GALOIS GROUPS OVER NONRIGID FIELDS

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Dedicated to Paulo Ribenboim

ABSTRACT. Let F be a field with $\operatorname{char} F \neq 2$. We show that F is a nonrigid field if and only if certain small 2-groups occur as Galois groups over F. These results provide new "automatic realizability" results for Galois groups over F. The groups we consider demonstrate the inequality of two particular metabelian 2-extensions of F which are unequal precisely when F is a nonrigid field. Using known results on connections between rigidity and existence of certain valuations, we obtain Galois-theoretic criteria for the existence of these valuations.

1. INTRODUCTION

Let F be a field with $\operatorname{char} F \neq 2$. The goal of this paper is to identify basic Galois groups that must occur if F is not a rigid field. If $|\dot{F}/\dot{F}^2| \geq 8$ then it is known that F is not rigid if and only if a certain group (often denoted DC) of order 16 occurs as a Galois group over F ([MS]). It has also been shown ([LS], simplifying an earlier proof in [AGKM]) that F is rigid if and only if $F^{(3)} = F^{\{3\}}$ (all notation defined below). In this paper we identify two groups, denoted G_1 and G_2, where $|G_1| = 32$, $|G_2| = 64$, with the property that a field F is nonrigid if and only if at least one of G_1 and G_2 is realizable as a Galois group over F. In each case the corresponding Galois extension lies in $F^{\{3\}}$ but not in $F^{(3)}$. Moreover, it is shown that extensions realizing G_1 and G_2 correspond, respectively, to extensions realizing the groups $D \wr C$ and $D \wr D$, which lie inside $F^{(3)}$. This correspondence gives new nontrivial "automatic realizability" results for Galois groups. We also point out how to detect nonrigid elements in F using the groups G_1 and G_2. Finally, since rigidity conditions on field elements are known to correspond to the existence of certain valuations ([W]), we can provide a Galois-theoretic interpretation of valuation-theoretic results.

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We begin in the following section with necessary notation and terminology, as well as an explanation of the extensions $F^{(3)}$ and $F^{\{3\}}$ and properties of their Galois groups. In section 3 we examine the group $D \wr C$ and its associated group G_1, and in section 4 the groups $D \wr D$ and G_2 are considered. Section 5 gives the main theorems connecting nonrigidity to the realizability of the groups $D \wr C$, G_1, $D \wr D$ and G_2, as well as the Galois-theoretic interpretation of the existence of a nonrigid element. We conclude in section 6 with a detailed analysis of the situation when $|\dot{F}/\dot{F}^2| = 4$. This is the case where the group DC is not realizable as a Galois group over F, and yet F may be nonrigid.

2. Notation, terminology and Preliminaries

We let $\dot{F} = F\backslash\{0\}$, the group of nonzero elements of F. If $a_1, \ldots a_n \in \dot{F}$, the notation $\langle a_1, \ldots, a_n\rangle$ denotes the quadratic form $a_1x_1^2 + \cdots + a_nx_n^2$. If q is a quadratic form defined over F, the notation $D_F(q)$ denotes the set of elements in $\dot{F}$ represented by q over F. Basic results on quadratic forms can be found in [La].

If G is a group and $\sigma, \tau \in G$, then $[\sigma, \tau]$ denotes the commutator $\sigma^{-1}\tau^{-1}\sigma\tau$.

The notion of a rigid field and a rigid element are important in what follows. An element $a \in \dot{F}\backslash \pm F^2$ is *rigid* if $D_F(\langle 1, a\rangle) = \dot{F}^2 \cup a\dot{F}^2$ and is *nonrigid* otherwise. An element $a \in \dot{F}\backslash \pm F^2$ is *double rigid* if a and $-a$ are both rigid. A field F is *rigid* if each $a \in \dot{F}\backslash \pm F^2$ is rigid and is *nonrigid* otherwise.

Lemma 2.1. *The following are equivalent.*

1. *F is nonrigid.*
2. *There exists $a \in \dot{F}\backslash F^2$ such that $|D_F(\langle 1, -a\rangle)/\dot{F}^2| \geq 4$.*
3. *There exists $a \in \dot{F}\backslash \pm F^2$ such that $|D_F(\langle 1, -a\rangle)/\dot{F}^2| \geq 4$.*

Proof. It is clear that (1) and (3) are equivalent and that (3) implies (2). Now assume that (2) holds but that each element in $\dot{F}\backslash \pm F^2$ is rigid. Then we can assume $a = -1$ in (2) and that $-1 \notin F^2$. Then there exists $b \in \dot{F}\backslash \pm F^2$ such that $b \in D_F(\langle 1, 1\rangle)$. Then $-1 \in D_F(\langle 1, -b\rangle) = \dot{F}^2 \cup -b\dot{F}^2$, which is a contradiction. □

An element $a \in \dot{F}$ is *basic* if either $a \in \pm F^2$ or a is not double rigid. The set of all basic elements in F is denoted $B(F)$. Thus $\dot{F} = B(F) \cup \{\text{double rigid elements}\}$ is a disjoint union and $B(F) = \dot{F}^2 \cup -\dot{F}^2$ if and only if F is a rigid field.

Let $v : F \to \Gamma \cup \{\infty\}$ be a valuation on F with valuation ring A, maximal ideal M, residue field $k = A/M$ and value group Γ, where

Γ is an ordered abelian group. Let $U = A\backslash M$ be the group of units of A. Then $\Gamma \cong \dot{F}/U$. The valuation v is called 2-henselian if v has a unique extension to the quadratic closure of F. If $\operatorname{char} F \neq 2$ and $1 + M \subseteq \dot{F}^2$, then v is 2-henselian ([W], Lemma 4.3). The valuation v is called 2-divisible if Γ is a 2-divisible group. Since $\dot{F}^2U$ is the set of elements in F whose valuation lies in 2Γ, it follows that v is 2-divisible if and only if $\dot{F} = \dot{F}^2U$.

