_1 and f_2 belong to these respective hereditary sub-C^*-algebras. This will ensure that $d_1 = u^*f_1u$ and $d_2 = v^*f_2v$ belong to A. Finally,

$$\begin{aligned} \|d_1^*ad_1 - a\| &= \|ud_1^*ad_1u^* - uau^*\| = \|f_1^*uau^*f_1 - uau^*\| < \varepsilon, \\ \|d_2^*bd_2 - b\| &= \|vd_2^*bd_2v^* - vbv^*\| = \|f_2^*vbv^*f_2 - vbv^*\| < \varepsilon, \\ \|d_2^*xd_1\| &= \|vd_2^*xd_1u^*\| = \|f_2^*vxu^*f_1\| < \varepsilon, \end{aligned}$$

as desired. $\square$

Remark 5.10 (Matrix diagonalization revisited) A C^*-algebra A is strongly purely infinite if and only if for each pair of positive element a, b in A and for each $\varepsilon > 0$ there are elements d_1, d_2 in A such that

$$\|d_1^*ad_1 - a\| \le \varepsilon, \qquad \|d_2^*bd_2 - b\| \le \varepsilon, \qquad \|d_2^*b^{1/2}a^{1/2}d_1\| \le \varepsilon. \tag{5.3}$$

To see this, note first that

$$\begin{pmatrix} a & a^{1/2}b^{1/2} \\ b^{1/2}a^{1/2} & b \end{pmatrix} = \begin{pmatrix} a^{1/2} & 0 \\ 0 & b^{1/2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a^{1/2} & 0 \\ 0 & b^{1/2} \end{pmatrix} \in M_2(A)^+.$$

The "only if" part of the claim now follows immediately from Definition 5.1.

To prove the "if" part it suffices to show that $\big((a-\eta)_+, (b-\eta)_+\big)$ has the matrix diagonalization property for every pair of positive elements a, b in A and for every $\eta > 0$, cf. Lemma 5.6. Let x in A be such that $\left(\begin{smallmatrix} (a-\eta)_+ & x^* \\ x & (b-\eta)_+ \end{smallmatrix}\right)$ is positive. Put $c = a+b$. It follows from (5.3), as in the proof of Proposition 5.4, that A must be strongly purely infinite. Hence, for some $\delta > 0$,

$$\begin{pmatrix} (a-\eta)_+ & x^* \\ x & (b-\eta)_+ \end{pmatrix} \precsim_0 \begin{pmatrix} (a-\eta/2)_+ & 0 \\ 0 & (b-\eta/2)_+ \end{pmatrix} \precsim_0 \begin{pmatrix} (c-\delta)_+ & 0 \\ 0 & (c-\delta)_+ \end{pmatrix} \precsim_0 c,$$

where $e \precsim_0 f$ means that $e = z^* f z$ for some z in (a rectangular matrix over) A. We need therefore only show that each positive matrix in $M_2(A)$ of the form y^*cy, where y belongs to $M_{1,2}(A)$ and c is a positive element in A, can be approximately matrix diagonalized.

Let $\left(\begin{smallmatrix} a & x^* \\ x & b \end{smallmatrix}\right) = y^*cy$ be given, so that $a = y_1^*cy_1$, $b = y_2^*cy_2$, and $x = y_2^*cy_1$. Write $y = (y_1, y_2)$, write $c^{1/2}y_1 = ua^{1/2}$ and $c^{1/2}y_2 = vb^{1/2}$, where u, v are partial isometries in A^{**}, and put $a_0 = uau^*$ and $b_0 = vbv^*$. Then a_0 and b_0 belong to A, and

$$\begin{pmatrix} u^* & 0 \\ 0 & v^* \end{pmatrix} \begin{pmatrix} a_0 & a_0^{1/2}b_0^{1/2} \\ b_0^{1/2}a_0^{1/2} & b_0 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} a & x^* \\ x & b \end{pmatrix}.$$

Find e_1, e_2 in A such that

$$\|e_1^*a_0e_1 - a_0\| < \varepsilon, \qquad \|e_2^*b_0e_2 - b_0\| < \varepsilon, \qquad \|e_2^*b_0^{1/2}a_0^{1/2}e_1\| < \varepsilon.$$

Upon replacing e_1 by ge_1g for a suitable positive contraction in the hereditary sub-C^*-algebra $\overline{a_0Aa_0}$ we may assume that e_1 belongs to this sub-C^*-algebra. Similarly, we may assume that e_2 belongs to $\overline{b_0Ab_0}$. Then $d_1 = ue_1$ and $d_2 = ve_2$ belong to A, and

$$\|d_1^*ad_1 - a\| < \varepsilon, \qquad \|d_2^*bd_2 - b\| < \varepsilon, \qquad \|d_2^*xd_1\| < \varepsilon.$$

Proposition 5.11 (Permanence properties)

(i) *If A is strongly purely infinite, then so is every non-zero quotient of A.*

(ii) *If A is strongly purely infinite, then so are all its non-zero hereditary sub-C^*-algebras.*

(iii) *If A and B are stably isomorphic and if A is strongly purely infinite, then so is B.*

(iv) *Any inductive limit of a system of strongly purely infinite C^*-algebras is again strongly purely infinite.*

Proof: (i). We must show that A/I is strongly purely infinite whenever I is a closed two-sided ideal in A. To see this take a positive element b in $M_2(A/I)$. Lift b to a positive element a in $M_2(A)$. Find d_1, d_2 in A that approximately matrix diagonalize a, cf. Definition 5.1. The image under the quotient mapping $A \to A/I$ of d_1, d_2 will then approximately matrix diagonalize b.

(ii). This follows from Lemma 5.2.

(iv). In the light of (i) it suffices to show that if A is a C^*-algebra with an upper directed family $\{A_i\}_{i\in\mathbb{I}}$ of strongly purely infinite sub-C^*-algebras A_i such that $\bigcup_{i\in\mathbb{I}} A_i$ is dense in A, then A is strongly purely infinite.

We must show that to each positive matrix $\left(\begin{smallmatrix} a & x^* \\ x & b \end{smallmatrix}\right)$ in $M_2(A)$ and to each $\varepsilon > 0$, there are d_1, d_2 in A such that $\|d_1^* a d_1 - a\| \leq \varepsilon$, $\|d_2^* b d_2 - b\| \leq \varepsilon$, and $\|d_2^* x d_1\| \leq \varepsilon$. It is no loss of generality to assume that the given positive matrix is a contraction.

Choose $\delta_1 > 0$ and $\delta_2 > 0$ such that $\delta_2 \leq \varepsilon/2$ and $(2\delta_2^{-1} + 1)\delta_1 \leq \varepsilon/2$. Find i in $\mathbb{I}$ and a positive element $\left(\begin{smallmatrix} a_0 & x_0^* \\ x_0 & b_0 \end{smallmatrix}\right)$ in $M_2(A_i)$ such that

$$\|a - a_0\| < \delta_1, \qquad \|b - b_0\| < \delta_1, \qquad \|x - x_0\| < \delta_1$$

Find next d_1, d_2 in A_i with $\|d_i\|^2 \leq 2\delta_2^{-1}$ and

$$d_1^* a_0 d_1 = (a_0 - \delta_2)_+, \qquad d_2^* b_0 d_2 = (b_0 - \delta_2)_+, \qquad \|d_2^* x_0 d_1\| < \delta_2,$$

cf. Lemma 5.6. Then

$$\|d_1^* a d_1 - a\| \leq \|d_1\|^2 \|a - a_0\| + \|a - a_0\| + \delta_2 \leq (2\delta_2^{-1} + 1)\delta_1 + \delta_2 \leq \varepsilon,$$

and, similarly, $\|d_2^* b d_2 - b\| < \varepsilon$. Also,

$$\|d_2^* x d_1\| \leq \|d_1\| \|d_2\| \|x - x_0\| + \|d_2^* x_0 d_1\| \leq 2\delta_2^{-1}\delta_1 + \delta_2 \leq \varepsilon.$$

This shows that A is strongly purely infinite.

(iii). By (i) it suffices to show that $A \otimes \mathcal{K}$ is strongly purely infinite when A is strongly purely infinite, and using (iv) it suffices to show that $M_n(A)$ is strongly purely infinite for all natural numbers n when A is strongly purely infinite. Let $\left(\begin{smallmatrix} a & x^* \\ x & b \end{smallmatrix}\right)$ be a positive element in $M_2(M_n(A))$ and let $\varepsilon > 0$. Then a and b are positive elements of $M_n(A)$. Denote by $\widehat{a}$ and $\widehat{b}$ the diagonal parts of a and b, i.e., $\widehat{a}$ is the diagonal matrix in $M_n(A)$ whose diagonal entries are equal to the diagonal entries of a, and similarly for $\widehat{b}$. Then $\mathrm{diag}(\widehat{a}, \widehat{b})$ is the diagonal part of $\left(\begin{smallmatrix} a & x^* \\ x & b \end{smallmatrix}\right)$. As remarked in Lemma 5.3, $a \precsim \widehat{a}$ and $b \precsim \widehat{b}$. Hence there is $\delta > 0$

and e_1, e_2 in $M_n(A)$ such that $e_1^*(\widehat{a}-\delta)_+e_1 = (a-\varepsilon)_+$ and $e_2^*(\widehat{b}-\delta)_+e_2 = (b-\varepsilon)_+$.

From Lemma 5.8 we can find (diagonal) matrices f_1, f_2 in $M_n(A)$ such that

$$f_1^* a f_1 = (\widehat{a}-\delta)_+, \qquad f_2^* b f_2 = (\widehat{b}-\delta)_+, \qquad \|f_2^* x f_1\| \le \|e_1\|^{-1}\|e_2\|^{-1}\varepsilon.$$

Put $d_1 = f_1 e_1$ and $d_2 = f_2 e_2$. Then $d_1^* a d_1 = (a-\varepsilon)_+$, $d_2^* b d_2 = (b-\varepsilon)_+$, and $\|d_2^* x d_1\| \le \|e_1\|\|e_2\|\|f_2^* x f_1\| \le \varepsilon$. This shows that $M_n(A)$ is strongly purely infinite. □

Proposition 5.12 *The following conditions are equivalent for every C^*-algebra A:*

(i) *A is strongly purely infinite.*

(ii) *$\ell^\infty(A)$ is strongly purely infinite.*

(iii) *A_ω is strongly purely infinite for every filter ω on $\mathbb{N}$.*

(iv) *A_ω is strongly purely infinite for some filter ω on $\mathbb{N}$.*

Proof: (i) $\Rightarrow$ (ii). Assume that A is strongly purely infinite. To show that $\ell^\infty(A)$ is strongly purely infinite take a positive matrix $\left(\begin{smallmatrix} a & x^* \\ x & b \end{smallmatrix}\right)$ in $\ell^\infty(A)$. Upon scaling this matrix, we can assume that a and b are contractions. Write

$$a = (a_1, a_2, \dots), \qquad b = (b_1, b_2, \dots), \qquad x = (x_1, x_2, \dots),$$

where $\{a_n\}$, $\{b_n\}$, and $\{x_n\}$ are sequences of contractions in A and $\left(\begin{smallmatrix} a_n & x_n^* \\ x_n & b_n \end{smallmatrix}\right)$ is positive in $M_2(A)$ for each n. Let $\varepsilon > 0$ be given. Use Lemma 5.6 to find elements $d_{1,n}, d_{2,n}$ in A with

$$d_{1,n}^* a_n d_{1,n} = (a_n - \varepsilon)_+, \quad d_{2,n}^* b_n d_{2,n} = (b_n - \varepsilon)_+, \quad \|d_{2,n}^* x_n d_{1,n}\| \le \varepsilon, \quad \|d_{j,n}\|^2 \le 2\varepsilon^{-1}.$$

Then $d_j = (d_{j,1}, d_{j,2}, \dots)$ belongs to $\ell^\infty(A)$ for $j = 1, 2$, and

$$d_1^* a d_1 = (a-\varepsilon)_+, \qquad d_2^* b d_2 = (b-\varepsilon)_+, \qquad \|d_2^* x d_1\| \le \varepsilon.$$

This shows that $\ell^\infty(A)$ is strongly purely infinite.

(ii) $\Rightarrow$ (iii). This follows from Proposition 5.11 (i).

(iii) $\Rightarrow$ (iv). Trivial!

(iv) $\Rightarrow$ (i). Assume that A_ω is strongly purely infinite for some free filter ω on $\mathbb{N}$. Let $\left(\begin{smallmatrix} a & x^* \\ x & b \end{smallmatrix}\right)$ be a positive element in $M_2(A)$ and let $\varepsilon > 0$ be given. Let $\pi_\omega \colon \ell^\infty(A) \to A_\omega$ denote the quotient mapping. Identify x in A with $\pi_\omega(x, x, \dots)$ (thus viewing A as a sub-C^*-algebra of A_ω). Then $\left(\begin{smallmatrix} a & x^* \\ x & b \end{smallmatrix}\right)$ is a positive element in $M_2(A_\omega)$. We can therefore find d_1, d_2 in A_ω

such that $\|d_1^*ad_1 - a\| < \varepsilon$, $\|d_2^*bd_2 - b\| < \varepsilon$, and $\|d_2^*xd_2\| < \varepsilon$. Write $d_j = \pi_\omega(d_{j,1}, d_{j,2}, \dots)$. Then

$$\limsup_\omega \|d_{1,n}^*ad_{1,n} - a\| < \varepsilon, \qquad \limsup_\omega \|d_{2,n}^*bd_{2,n} - b\| < \varepsilon, \qquad \limsup_\omega \|d_{2,n}^*xd_{1,n}\| < \varepsilon.$$

Hence $\|d_{1,n}^*ad_{1,n} - a\| < \varepsilon$, $\|d_{2,n}^*bd_{2,n} - b\| < \varepsilon$, and $\|d_{2,n}^*xd_{1,n}\| < \varepsilon$ for each n in some subset belonging to ω, and hence for at least one n. This completes the proof. $\square$

If ω is a free filter on $\mathbb{N}$ and if A is a strongly purely infinite C^*-algebra, then A_ω is strongly purely infinite. Combining Lemma 5.6 and Lemma 2.5 we get:

Lemma 5.13 *In the notation above, Lemma 5.8 holds with $\delta = 0$ when A is replaced with the C^*-algebra A_ω.*

Proposition 5.14 *Every approximately divisible purely infinite C^*-algebra is strongly purely infinite.*

Proof: Let A be an approximately divisible purely infinite C^*-algebra, and let ω be a free filter on $\mathbb{N}$. We show that A_ω is strongly purely infinite, and it will then follow from Proposition 5.12 that A is strongly purely infinite.

Let $T = \left(\begin{smallmatrix} a & x^* \\ x & b \end{smallmatrix}\right)$ be a positive matrix in $M_2(A_\omega)$. Lift T to a positive matrix $(T_1, T_2, \dots)$ in $M_2(\ell^\infty(A))$ and write $T_n = \left(\begin{smallmatrix} a_n & x_n^* \\ x_n & b_n \end{smallmatrix}\right)$. With $E = M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$ we can find unital *-homomorphisms $\varphi_n \colon E \to \mathcal{M}(A)$ such that

$$\lim_{n\to\infty} \|\varphi_n(e)a_n - a_n\varphi_n(e)\| = \lim_{n\to\infty} \|\varphi_n(e)b_n - b_n\varphi_n(e)\| = \lim_{n\to\infty} \|\varphi_n(e)x_n - x_n\varphi_n(e)\| = 0$$

for all e in E.

With $\pi_\omega \colon \ell^\infty(\mathcal{M}(A)) \to \mathcal{M}(A)_\omega$ the quotient mapping, define

$$\varphi \colon E \to \mathcal{M}(A)_\omega \subseteq \mathcal{M}(A_\omega) \;\text{ by }\; \varphi(e) = \pi_\omega(\varphi_1(e), \varphi_2(e), \dots).$$

Then a, b, x commute with the image of φ.