The following theorem first appeared in [W], Theorem 4.4 (3).

Theorem 2.2. *Suppose F is nonrigid. Then there exists a valuation v on F such that $1 + M \subseteq \dot{F}^2$ and $B(F) = \dot{F}^2U$.*

Corollary 2.3. *Suppose F is nonrigid. The following are equivalent.*

1. *There exists a valuation v on F such that $1 + M \subseteq \dot{F}^2$, $B(F) = \dot{F}^2U$ and v is not 2-divisible.*
2. *F contains a double rigid element.*

In particular, if v is a valuation on F such that $1 + M \subseteq \dot{F}^2$ and $B(F) = \dot{F}^2U$, then an element $f \in F$ is double rigid if and only if $v(f) \notin 2\Gamma$.

Proof. Since F is nonrigid, there exists a valuation v on F such that $1 + M \subseteq \dot{F}^2$ and $B(F) = \dot{F}^2U$. We have v is not 2-divisible if and only if $\dot{F}^2U \subsetneq \dot{F}$, which is equivalent to $B(F) \subsetneq \dot{F}$, in other words, F contains a double rigid element. □

We now define some of our main objects of study.

1. Let $F^{(2)} = F(\sqrt{F})$, the maximal multiquadratic extension of F. Then $\operatorname{Gal}(F^{(2)}/F)$ is an elementary abelian 2-group isomorphic to $\prod_I \mathbb{Z}/2\mathbb{Z}$ where $|I| = \dim_{\mathbb{Z}/2\mathbb{Z}} \dot{F}/\dot{F}^2$. We denote $\operatorname{Gal}(F^{(2)}/F)$ by $G_F^{[2]}$.
2. Let $F^{\{3\}} = F^{(2)}(\sqrt{F^{(2)}}) = (F^{(2)})^{(2)}$. Then $F^{\{3\}}$ is a Galois extension of F and we denote $\operatorname{Gal}(F^{\{3\}}/F)$ by $G_F^{\{3\}}$.
3. Let $F^{(3)}$ denote the subfield of $F^{\{3\}}$ generated by all fields $F^{(2)}(\sqrt{g})$ such that $g \in F^{(2)}$ and $F^{(2)}(\sqrt{g})/F$ is a Galois extension. Denote $\operatorname{Gal}(F^{(3)}/F)$ by $G_F^{[3]}$.

Thus $F \subseteq F^{(2)} \subseteq F^{(3)} \subseteq F^{\{3\}}$ and $F^{(2)}/F$, $F^{(3)}/F$, $F^{\{3\}}/F$ are each Galois extensions. The groups $G_F^{[3]}$, $G_F^{[2]}$ are quotients of $G_F^{\{3\}}$.

For the connection between $G_F^{[3]}$ and the Witt ring of quadratic forms over F, see [MSp].

Lemma 2.4. 1. *The group $G_F^{\{3\}}$ has exponent dividing 4.*

2. *For each* $\sigma, \tau \in G_F^{\{3\}}$, *the commutator* $[\sigma, \tau]$ *has order dividing* 2, *and* $[\sigma, \tau] = [\tau, \sigma]$.
3. *All commutators and squares in* $G_F^{\{3\}}$ *commute with each other.*

Proof. Let $\sigma, \tau \in G_F^{\{3\}}$. Then $\sigma^2|_{F^{(2)}} = 1$ since $G_F^{[2]}$ has exponent dividing 2 and $[\sigma, \tau]|_{F^{(2)}} = 1$ since $G_F^{[2]}$ is abelian. Thus $\sigma^2, [\sigma, \tau] \in \mathrm{Gal}(F^{\{3\}}/F^{(2)})$, a group of exponent dividing 2. Therefore $\sigma^4 = 1$ and $[\sigma, \tau]^2 = 1$. Note that $[\tau, \sigma] = [\sigma, \tau]^{-1} = [\sigma, \tau]$ This proves (1) and (2).

For (3), we have from above that each commutator and square in $G_F^{\{3\}}$ lies in $\mathrm{Gal}(F^{\{3\}}/F^{(2)})$, which is an abelian group of exponent dividing 2. □

Lemma 2.5. *For every* $\sigma, \tau \in G_F^{\{3\}}$, *the commutator* $[\,[\sigma, \tau], \sigma]$ *commutes with both* σ *and* τ.

Proof. In any group, we have the identity

$$[a, bc] = [a, c][a, b][\,[a, b], c].$$

Let $b = c$ and assume that $a, b \in G_F^{\{3\}}$. Then $[a, b^2] = [\,[a, b], b]$ by Lemma 2.4. Let $a = [\sigma, \tau]$ and $b = \sigma$. We have $[\,[\sigma, \tau], \sigma^2] = 1$ by Lemma 2.4. Therefore, $[\,[\,[\sigma, \tau], \sigma], \sigma] = 1$. This implies $[\,[\sigma, \tau], \sigma]$ commutes with σ.

Now let $a = \sigma^2$ and $b = \tau$. Since $[\sigma^2, \tau^2] = 1$ by Lemma 2.4, we have $[\,[\sigma^2, \tau], \tau] = 1$ and so $[\sigma^2, \tau]$ commutes with τ. But $[\sigma^2, \tau] = [\,[\sigma, \tau], \sigma]$ by the following argument. Let $a = \tau$ and $b = \sigma$ in the identity to see

$$[\sigma^2, \tau] = [\tau, \sigma^2] = [\,[\tau, \sigma], \sigma] = [\,[\sigma, \tau], \sigma].$$

□

Remark 1. The results in Lemmas 2.4, 2.5 hold in any homomorphic image of $G_F^{\{3\}}$.