Choose a full (non-unital) *-homomorphism $\iota \colon M_2(\mathbb{C}) \to E$, let $\{e_{ij}\}$ be the matrix units for $M_2(\mathbb{C})$, and put $f_{ij} = \iota(e_{ij})$. There are elements e_1, e_2, e_3 in E such that $1 = \sum_{j=1}^3 e_j^* f_{11} e_j$. Hence

$$a = \sum_{j=1}^3 \varphi(e_j^*)\varphi(f_{11})a\varphi(e_j)$$

and so a belongs to the ideal generated by $\varphi(f_{11})a$ $(= \varphi(f_{11})a\varphi(f_{11}))$. Similarly, b belongs to the ideal generated by $\varphi(f_{22})b\varphi(f_{22})$. Let $\varepsilon > 0$ be given. Since A_ω is purely infinite, cf. Proposition 3.5, there are elements c_1, c_2 in A_ω such that

$$c_1^*\varphi(f_{11})a\varphi(f_{11})c_1 = (a-\varepsilon)_+, \qquad c_2^*\varphi(f_{22})b\varphi(f_{22})c_2 = (b-\varepsilon)_+.$$

Put $d_1 = \varphi(f_{11})c_1$ and $d_2 = \varphi(f_{22})c_2$. Then $d_2^*xd_1 = 0$ because x commutes with the image of φ, $d_1^*ad_1 = (a-\varepsilon)_+$, and $d_2^*bd_2 = (b-\varepsilon)_+$. Hence T can be matrix diagonalized, and this proves that A_ω is strongly purely infinite. □

6 Purely infinite C^*-algebras of real rank zero

It is shown in this section that every purely infinite C^*-algebra of real rank zero is strongly purely infinite.

Call an element a in a C^*-algebra A *locally central* if it belongs to the center of the hereditary sub-C^*-algebra $\overline{aAa}$. Every projection and every multiple of a projection are locally central. If a is locally central, then so is $(a-\rho)_+$ for every $\rho \geq 0$.

Definition 6.1 A C^*-algebra A is said to have the *locally central decomposition property* if for every a in A^+ and for every $\varepsilon > 0$ there exist e, f in A and locally central elements $a_1, a_2, \dots, a_n$ in A^+ such that

(i) $e^*ae = \sum_{j=1}^n a_j$,

(ii) $f^*(e^*ae)f = (a-\varepsilon)_+$.

Remark 6.2 A purely infinite C^*-algebra A satisfies the locally central decomposition property if it has the following property: for every a in A^+ and for every $\varepsilon > 0$ there exist locally central elements $a_1, a_2, \dots, a_n$ in A^+ such that

(i) $a_1, \dots, a_n$ belong to $\overline{AaA}$,

(ii) $(a-\varepsilon)_+$ belongs to $\overline{A(\sum_{j=1}^n a_j)A}$.

To see that (i) and (ii) above imply the locally central decomposition property, take a in A^+ and $\varepsilon > 0$. Applying the conditions above to $(a-\varepsilon/3)_+$ and to $\varepsilon/3$, and using the assumption that A is purely infinite, we find locally central elements $a_1, a_2, \dots, a_n$ in A^+ such that

$$\sum_{j=1}^n a_j \precsim (a-\varepsilon/3)_+, \qquad (a-2\varepsilon/3)_+ \precsim \sum_{j=1}^n a_j.$$

We can now use Lemma 2.4 to find e, f in A such that (i) and (ii) in Definition 6.1 hold.

Proposition 6.3 *Every purely infinite C^*-algebra of real rank zero has the locally central decomposition property.*

Proof: Let a be a non-zero positive element in a purely infinite C^*-algebra A of real rank zero. Since $\overline{aAa}$ has an approximate unit consisting of projections, it contains for each $\varepsilon > 0$ a projection p such that $(a-\varepsilon)_+$ belongs to the ideal generated by p. Hence the two conditions of Remark 6.2 are satisfied. $\square$

It is shown in [4] that continuous field C^*-algebras, whose fibers are purely infinite and simple, have the locally central decomposition property.

Lemma 6.4 *Let A be a purely infinite C^*-algebra. Let $\delta > 0$, let x, f in A and e in $\overline{x^*Ax}$ be given such that x is a contraction and*

$$\|f^*xe\| \leq \delta^2, \qquad (|x| - \delta)_+ \leq |e|.$$

Let D be a hereditary sub-C^*-algebra of A which contains x^*x. Then for each positive element b in D and for each $\varepsilon > 0$ there exists d in D with $d^*d = (b-\varepsilon)_+$ and

$$\|f^*xd\| \leq \big(2\|b\|\|f\|(1 + 5\|f\|)\big)^{1/2}\delta^{1/2}.$$

Proof: We may assume that $\delta < 1$. Put $b_0 = (b-\varepsilon)_+$ and find positive contractions u_1, u_2 in D such that $u_1u_2 = u_2u_1 = u_2$ and $u_2b_0 = b_0u_2 = b_0$.

Let J be the closed two-sided ideal in A generated by $(|e^*| - \delta)_+$ and let $\pi_J \colon A \to A/J$ be the quotient mapping. Then $\|\pi_J(e)\| \leq \delta$, and hence $\|\pi_J(|x|)\| \leq 2\delta$. Accordingly, we can find a positive contraction g in J such that $\|xu_1^{1/2}(1-g)\| \leq \sqrt{5}\delta$.

Put $a = u_1^{1/2}(1-g)^2u_1^{1/2} + |e^*|$. Then a belongs to D and $(u_1 - \delta)_+$ belongs to the closed two-sided ideal I in A generated by $(a-\delta)_+$. To see the latter, observe first that $\|\pi_I(|e^*|)\| \leq \|\pi_I(a)\| \leq \delta$, which shows that $(|e^*| - \delta)_+$ belongs to I. This entails that I contains J. Hence

$$\pi_I(u_1) = \pi_I(u_1^{1/2}(1-g)^2u_1^{1/2}) \leq \pi_I(a) \leq \delta,$$

whence $(u_1 - \delta)_+$ belongs to I.

Since $(a-\delta)_+$ is properly infinite, we have $(u_1 - \delta)_+ \precsim (a-\delta)_+$ (in A and hence in D), and $u_2 \precsim (u_1 - \delta)_+$. By Lemma 2.4 (ii) we can find y in D such that $\|y\|^2 \leq 2/\delta$ and

$y^*ay = u_2$. Put $d = a^{1/2}yb_0^{1/2}$. Then d belongs to D, $d^*d = b_0$, and

$$\begin{aligned}
\|f^*xd\|^2 &= \|f^*xdd^*x^*f\| \le \|b_0\|\|y\|^2\|f^*xax^*f\| \\
&\le 2\delta^{-1}\|b\|\big(\|f^*xu^{1/2}(1-g)^2u^{1/2}x^*f\| + \|f^*x|e^*|x^*f\|\big) \\
&\le 2\delta^{-1}\|b\|\big(\|f\|^2\|xu^{1/2}(1-g)\|^2 + \|f^*x|e^*|\|\|x^*\|\|f\|\big) \\
&\le 2\delta^{-1}\|b\|\big(5\delta^2\|f\|^2 + \delta^2\|f\|\big) \\
&= 2\delta\|b\|\|f\|(5\|f\|+1).
\end{aligned}$$

□

Proposition 6.5 *Let A be a purely infinite C^*-algebra, let D_1 and D_2 be hereditary sub-C^*-algebras of A, and let x be an element in A such that $x^*x \in D_1$ and $xx^* \in D_2$. Let a be a positive element in D_1, let b be a positive element in D_2, and let $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ be given. It follows that for each $\delta > 0$ there exist $d_1 \in D_1$ and $d_2 \in D_2$ such that $d_1^*d_1 = (a-\varepsilon_1)_+$, $d_2^*d_2 = (b-\varepsilon_2)_+$, and $\|d_2^*xd_1\| < \delta$.*

Proof: Since $\delta > 0$ is arbitrary, we may assume that a, b, and x are contractions. Choose $\eta > 0$ such that $12^3\eta \le \delta^4$. Using that A is purely infinite we can find h_1, h_2 in $\overline{x^*Ax}$ such that

$$\begin{pmatrix} (|x|-\eta)_+^2 & 0 \\ 0 & (|x|^{1/2}-\eta)_+^2 \end{pmatrix} = \begin{pmatrix} h_1^* & 0 \\ h_2^* & 0 \end{pmatrix}\begin{pmatrix} |x| & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} h_1 & h_2 \\ 0 & 0 \end{pmatrix},$$

or, equivalently, such that

$$h_1^*|x|h_1 = (|x|-\eta)_+^2, \qquad h_2^*|x|h_2 = (|x|^{1/2}-\eta)_+^2, \qquad h_1^*|x|h_2 = 0.$$

Let $x = v|x| = |x^*|v$ be the polar decomposition for x, where v is a partial isometry in A^{**}. Put $x_1 = v|x|^{1/2}$, $f_1 = x_1h_1v^*$, and $e_1 = h_2$. Then e_1 belongs to $\overline{x^*Ax} = \overline{x_1^*Ax_1}$, $f_1^*x_1e_1 = vh_1^*x_1^*x_1h_2 = vh_1^*|x|h_2 = 0$,

$$|e_1| = (h_2^*h_2)^{1/2} \ge (h_2^*|x|h_2)^{1/2} = (|x|^{1/2}-\eta)_+ = (|x_1|-\eta)_+,$$

and f_1 is a contraction. It follows from Lemma 6.4 that there exists d_1 in D_1 with $d_1^*d_1 = (a-\varepsilon_1)_+$ and $\|f_1^*x_1d_1\| \le (12\eta)^{1/2}\,(=\eta_1)$.

Put $x_2 = x^*$, $f_2 = d_1$, and $e_2 = vh_1v^*$. Then

$$\begin{aligned}
f_1^*x_1d_1 &= vh_1x_1^*x_1f_2 = vh_1|x|f_2 \\
&= (f_2^*|x|h_1^*v^*)^* = (f_2^*x^*vh_1v^*)^* = (f_2^*x_2e_2)^*,
\end{aligned}$$

and so $\|f_2^* x_2 e_2\| \le \eta_1$. Moreover, e_2 belongs to $v(\overline{x^*Ax})v^* = \overline{xAx^*} = \overline{x_2^*Ax_2}$, $x_2^*x_2 = xx^*$ belongs to D_2, and

$$\begin{aligned} |e_2| &= v(h_1^*h_1)^{1/2}v^* \ \ge\ v(h_1^*|x|h_1)^{1/2}v^* \ =\ v(|x|-\eta)_+v^* \\ &= (|x^*|-\eta)_+ \ =\ (|x_2|-\eta)_+ \ \le\ (|x_2|-\eta_1)_+. \end{aligned}$$

Apply Lemma 6.4 once more to get an element d_2 in D_2 satisfying $d_2^*d_2 = (b-\varepsilon_2)_+$, and

$$\|f_2^*x_2d_2\| \le (12\eta_1\|b\|)^{1/2} \le (12\eta_1)^{1/2} \le (12(12\eta)^{1/2})^{1/2} \le \delta.$$

This proves the lemma because $\|d_2^*xd_1\| = \|d_1^*x^*d_2\| = \|f_2^*x_2d_2\|$. □

Remark 6.6 (Pure versus strong pure infiniteness) At a first glance it would seem that Proposition 6.5 proves that every purely infinite C^*-algebra is strongly purely infinite. Here is what actually follows from this proposition: If a, b are positive elements in a purely infinite C^*-algebra A and if $x = b^{1/2}x_0a^{1/2}$, where x_0 belongs to $\overline{(b-\rho)_+A(a-\rho)_+}$ for some $\rho > 0$, then for each $\varepsilon > 0$ and $\delta > 0$ there are elements d_1, d_2 in A such that $d_1^*ad_1 = (a-\varepsilon)_+$, $d_2^*bd_2 = (b-\varepsilon)_+$, and $\|d_2^*xd_1\| \le \delta$.

Indeed, assume as we may that $\rho < \varepsilon$, and put

$$a_0 = (a-\rho)_+, \quad b_0 = (b-\rho)_+, \qquad D_1 = \overline{a_0Aa_0}, \quad D_2 = \overline{b_0Ab_0},$$

so that $x_0^*x_0$ belongs to D_1, $x_0x_0^*$ belongs to D_2, $(a-\varepsilon)_+ = (a_0-(\varepsilon-\rho))_+$, and $(b-\varepsilon)_+ = (b_0-(\varepsilon-\rho))_+$. We can apply Proposition 6.5 to get e_1 in D_1 and e_2 in D_2 satisfying $e_1^*e_1 = (a-\varepsilon)_+$, $e_2^*e_2 = (b-\varepsilon)_+$, and $\|e_2^*x_0e_1\| \le \delta$. Now, find d_1, d_2 in A such that $e_1 = a^{1/2}d_1$ and $e_2 = b^{1/2}d_2$. Then $d_1^*ad_1 = e_1^*e_1 = (a-\varepsilon)_+$, $d_2^*bd_2 = e_2^*e_2 = (b-\varepsilon)_+$, and

$$\|d_2^*xd_1\| = \|d_2^*b^{1/2}x_0a^{1/2}d_1\| = \|e_2^*x_0e_1\| \le \delta.$$

In general we cannot take x to be of the form $b^{1/2}x_0a^{1/2}$, with x_0 as above, although, by Remark 5.10, to prove that A is strongly purely infinite, it would suffice to find d_1, d_2 as above for $x = b^{1/2}a^{1/2}$.

The two next lemmas concern the matrix diagonalization property of n-tuples as in Definition 5.5 and locally central elements (defined above Definition 6.1).

Lemma 6.7 *Each n-tuple $(a_1, \dots, a_n)$ of locally central positive elements has the matrix diagonalization property.*

Proof: By Lemma 5.7 it suffices to prove the lemma for $n = 2$. Recall that $(a_j - \eta)_+$ is locally central for each $\eta \geq 0$.

Let x in A be such that
$$\begin{pmatrix} a_1 & x^* \\ x & a_2 \end{pmatrix} \in M_2(A)^+.$$
Choose a continuous function $f\colon \mathbb{R}^+ \to [0,1]$ satisfying that $f(t) = 0$ when $t \leq \delta/2$ and $|f(t)t - t| < \delta$ for all $t \geq 0$. Put $D_j = \overline{a_j A a_j}$, $j = 1, 2$, so that x^*x belongs to D_1 and xx^* belongs to D_2. It follows from Proposition 6.5 that there exist d_j in D_j, $j = 1, 2$, satisfying $d_j^* d_j = f(a_j)$ and $\|d_2^* x d_1\| < \delta$. Notice that $\|d_j\|^2 = \|f(a_j)\| \leq 1$. Using that a_1 and a_2 are locally central, we get $d_j^* a_j d_j = d_j^* d_j a_j = f(a_j)a_j$, and hence $\|d_j^* a_j d_j - a_j\| \leq \delta$ and $\|d_2^* x d_1\| \leq \delta$. Since $\delta > 0$ was arbitrary we can use Lemma 2.2 to conclude that (a_1, a_2) has the matrix diagonalization property. □

Theorem 6.8 *Every purely infinite C^*-algebra with the locally central decomposition property is strongly purely infinite.*

Proof: Let A be a C^*-algebra with the locally central decomposition property, and let
$$\begin{pmatrix} a & x^* \\ x & b \end{pmatrix} \in M_2(A)^+$$
and $\varepsilon > 0$ be given. By the assumption that A satisfies the locally central decomposition property and by Lemmas 6.7 and 5.9 there are elements e_1, f_1, e_2, f_2 in A such that $(e_1^* a e_1, e_2^* b e_2)$ satisfies the matrix diagonalization property, and such that
$$\|f_1^*(e_1^* a e_1) f_1 - a\| \leq \varepsilon/2, \qquad \|f_2^*(e_2^* b e_2) f_2 - b\| \leq \varepsilon/2.$$
Now,
$$\begin{pmatrix} e_1^* & 0 \\ 0 & e_2^* \end{pmatrix} \begin{pmatrix} a & x^* \\ x & b \end{pmatrix} \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} = \begin{pmatrix} e_1^* a e_1 & x_1^* \\ x_1 & e_2^* b e_2 \end{pmatrix},$$
where $x_1 = e_2^* x e_1$. Find g_1, g_2 in A such that
$$\|g_1^*(e_1^* a e_1) g_1 - e_1^* a e_1\| \leq \frac{\varepsilon}{2\|f_1\|^2}, \quad \|g_2^*(e_2^* b e_2) g_2 - e_2^* b e_2\| \leq \frac{\varepsilon}{2\|f_2\|^2}, \quad \|g_2^* x_1 g_1\| \leq \frac{\varepsilon}{\|f_1\|\|f_2\|}.$$
Put $d_j = e_j g_j f_j$ for $j = 1, 2$. Then
$$\begin{aligned} \|d_1^* a d_1 - a\| &= \|f_1^* g_1^* (e_1^* a e_1) g_1 f_1 - a\| \\ &\leq \|f_1\|^2 \|g_1^*(e_1^* a e_1) g_1 - e_1^* a e_1\| + \|f_1^*(e_1^* a e_1) f_1 - a\| \ \leq \ \varepsilon, \end{aligned}$$

and, similarly, $\|d_2^* b d_2 - b\| \le \varepsilon$. Finally,

$$\|d_2^* x d_1\| = \|f_2^* g_2^* e_2^* x e_1 g_1 f_1\| = \|f_2^* g_2^* x_1 g_1 f_1\| \le \|f_1\| \|f_2\| \|g_2^* x_1 g_1\| \le \varepsilon,$$

as desired. $\square$

Combining Theorem 6.8 with Proposition 6.3 yields:

Corollary 6.9 *Every purely infinite C^*-algebra of real rank zero is strongly purely infinite.*

7 Approximately inner completely positive contraction on strongly purely infinite C^*-algebras

The main result of this section says that all approximately inner completely positive contractions from a sub-C^*-algebra of a nuclear strongly purely infinite A into A are approximately 1-step inner. This result will in Section 8 be used to showing that A is isomorphic to $A \otimes \mathcal{O}_\infty$, if A is also separable. The section is divided into three, the last of which contains a digression that will not be used in the rest of the paper.

Some preliminary results

We begin by defining what it means for a completely positive mapping to be (approximately) inner:

Definition 7.1 (Inner and approximately inner maps) Let A and B be C^*-algebras both contained in a C^*-algebra E, and let $T\colon B \to A$ be a completely positive map. The map T is said to be *n-step inner* (relatively to E) if there are elements $e_1, \dots, e_n$ in E such that

$$T(b) = \sum_{j=1}^{n} e_j^* b e_j$$

for all b in B. We say that T is *inner* if T is n-step inner for some natural number n.

If for each finite subset F of B and for each $\varepsilon > 0$ there is an n-step inner, respectively, an inner completely positive map $S\colon B \to A$ such that $\|T(b) - S(b)\| \le \varepsilon$ for all b in F, then T is called *approximately n-step inner*, respectively, *approximately inner*.

The C^*-algebra E in the definition above will usually be either A, the multiplier algebra of A, or a limit algebra, A_ω, of A corresponding to some free filter ω on $\mathbb{N}$. If $E = A$ or if it is clear from the context which ambient C^*-algebra we are considering, then we might omit the reference "relatively to E". Lemma 7.3 below says that (approximate) innerness is not really dependent of the ambient C^*-algebra. First we need the following lemma:

Lemma 7.2 *Let $T\colon B \to A$ be a completely positive contraction that is approximately n-step inner relatively to a C^*-algebra E containing both A and B. Then for each finite subset F of B and for each $\varepsilon > 0$ there are elements $e_1, \dots, e_n$ in E such that*

$$\|T(b) - \sum_{j=1}^{n} e_j^* b e_j\| \le \varepsilon, \quad b \in F, \qquad \|\sum_{j=1}^{n} e_j^* e_j\| \le 1.$$

Proof: Take an n-step inner completely positive contraction S with $\|T(b) - S(b)\| \le \varepsilon/2$ for all b in F. Write $S(b) = \sum_{j=1}^n d_j^* b d_j$, where d_j belong to E. Find a positive contraction f in B such that $\|f^{1/2} b f^{1/2} - b\| \le \varepsilon/2$ for all b in F, and set $e_j = f^{1/2} d_j$. Then $\sum_{j=1}^n e_j^* e_j = S(f)$, and $\|S(f)\| \le \|f\| \le 1$. Also,

$$\sum_{j=1}^{n} e_j^* b e_j - T(b) = \big(S(f^{1/2} b f^{1/2}) - S(b)\big) + \big(S(b) - T(b)\big), \qquad b \in B,$$

and this proves the lemma. □

Lemma 7.3 *Let A and B be C^*-algebras with B separable, and let $T\colon B \to A$ be a completely positive contraction.*

(i) *Suppose that B is a sub-C^*-algebra of A_ω for some free filter ω on $\mathbb{N}$, and that T is approximately n-step inner relatively to A_ω. Then T is (exactly) n-step inner relatively to A_ω and T is approximately n-step inner relatively to A.*

(ii) *Suppose that B is a sub-C^*-algebra of $\mathcal{M}(A)$ and that T is approximately n step inner relatively to $\mathcal{M}(A)$. Then T is approximately n step inner relatively to A.*

Proof: (i). Assume that T is approximately n-step inner relatively to A_ω. Then for each natural number k there are contractions $d_{1,k}, \dots, d_{n,k}$ in A_ω such that

$$\lim_{k\to\infty} \sum_{j=1}^{n} d_{j,k}^* b d_{j,k} = T(b)$$

for all b in B. By (an obvious modification of) Lemma 2.5 we find contractions $d_1, \dots, d_n$ in A_ω satisfying $\sum_{j=1}^n d_j^* b d_j = T(b)$ for all b in B.

Write $d_j = \pi_\omega(d_j^{(1)}, d_j^{(2)}, \dots)$ where each $d_j^{(k)}$ is a contraction in A. Then

$$\lim_\omega \|\sum_{j=1}^n (d_j^{(k)})^* b d_j^{(k)} - T(b)\| = 0, \qquad b \in B.$$

For each finite subset F of B and for each $\varepsilon > 0$ we can therefore find a natural number k such that

$$\|\sum_{j=1}^n (d_j^{(k)})^* b d_j^{(k)} - T(b)\| \le \varepsilon, \qquad b \in F.$$

Hence T is approximately inner relatively to A.

(ii). Assume that T is approximately n-step inner relatively to $\mathcal{M}(A)$. Take a finite subset F of B and $\varepsilon > 0$. Find an n-step inner completely positive contraction $S\colon B \to \mathcal{M}(A)$ such that $\|T(b) - S(b)\| \le \varepsilon/2$ for all b in F. Write $S(b) = \sum_{j=1}^n d_j^* b d_j$ for suitable d_j in $\mathcal{M}(A)$. Find a positive contraction f in A with $\|f^{1/2}T(b)f^{1/2} - T(b)\| \le \varepsilon/2$ for all b in F, and put $e_j = d_j f^{1/2}$. Then each e_j belongs to A, and

$$\|\sum_{j=1}^n e_j^* b e_j - T(b)\| \le \|f^{1/2}\big(S(b) - T(b)\big)f^{1/2}\| + \|f^{1/2}T(b)f^{1/2} - T(b)\| \ \le \ \varepsilon$$

for all b in F. □

Part (i) of our next preliminary result describes (approximately) 1-step inner maps, and part (ii) is a related result that shall be used in Section 8.

Lemma 7.4 *Let A be a stable C^*-algebra.*

(i) *Let B be a separable sub-C^*-algebra of A, and let $V\colon B \to A$ be an approximately 1-step inner completely positive contraction. Then there is a sequence $\{t_n\}_{n=1}^\infty$ of isometries in $\mathcal{M}(A)$ such that $\|t_n^* b t_n - V(b)\| \to 0$ for all b in B.*

(ii) *If there is a sequence $\{d_n\}$ of elements in $M_2(A)$ such that*

$$\lim_{n\to\infty} \left\| d_n^* \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} d_n - \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right\| = 0 \qquad (7.1)$$

for all a in A, then there are isometries u_n, v_n in $\mathcal{M}(A)$ such that $u_n u_n^* + v_n v_n^* \le 1$

and

$$\lim_{n\to\infty} \Big\| \begin{pmatrix} u_n^* & 0 \\ v_n^* & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_n & v_n \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \Big\| = 0 \tag{7.2}$$

for all a in A.

Proof: Because A is stable, we can write $A = A_0 \otimes \mathcal{K}(\ell^2(\mathbb{N}))$. View $1 \otimes B(\ell^2(\mathbb{N}))$ as a sub-C^*-algebra of $\mathcal{M}(A)$. Choose an increasing approximate unit $\{e_n\}_{n=1}^\infty$ consisting of projections for $\mathcal{K}(\ell^2(\mathbb{N}))$. Then $\|a(1 - 1 \otimes e_n)\| \to 0$ for every a in A. Take isometries $s_{n,1}, s_{n,2}$ in $1 \otimes B(\ell^2(\mathbb{N}))$ satisfying $s_{n,1}s_{n,1}^* + s_{n,2}s_{n,2}^* = 1$ and $s_{n,2}s_{n,2}^* \leq 1 - 1 \otimes e_n$. Then $\|as_{n,2}s_{n,2}^*\| \to 0$ for all a in A.

(i). Use Lemma 7.2 to find a sequence $\{d_n\}_{n=1}^\infty$ of contractions in A such that $d_n^* b d_n \to V(b)$ for all b in B, and put

$$t_n = s_{n,1}s_{n,1}^* d_n + s_{n,2}(1 - d_n^* s_{n,1}s_{n,1}^* d_n)^{1/2}. \tag{7.3}$$

Then each t_n is an isometry, and $t_n^* b t_n \to V(b)$ as desired.

(ii). Observe first that $\mathcal{M}(M_2(A)) = M_2(\mathcal{M}(A))$. Let e in $M_2(\mathcal{M}(A))$ be $\mathrm{diag}(1,0)$. By Lemma 7.2 we can assume that each d_n is a contraction. Upon replacing d_n by ed_n we may also assume that $ed_n = d_n$. Let $s_{1,n}, s_{2,n}$ be as above with $s_{1,n}s_{1,n}^* + s_{2,n}s_{2,n}^* = e = s_{j,n}^* s_{j,n}$, and let t_n in $M_2(\mathcal{M}(A))$ be given as in (7.3). Then t_n is an isometry satisfying (7.1) (with t_n in the place of d_n), $t_n = et_n$, and hence

$$t_n = \begin{pmatrix} u_n & v_n \\ 0 & 0 \end{pmatrix}.$$

for some u_n, v_n that necessarily are isometries with $u_n u_n^*$ orthogonal to $v_n v_n^*$ because t_n is an isometry. □

A local Weyl–von Neumann theorem

The main result of this subsection is Theorem 7.21 which will be proved in a series of lemmas. Two of these lemmas (Lemma 7.7 and Lemma 7.12) will use the following:

Remark 7.5 (A construction with commuting elements) Let $a_1, a_2, \ldots, a_n$ be commuting, positive contractions in a strongly purely infinite C^*-algebra A, and let $\eta_1 > 0$, $\eta_2 > 0$, and $\eta_3 > 0$ be given.

Let X denote the primitive ideal spectrum of $D = C^*(a_1, a_2, \dots, a_n)$, so that D is isomorphic to $C_0(X)$. For each x in X let $\rho_x \colon D \to \mathbb{C}$ denote the corresponding character on D. Identifying X with the image of the (injective) map

$$x \mapsto (\rho_x(a_1), \rho_x(a_2), \dots, \rho_x(a_n)),$$

X becomes a bounded subset of $\mathbb{R}^n$. Put

$$\Omega = \{x \in X : \max\{a_1(x), \dots, a_r(x)\} \geq \eta_2\},$$

so that Ω is a compact subset of X. It is a standard fact from dimension theory that every open cover of $\Omega \subseteq \mathbb{R}^n$ has an open sub-cover such that each point in $\mathbb{R}^n$ belongs to at most $n+1$ of the open sets in the sub-cover. Use this to find open sets $U_1, \dots, U_r$ and $V_1, \dots, V_r$ such that

(i) $\Omega \subseteq V_1 \cup V_2 \cup \dots \cup V_r \subseteq U_1 \cup U_2 \cup \dots \cup U_r \subseteq X$, and $\overline{V}_j \subseteq U_j$,

(ii) $|\rho_x(a_k) - \rho_{x'}(a_k)| \leq \eta_2$ for all $j = 1, 2, \dots, r$, for all x, x' in U_j, and for all k,

(iii) for each x in X is contained in at most $n+1$ of the open sets $U_1, \dots, U_r$.

Choose x_j in V_j for each j. Let $g_1, \dots, g_r$ be positive contractions in D such that $g_1 + \dots + g_r \leq 1$, $\rho_x(g_1 + \dots + g_r) = 1$ for x in Ω, and $\rho_x(g_j) = 0$ when $x \notin V_j$. Choose next positive contractions $f_1, \dots, f_r$ in D such that $f_j g_j = g_j$ and $\rho_x(f_j) = 0$ when $x \notin U_j$. Then

$$\|c a_k c - \rho_{x_j}(a_k) c^2\| \leq \eta_2, \qquad \|a_k - \sum_{j=1}^{r} \rho_{x_j}(a_k) g_j\| \leq \eta_2 \tag{7.4}$$

for each contraction c in $\overline{f_j D f_j}$ and for $k = 1, \dots, n$. Put

$$f = \begin{pmatrix} f_1^{1/2} & \cdots & f_r^{1/2} \end{pmatrix} \in M_{1,r}(A), \qquad g = \begin{pmatrix} g_1^{1/2} \\ \vdots \\ g_r^{1/2} \end{pmatrix} \in M_{r,1}(A).$$

For each $\delta \geq 0$ define the map $S_\delta \colon D \to M_r(D)$ by

$$S_\delta(c) = \begin{pmatrix} \rho_{x_1}(c)(f_1 - \delta)_+ & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \rho_{x_r}(c)(f_r - \delta)_+ \end{pmatrix}$$

Applying Lemma 5.8 (with $\varepsilon_j = \eta_1$ and δ small enough) to the positive matrix

$$f^*f = \begin{pmatrix} f_1 & \cdots & f_1^{1/2} f_r^{1/2} \\ \vdots & & \vdots \\ f_r^{1/2} f_1^{1/2} & \cdots & f_r \end{pmatrix}$$

we obtain elements $d_1, d_2, \dots, d_r$ in A such that $d_j^* f_j d_j = (f_j - \eta_1)_+$, $\|d_j\|^2 \le 2/\eta_1$, and

$$\left\| \begin{pmatrix} d_1^* & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_r^* \end{pmatrix} f^*f \begin{pmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_r \end{pmatrix} - \begin{pmatrix} (f_1 - \eta_1)_+ & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (f_r - \eta_1)_+ \end{pmatrix} \right\| \le \eta_3. \tag{7.5}$$

Set $h_j = f_j^{1/2} d_j$, and set $h = (h_1, \dots, h_r)$ in $M_{1,r}(A)$, so that $h_j^* h_j = d_j^* f_j d_j = (f_j - \eta_1)_+$.

Lemma 7.6 *In the notation of Remark 7.5,*

$$\|h^* a_k h - S_{\eta_1}(a_k)\| \le 2(n+1)\eta_2 \eta_1^{-1} + \eta_3, \qquad \|g^* S_{2\eta_1}(a_k) g - a_k\| \le 2\eta_1 + \eta_2,$$

g is a contraction, and $\|f\|^2 \le n+1$.

Proof: Since $ff^* = \sum_{j=1}^n f_j \le n+1$ by condition (iii) in Remark 7.5, we have $\|f\|^2 \le n+1$.