Proposition 2.6. *Every square and commutator of elements in* $G_F^{[3]}$ *lies in* $Z(G_F^{[3]})$, *the center of* $G_F^{[3]}$.

Proof. Let $\sigma, \tau, \psi \in G_F^{[3]}$. We must show σ^2 and $[\sigma, \tau]$ commute with ψ. Since $F^{(3)}$ is the compositum of all fields $F^{(2)}(\sqrt{g})$ where $g \in F^{(2)}$ and $F^{(2)}(\sqrt{g})/F$ is a Galois extension, it is sufficient to prove the result for elements of the group $\mathrm{Gal}(F^{(2)}(\sqrt{g})/F)$ whenever $F^{(2)}(\sqrt{g})/F$ is a Galois extension. Since $G_F^{[2]}$ is an abelian group of exponent dividing 2, it follows σ^2 and $[\sigma, \tau]$ both lie in $\mathrm{Gal}(F^{(2)}(\sqrt{g})/F^{(2)})$, a group of order dividing 2. As $F^{(2)}(\sqrt{g})/F$ is a Galois extension, the group $\mathrm{Gal}(F^{(2)}(\sqrt{g})/F^{(2)})$ is a normal subgroup of $\mathrm{Gal}(F^{(2)}(\sqrt{g})/F)$ of order

1 or 2 and thus lies in the center of $\mathrm{Gal}(F^{(2)}(\sqrt{g})/F)$. Therefore σ^2 and $[\sigma,\tau]$ lie in $Z(\mathrm{Gal}(F^{(2)}(\sqrt{g})/F))$. □

Remark 2. In Lemma 2.4(3) and Proposition 2.6, the result for commutators is a consequence of the result for squares since the identity $[\sigma,\tau] = (\sigma^{-1})^2(\sigma\tau^{-1})^2\tau^2$ shows that a commutator is a product of three squares.

Proposition 2.7. *Let G_0 be a group with generators x, y and assume*

1. $x^4 = y^4 = 1$,
2. $[x,y]^2 = [\,[x,y],x\,]^2 = [\,[x,y],y\,]^2 = 1$,
3. $[\,[x,y],x\,]$ *and* $[\,[x,y],y\,]$ *each commute with x and y.*

Then the following identities hold in G_0, and $|G_0| \le 128$.

(a) $yx = xy[x,y]$,
(b) $[x,y]x = x[x,y][\,[x,y],x\,]$,
(c) $[x,y]y = y[x,y][\,[x,y],y\,]$.

If in addition one assumes

4. $y^2 = 1$,

then $[\,[x,y],y\,] = 1$ and $|G_0| \le 32$.

Proof. Since $[x,y] = [x,y]^{-1} = [y,x]$, we have $yx = xy[y,x] = xy[x,y]$, which is (a). For (b), let $z = [x,y]$ in the identity $zx = xz[z,x]$ and prove (c) similarly.

We will next show that each element $g \in G_0$ can be written in the form

$$g = x^{e_1}y^{e_2}[x,y]^{e_3}[\,[x,y],x\,]^{e_4}[\,[x,y],y\,]^{e_5},$$

where $e_1, e_2 \in \{0,1,2,3\}$ and $e_3, e_4, e_5 \in \{0,1\}$. This will show $|G_0| \le 4\cdot 4\cdot 2\cdot 2\cdot 2 = 128$. Let $g \in G_0$. Since $x^4 = 1$ and $[\,[x,y],x\,]$ commutes with x, (a) and (b) allow us to write g in the form $g = x^{e_1}z$ where $e_1 \in \{0,1,2,3\}$ and z involves factors of the type $y, [x,y], [\,[x,y],x\,]$. Since $y^4 = 1$ and $[\,[x,y],x\,]$ and $[\,[x,y],y\,]$ both commute with y, (c) allows us to write $g = x^{e_1}y^{e_2}w$ where $e_2 \in \{0,1,2,3\}$ and w involves factors of the type $[x,y], [\,[x,y],x\,], [\,[x,y],y\,]$. We now use (2) and (3) to show g has the required form.

Now assume (4) also holds. Then

$$[x,y]y = x^{-1}y^{-1}xy^2 = x^{-1}yx = x^{-1}(xy[x,y]) = y[x,y]$$

and so $[\,[x,y],y\,] = 1$. In this case we may assume $e_2 \in \{0,1\}$ and $e_5 = 0$, and therefore $|G_0| \le 4\cdot 2\cdot 2\cdot 2\cdot 1 = 32$. □

Remark 3. The proof that $[\,[x,y],y\,] = 1$ in (4) holds in any group where $y^2 = [x,y]^2 = 1$.

Corollary 2.8. 1. *Any subgroup of $G_F^{\{3\}}$ or any subgroup of a homomorphic image of $G_F^{\{3\}}$ that is generated by two elements x, y satisfies (1)-(3) of Proposition 2.7.*

2. *Any subgroup of $G_F^{[3]}$ or any subgroup of a homomorphic image of $G_F^{[3]}$ that is generated by two elements x, y has order at most 32.*

Proof. (1). Lemmas 2.4, 2.5 imply (1)-(3) hold in $G_F^{\{3\}}$, and these properties are preserved in any homomorphic image of $G_F^{\{3\}}$.

(2). As $G_F^{[3]}$ is a homomorphic image of $G_F^{\{3\}}$, we know (1)-(3) hold. Since $[x, y] \in Z(G_F^{[3]})$ by Proposition 2.6, it follows $[\,[x, y], x\,] = [\,[x, y], y\,] = 1$. Therefore the subgroup has order at most $4 \cdot 4 \cdot 2 \cdot 1 \cdot 1 = 32$. □

The next proposition and corollary are needed in section 4.

Proposition 2.9. *Let G be a metabelian group. If $x, y, z \in G$ then*

$$[\,[x, y],\, z][\,[y, z],\, x][\,[z, x],\, y] = 1.$$

Proof. Since G is metabelian, there is a normal subgroup H such that H and G/H are abelian. Thus every commutator of G lies in H and any two commutators commute.

After completely expanding $[\,[x, y],\, z][\,[y, z],\, x][\,[z, x],\, y]$ and making obvious cancellations, one is left with

$$y^{-1}x^{-1}yxz^{-1}x^{-1}y^{-1}(xzyx^{-1}y^{-1}z^{-1})(yxzy^{-1}z^{-1}x^{-1})zxy.$$

The two expressions in parentheses commute since they are the commutators $[x^{-1}, y^{-1}z^{-1}]$ and $[y^{-1}, z^{-1}x^{-1}]$. After transposing these expressions and cancelling the remaining terms, one is left with the identity. □

Corollary 2.10. *Let $\sigma, \tau, \psi \in G_F^{\{3\}}$ and assume $[\tau, \psi] = 1$. Then $[\tau,\ [\sigma, \psi]\,] = [\psi,\ [\sigma, \tau]\,]$.*

Proof. Since $G_F^{\{3\}}$ is metabelian, Proposition 2.9 implies

$$1 = [\,[\sigma, \tau],\, \psi][\,[\tau, \psi],\, \sigma][\,[\psi, \sigma],\, \tau] = [\,[\sigma, \tau],\, \psi][\,[\psi, \sigma],\, \tau].$$

Since all commutators have order dividing 2 by Lemma 2.4, the last expression equals $[\psi,\ [\sigma, \tau]\,][\tau,\ [\sigma, \psi]\,]$ and the result follows. □

3. The groups $D \wr C$ and G_1 as Galois groups

Let D denote the dihedral group of order 8 and let C denote the cyclic group of order 4. Let $\lambda : D \to \mathbb{Z}/2\mathbb{Z}$ be a group homomorphism with $\ker(\lambda) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and let $\eta : C \to \mathbb{Z}/2\mathbb{Z}$ be the unique nontrivial group homomorphism. We let $D \wr C$ denote the pullback of

this pair of homomorphisms. Thus $D \curlywedge C$ is the subgroup of $D \times C$ of order 16 consisting of elements (u, v) such that $\lambda(u) = \eta(v)$. Note that the subgroup $D \curlywedge C$ projects onto both D and C with maps $\pi_1 : D \curlywedge C \to D$ and $\pi_2 : D \curlywedge C \to C$ such that $\lambda \circ \pi_1 = \eta \circ \pi_2$.

Let $a, b \in F$ and let $E = F(\sqrt{a}, \sqrt{b})$. Assume $[E : F] = 4$. We let $D^{a,b}$ denote a Galois extension of F (if one exists) such that $F \subseteq E \subseteq D^{a,b}$, $\mathrm{Gal}(D^{a,b}/F) \cong D$, and $\mathrm{Gal}(D^{a,b}/F(\sqrt{ab})) \cong C$.

We let C^a denote a cyclic quartic extension of F (if one exists) such that $F \subseteq F(\sqrt{a}) \subseteq C^a$.

It is known that $D^{a,b}$ exists if and only if the quaternion algebra $(a, b)_F = 0$ in the Brauer group $Br(F)$, and this is equivalent to $a \in \mathrm{im}(N_{F(\sqrt{b})/F})$ where $N_{F(\sqrt{b})/F}$ is the norm map from $F(\sqrt{b})$ to F. The field C^a exists if and only if the quaternion algebra $(a, a)_F = 0$, which is equivalent to $a \in \mathrm{im}(N_{F(\sqrt{a})/F})$ and also equivalent to a being a sum of two squares in F.

Now assume C^a and $D^{a,b}$ exist, and let K denote a composite of C^a and $D^{a,b}$. We have $F(\sqrt{a}) = C^a \cap D^{a,b}$ since D does not admit C as a quotient group. Thus K/F is a Galois extension and $[K : F] = 16$.

Proposition 3.1. $\mathrm{Gal}(K/F) \cong D \curlywedge C$.

Proof. Let $r : \mathrm{Gal}(K/F) \longrightarrow D \times C$ be the group homomorphism defined by $r(\sigma) = (\sigma|_{D^{a,b}}, \sigma|_{C^a})$. Then r is injective since $K = D^{a,b}C^a$. Since $F(\sqrt{a}) = C^a \cap D^{a,b}$ and $\mathrm{Gal}(D^{a,b}/F(\sqrt{a})) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, it follows that $r(\sigma) \in D \curlywedge C$. Since $|\mathrm{Gal}(K/F)| = 16 = |D \curlywedge C|$, it follows that r maps $\mathrm{Gal}(K/F)$ isomorphically onto $D \curlywedge C$. □

Proposition 3.2. *The following statements are equivalent.*

1. *There is a Galois extension K/F such that* $\mathrm{Gal}(K/F) \cong D \curlywedge C$.
2. *There exist elements $a, b \in F$ such that $[F(\sqrt{a}, \sqrt{b}) : F] = 4$ and $(a, a)_F = (a, b)_F = 0$ in the Brauer group $Br(F)$.*