For each $j = 1, \dots, r$ let p_j in A^{**} correspond to the indicator function 1_{U_j} so that $f_j p_j = f_j$ and $\|p_j a_k - \rho_{x_j}(a_k) p_j\| \le \eta_2$, cf. (7.4). Then, using (7.5), we get

$$\begin{aligned}
&\|h^* a_k h - S_{\eta_1}(a_k)\| \\
=\ & \left\| \begin{pmatrix} d_1^* & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_r^* \end{pmatrix} \left(f^* a_k f - f^* f \begin{pmatrix} \rho_{x_1}(a_k) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \rho_{x_r}(a_k) \end{pmatrix} \right) \begin{pmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_r \end{pmatrix} \right\| + \eta_3 \\
\le\ & 2\eta_1^{-1} \left\| f^* f \begin{pmatrix} a_k & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_k \end{pmatrix} - f^* f \begin{pmatrix} \rho_{x_1}(a_k) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \rho_{x_r}(a_k) \end{pmatrix} \right\| + \eta_3 \\
=\ & 2\eta_1^{-1} \left\| f^* f \begin{pmatrix} p_1 a_k & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & p_r a_k \end{pmatrix} - f^* f \begin{pmatrix} \rho_{x_1}(a_k) p_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \rho_{x_r}(a_k) p_r \end{pmatrix} \right\| + \eta_3 \\
\le\ & 2\eta_1^{-1} \|f^* f\| \eta_2 + \eta_3 \ \le\ 2(n+1)\eta_2 \eta_1^{-1} + \eta_3.
\end{aligned}$$

Next, $g^*g = \sum_{j=1}^r g_j$, so that g is a contraction, and

$$g^* S_{2\eta_1}(a_k) g = \sum_{j=1}^r \rho_{x_j}(a_k) g_j^{1/2}(f_j - 2\eta_1)_+ g_j^{1/2} = (1-2\eta_1)\sum_{j=1}^r \rho_{x_j}(a_k) g_j.$$

This together with the second identity in (7.4) yield $\|g^*S_{2\eta_1}(a_k)g - a_k\| \le 2\eta_1 + \eta_2$. $\square$

Lemma 7.7 *Let $a_1, a_2, \dots, a_n$ be commuting, positive elements in a strongly purely infinite C^*-algebra A. Then for each $\varepsilon > 0$ and for each m in $\mathbb{N}$ there exists b in $M_m(A)$ such that*

$$\Big\| b^* \begin{pmatrix} a_k & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} b - \begin{pmatrix} a_k & 0 & \cdots & 0 \\ 0 & a_k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_k \end{pmatrix} \Big\| \le \varepsilon.$$

for $k = 1, \dots, n$.

Proof: Upon replacing the elements $a_1, \dots, a_n$ with $ta_1, \dots, ta_n$ for some positive real number t we may assume that each a_k is a contraction.

Apply Remark 7.5 and Lemma 7.6 with

$$\eta_1 = \varepsilon/4, \qquad \eta_2 = \min\Big\{\frac{\varepsilon^3}{320(n+1)}, \frac{\varepsilon}{4}\Big\}, \qquad \eta_3 = \varepsilon/40.$$

Then by Lemma 7.6,

$$\|h^*a_kh - S_{\varepsilon/4}(a_k)\| \le \frac{\varepsilon^2}{20}, \qquad \|g^*S_{\varepsilon/2}(a_k)g - a_k\| \le \frac{3\varepsilon}{4}, \quad \text{for } k = 1, \dots, n. \tag{7.6}$$

By Lemma 2.4 (i) there are $t_1, \dots, t_r$ in $M_{m,1}(A)$ with

$$\|t_j\|^2 \le 5/\varepsilon, \qquad t_j^*(f_j - \varepsilon/4)_+ t_j = (f_j - \varepsilon/2)_+ \otimes 1_m.$$

Write $t_j = (t_{j,i})$ and put

$$t = \begin{pmatrix} t_{1,1} & \dots & 0 & t_{1,2} & \dots & 0 & \dots & t_{1,m} & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \ddots & \vdots \\ 0 & \dots & t_{r,1} & 0 & \dots & t_{r,2} & \dots & 0 & \dots & t_{r,m} \end{pmatrix} \in M_{r,rm}(A).$$

Then $\|t\|^2 \leq 5/\varepsilon$ and $t^* S_{\varepsilon/4}(c) t = S_{\varepsilon/2}(c) \otimes 1_m$ for all c in D, and hence

$$\|t^* h^* a_k h t - S_{\varepsilon/2}(a_k) \otimes 1_m\| \leq \varepsilon/4 \quad \text{for} \quad k = 1, \dots, n, \tag{7.7}$$

by the first inequality in (7.6). Put $b_0 = ht(g \otimes 1_m) \in M_{1,m}(A)$. By (7.7) and the second inequality in (7.6) we get the estimate:

$$\begin{aligned} \|b_0^* a_k b_0 - a_k \otimes 1_m\| &\leq \|(g^* \otimes 1_m)\big(t^* h^* a_k h t - (S_{\varepsilon/2}(a_k) \otimes 1_m)\big)(g \otimes 1_m)\| \\ &\quad + \|(g^* \otimes 1_m)(S_{\varepsilon/2}(a_k) \otimes 1_m)(g \otimes 1_m) - a_k \otimes 1_m\| \\ &\leq \|t^* h^* a_k h t - (S_{\varepsilon/2}(a_k) \otimes 1_m)\| + \|g^* S_{\varepsilon/2}(a_k) g - a_k\| \;\leq\; \varepsilon. \end{aligned}$$

Taking b in $M_m(A)$ to be the matrix whose first row is b_0 and whose other rows are zero, the lemma is proved. □

Proposition 7.8 *Let A be a strongly purely infinite C^*-algebra.*

(i) *Let $a_1, a_2, \dots, a_n$ be commuting, positive elements in A. For each natural number m and for each $\varepsilon > 0$ there is a* contraction *b such that*

$$\left\| b^* \begin{pmatrix} a_k & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} b - \begin{pmatrix} a_k & 0 & \cdots & 0 \\ 0 & a_k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_k \end{pmatrix} \right\| \leq \varepsilon$$

for $k = 1, \dots, n$.

(ii) *Let ω be a free filter on $\mathbb{N}$. Let $a_1, a_2, \dots, a_n$ be commuting, positive elements in A_ω, and let m be a natural number. Then there exists a contraction b in $M_m(A_\omega)$ such that*

$$b^* \begin{pmatrix} a_k & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} b = \begin{pmatrix} a_k & 0 & \cdots & 0 \\ 0 & a_k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_k \end{pmatrix}$$

for $k = 1, \dots, n$.

Proof: (i). As in the proof of Lemma 7.7 we may assume that each a_k is a contraction. Choose a positive contraction e in the Abelian C^*-algebra generated by $a_1, \dots, a_n$ with

$\|ea_ke - a_k\| \le \varepsilon/3$. Use Lemma 7.7 to find d in $M_{1,m}(A)$ satisfying $\|d^*cd - (c\otimes 1_m)\| \le \varepsilon/3$ for all c in the set $\{ea_1e, \dots, ea_ke, e^2\}$. Put $b_0 = (1+\varepsilon/3)^{-1/2}ed$. Then

$$b_0^*b_0 = (1+\varepsilon/3)^{-1}d^*e^2d \le 1,$$

so b_0 is a contraction. Also,

$$\begin{aligned}
\|b_0^*a_kb_0 - a_k\otimes 1_m\| &= \|(1+\varepsilon/3)^{-1}d^*ea_ked - a_k\otimes 1_m\| \\
&\le \|(1+\varepsilon/3)^{-1}d^*ea_ked - ea_ke\otimes 1_m\| + \varepsilon/3 \\
&\le (1+\varepsilon/3)^{-1}\|d^*ea_ked - ea_ke\otimes 1_m\| + \varepsilon/3 + \varepsilon/3 \ \le\ \varepsilon.
\end{aligned}$$

The element b in $M_m(A)$ whose first row is b_0 and whose other rows are zero will therefore be as desired.

(ii). This follows from (i) and from Lemma 2.5. $\square$

For any positive element a in a *purely infinite* C^*-algebra A, for any m in $\mathbb{N}$, and for every $\varepsilon > 0$, one can find d in $M_{1,m}(A)$ such that $\|d^*ad - a\otimes 1_m\| \le \varepsilon$. It is, however, not know to us if one always can choose d to be a *contraction*. One of the offsprings of Proposition 7.8 above is that d can be chosen to be a contraction in a *strongly purely infinite* C^*-algebra.

Lemma 7.9 *Let D be a C^*-algebra, let a be a positive element in D, and let d be a contraction in D. Then the following two conditions are equivalent:*

(i) *d commutes with a, and $d^*da = a$,*

(ii) *$d^*ad = a$ and $d^*a^2d = a^2$.*

Proof: (ii) $\Rightarrow$ (i). Set $x = ad - da$. Then

$$x^*x = d^*a^2d - d^*ada - ad^*ad + ad^*da = ad^*da - a^2 \le 0$$

(because d is a contraction). This shows that x must be zero. Hence d commutes with a, and $d^*da = d^*ad = a$. (i) $\Rightarrow$ (ii) is trivial. $\square$

Lemma 7.10 *Suppose that A is a strongly purely infinite C^*-algebra. Let ω be a free filter on $\mathbb{N}$, and let $a_1, a_2, \dots, a_n$ be a set of commuting normal elements in A_ω. Then for each natural number m there are contractions $r_1, r_2, \dots, r_m$ in $A_\omega \cap \{a_1, a_2, \dots, a_n\}'$ such that*

$$r_i^*r_j = \delta_{ij}r_1^*r_1, \qquad r_1^*r_1a_k = a_k, \quad \textit{for all } i,j,k.$$

Proof: We can without loss of generality assume that each a_j is positive. Use Lemma 2.5 to find a positive contraction e in A_ω such that $ea_k = a_k e = a_k$. Given m, take a contraction b in $M_m(A_\omega)$ as in Proposition 7.8 (ii) with respect to the set

$$\{a_1, a_2, \dots, a_n, a_1^2, a_2^2, \dots, a_n^2, e, e^2\}.$$

We can assume that b is a row matrix, i.e., that

$$b = \begin{pmatrix} b_1 & b_2 & \cdots & b_m \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then each b_j is a contraction, $b_i^* a_k b_j = \delta_{ij} a_k$, and $b_i^* a_k^2 b_j = \delta_{ij} a_k^2$.

We conclude from Lemma 7.9 that each b_i belongs to $A_\omega \cap \{a_1, a_2, \dots, a_n\}'$, and that $b_i^* b_j a_k = \delta_{ij} a_k$. Similarly we find that each b_i commutes with e and that $b_i^* b_i e = e$. Set $r_i = b_i e^{1/2}$. Then each r_i is a contraction in $A_\omega \cap \{a_1, a_2, \dots, a_n\}'$ and $r_i^* r_j = \delta_{ij} e$. This proves the lemma. □

Lemma 7.11 *Let D, E be C^*-algebras, and let a be a positive element in D.*

(i) *Let $S\colon D \to E$ be a completely positive contraction. If $\|S(a^2) - S(a)^2\| \le \varepsilon$, then $\|S(ab) - S(a)S(b)\| \le \varepsilon^{1/2}$ and $\|S(ba) - S(b)S(a)\| \le \varepsilon^{1/2}$ for all contraction b in D.*

(ii) *If $T\colon E \to D$ is a completely positive contraction and if $\|S(T(a)) - a\| \le \varepsilon$ and $\|S(T(a^2)) - a^2\| \le \varepsilon$, then $\|S(T(a)^2) - S(T(a))^2)\| \le (1 + 2\|a\|)\varepsilon$.*

Proof: (i). By Stinespring's theorem we can find a representation of E on a Hilbert space H, a *-homomorphism $\pi\colon D \to B(H)$, and a projection p on H such that $S(d) = p\pi(d)p$ (when viewing E as a sub-C^*-algebra of $B(H)$). Calculate

$$S(a^2) - S(a)^2 = p\pi(a)\pi(a)p - p\pi(a)p\pi(a)p = \big(p\pi(a)(1-p)\big)\big(p\pi(a)(1-p)\big)^*.$$

We deduce that $\|p\pi(a)(1-p)\| \le \varepsilon^{1/2}$, and hence that $\|p\pi(a) - \pi(a)p\| \le \varepsilon^{1/2}$.

This leads to the estimate

$$\begin{aligned} \|S(ab) - S(a)S(b)\| &= \|p\pi(ab)p - p\pi(a)p\pi(b)\| = \|p\big(p\pi(a) - \pi(a)p\big)\pi(b)p\| \\ &\le \|b\|\|\pi(a)p - p\pi(a)\| \le \|b\|\varepsilon^{1/2}. \end{aligned}$$

The other inequality follows by taking adjoints.

(ii). A completely positive contraction V between two C^*-algebras satisfies the inequality $V(x)^*V(x) \le V(x^*x)$. (This can for example be proved using Stinespring's theorem as in the proof of (i) above.) Using this fact twice together with the estimate $\|S(T(a))^2 - a^2\| \le 2\|a\|\varepsilon$ yields:

$$\begin{aligned} 0 \;&\le\; S(T(a)^2) - S(T(a))^2 \;\le\; S(T(a^2)) - S(T(a))^2 \\ &\le\; (a^2 + \varepsilon\cdot 1) - (a^2 - 2\|a\|\varepsilon\cdot 1) \;=\; (1 + 2\|a\|)\varepsilon. \end{aligned}$$

This proves (ii). □

Lemma 7.12 *Suppose that A is a strongly purely infinite C^*-algebra. Let ω be a free filter on $\mathbb{N}$, let $a_1, a_2, \dots, a_n$ be a set of commuting normal elements in A_ω, and let $b_1, b_2, \dots, b_m$ be any finite subset of A_ω. Then there exists a contraction d in A_ω with*

$$d^*da_k = a_k, \quad [d, a_k] = 0, \quad [d^*b_jd, a_k] = 0, \quad [d^*b_jd, d^*b_id] = 0$$

for all i, j, and k.

Proof: It is no loss of generality to assume that each a_k and each b_j is a positive contraction. Next, it suffices to prove the lemma in the case where $m = 1$. Indeed, for $m \ge 2$ use the case $m = 1$ to find a contraction d_1 satisfying

$$d_1^*d_1a_k = a_k, \qquad [d_1, a_k] = 0, \qquad [d_1^*b_1d, a_k] = 0.$$

Repeat the process to the $n+1$ commuting elements

$$\tilde{a}_1 = d_1^*a_1d_1, \quad \tilde{a}_2 = d_2^*a_2d_2, \quad \dots, \quad \tilde{a}_n = d_1^*a_nd_1, \quad \tilde{a}_{n+1} = d_1^*b_1d_1$$

and to $\tilde{b}_1 = d_1^*b_2d_1$. The lemma is proved after m such steps.

Assume accordingly that $m = 1$ and that $b = b_1$ is a positive contraction. By Lemma 7.9 it suffices to find a contraction d in A_ω such that $d^*a_kd = a_k$, $d^*a_k^2d = a_k^2$, and $[d^*bd, a_k] = 0$ for all k. To find d replace the set $\{a_1, \dots, a_n\}$ with the set $\{a_1, \dots, a_n, a_1^2, \dots, a_n^2\}$, show that for each $\varepsilon > 0$ there is a contraction d in A_ω such that

$$\|d^*a_kd - a_k\| \le \varepsilon, \qquad \|[d^*bd, a_k]\| \le \varepsilon, \tag{7.8}$$

and invoke Lemma 2.5 to complete the proof.

We apply Remark 7.5 and Lemma 7.6 with A_ω in the place of A, with $\eta_1 = \varepsilon/4$, with $\eta_2 > 0, \eta_3 > 0$ chosen such that the right-hand estimates in (7.9) and (7.10) hold, and with $\{a_1, \dots, a_n, a_1^2, \dots, a_n^2\}$ in the place of the set $\{a_1, \dots, a_n\}$. Hence n (from Remark 7.5) is replaced with $2n$.

In addition to the positive contractions $f_1, \dots, f_r$, $g_1, \dots g_r$, and the elements $h_1, \dots, h_r$ from Remark 7.5 we shall need positive contractions $k_1, \dots, k_r$ in D such that $f_j k_j = k_j$ and $k_j g_j = g_j$.

Observe that $h_j^* b h_j \le h_j^* h_j = (f_j - \eta_1)_+$ because b is a contraction. Apply Lemma 5.13 to the positive matrix

$$h^* b h + \begin{pmatrix} (f_1 - \eta_1)_+ - h_1^* b h_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (f_r - \eta_1)_+ - h_r^* b h_r \end{pmatrix}$$

to obtain positive elements $e_1, \dots, e_r$ in A_ω such that

$$\|e_j\|^2 \le 2/\eta_1, \qquad e_j^*(f_j - \eta_1)_+ e_j = (f_j - 2\eta_1)_+, \qquad e_i^* h_i^* b h_j e_j = 0 \quad \text{when} \quad i \ne j.$$

Put $d = \sum_{j=1}^r h_j e_j g_j^{1/2}$. Then

$$d^* d = \sum_{j=1}^r g_j^{1/2} (f_j - 2\eta_1)_+ g_j^{1/4} = (1 - 2\eta_1) \sum_{j=1}^r g_j,$$

and so d is a contraction. We proceed to verify (7.8).