Proof. We have that (2) implies (1) by Proposition 3.1 since (2) implies that $D^{a,b}$ and C^a exist. Now suppose (1) holds and let $\mathrm{Gal}(K/F) \cong D \curlywedge C$. The kernel of $\lambda \circ \pi_1 = \eta \circ \pi_2$ has order 8 and thus the subfield of K corresponding to $\ker(\lambda \circ \pi_1)$ is a quadratic exension of F that we denote $F(\sqrt{a})$. It follows that $\ker(\pi_1)$ is a subgroup of $D \curlywedge C$ of order 2 which corresponds to a subfield of K of the form $D^{a,b}$ for some $b \in F$ and $\ker(\pi_2)$ is a subgroup of $D \curlywedge C$ of order 4 which corresponds to a subfield of K of the form C^a. The existence of both $D^{a,b}$ and C^a implies (2). □

See [GSS] and [GS] for more information on the realizability of $D \curlywedge C$ as a Galois group.

Corollary 3.3. *Let K/F be a Galois extension with* $\mathrm{Gal}(K/F) \cong D \rtimes C$. *Then K contains a unique quadratic extension of F that imbeds into a cyclic quartic extension contained in K/F. This quadratic extension imbeds into two different cyclic quartic extensions contained in K/F.*

Proof. This follows from the observation that $D \rtimes C$ has just two normal subgroups N_1, N_2 with the property that $(D \rtimes C)/N_i$ is a cyclic group of order 4 and that $|N_1 \cap N_2| = 2$. Indeed if $D \rtimes C/N \cong C$, then N contains the commutator subgroup $(D \rtimes C)'$. Since $(D \times C)'$ has order 2, it follows that $(D \rtimes C)'$ has order 2 and $D \rtimes C/(D \rtimes C)' \cong C \times \mathbb{Z}/2\mathbb{Z}$. One checks that $C \times \mathbb{Z}/2\mathbb{Z}$ has exactly two subgroups M of order 2 such that $(C \times \mathbb{Z}/2\mathbb{Z})/M \cong C$. Then N is one of the two inverse images of M in $D \rtimes C$. Since $|N_1 \cap N_2| = |(D \rtimes C)'| = 2$, the subfields corresponding to N_i do not generate K, and so they must intersect in a quadratic extension of F. □

Now we consider a group G_1 (defined below) of order 32 and study how to contruct a Galois extension with Galois group isomorphic to G_1.

Definition 3.4. *Let G_1 be the group generated by two symbols x, y subject to the relations*

1. $x^4 = y^2 = 1$, $[x,y]^2 = 1$, $[\,[x,y],x\,]^2 = 1$.
2. $[[x,y],x]$ *commutes with x and y.*

Proposition 3.5. $|G_1| \leq 32$.

Proof. Since $[\,[x,y],y\,] = 1$ by Remark 3 after Proposition 2.7, the result now follows from Proposition 2.7. □

We now give a construction that will show $|G_1| = 32$. Let $a, b \in F$, let $E = F(\sqrt{a}, \sqrt{b})$ and assume $[E : F] = 4$. Assume in addition that $D^{a,b}$ and C^a exist. Then $a \in \mathrm{im}(N_{F(\sqrt{b})/F}) \cap \mathrm{im}(N_{F(\sqrt{a})/F})$ and so there exist $\alpha \in F(\sqrt{a})$ and $\beta \in F(\sqrt{b})$ such that $N_{F(\sqrt{a})/F}(\alpha) = N_{F(\sqrt{b})/F}(\beta) = a$. Then [Wd], Lemma 2.14, implies there exist $\gamma \in E$ and $d \in F$ such that $N_{E/F(\sqrt{a})}(\gamma) = \alpha d$ and $N_{E/F(\sqrt{b})}(\gamma) = \beta d$. Let $L = E(\sqrt{\alpha d}, \sqrt{\beta d}, \sqrt{\gamma})$ and $K = E(\sqrt{\alpha d}, \sqrt{\beta d})$.

Proposition 3.6.
1. *K/F is a Galois extension with* $\mathrm{Gal}(K/F) \cong D \rtimes C$.
2. *L/F is a Galois extension with* $\mathrm{Gal}(L/F) \cong G_1$.
3. $K \subseteq F^{(3)}$, $L \subseteq F^{\{3\}}$, $L \not\subseteq F^{(3)}$.

Proof. The extensions $F(\sqrt{\alpha d})/F$ and $E(\sqrt{\beta d})/F$ are Galois with

$$\mathrm{Gal}(F(\sqrt{\alpha d})/F) \cong C, \quad \mathrm{Gal}(E(\sqrt{\beta d})/F) \cong D,$$

because $N_{F(\sqrt{a})/F}(\alpha d) = ad^2$ and $N_{F(\sqrt{b})/F}(\beta d) = ad^2$. We also note that $\mathrm{Gal}(E(\sqrt{\beta d})/F(\sqrt{ab})) \cong C$ because $F(\sqrt{ab}) \subseteq F(\sqrt{ab})(\sqrt{a}) = E \subseteq E(\sqrt{\beta d})$ and $N_{E/F(\sqrt{ab})}(\beta d) = N_{F(\sqrt{b})/F}(\beta d) = ad^2$. We see that K is the composite of $F(\sqrt{\alpha d})$ and $E(\sqrt{\beta d})$, so K/F is Galois and Proposition 3.1 implies $\mathrm{Gal}(K/F) \cong D \leftthreetimes C$.