Let g be as in Remark 7.5 and define $R\colon M_r(A_\omega) \to A_\omega$ and $T\colon A_\omega \to M_r(A_\omega)$ by $R(c) = g^* c g$ and

$$T(c) \;=\; \begin{pmatrix} k_1 e_1^* & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & k_r e_r^* \end{pmatrix} h^* c h \begin{pmatrix} e_1 k_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & e_r k_r \end{pmatrix}.$$

Use the relation $g_j^{1/2} k_j = g_j^{1/2}$ to see that $d^* b d = R(T(b))$. The (ij)th entry of $T(b)$ is given by

$$\big(T(b)\big)_{ij} = k_i e_i^* h_i^* b h_j e_j k_j = \begin{cases} k_j e_j^* h_j^* b h_j e_j k_j, & \text{if } i = j \\ 0, & \text{if } i \ne j \end{cases},$$

so that $S(a_k) T(b) = T(b) S(a_k)$ for all k, where $S = S_0\colon D \to M_r(D)$ is as in Remark 7.5.

Because $g_j^{1/2} f_j g_j^{1/2} = g_j$ we get $R(S(a_k^\nu)) = \sum_{j=1}^r \rho_{x_j}(a_k^\nu) g_j$. Hence $\|R(S(a_k^\nu)) - a_k^\nu\| \le \eta_2$ for all k and for $\nu = 1, 2$ by (7.4). Lemma 7.11 (ii) now yields

$$\|R(S(a_k)^2) - R(S(a_k))^2\| \le (1 + 2\|a_k\|)\eta_2 \le 3\eta_2,$$

whence

$$\begin{aligned} \|R\big(S(a_k)T(b)\big) - R(S(a_k))R(T(b))\| &\le \sqrt{3\eta_2}, \\ \|R\big(T(b)S(a_k)\big) - R(T(b))R(S(a_k))\| &\le \sqrt{3\eta_2}, \end{aligned}$$

by Lemma 7.11 (i). Recalling that $S(a_k)$ commutes with $T(b)$ and that $R(T(b)) = d^*bd$, this leads to the estimate $\|R(S(a_k))d^*bd - d^*bdR(S(a_k))\| \le 2\sqrt{3\eta_2}$, which in turns implies that

$$\|[d^*bd, a_k]\| \le 2\sqrt{3\eta_2} + 2\eta_2 \le \varepsilon. \tag{7.9}$$

With $S_\delta \colon D \to M_r(D)$ as in Remark 7.5, we have

$$\begin{aligned} d^* a_k d &= g^* \begin{pmatrix} e_1^* & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e_r^* \end{pmatrix} h^* a_k h \begin{pmatrix} e_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e_r \end{pmatrix} g, \\ g^* S_{2\eta_1}(a_k) g &= g^* \begin{pmatrix} e_1^* & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e_r^* \end{pmatrix} S_{\eta_1}(a_k) \begin{pmatrix} e_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e_r \end{pmatrix} g. \end{aligned}$$

From Lemma 7.6 we know that

$$\|h^* a_k h - S_{\eta_1}(a_k)\| \le 2(2n+1)\eta_2\eta_1^{-1} + \eta_3, \qquad \|g^* S_{2\eta_1}(a_k) g - a_k\| \le 2\eta_1 + \eta_2.$$

By the choice of η_2 and η_3, this shows that

$$\|d^* a_k d - a_k\| \le 2\eta_1^{-1}\big(2(2n+1)\eta_2\eta_1^{-1} + \eta_3\big) + 2\eta_1 + \eta_2 \le \varepsilon. \tag{7.10}$$

as desired. □

Proposition 7.13 (Extension) *Let A be a strongly purely infinite C^*-algebra, let ω be a free filter on $\mathbb{N}$, let B be a separable sub-C^*-algebra of A_ω, and let C be an Abelian sub-C^*-algebra of B. It follows that there is a 1-step inner completely positive contraction*

$T\colon B \to A_\omega$ such that the image of T is contained in an Abelian sub-C^*-algebra of A_ω and such that $T(c) = c$ for all c in C.

Proof: Choose countable subsets $\{b_1, b_2, \dots\}$ and $\{c_1, c_2, \dots\}$ of positive contractions in B, respectively, C, which span dense subspaces of B and C, respectively.

By Lemma 7.12 there is for each natural number n a contraction d_n in A_ω such that

$$d_n^* c_k d_n = c_k, \quad [d_n^* b_j d_n, c_k] = 0, \quad [d_n^* b_j d_n, d_n^* b_i d_n] = 0 \quad \text{for all } i, j, k = 1, 2, \dots, n. \tag{7.11}$$

Lemma 2.5 then shows that there is a contraction d in A_ω such that (7.11) holds for all i, j, and k in $\mathbb{N}$. Now, the completely positive contraction given by $T(b) = d^*bd$ has the desired properties. □

Proposition 7.14 *Let A be a strongly purely infinite C^*-algebra.*

(i) *Let ω be a free filter on $\mathbb{N}$, let B be a separable sub-C^*-algebra of A_ω, and let $V\colon B \to A_\omega$ be an approximately inner completely positive contraction whose image is contained in an Abelian sub-C^*-algebra of A_ω. It follows that V is 1-step inner relatively to A_ω.*

(ii) *Let B be a separable sub-C^*-algebra of $\mathcal{M}(A)$, the multiplier algebra of A, and let $V\colon B \to A$ be an approximately inner completely positive contraction whose image is contained in an Abelian sub-C^*-algebra of A. Then V is approximately 1-step inner relatively to A.*

Proof: We prove (i) and (ii) simultaneously. It suffices to show that if B is either a separable subalgebra of A_ω or of $\mathcal{M}(A)$, if $V\colon B \to A_\omega$ is approximately inner (relatively to A_ω), then V is 1-step inner relatively to A_ω. If $V\colon B \to A$ is 1-step inner relatively to A_ω, then V is approximately 1-step inner relatively to A by Lemma 7.3 (i).

Choose a set $\{b_1, b_2, b_3, \dots\}$ of positive elements that span a dense subset of B. By Lemma 2.5 it suffices, for each n and for each $\varepsilon > 0$, to find a contraction f in A_ω such that $\|V(b_k) - f^* b_k f\| \le \varepsilon$ for $k = 1, 2, \dots, n$. By assumption there is an m-step inner completely positive contraction $W\colon B \to A_\omega$ such that $\|V(b_k) - W(b_k)\| \le \varepsilon$ for all $k = 1, \dots, n$. Find $d_1, d_2, \dots, d_m$ in A_ω such that $W(b) = \sum_{i=1}^m d_j^* b d_j$.

By Lemma 7.12 there is a contraction d in A_ω such that each pair of elements in the finite set

$$\{d^* d_i^* b_k d_j d : i, j = 1, \dots, m,\ k = 1, \dots, n\}$$

commute, and such that $d^* V(b_k) d = V(b_k)$ for $k = 1, \dots, n$, cf. Lemma 7.9.

In particular, each $d^* d_i^* b_k d_j d$ is normal. Use Lemma 7.10 to find contractions $r_1, \dots, r_m$ in A_ω commuting with each $d^* d_i^* b_k d_j d$, and such that $r_j^* d^* d_j^* b_k d_i d r_i = \delta_{ij} d^* d_j^* b_k d_i d$. Put $f = \sum_{j=1}^m d_j d r_j$. Then

$$f^* b_k f \;=\; \sum_{i,j=1}^m r_j^* d^* d_j^* b_k d_i d r_i \;=\; \sum_{j=1}^m d^* d_j^* b_k d_j d \;=\; d^* W(b_k) d, \qquad k = 1, \dots, n.$$

As $\|W(b_k) - V(b_k)\| \le \varepsilon$, and d is a contraction satisfying $d^* V(b_k) d = V(b_k)$, we conclude that $\|V(b_k) - f^* b_k f\| \le \varepsilon$. $\square$

Definition 7.15 For each pair of C^*-algebras A and B define $\mathscr{C}_0(B, A)$ to be the set of all approximately inner completely positive maps from B to A whose image is contained in an Abelian sub-C^*-algebra of A. Define $\mathscr{C}(B, A)$ to be the set of all completely positive maps $T \colon B \to A$ such that

$$T(b) = \sum_{i,j=1}^n a_j^* T_0(c_j^* b c_i) a_i, \quad b \in B \tag{7.12}$$

for some natural number n, for some $a_1, \dots, a_n$ in $\mathcal{M}(A)$, and for some $c_1, \dots, c_n$ in $\mathcal{M}(B)$. Let $\overline{\mathscr{C}}(B, A)$ denote the pointwise–norm closure of $\mathscr{C}(B, A)$.

Let A be a C^*-algebra such that its multiplier algebra $\mathcal{M}(A)$ contains a unital copy of the Cuntz algebra $\mathcal{O}_2$. Each stable C^*-algebra has this property, and more generally, if D is stable and if ω is any filter on $\mathbb{N}$, the $A = D_\omega$ has this property. Indeed, $\mathcal{M}(D)$ contains $B(\ell^2)$ and therefore $\mathcal{O}_2$ as a unital sub-C^*-algebra, and there is a (canonical) unital embedding of $\mathcal{M}(D)$ into $\mathcal{M}(D_\omega)$. (The C^*-algebra D_ω need not be stable, even if D is stable.)

By assumption there are isometries s_1, s_2 in $\mathcal{M}(A)$ satisfying the Cuntz relation: $s_1 s_1^* + s_2 s_2^* = 1$. Fixing two such isometries s_1, s_2 we can for each C^*-algebra B define the Cuntz sum of two maps $T_1, T_2 \colon B \to A$ by

$$(T_1 \oplus T_2)(b) = s_1 T_1(b) s_1^* + s_2 T_2(b) s_2^*, \qquad b \in B. \tag{7.13}$$

The operation $\oplus$ depends on the choice of s_1, s_2, but only up to unitary equivalence; $\oplus$ is not associative, but $(T_1 \oplus T_2) \oplus T_3$ is unitarily equivalent to $T_1 \oplus (T_2 \oplus T_3)$. With this in mind, define inductively $T_1 \oplus T_2 \oplus \cdots \oplus T_n$ to be $(T_1 \oplus T_2) \oplus \cdots \oplus T_n$. Then

$$(T_1 \oplus T_2 \oplus \cdots \oplus T_n)(b) = s_{1,n} T_1(b) s_{1,n}^* + \cdots + s_{n,n} T_n(b) s_{n,n}^* \tag{7.14}$$

for some isometries $s_{1,n}, \dots, s_{n,n}$ in A_ω satisfying $1 = s_{1,n}s_{1,n}^* + \cdots + s_{n,n}s_{n,n}^*$.

Lemma 7.16 *Let A and B be C^*-algebras, and assume that there is a unital embedding of $\mathcal{O}_2$ into $\mathcal{M}(A)$. Then the following holds:*

(i) *If T_1, T_2 belong to $\mathscr{C}(B, A)$, then so does their Cuntz sum $T_1 \oplus T_2$.*

(ii) *If $T_1, \dots, T_n$ belong to $\mathscr{C}(B, A)$, if $a_1, \dots, a_n$ are elements in $\mathcal{M}(A)$, then $T\colon B \to A$, given by $T(b) = \sum_{j=1}^n a_j^* T_j(b) a_j$, belongs to $\mathscr{C}(B, A)$. In particular, $\mathscr{C}(B, A)$ is a cone.*

(iii) *If T belongs to $\mathscr{C}(B, A)$, if $a_1, \dots, a_n$ belong to $\mathcal{M}(A)$, $c_1, \dots, c_n$ belong to $\mathcal{M}(B)$, then $S\colon B \to A$, given by $S(b) = \sum_{i,j=1}^n a_j^* T(c_j^* b c_i) a_i$, belongs to $\mathscr{C}(B, A)$.*

Straightforward continuity considerations show that Lemma 7.16 holds with $\mathscr{C}(B, A)$ replaced by $\overline{\mathscr{C}}(B, A)$.

Proof: (i). Suppose first that T_1, T_2 belong to $\mathscr{C}_0(A, B)$. The image of T_j is then contained in an Abelian sub-C^*-algebra D_j of A. Now, $D = s_1 D_1 s_1^* + s_2 D_2 s_2^*$ is an Abelian sub-C^*-algebra of A which contains the image of $T_1 \oplus T_2$. Hence $T_1 \oplus T_2$ belongs to $\mathscr{C}_0(B, A)$. Assume now that T_1, T_2 belong to $\mathscr{C}(B, A)$. Then

$$T_k(b) = \sum_{i,j=1}^{n_k} a_{j,k}^* S_k(c_{j,k}^* b c_{i,k}) a_{i,k}$$

for suitable $a_{j,k}$ in $\mathcal{M}(A)$, $c_{j,k}$ in $\mathcal{M}(B)$, and with S_1, S_2 in $\mathscr{C}_0(B, A)$. We saw above that $S = S_1 \oplus S_2$ belongs to $\mathscr{C}_0(B, A)$. The Cuntz relation for the isometries s_1, s_2 implies that $s_k^* S(b) s_l = \delta_{k,l} S_k(b)$. Hence

$$\begin{aligned}
(T_1 \oplus T_2)(b) &= \sum_{k=1}^{2} \sum_{i,j=1}^{n_k} s_k a_{j,k}^* S_k(c_{j,k}^* b c_{i,k}) a_{i,k} s_k^* \\
&= \sum_{k,l=1}^{2} \sum_{i,j=1}^{n_k} s_k a_{j,k}^* s_k^* S(c_{j,k}^* b c_{i,l}) s_l a_{i,l} s_l^*,
\end{aligned}$$

and it follows from the latter expression that $T_1 \oplus T_2$ belongs to $\mathscr{C}(B, A)$.

(ii). Let $T_1, \dots, T_n$ be in $\mathscr{C}(B, A)$, let $a_1, \dots, a_n$ be elements in $\mathcal{M}(A)$, and assume that T given by $T(b) = \sum_{j=1}^n a_j^* T_j(b) a_j$ is a contraction. In the notation of (7.14), put $a = \sum_{j=1}^n s_{j,n} a_j \in \mathcal{M}(A)$. Then

$$T(b) = a^*(T_1 \oplus \cdots \oplus T_n)(b) a.$$

From (i) we know that $T_1 \oplus \cdots \oplus T_n$ belongs to $\mathscr{C}$. It is now clear from (7.12) that T belongs to $\mathscr{C}(B, A)$.

(iii). Let S, T be as in (iii). Find R in $\mathscr{C}_0(B, A)$, elements $e_1, \dots, e_m$ in $\mathcal{M}(A)$, and elements $f_1, \dots, f_m$ in $\mathcal{M}(B)$ such that

$$T(b) = \sum_{k,l=1}^{m} e_k^* R(f_k^* b f_l) e_l, \qquad b \in B.$$

Then

$$S(b) = \sum_{i,j=1}^{n} a_j^* T(c_j^* b c_i) a_i = \sum_{i,j=1}^{n} \sum_{k,l=1}^{m} a_j^* e_k^* R(f_k^* c_j^* b c_i f_l) e_l a_i.$$

This expression agrees with (7.12), and so S belongs to $\mathscr{C}(B, A)$. □

The following two (well-known) facts, formulated as a lemma, are needed for the proof of Lemma 7.18 below. If ρ is a positive functional on a C^*-algebra D and if d is an element in D, then $d^*\rho d$ denotes the positive functional on D given by $(d^*\rho d)(x) = \rho(d^* x d)$.

Lemma 7.17 *Let D be a C^*-algebra.*

(i) *Let $f_1, \dots, f_n$ be elements in the dual of D. Then there is a cyclic representation π of D on some Hilbert space H, say with cyclic vector ξ, and elements $c_1, \dots, c_n$ in $\pi(D)' \cap B(H)$ such that $f_j(d) = \langle \pi(d)\xi, c_j^*\xi \rangle$.*

(ii) *Suppose that $\mathscr{K}$ is a weak–* closed sub-cone of the cone of all positive linear functionals on D, and suppose that $d^*\rho d$ belongs to $\mathscr{K}$ for every d in D and for every ρ in $\mathscr{K}$. Let J be the set of all d in D such that $\rho(d^*d) = 0$ for all ρ in $\mathscr{K}$. Then J is a closed two-sided ideal in D; and if ρ_0 is a positive functional on D such that $\rho_0(d^*d) = 0$ for all d in J, then ρ_0 belongs to $\mathscr{K}$.*

Proof: (i). Since each state on D is a linear combination of positive functionals on D it suffices to consider the case where each f_j is a positive. Put $f = f_1 + \cdots + f_n$, and let (π, H, ξ) be the GNS-representation with respect to f, so that $f(d) = \langle \pi(d)\xi, \xi \rangle$. Now f and each f_j extend to positive functionals g, respectively, g_j on the weak closure of $\pi(D)$, i.e., $g(\pi(d)) = f(d)$ and $g_j(\pi(d)) = f_j(d)$, and $g_j \le g$. By (a linear version of) Sakai's Radon–Nikodỳm theorem (see [12, Proposition 7.3.5]) there are elements c_j in $B(H) \cap \pi(D)'$ such that $g_j(d) = g(dc_j)$ for all d in the weak closure of $\pi(D)$. It follows that

$$f_j(d) = g_j(\pi(d)) = g(\pi(d)c_j) = \langle \pi(d)c_j\xi, \xi \rangle = \langle \pi(d)\xi, c_j^*\xi \rangle, \qquad d \in D,$$

as desired.