We have $K \subseteq F^{(3)}$ since $F(\sqrt{\alpha d}) = F(\sqrt{a})(\sqrt{\alpha d}) \subseteq F^{(3)}$ and $E(\sqrt{\beta d}) \subseteq F^{(3)}$. This proves (1) and the first part of (3).

Since $L = K(\sqrt{\gamma})$, in order to show L/F is Galois, it is sufficient to show $\gamma\sigma(\gamma) \in K^2$ for all $\sigma \in \mathrm{Gal}(K/F)$. Since $\gamma \in E$, we need only consider $\sigma \in \mathrm{Gal}(E/F)$. This is clear for $\sigma = 1$, and the others follow from the calculations $N_{E/F(\sqrt{a})}(\gamma) = \alpha d \in K^2$, $N_{E/F(\sqrt{b})}(\gamma) = \beta d \in K^2$ and

$$N_{E/F(\sqrt{ab})}(\gamma) = \frac{N_{E/F}(\gamma)\gamma^2}{N_{E/F(\sqrt{a})}(\gamma)N_{E/F(\sqrt{b})}(\gamma)} = \frac{ad^2\gamma^2}{\alpha d \beta d} \in K^2.$$

Therefore L/F is a Galois extension.

We have $L \subseteq F^{\{3\}}$ since αd, βd, $\gamma \in E \subseteq F^{(2)}$. To show $L \nsubseteq F^{(3)}$ it is sufficient to show $F^{(2)}(\sqrt{\gamma})/F$ is not a Galois extension. There exists $\sigma \in \mathrm{Gal}(F^{(2)}/F)$ such that $\gamma\sigma(\gamma) = N_{E/F(\sqrt{a})}(\gamma) = \alpha d$. But $\alpha d \notin (F^{(2)})^2$ since $\mathrm{Gal}(F(\sqrt{\alpha d})/F) \cong C$, and thus $F^{(2)}(\sqrt{\gamma})/F$ is not a Galois extension. This finishes the proof of (3).

We now show $\mathrm{Gal}(L/F) \cong G_1$. We have $[L : F] = 32$ since $[K : F] = 16$ and $K \subsetneq L$ by (3). There exist $\sigma_a, \sigma_b \in \mathrm{Gal}(L/F)$ such that $\sigma_a(\sqrt{a}) = -\sqrt{a}$, $\sigma_a(\sqrt{b}) = \sqrt{b}$, $\sigma_b(\sqrt{a}) = \sqrt{a}$, $\sigma_b(\sqrt{b}) = -\sqrt{b}$ and $\sigma_b(\sqrt{\alpha d}) = \sqrt{\alpha d}$. (To see that σ_b exists, note that an automorphism with this property lies in $\mathrm{Gal}(K/F)$ and that it can be extended to $\mathrm{Gal}(L/F)$.) The group generated by σ_a and σ_b equals $\mathrm{Gal}(L/F)$, otherwise the fixed field of this subgroup would contain a quadratic extension of F, but none of $\sqrt{a}, \sqrt{b}, \sqrt{ab}$ is fixed by both σ_a and σ_b. We will now show that σ_a and σ_b satisfy the relations in Definition 3.4. Since the group generated by σ_a and σ_b has order 32 and $|G_1| \leq 32$, this will imply that $\mathrm{Gal}(L/F) \cong G_1$.

Lemmas 2.4, 2.5 and Remark 1 following these lemmas, along with the fact that σ_a and σ_b generate $\mathrm{Gal}(L/F)$, imply that $\sigma_a^4 = [\sigma_a, \sigma_b]^2 = [[\sigma_a, \sigma_b], \sigma_a]^2 = 1$, and $[[\sigma_a, \sigma_b], \sigma_a] \in Z(G_1)$. It remains to prove $\sigma_b^2 = 1$.

Since we know $\mathrm{Gal}(E(\sqrt{\beta d})/F) \cong D$, $\mathrm{Gal}(E(\sqrt{\beta d})/F(\sqrt{ab})) \cong C$ and $\sigma_b(\sqrt{ab}) = -\sqrt{ab}$, it follows that $\sigma_b|_{E(\sqrt{\beta d})}$ has order 2. Thus $\sigma_b|_K$ has order 2. We now show $\sigma_b^2(\sqrt{\gamma}) = \sqrt{\gamma}$. We have $\gamma\sigma_b(\gamma) = \alpha d$ and thus $\sqrt{\gamma}\sigma_b(\sqrt{\gamma}) = (-1)^\epsilon\sqrt{\alpha d}$, where $\epsilon \in \{0, 1\}$. Then $\sigma_b(\sqrt{\gamma})\sigma_b^2(\sqrt{\gamma}) = \sigma_b((-1)^\epsilon\sqrt{\alpha d}) = (-1)^\epsilon\sqrt{\alpha d}$. These equations imply $\sigma_b^2(\sqrt{\gamma}) = \sqrt{\gamma}$.

It follows that $\mathrm{Gal}(L/F) \cong G_1$ since the relations in Definition 3.4 hold and $[L : F] = 32$. □

Theorem 3.7. *The following statements are equivalent.*

1. *There is a Galois extension K/F such that* $\mathrm{Gal}(K/F) \cong D \leftthreetimes C$.
2. *There is a Galois extension L/F such that* $\mathrm{Gal}(L/F) \cong G_1$.
3. *There exist $a, b \in F$ such that $[F(\sqrt{a}, \sqrt{b}) : F] = 4$ and $(a, a)_F = (a, b)_F = 0$.*
4. *There exists $a \in D_F(\langle 1, 1\rangle) \backslash \dot{F}^2$ such that $|D_F(\langle 1, -a\rangle)| \geq 4$.*
5. *F is not rigid and $D_F(\langle 1, 1\rangle) \not\subseteq F^2 \cup -F^2$.*
6. *Either $-1 \in F^2$ and F is not rigid or $-1 \notin F^2$ and $D_F(\langle 1, 1\rangle) \not\subseteq F^2 \cup -F^2$.*

If F is formally real, then statements (1)-(6) hold if and only if F is not a Pythagorean field.