(ii). The set J is a closed left-ideal of D (because the left-kernel of each ρ in $\mathscr{K}$ is a closed left-ideal). Also, for each z in J and each d in D we have $\rho((zd)^*(zd)) = (d^*\rho d)(z^*z) = 0$ for each ρ in $\mathscr{K}$ because $d^*\rho d$ belongs to $\mathscr{K}$. This proves that zd belongs to J, and hence that J is a closed two-sided ideal in D.

To prove the second part of (ii) we may assume that $J = 0$. (Otherwise just replace A by A/J and view ρ_0 and each functional in $\mathscr{K}$ as a functional on A/J.) Suppose that ρ_0 does not belong to $\mathscr{K}$. Then, by Hahn–Banach's theorem and the characterization of weak-* continuous linear mappings on the dual space of A as being evaluation maps $\rho \mapsto \rho(a)$ for some a in A, there is a an element a in A such that $\rho_0(a) < 0$ and $\rho(a) \geq 0$ for all ρ in $\mathscr{K}$. Upon replacing a by $(a+a^*)/2$ we may assume that a is self-adjoint. Write $a = a_+ - a_-$ where a_+ and a_- are the positive and negative parts of a. Let $\{e_n\}$ be an approximate unit consisting of positive contractions for the C^*-algebra generated by a_-. Then $e_n a e_n \to -a_-$ and $e_n a e_n \leq 0$ for all n. For each ρ in $\mathscr{K}$, $e_n^* \rho e_n$ belongs to $\mathscr{K}$, and hence

$$0 \leq (e_n^* \rho e_n)(a) = \rho(e_n a e_n) \leq 0,$$

so that $\rho(-e_n a e_n) = 0$ for ρ in $\mathscr{K}$ and for all n. Since $-e_n a e_n$ is positive, this entails that $e_n a e_n = 0$ for all n, and hence that $a_- = 0$. But this contradict the fact that $\rho_0(a) < 0$. $\square$

Lemma 7.18 *Let B be a separable nuclear sub-C^*-algebra of a stable C^*-algebra A, and suppose that for each positive b in B there is V in $\overline{\mathscr{C}}(B, A)$ such that $V(b) = b$. Then the inclusion mapping $\iota \colon B \to A$ belongs to $\overline{\mathscr{C}}(B, A)$.*

Proof: For each finite set $b_1, \dots, b_n$ in B and for each $\varepsilon > 0$ we must find V in $\overline{\mathscr{C}}(B, A)$ such that $\|V(b_j) - b_j\| \leq \varepsilon$ for all j. Equivalently, we must show that

$$(b_1, \dots, b_n) \in \overline{\{\big(V(b_1), \dots, V(b_n)\big) : V \in \mathscr{C}(B, A)\}}. \tag{7.15}$$

By a Hahn–Banach separation argument, and because the set displayed above is convex, cf. Lemma 7.16 (ii), it will suffice to show the following: For each finite set $b_1, \dots, b_n$ in B, for each $f_1, \dots, f_n$ in A^*, and for each $\varepsilon > 0$ there is V in $\mathscr{C}(B, A)$ such $|f_j(b_j) - f_j(V(b_j))| \leq \varepsilon$ for all j. Choose a cyclic representation $\pi \colon A \to B(H)$, a cyclic vector ξ in H, and elements $c_1, \dots, c_n$ in $\pi(A)' \cap B(H)$ such that $f_j(b) = \langle \pi(b)\xi, c_j^* \xi \rangle$ for all b in B and for all j, cf. Lemma 7.17 (i). Let C be sub-C^*-algebra of $B(H) \cap \pi(A)'$ generated by $c_1, \dots, c_n$. Keeping

π and ξ fixed, let φ_V be the positive functional on $B \otimes C$ defined by

$$\varphi_V(b \otimes c) = \big\langle \pi(V(b))\xi, c^*\xi \big\rangle, \qquad b \in B, \;\; c \in C. \tag{7.16}$$

(A priori, φ_V defines a functional on the maximal tensor product $B \otimes_{\max} C$, but the maximal and the minimal tensor products on $B \odot C$ coincide because B is nuclear.) Let $\mathscr{K}$ be the weak–* closure of the cone $\{\varphi_V : V \in \overline{\mathscr{C}}(B, A)\}$. Observe that

$$|f_j(b_j) - f_j(V(b_j))| = \big|\big\langle \pi(b_j)\xi, c_j^*\xi \big\rangle - \big\langle \pi(V(b_j))\xi, c_j^*\xi \big\rangle\big| = |\varphi_\iota(b_j \otimes c_j) - \varphi_V(b_j \otimes c_j)|.$$

Hence it will suffice to show that φ_ι belongs to $\mathscr{K}$.

We proceed to check that $d^*\rho d$ belongs to $\mathscr{K}$ for all ρ in $\mathscr{K}$ and for all d in $B \otimes C$. By continuity of the maps $\rho \mapsto d^*\rho d$ and $d \mapsto d^*\rho d$ (with d, respectively, ρ fixed) it suffices to show that $d^*\varphi_V d$ belongs to $\mathscr{K}$ for $d = \sum_{j=1}^n x_j \otimes y_j$ in the algebraic tensor product and for V in $\mathscr{C}(B, A)$. Now,

$$\begin{aligned} (d^*\varphi_V d)(b \otimes c) &= \sum_{i,j=1}^n \varphi_V(x_i^* b x_j \otimes y_i^* c y_j) = \sum_{i,j=1}^n \big\langle \pi(V(x_i^* b x_j))\xi, (y_i^* c y_j)^*\xi \big\rangle \\ &= \sum_{i,j=1}^n \big\langle \pi(V(x_i^* b x_j)) y_j \xi, c^* y_i \xi \big\rangle. \end{aligned}$$

The vectors $y_j\xi$ can be approximated arbitrarily well by vectors of the form $\pi(a_j)\xi$ for suitable elements a_j in A. Let $W \colon B \to A$ be given by $W(b) = \sum_{i,j=1}^n a_i^* V(x_i^* b x_j) a_j$. Then W belongs to $\mathscr{C}(B, A)$ by Lemma 7.16 (iii), and

$$\varphi_W(b \otimes c) = \sum_{i,j=1}^n \big\langle \pi(V(x_i^* b x_j))\pi(a_j)\xi, c\pi(a_i)\xi \big\rangle.$$

We conclude that $d^*\varphi_V d$ belongs to the weak-* closure of the set of elements of the form φ_W with W in $\mathscr{C}(B, A)$.

As in Lemma 7.17 (ii), let J be the closed two-sided ideal in $B \otimes C$ consisting of those elements z such that $\varphi_V(z^*z) = 0$ for all V in $\overline{\mathscr{C}}(B, A)$. By Lemma 7.17 (ii) the present lemma is proved if we can show that $\varphi_\iota(z^*z) = 0$ for all z in J. It is a consequence of a theorem of Blackadar, [1, Theorem 3.3], see also [14, Proposition 2.13], that because B is nuclear, then J is the closed linear span of the set of elementary tensors $x \otimes y$ in J. The left kernel L of φ_ι, consisting of all z in $B \otimes C$ such that $\varphi_\iota(z^*z) = 0$, is a closed linear subspace of $B \otimes C$, and so it suffices to show that $\varphi_\iota(x^*x \otimes y^*y) = 0$ whenever $x \in B$ and

$y \in C$ are such that $x \otimes y$ belongs to J. Now, if $x \otimes y$ belongs to J, then $\pi(x)$ vanishes on the range of y. Indeed, by assumption there is V_x in $\overline{\mathscr{C}}(B, A)$ such that $V_x(x^*x) = x^*x$, and by Lemma 7.16 (ii) there is for each a in A an element $V_{x,a}$ in $\overline{\mathscr{C}}(B, A)$ such that $V_{x,a}(x^*x) = a^*x^*xa$. Hence

$$0 = \varphi_{V_{x,a}}(x^*x \otimes y^*y) = \big\langle \pi(a^*x^*xa)\xi, y^*y\xi \big\rangle = \|\pi(x)y\pi(a)\xi\|^2.$$

Since ξ is cyclic for the representation A, we conclude that $\pi(x)y\eta = 0$ for all η in H. It follows that

$$\varphi_\iota(x^*x \otimes y^*y) = \big\langle \pi(x^*x)\xi, y^*y\xi \big\rangle = \big\langle \pi(x)y\xi, \pi(x)y\xi \big\rangle = 0,$$

as desired. □

Lemma 7.19 *Let B be a sub-C^*-algebra of a stable C^*-algebra A and assume that the inclusion mapping $B \to A$ belongs to $\overline{\mathscr{C}}(B, A)$. Then $\overline{\mathscr{C}}(B, A)$ contains every approximately inner completely positive map from B to A.*

Proof: Because $\overline{\mathscr{C}}(B, A)$ is closed in the pointwise–norm topology, it suffices to show that each inner completely positive contraction $T\colon B \to A$ belongs to $\overline{\mathscr{C}}(B, A)$. Let $\iota\colon B \to A$ denote the inclusion mapping (that belongs to $\overline{\mathscr{C}}(B, A)$), and find $d_1, \dots, d_n$ in A such that

$$T(b) = \sum_{j=1}^{n} d_j^* b d_j = \sum_{j=1}^{n} d_j^* \iota(b) d_j.$$

It now follows from Lemma 7.16 (ii) (and the remark below Lemma 7.16) that T belongs to $\overline{\mathscr{C}}(B, A)$. □

Lemma 7.20 *Let A be a C^*-algebra, let B be a sub-C^*-algebra of $\mathcal{M}(A)$, and assume that every map in $\mathscr{C}_0(B, A)$ is approximately 1-step inner. Then every map in $\overline{\mathscr{C}}(B, A)$ is approximately 1-step inner.*

Proof: By the definition of being approximately 1-step inner, it suffices to show that every map in $\mathscr{C}(B, A)$ is approximately 1-step inner. Take T in $\mathscr{C}(B, A)$ and find S in $\mathscr{C}_0(B, A)$, $a_1, \dots, a_n$ in $\mathcal{M}(A)$ and $c_1, \dots, c_n$ in $\mathcal{M}(B)$ such that (7.12) holds. Put

$$C = \max\{\|a_1\|, \|a_2\|, \dots, \|a_n\|\}.$$

Let F be a finite subset of B and let $\varepsilon > 0$. Choose a positive contraction f in B such that $\|fbf - b\| \le \varepsilon/2$ for all b in F. By the assumption that S is approximately 1-step inner, there is d in $\mathcal{M}(A)$ such that

$$\|S(c_j^* fbfc_i) - d^* c_j^* fbfc_i d\| \leq \varepsilon/(2C^2 n^2), \qquad b \in F, \quad i,j = 1, \dots n.$$

Put $e = \sum_{i=1}^n fc_i d a_i \in \mathcal{M}(A)$. Then

$$\begin{aligned} \|T(b) - e^* be\| &\leq \|T(b) - T(fbf)\| + \|T(fbf) - e^* be\| \\ &\leq \varepsilon/2 + \sum_{i,j=1}^n \|a_j^* S(c_j^* fbfc_i) a_i - a_j^* d^* c_j^* fbfc_i d a_i\| \\ &\leq \varepsilon/2 + \sum_{i,j=1}^n \|a_j\| \|a_i\| \varepsilon (2C^2 n^2)^{-1} \ \leq \ \varepsilon. \end{aligned}$$

This shows that T is approximately 1-step inner. □

Theorem 7.21 *Let A be a strongly purely infinite C^*-algebra and let B be a nuclear, separable sub-C^*-algebra of A. Then each approximately inner completely positive map from B to A is approximately 1-step inner.*

Proof: It suffices to prove the theorem in the case where A is stable. Indeed, we may view any C^*-algebra A as being a sub-C^*-algebra of $A \otimes \mathcal{K}$. If $V \colon B \to A$ is approximately inner, then clearly so is $V \colon B \to A \otimes \mathcal{K}$. Next, if $\{d_n\}_{n=1}^\infty$ is a sequence of elements in $A \otimes \mathcal{K}$ satisfying $d_n^* b d_n \to V(b)$, then we will also have $g_n^* b g_n \to V(b)$, when $g_n = e_n d_n e_n \in A$ for a suitable approximate unit $\{e_n\}_{n=1}^\infty$ for A.

Let ω be a free filter on $\mathbb{N}$, and view A as a sub-C^*-algebra of the limit algebra A_ω. By Lemma 7.3 (i) it suffices to show that each approximately inner completely positive map from B to A_ω is approximately 1-step inner. The inclusion mapping $\iota \colon B \to A_\omega$ belongs to $\overline{\mathscr{C}}(B, A_\omega)$ by Lemma 7.18 and Proposition 7.13, and $\overline{\mathscr{C}}(B, A_\omega)$ therefore contains all approximately inner completely positive maps by Lemma 7.19. Proposition 7.14 (i) says that each map in $\mathscr{C}_0(B, A_\omega)$ is 1-step inner. Lemma 7.20 then shows that each map in $\overline{\mathscr{C}}(B, A_\omega)$, and hence each approximately inner completely positive contraction from B to A_ω, is approximately 1-step inner. □

Other applications

In the definition of strongly purely infinite there is no assumptions made on the norm of the elements d_1, d_2. A norm estimate for d_1 and d_2 was proved in Lemma 5.6. Now we can show that d_1 and d_2 can be chosen to be contractions:

Corollary 7.22 *Let A be a strongly purely infinite C^*-algebra. For each*

$$\begin{pmatrix} a & x^* \\ x & b \end{pmatrix} \in M_2(A)^+$$

and for each $\varepsilon > 0$ there are contractions *d_1, d_2 in A such that*

$$\Big\| \begin{pmatrix} d_1^* & 0 \\ 0 & d_2^* \end{pmatrix} \begin{pmatrix} a & x^* \\ x & b \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \Big\| \leq \varepsilon.$$

If $A = B_\omega$ for some C^*-algebra B and for some free filter ω on $\mathbb{N}$, then there are contractions d_1, d_2 in A such that

$$\begin{pmatrix} d_1^* & 0 \\ 0 & d_2^* \end{pmatrix} \begin{pmatrix} a & x^* \\ x & b \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

Proof: The second statement (concerning the case $A = B_\omega$) follows from the first statement and from Lemma 2.5. We proceed to prove the first statement of the corollary. Let $\left(\begin{smallmatrix} a & x^* \\ x & b \end{smallmatrix}\right)$ in $M_2(A)^+$ and let $\varepsilon > 0$ be given. Choose positive contractions e in $C^*(a)$ and f in $C^*(b)$ such that $\|a - eae\| < \varepsilon/2$ and $\|b - fbf\| < \varepsilon/2$. Put $B_0 = C^*(a, b, x)$, put $B = M_2(B_0)$, and put

$$C = \Big\{ \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} : c_1 \in C^*(a), \quad c_2 \in C^*(b) \Big\}.$$

Then C is an Abelian sub-C^*-algebra of the separable C^*-algebra B. Let ω be a free ultrafilter on $\mathbb{N}$. Apply Proposition 7.13 (and its proof) to find a contraction d in B_ω and an Abelian sub-C^*-algebra D of B_ω such that d^*bd belongs to D for all b in B and $d^*cd = c$ for all c in C. Write

$$d = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \in B_\omega = M_2((B_0)_\omega).$$

For each c_1 in $C^*(a)$ and for each c_2 in $C^*(b)$ we have

$$\begin{aligned} \begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} &= d^* \begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} d = \begin{pmatrix} d_{11}^* c_1 d_{11} & d_{11}^* c_1 d_{12} \\ d_{12}^* c_1 d_{11} & d_{12}^* c_1 d_{12} \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix} &= d^* \begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix} d = \begin{pmatrix} d_{21}^* c_2 d_{21} & d_{21}^* c_2 d_{22} \\ d_{22}^* c_2 d_{21} & d_{22}^* c_2 d_{22} \end{pmatrix}. \end{aligned}$$

We conclude that

$$d_{11}^* c_1 d_{11} = c_1, \qquad d_{22}^* c_2 d_{22} = c_2, \qquad e^{1/2} d_{12} = 0, \qquad f^{1/2} d_{21} = 0, \tag{7.17}$$

for all n. Using this, we get

$$d^* \begin{pmatrix} 0 & 0 \\ f^{1/2} x e^{1/2} & 0 \end{pmatrix} d = \begin{pmatrix} 0 & 0 \\ d_{22}^* f^{1/2} x e^{1/2} d_{11}^* & 0 \end{pmatrix},$$

and this element commutes with $\mathrm{diag}(e^{1/2}, 0)$ and with $\mathrm{diag}(0, f^{1/2})$. Hence

$$0 = d_{22}^* f^{1/2} x e^{1/2} d_{11}^* e^{1/2} = f^{1/2} d_{22}^* f^{1/2} x e^{1/2} d_{11}^*. \tag{7.18}$$

Put $s = e^{1/2} d_{11} e^{1/2}$ and $t = f^{1/2} d_{22} f^{1/2}$. Then s and t are contractions, $t^* x s = 0$ by (7.18), and

$$s^* a s = e^{1/2} d_{11}^* \big(e^{1/2} a e^{1/2}\big) d_{11} e^{1/2} = eae, \qquad t^* b t = f^{1/2} d_{22}^* \big(f^{1/2} b f^{1/2}\big) d_{22} f^{1/2} = fbf$$

by (7.17). Write

$$s = \pi_\omega(s_1, s_2, \dots), \qquad t = \pi_\omega(t_1, t_2, \dots),$$

where s_n and t_n are contractions in $B_0 \subseteq A$. Then

$$\lim_\omega \|t_n^* x s_n\| = 0, \qquad \lim_\omega \|s_n^* a s_n - a\| < \varepsilon/2, \qquad \lim_\omega \|t_n^* b t_n - b\| < \varepsilon/2.$$

We can therefore find n such that $\|t_n^* x s_n\| \leq \varepsilon/2$, $\|s_n^* a s_n - a\| \leq \varepsilon/2$, and $\|t_n^* b t_n - b\| \leq \varepsilon/2$, and we can take $d_1 = s_n$ and $d_2 = t_n$. □

8 Tensor products with $\mathcal{O}_\infty$

Corollary 8.3 below gives a McDuff type description of when a C^*-algebra A satisfies $A \cong A \otimes \mathcal{O}_\infty$. This corollary is proved in [14] by the first named author. For the convenience of the reader, we give here the proof in a version taken almost vertabim from [19, Chapter 7] (a work in progress at the time where this is written).

Proposition 8.1 (An approximate intertwining) *Let A and B be separable C^*-algebras, and let $\varphi\colon A \to B$ be an injective *-homomorphism. Suppose that there is a sequence*

$\{v_n\}_{n=1}^\infty$ of unitaries in $\mathcal{M}(B)$ such that

$$\lim_{n\to\infty} \|v_n\varphi(a) - \varphi(a)v_n\| = 0, \qquad \lim_{n\to\infty} \operatorname{dist}(v_n^* b v_n, \varphi(A)) = 0,$$

for all a in A and all b in B. Then A and B are isomorphic, and there is an isomorphism $\psi\colon A \to B$ which is approximately unitarily equivalent to φ.

Proof: Let $\{a_1, a_2, a_3, \dots\}$ and $\{b_1, b_2, b_3, \dots\}$ be (countable) dense subsets of A, respectively, of B. Put $v_0 = 1$, and find next, inductively, unitaries v_n in $\mathcal{M}(B)$ and elements $a_{j,n}$ in A such that

$$\|v_n^*(v_{n-1}^* \cdots v_1^* v_0^* b_j v_0 v_1 \cdots v_{n-1})v_n - \varphi(a_{n,j})\| \le 1/n,$$
$$\|v_n\varphi(a_j) - \varphi(a_j)v_n\| \le 2^{-n}, \qquad \|v_n\varphi(a_{m,j}) - \varphi(a_{m,j})v_n\| \le 2^{-n},$$

for $j = 1, 2\dots, n$ and $m = 1, 2, \dots, n-1$. Being a Cauchy sequence the limit

$$\psi(a) = \lim_{n\to\infty} v_1 v_2 \cdots v_n \varphi(a) v_n^* \cdots v_2^* v_1^*$$

exists for all a in $\{a_1, a_2, a_3, \dots\}$, and hence for all a in A; and $\psi\colon A \to B$ is a *-homomorphism. Clearly ψ is injective, because $\|\psi(a_j)\| = \|a_j\|$ for all j, and approximately unitarily equivalent to φ. Observe that

$$\|\psi(a_{n,j}) - v_1 v_2 \cdots v_n \varphi(a_{n,j}) v_n^* \cdots v_2^* v_1^*\| < 2^{-n},$$

and use this to deduce

$$\|b_j - \psi(a_{n,j})\| \le 2^{-n} + \|v_n^* v_{n-1}^* \cdots v_1^* v_0^* b_j v_0 v_1 \cdots v_{n-1} v_n - \varphi(a_{n,j})\| \le 2^{-n} + 1/n.$$

Since $\psi(A)$ is closed and $\{b_1, b_2, b_3, \dots\}$ is dense in B we conclude that $\psi(A) = B$. $\square$

Theorem 8.2 ([14, Corollary 10.8]) *Let A be a separable C^*-algebra and let B be a unital and separable C^*-algebra. Then A is isomorphic to $A \otimes B$ if*

(i) *there is a sequence $\{\varphi_n\}_{n=1}^\infty$ of unital injective *-homomorphisms from B into $\mathcal{M}(A)$ satisfying $\|\varphi_n(x)a - a\varphi_n(x)\| \to 0$ for all a in A and all b in B, and*

(ii) *the two *-homomorphisms $\alpha, \beta\colon B \to B \otimes B$ given by $\alpha(b) = b \otimes 1$ and $\beta(b) = 1 \otimes b$, $b \in B$, are approximately unitarily equivalent.*

It is shown in [15], based on ideas from the paper [7] by Effros, that a C^*-algebra B that satisfies (ii) must necessarily be simple and nuclear.

Proof: We shall show that conditions of Proposition 8.1 are satisfied with respect to the injective *-homomorphism $\varphi\colon A \to A \otimes B$ given by $a \mapsto a \otimes 1_B$. More specifically we shall for each finite subset F of A, for each finite subset G of B, and for each $\varepsilon > 0$ find a unitary v in $\mathcal{M}(A \otimes B)$ such that

$$\|v\varphi(a) - \varphi(a)v\| \leq \varepsilon, \qquad \operatorname{dist}(v^*(a \otimes b)v, \varphi(A)) \leq \varepsilon, \qquad a \in F,\ b \in G. \tag{8.1}$$

Let ω be a free filter on $\mathbb{N}$. Let $\alpha_n\colon B \to \mathcal{M}(A \otimes B)$ be given by $\alpha_n(b) = \varphi_n(b) \otimes 1$, and define $\alpha\colon B \to \mathcal{M}(A \otimes B)_\omega$ to be $\alpha(b) = \pi_\omega(\alpha_1(b), \alpha_2(b), \dots)$. By the assumption on the *-homomorphisms φ_n the image of α commutes with the image of φ (when viewing $\mathcal{M}(A \otimes B)$ as a sub-C^*-algebra of $\mathcal{M}(A \otimes B)_\omega$). Let $\beta\colon B \to \mathcal{M}(A \otimes B) \subseteq \mathcal{M}(A \otimes B)_\omega$ be given by $\beta(b) = 1 \otimes b$. The images of α and β commute with each other and with the image of φ. The C^*-algebra generated by $\alpha(B)$ and $\beta(B)$ is isomorphic to $B \otimes B$ (because B is nuclear and simple). By (ii) there is a unitary w in $C^*(\alpha(B), \beta(B))$ with $\|w^*\beta(b)w - \alpha(b)\| \leq \varepsilon/(2C)$ for all b in G, where $C = \max\{\|a\| : a \in F\}$. Since w commutes with the image of φ we have

$$w^*(a \otimes b)w = w^*\varphi(a)\beta(b)w = \varphi(a)w^*\beta(b)w, \qquad a \in A,\ b \in B.$$

Write $w = \pi_\omega(w_1, w_2, \dots)$ where each w_n is a unitary in $\mathcal{M}(A \otimes B)$. Then

$$\lim_\omega \|w_n\varphi(a) - \varphi(a)w_n\| = 0, \qquad \lim_\omega \|w_n^*(a \otimes b)w_n - \varphi(a)w_n^*\beta(b)w_n\| = 0,$$
$$\lim_\omega \|w_n^*\beta(b)w_n - \alpha_n(b)\| = \|w^*\beta(b)w - \alpha(b)\| \leq \varepsilon/(2C).$$

for all a in A and all b in B. It follows that

$$\|w_n\varphi(a) - \varphi(a)w_n\| \leq \varepsilon, \qquad \|w_n^*(a \otimes b)w_n - \varphi(a)\alpha_n(b)\| \leq \varepsilon, \qquad a \in F,\ \in G,$$

for all $n \in X$ for some $X \in \omega$, and hence for at least one n. The element $\varphi(a)\alpha_n(b)$ belongs to $A \otimes 1 = \varphi(A)$ and therefore (8.1) is satisfied with $v = w_n$. $\square$

Corollary 8.3 *Let A be a separable C^*-algebra. Then A is isomorphic to $A \otimes \mathcal{O}_\infty$ if and only if there is a sequence of unital *-homomorphisms $\varphi_n\colon \mathcal{O}_\infty \to \mathcal{M}(A)$ such that $\|\varphi_n(x)a - a\varphi_n(x)\| \to 0$ for all x in $\mathcal{O}_\infty$ and for all a in A.*

Proof: The "if" part follows immediately from Theorem 8.2 together with the theorem of Lin and Phillips, [17], that any pair of unital *-homomorphisms $\mathcal{O}_\infty \to \mathcal{O}_\infty \otimes \mathcal{O}_\infty$ are approximately unitarily equivalent.

It is shown in Lin and Phillips paper, [17], that $\mathcal{O}_\infty$ is isomorphic to $\otimes_{n=1}^\infty \mathcal{O}_\infty$. As a consequence of this isomorphism we get a sequence $\psi_n \colon \mathcal{O}_\infty \to \mathcal{O}_\infty$ of unital *-homomorphisms satisfying $\|\psi_n(x)y - y\psi_n(x)\| \to 0$ for all x, y in $\mathcal{O}_\infty$. Let A be any C^*-algebra. The C^*-algebra $\mathcal{M}(A) \otimes \mathcal{O}_\infty$ is a sub-C^*-algebra of $\mathcal{M}(A \otimes \mathcal{O}_\infty)$, and so $\varphi_n(x) = 1 \otimes \psi_n(x)$ defines a sequence of *-homomorphisms from $\mathcal{O}_\infty$ into $\mathcal{M}(A \otimes \mathcal{O}_\infty)$ which satisfies $\|\varphi_n(x)a - a\varphi_n(x)\| \to 0$ for all x in $\mathcal{O}_\infty$ and for all a in $A \otimes \mathcal{O}_\infty$. $\square$

It follows easily from Corollary 8.3 that any C^*-algebra that absorbs $\mathcal{O}_\infty$ is approximately divisible.

Proposition 8.4 *Let A be a separable C^*-algebra which is either unital or stable. Then the following conditions are equivalent:*

(i) $A \cong A \otimes \mathcal{O}_\infty$,

(ii) *for each natural number m, every approximately inner completely positive contraction from a sub-C^*-algebra B of $M_m(A)$ into $M_m(A)$ is approximately 1-step inner,*

(iii) *there is sequence $\{d_n\}_{n=1}^\infty$ in $M_2(A)$ such that*

$$\lim_{n\to\infty} \left\| d_n^* \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} d_n - \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right\| = 0 \tag{8.2}$$

for all a in A.

Proof: (i) $\Rightarrow$ (ii). If A satisfies (i), then so does $M_m(A)$ for all natural numbers m. We need therefore only consider the case $m = 1$. Let B be a sub-C^*-algebra of A. It suffices to show that each n-step inner completely positive contraction $V \colon B \to A$ is approximately 1-step inner. Find $d_1, \dots, d_n$ in $\mathcal{M}(A)$ with $V(b) = \sum_{j=1}^n d_j^* b d_j$. By Corollary 8.3 we can find sequences of isometries $\{t_{j,k}\}_{k=1}^\infty$ in $\mathcal{M}(A)$ for $j = 1, \dots, n$ satisfying

$$t_{1,k}t_{1,k}^* + \cdots + t_{n,k}t_{n,k}^* \le 1, \qquad \lim_{k\to\infty} \|t_{j,k}a - at_{j,k}\| = 0,$$

for all a in A and all j. Put $f_k = \sum_{j=1}^n t_{j,k} d_j$. Then $f_k^* a f_k \to V(a)$, and this shows that V is approximately 1-step inner.

(ii) $\Rightarrow$ (iii). Let B be the sub-C^*-algebra of $M_2(A)$ consisting of all elements of the form $\operatorname{diag}(a, 0)$ with a in A. Let $V\colon B \to M_2(A)$ be given by

$$V\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

The map V is approximately 2-step inner, and hence V is approximately 1-step inner if (ii) holds.

(iii) $\Rightarrow$ (i). If A is stable, then by Lemma 7.4 (ii) we can take d_n in $M_2(\mathcal{M}(A))$ satisfying (8.2) to be isometries of the form

$$d_n = \begin{pmatrix} u_n & v_n \\ 0 & 0 \end{pmatrix}. \tag{8.3}$$

Notice that (8.3) and (8.2) imply that u_n and v_n are isometries in $\mathcal{M}(A)$ with orthogonal range projections, that $u_n^* a u_n \to a$ and $v_n^* a v_n \to a$ for all a in A, and hence (by Lemma 7.9) that

$$\lim_{n\to\infty} \|u_n a - a u_n\| = 0, \qquad \lim_{n\to\infty} \|v_n a - a v_n\| = 0 \tag{8.4}$$

for all a in A.

We show next that we also can take the d_n's to be of the form in (8.3) when A is unital. Put $e = \operatorname{diag}(1, 0)$ in $M_2(A)$. Upon replacing d_n by ed_n we may assume that $d_n = ed_n$. Applying (8.2) to $a = 1$ we find that $d_n^* d_n \to 1$. Hence $d_n^* d_n$ is invertible (for n large enough), and $d_n = v_n |d_n|$ for some isometry v_n. Since $\|v_n - d_n\| \to 0$ and since $v_n = ev_n$ we obtain (8.3) by replacing d_n by v_n.

Let $\mathcal{E}_2$ be the universal C^*-algebra generated by two isometries with orthogonal range projections. The two isometries u_n, v_n have orthogonal range projections (for each fixed n), and so there is a unital *-homomorphism $\psi_n\colon \mathcal{E}_2 \to \mathcal{M}(A)$ mapping the two canonical generators of $\mathcal{E}_2$ onto u_n and v_n. By (8.4) we see that $\psi_n(x)a - a\psi_n(x) \to 0$ for all x in $\mathcal{E}_2$ and all a in A. The C^*-algebra $\mathcal{E}_2$ has a unital sub-C^*-algebra isomorphic to $\mathcal{O}_\infty$. Taking $\varphi_n\colon \mathcal{O}_\infty \to \mathcal{M}(A)$ to be the restriction of ψ_n, an application of Corollary 8.3 yields that A is isomorphic to $A \otimes \mathcal{O}_\infty$. $\square$

Recall that a C^*-algebra A is called $\mathcal{O}_\infty$-absorbing if $A \otimes \mathcal{O}_\infty$ is isomorphic to A.