Proof. We have already seen in Proposition 3.2 and the proof of Proposition 3.6 that (1)-(3) are equivalent. Note that (2) implies (1) because $D \leftthreetimes C$ is a quotient of G_1. The equivalence of (3) and (4) is easy to check. It is clear that (5) implies (6).

(6) $\Rightarrow$ (3): First assume $-1 \in F^2$ and F is not rigid. Then there exists $a \in F$ with $a \notin \pm F^2$ such that $b \in D_F(\langle 1, a\rangle)$ and $b \notin F^2 \cup aF^2$. Then (3) holds since $(a, b)_F = 0$ and $-1 \in F^2$ implies $(a, a)_F = (a, -a)_F = 0$. Now assume $-1 \notin F^2$ and $D_F(\langle 1, 1\rangle) \not\subseteq F^2 \cup -F^2$. Let $a \in D_F(\langle 1, 1\rangle)$ with $a \notin \pm F^2$. Then $[F(\sqrt{a}, \sqrt{-1}) : F] = 4$ and $(a, a)_F = 0$. Now (3) holds with $b = -a$.

(3) $\Rightarrow$ (5): Suppose $D_F(\langle 1, 1\rangle) \subseteq F^2 \cup -F^2$. We have $a \in D_F(\langle 1, 1\rangle)$ since $(a, a)_F = 0$. Since $a \notin F^2$, we have $a \in -F^2$. Then $(-1, b)_F = 0$ and thus $b \in D_F(\langle 1, 1\rangle)$. This is a contradiction since $b \notin F^2 \cup aF^2 = F^2 \cup -F^2$. Therefore $D_F(\langle 1, 1\rangle) \not\subseteq F^2 \cup -F^2$. Since $(a, a)_F = (a, b)_F = 0$ implies $a, b \in D_F(\langle 1, -a\rangle)$, we have $1, a, b, ab \in D_F(\langle 1, -a\rangle)$ and so $D_F(\langle 1, -a\rangle)$ contains at least four square classes. Thus F is not rigid by Lemma 2.1.

Now assume F is formally real. If (6) holds, then $-1 \notin F^2$ and $D_F(\langle 1, 1\rangle) \not\subseteq F^2 \cup -F^2$. Thus F is not a Pythagorean field. If F is not a Pythagorean field, then $D_F(\langle 1, 1\rangle) \not\subseteq F^2 \cup -F^2$, since F is formally real. Thus (6) holds. □

Nondyadic local fields are rigid. Every dyadic local field F satisfies conditions (1)-(6). In fact, a can be chosen as any nonsquare element in $D_F(\langle 1, 1\rangle)$. All global fields (of characteristic different from 2) also satisfy conditions (1)-(6).

Proposition 3.8. *Suppose* $\mathrm{Gal}(L/F) \cong G_1$. *Then there is a unique quadratic extension* $F(\sqrt{a})$ *of* F *such that*

$$F \subseteq F(\sqrt{a}) \subseteq E \subseteq L,$$

for some subfield E *of* L *where* $\mathrm{Gal}(E/F) \cong C$. *The element* $-a$ *is not rigid.*

Proof. Let $K = F^{(3)} \cap L$. Then $\mathrm{Gal}(K/F) \cong D \wr C$. Corollary 3.3 implies there is a unique quadratic extension $F(\sqrt{a})$ of F in K such that F is contained in a cyclic quartic extension E lying in K. The construction in Proposition 3.1 along with Proposition 3.2 and Theorem 3.7 imply $-a$ is not rigid. □

In the notation of this section, the two cyclic quartic extensions that appear in Corollary 3.3 are easily seen to be $F(\sqrt{a}, \sqrt{\alpha d})$ and $F(\sqrt{a}, \sqrt{\alpha d b})$.

4. The groups $D \wr D$ and G_2 as Galois groups

Let $\lambda_1 : D \to \mathbb{Z}/2\mathbb{Z}$ be a group homomorphism with $\ker(\lambda_1) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and let $\lambda_2 : D \to \mathbb{Z}/2\mathbb{Z}$ be another group homomorphism with $\ker(\lambda_2) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Let $D \wr D$ denote the pullback of this pair of homomorphisms. Thus $D \wr D$ is the subgroup of $D \times D$ of order 32 consisting of elements (u, v) such that $\lambda_1(u) = \lambda_2(v)$. Note that the subgroup $D \wr D$ projects onto D with maps $\pi_1 : D \wr D \to D$ and $\pi_2 : D \wr D \to D$ such that $\lambda_1 \circ \pi_1 = \lambda_2 \circ \pi_2$.

Proposition 4.1. *The following statements are equivalent.*

1. *There is a Galois extension* K/F *such that* $\mathrm{Gal}(K/F) \cong D \wr D$.
2. *There exist elements* $a, b, c \in F$ *such that* $[F(\sqrt{a}, \sqrt{b}, \sqrt{c}) : F] = 8$ *and* $(a, b)_F = (a, c)_F = 0$ *in the Brauer group* $Br(F)$.