Proposition 8.5 (Permanence properties)

(i) *If A is an $\mathcal{O}_\infty$-absorbing C^*-algebra, then so is every closed two-sided ideal in A.*

(ii) *If A is an $\mathcal{O}_\infty$-absorbing C^*-algebra, then so is every quotient of A.*

(iii) *If A is a separable $\mathcal{O}_\infty$-absorbing C^*-algebra, and if B is a hereditary sub-C^*-algebra of A admitting an approximate unit consisting of projections, then B is $\mathcal{O}_\infty$-absorbing.*

(iv) *If A is an inductive limit of a sequence $A_1 \to A_2 \to A_3 \to \cdots$ of separable C^*-algebras A_n, each of which absorbs $\mathcal{O}_\infty$, and if each connecting map $A_n \to A_{n+1}$ is non-degenerate* [3]*, then A absorbs $\mathcal{O}_\infty$.*

Proof: (i) and (ii). Write $A = A_0 \otimes \mathcal{O}_\infty$ for some C^*-algebra A_0. Suppose that I is an ideal in A. Because $\mathcal{O}_\infty$ is exact and simple, $I = I_0 \otimes \mathcal{O}_\infty$ for some ideal I_0 in I, cf. [1, Theorem 3.3] and [14, Proposition 2.13]. Hence I is $\mathcal{O}_\infty$-absorbing, and $A/I \cong (A_0/I_0) \otimes \mathcal{O}_\infty$ because $\mathcal{O}_\infty$ is exact.

(iii). Let $\{p_k\}_{k=1}^\infty$ be an increasing approximate unit for B where each p_k is a projection. Observing that there are unital embeddings of $\mathcal{E}_2$ into $\mathcal{O}_\infty$ and vice versa, we may use Corollary 8.3 — with $\mathcal{E}_2$ in the place of $\mathcal{O}_\infty$ — to find an asymptotically central sequence of unital *-homomorphisms $\varphi_n \colon \mathcal{E}_2 \to \mathcal{M}(A)$. Let s, t be the two canonical generators of $\mathcal{E}_2$, i.e., s, t are isometries with $ss^* \perp tt^*$. Put $p_0 = 0$ and put $q_k = p_k - p_{k-1}$ for all k in $\mathbb{N}$. Then $1 = \sum_{j=1}^\infty q_k$ in $\mathcal{M}(B)$ (the sum is strictly convergent). Also,

$$\lim_{n\to\infty} \|\varphi_n(s)q_k - q_k\varphi_n(s)\| = \lim_{n\to\infty} \|\varphi_n(t)q_k - q_k\varphi_n(t)\| = 0.$$

The relations satisfied by s, t are stable, and we can therefore for each k in $\mathbb{N}$ find sequences $\{s_{n,k}\}_{n=1}^\infty$ and $\{t_{n,k}\}_{n=1}^\infty$ of isometries in $q_kAq_k = q_kBq_k$ such that $s_{n,k}s_{n,k}^* \perp t_{n,k}t_{n,k}^*$, and such that

$$\lim_{n\to\infty} \|q_k\varphi_n(s)q_k - s_{n,k}\| = \lim_{n\to\infty} \|q_k\varphi_n(t)q_k - t_{n,k}\| = 0.$$

For each n there is a (unique) unital *-homomorphism $\psi_n \colon \mathcal{E}_2 \to \mathcal{M}(B)$ that satisfies

$$\psi_n(s) = \sum_{k=1}^\infty s_{n,k}, \qquad \psi_n(t) = \sum_{k=1}^\infty t_{n,k},$$

[3] A *-homomorphism $\varphi \colon D \to E$ is said to be *non-degenerate* if $\varphi(D)E\varphi(D)$ is dense in E.

(the sums are strictly convergent). We have $\|\psi_n(x)b - b\psi_n(x)\| \to 0$ for each x in $\mathcal{E}_2$ and for each b in B. (To see this, consider first b in $p_k B p_k$ for some k.) Hence the conditions of Corollary 8.3 are satisfied, and so B is isomorphic to $B \otimes \mathcal{O}_\infty$.

(iv). Let $\mu_n\colon A_n \to A$ be the inductive limit map. Then μ_n is non-degenerate for each n and therefore extends to a unital *-homomorphism $\widehat{\mu}_n\colon \mathcal{M}(A_n) \to \mathcal{M}(A)$. Use Corollary 8.3 to find a sequence of unital *-homomorphisms $\varphi_{k,n}\colon \mathcal{O}_\infty \to \mathcal{M}(A_n)$ such that

$$\lim_{k\to\infty} \|\varphi_{k,n}(x)a - a\varphi_{k,n}(x)\| = 0, \qquad x \in \mathcal{O}_\infty,\ a \in A_n.$$

Let $\psi_{k,n}\colon \mathcal{O}_\infty \to \mathcal{M}(A)$ be the composition $\widehat{\mu}_n \circ \varphi_{k,n}$. Then

$$\lim_{k,n\to\infty} \|\psi_{k,n}(x)a - a\psi_{k,n}(x)\| = 0, \qquad x \in \mathcal{O}_\infty,\ a \in A.$$

Corollary 8.3 now shows that A absorbs $\mathcal{O}_\infty$. □

Theorem 8.6 *Let A be a separable C^*-algebra. If A is strongly purely infinite and nuclear, and if A is either stable or has an approximate unit consisting of projections, then A is isomorphic to $A \otimes \mathcal{O}_\infty$. Conversely, for any C^*-algebra A, if A is isomorphic to $A \otimes \mathcal{O}_\infty$, then A is strongly purely infinite.*

Proof: Assume first that A is isomorphic to $A \otimes \mathcal{O}_\infty$. Then, by Corollary 8.3, there are sequences $\{u_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ of isometries in $\mathcal{M}(A)$ such that $u_n u_n^* \perp v_n v_n^*$ and

$$\lim_{n\to\infty} \|u_n a - a u_n\| = 0, \qquad \lim_{n\to\infty} \|v_n a - a v_n\| = 0$$

for all a in A. Hence

$$\begin{pmatrix} u_n^* & 0 \\ 0 & v_n^* \end{pmatrix} \begin{pmatrix} a & x^* \\ x & b \end{pmatrix} \begin{pmatrix} u_n & 0 \\ 0 & v_n \end{pmatrix} \to \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

for all a, b, x in A, and this shows that A is strongly purely infinite, cf. Lemma 5.2.

Suppose next that A is strongly purely infinite, nuclear, and that A either has an approximate unit consisting of projections or is stable. If $A \otimes \mathcal{K}$ absorbs $\mathcal{O}_\infty$, then so does A by Proposition 8.5 (in either case), and $A \otimes \mathcal{K}$ is strongly purely infinite by Proposition 5.11 (iii). It therefore suffices to consider the case, where A is stable.

Let B and $V\colon B \to M_2(A)$ be as in the proof of (ii) $\Rightarrow$ (iii) of Proposition 8.4. Being isomorphic to A, B is nuclear, and V is approximately 2-step inner. Thus V is approximately 1-step inner by Theorem 7.21, and so A is isomorphic to $A \otimes \mathcal{O}_\infty$ by Proposition 8.4.

$\square$

9 Summary and open problems

The main results of this paper are contained in Theorem 7.21 and in the theorem below:

Theorem 9.1 *Consider the following six properties of a separable C^*-algebra A:*

(i) $A \cong A \otimes \mathcal{O}_\infty$.

(ii) *A is strongly purely infinite* [4].

(iii) *A is purely infinite* [5].

(iv) *A is weakly purely infinite* [6].

(v) *A_ω is traceless* [7] *for some free filter ω on $\mathbb{N}$.*

(vi) *A is traceless.*

Then

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(iv)} \Leftrightarrow \text{(v)} \Rightarrow \text{(vi)},$$

and

(ii) $\Rightarrow$ (i) *if A is nuclear, and either stable or with an approximate unit consisting of projections,*

(iii) $\Rightarrow$ (ii) *if A is either simple, of real rank zero, or approximately divisible* [8]*,*

(iv) $\Rightarrow$ (iii) *if A has the global Glimm property* [9]*, and in particular if A is either simple, of real rank zero, or approximately divisible,*

(vi) $\Rightarrow$ (v) *if A is approximately divisible,*

Proof: The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) are treated in Theorem 8.6.

(ii) $\Rightarrow$ (iii) is proved in Proposition 5.4. It is shown in Corollary 6.9 that (iii) $\Rightarrow$ (ii) if A is of real rank zero, and hence in particular if A is simple, because all simple purely infinite

[4]A definition of being strongly purely infinite is given in Definition 5.1.
[5]A definition of being purely infinite is given in Definition 3.4.
[6]A definition of being weakly purely infinite is given in Definition 4.3.
[7]Traceless C^*-algebras are defined in Definition 4.2.
[8]A definition of being approximately divisible can be found in Definition 4.1.
[9]A definition of the global Glimm property can be found in Definition 4.12

C^*-algebras are of real rank zero. Proposition 5.14 shows that (iii) $\Rightarrow$ (ii) for approximately divisible C^*-algebras.

(iii) $\Rightarrow$ (iv) follows from [16, Theorem 4.16], cf. the remark below Definition 4.3. The implication (iv) $\Rightarrow$ (iii) is treated in Proposition 4.15, Corollary 4.16 and Proposition 4.18.

The equivalence (iv) $\Leftrightarrow$ (v) is proved in Theorem 4.8 (i), and by Theorem 4.8 (ii) we have (iv) $\Rightarrow$ (vi). Finally, the implication (vi) $\Rightarrow$ (iv) is proved to hold for approximately divisible C^*-algebras in [16, Theorem 5.9]. $\square$

The implication (ii) $\Rightarrow$ (i) does not hold in general. There is in [11, Theorem 1.4] an example of a simple, unital, purely infinite, separable C^*-algebra A which is not approximately divisible. Hence A is not isomorphic to $A \otimes \mathcal{O}_\infty$ (as noted below Corollary 8.3). We do not have counter examples to any other implication of Theorem 9.1.

We summarize the situation when A is either simple, approximately divisible or of real rank zero. The conclusions of Corollary 9.2 below was obtained in 1994 by the first named author (and is published in [15]).

Corollary 9.2 *Let A be a simple, separable, nuclear C^*-algebra. Then $A \cong A \otimes \mathcal{O}_\infty$ if and only if A is purely infinite.*

Proof: If A is simple and purely infinite, then A is either stable or unital by [22]. Hence conditions (i) – (v) in Theorem 9.1 are equivalent for A. $\square$

Corollary 9.3 *Let A be a separable, nuclear C^*-algebra that is either stable or admits an approximate unit consisting of projections. Then the following conditions are equivalent:*

(i) $A \cong A \otimes \mathcal{O}_\infty$,

(ii) *A is approximately divisible and purely infinite,*

(iii) *A is approximately divisible and traceless.*

Proof: The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) follows from Theorem 9.1. If (i) holds then A is approximately divisible (this is easy to see from Corollary 8.3), and A is traceless (by Theorem 9.1). $\square$

It is a consequence of the next corollary that all non-degenerate (simple and non-simple) Cuntz–Krieger algebras absorb $\mathcal{O}_\infty$. A Cuntz–Krieger algebra $\mathcal{O}_A$ is non-degenerate if its corresponding matrix has no irreducible component which is a permutation matrix. Non-degenerate Cuntz–Krieger algebras have real rank zero and all their non-zero projections are properly infinite.

Corollary 9.4 *Let A be a separable, nuclear C^*-algebra of real rank zero. Then the following conditions are equivalent:*

(i) $A \cong A \otimes \mathcal{O}_\infty$,

(ii) *A is strongly purely infinite,*

(iii) *A is purely infinite,*

(iv) *A is weakly purely infinite,*

(v) *A_ω is traceless for all free filters ω on $\mathbb{N}$,*

(vi) *all non-zero projections in A are properly infinite.*

Proof: Each C^*-algebra of real rank zero admits an approximate unit consisting of projections, and so conditions (i) – (v) of Theorem 9.1 (and hence of the present corollary) are equivalent for separable, nuclear C^*-algebras of real rank zero. The implication (iii) $\Rightarrow$ (vi) follows from [16, Theorem 4.16] (saying that all non-zero positive elements in a purely infinite C^*-algebra are properly infinite), and (vi) $\Rightarrow$ (iii) follows from [16, Proposition 4.7] (since every hereditary sub-C^*-algebra of a quotient of a C^*-algebra of real rank zero again is of real rank zero). $\square$

There are still several unanswered questions regarding the structure of infinite C^*-algebras. We list some of the more intriguing of these open problems below:

Question 9.5 (Three kinds of pure infiniteness) Do we have

$$A \text{ strongly purely infinite} \iff A \text{ purely infinite} \iff A \text{ weakly purely infinite}$$

for all C^*-algebras A?

The two right-implications "$\Rightarrow$" in Question 9.5 are true and easy to prove (see Theorem 9.1). All weakly purely infinite C^*-algebras are purely infinite if and only if all weakly purely infinite C^*-algebras have the global Glimm property (see Proposition 4.15), or, equivalently if and only if all non-zero projections in a weakly purely infinite C^*-algebra are infinite (see Proposition 4.19).

We do not know if the multiplier algebra of a purely infinite C^*-algebra is purely infinite. We do not even know if its unit is infinite. Following the proof of Proposition 4.11, this would follow if we have an affirmative answer to the following:

Question 9.6 (Sums of properly infinite elements) Let a and b be positive elements in a C^*-algebra A such that, for some $\delta > 0$, the elements $(a-\varepsilon)_+$ and $(b-\varepsilon)_+$ are properly infinite for all $\varepsilon \in [0, \delta]$. Does it follow that their sum $a + b$ is properly infinite?

A partial answer to this question can be found in Lemma 4.9. Note that our assumption on a and b is slightly stronger than just asking these two elements to be properly infinite. For example, any strictly positive element a in the C^*-algebra $\mathcal{K}$ of compact operators on an infinite dimensional Hilbert space is properly infinite (see [16, Proposition 3.7]), but $(a - \varepsilon)_+$ is not properly infinite for $\varepsilon > 0$.

There are strongly purely infinite C^*-algebras that are neither stable nor have an approximate unit consisting of projections. Take for example $C_0(\mathbb{R}) \otimes \mathcal{O}_\infty$ (which by the way clearly is $\mathcal{O}_\infty$-absorbing). The implication (ii) $\Rightarrow$ (i) of Theorem 9.1 therefore does not apply to all separable, nuclear C^*-algebras. Nonetheless, we have no (nuclear, separable) counter example to this implication. We therefore ask:

Question 9.7 (Morita equivalence of $\mathcal{O}_\infty$-absorbsion) Suppose that A and B are stably isomorphic C^*-algebras and that $A \cong A \otimes \mathcal{O}_\infty$. Does it follow that $B \cong B \otimes \mathcal{O}_\infty$?

One can answer Question 9.7 in the affirmative if one can prove that the inductive limit of any sequence $A_1 \to A_2 \to \cdots$ of $\mathcal{O}_\infty$-absorbing, separable C^*-algebras (not necessarily with non-degenerate connecting mappings) is $\mathcal{O}_\infty$-absorbing.

We know that extensions of weakly purely infinite and of purely infinite C^*-algebras again are weakly purely infinite, respectively, purely infinite. What is the situation for strongly purely infinite C^*-algebras?

Question 9.8 (Extensions of $\mathcal{O}_\infty$-absorbing C^*-algebras) Given an extension

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

of C^*-algebras. Suppose that I and B are strongly purely infinite. Does it follow that A is strongly purely infinite? Can one conclude that $A \cong A \otimes \mathcal{O}_\infty$ if we know that $I \cong I \otimes \mathcal{O}_\infty$ and $B \cong B \otimes \mathcal{O}_\infty$?

It is shown in Proposition 8.5 that I and B are $\mathcal{O}_\infty$-absorbing if A is $\mathcal{O}_\infty$-absorbing.

Question 9.9 (Traceless C^*-algebras) Let ω be (any) free filter on $\mathbb{N}$, and let A be a C^*-algebra. Can one conclude that A_ω is traceless if A is traceless?

With a positive answer to Question 9.9, a C^*-algebra is weakly purely infinite if and only if it is traceless. This would have particularly interesting consequences for simple C^*-algebras. Indeed, all simple, weakly purely infinite C^*-algebras are purely infinite, and a simple C^*-algebra is traceless if it admits no (densely defined) quasitrace. Hence, an affirmative answer to Question 9.9 would also answer in the affirmative the old problem whether all simple C^*-algebras must either have a quasitrace or be purely infinite. It is the guess of the authors that there could be a (non-nuclear) counter example to Question 9.9.

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