Proof. First assume that (2) holds. Then there exist Galois extensions $D^{a,b}$ and $D^{a,c}$ of F. Let K denote a composite of $D^{a,b}$ and $D^{a,c}$. Then $D^{a,b} \cap D^{a,c} = F(\sqrt{a})$ since the intersection is a Galois extension of F and the only Galois subextension of $D^{a,b}$ of degree 4 over F is $F(\sqrt{a}, \sqrt{b})$. Therefore, K/F is a Galois extension of F with $[K : F] = 32$. Now a proof that is very similar to the proof of Proposition 3.1 shows that $\mathrm{Gal}(K/F) \cong D \wr D$.

Now assume that (1) holds and let $\mathrm{Gal}(K/F) \cong D \wr D$. The kernel of $\lambda_1 \circ \pi_1 = \lambda_2 \circ \pi_2$ has order 16 and thus the subfield of K corresponding to this kernel is a quadratic extension of F which we denote $F(\sqrt{a})$. Since both π_1 and π_2 are surjective, it follows that $\ker(\pi_1)$ and $\ker(\pi_2)$ are each normal subgroups of $D \wr D$ of order 4 which correspond to subfields E_1, E_2 of K which are Galois over F with $\mathrm{Gal}(E_1/F) \cong D \cong$

$\mathrm{Gal}(E_2/F)$. Since $\ker(\pi_1) \cap \ker(\pi_2) = 1$, it follows that $\ker(\pi_1)$ and $\ker(\pi_2)$ generate a subgroup of order 16. This subgroup is $\ker(\lambda_1 \circ \pi_1)$ since $\ker(\pi_i) \subseteq \ker(\lambda_1 \circ \pi_1) = \ker(\lambda_2 \circ \pi_2)$. Thus $E_1 \cap E_2 = F(\sqrt{a})$. Since $\ker(\lambda_i) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, it follows E_1 has the form $D^{a,b}$ and E_2 has the form $D^{a,c}$ for some $b, c \in \dot{F}$. Thus $(a,b)_F = (a,c)_F = 0$. We have $[F(\sqrt{a}, \sqrt{b}, \sqrt{c}) : F] = 8$ since $D^{a,b} \cap D^{a,c} = F(\sqrt{a})$. □

Proposition 4.2. *The group $D \curlywedge D$ contains a unique abelian subgroup of order 16. This subgroup equals $\ker(\lambda_1) \times \ker(\lambda_2)$ and is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$.*

Proof. We have that $\ker(\lambda_1) \times \ker(\lambda_2)$ is an abelian subgroup of $D \curlywedge D$ of order 16 that is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$. Now let H be any abelian subgroup of $D \curlywedge D$ of order 16. Since $\pi_i(H)$ is an abelian subgroup of D, we have $\pi_i(H)$ is a subgroup of order at most 4. Since $H \subseteq \pi_1(H) \times \pi_2(H)$ and H has order 16, it follows $H = \pi_1(H) \times \pi_2(H)$. Since $\pi_1(H) \times \pi_2(H) \subseteq D \curlywedge D$, it follows $\pi_1(H) = \ker(\lambda_1)$ and $\pi_2(H) = \ker(\lambda_2)$. □

Corollary 4.3. *In the notation of Proposition 4.1,*

$$\mathrm{Gal}(K/F(\sqrt{a})) = \ker(\lambda_1 \circ \pi_1) = \ker(\lambda_1) \times \ker(\lambda_2) \cong (\mathbb{Z}/2\mathbb{Z})^4.$$

Proof. The first equality follows from the proof of Proposition 4.1. Since $\ker(\lambda_1) \times \ker(\lambda_2) \subseteq \ker(\lambda_1 \circ \pi_1)$ and each has order 16, we have equality. □

Now we consider a group G_2 (defined below) of order 64 and study how to construct a Galois extension over a field F with Galois group isomorphic to G_2.

Definition 4.4. *Let G_2 be the group generated by three symbols x, y, z subject to the relations*

1. $x^2 = y^2 = z^2 = 1$, $[x,y]^2 = [x,z]^2 = [y,z] = 1$.
2. $[y,[x,z]] = [z,[x,y]]$ *has order dividing 2 and commutes with x, y and z.*

Lemma 4.5. *The following identities hold in G_2 and $|G_2| \leq 64$.*

1. $yx = xy[x,y]$, $zx = xz[x,z]$.
2. $[x,y]$ *commutes with both x and y and $[x,z]$ commutes with both x and z.*
3. $[x,z]y = y[x,z][y,\,[x,z]\,]$ *and* $[x,y]z = z[x,y][z,\,[x,y]\,]$.
4. $[x,z][x,y] = [x,y][x,z]$.

Proof. Statements (1)-(3) are proved as in Proposition 2.7. (See Remark 3 following the proof of Proposition 2.7.) Since $[x,y] = xyxy$,

and $[y,[x,z]]$ is in the center, we have

$$\begin{aligned}[x,z][x,y] &= [x,z]xyxy = x[x,z]yxy = xy[x,z][y,\,[x,z]\,]xy \\ &= xy[x,z]xy[y,\,[x,z]\,] = xyxy[x,z][y,\,[x,z]\,]^2 \\ &= [x,y][x,z],\end{aligned}$$

which proves (4).

To show $|G_2| \leq 64$, it is sufficient to show that each element of G_2 can be written in the form

$$x^{e_1}y^{e_2}z^{e_3}[x,y]^{e_4}[x,z]^{e_5}[y,\,[x,z]\,]^{e_6},$$

where each $e_i \in \{0,1\}$, $1 \leq i \leq 6$. The relations defining G_2 and statements (1)-(4) allow exactly this. □

We now give a construction of a Galois extension L/F with $[L:F] = 64$ and $\mathrm{Gal}(L/F) \cong G_2$.

Assume $a,b,c \in F$ such that $[F(\sqrt{a},\sqrt{b},\sqrt{c}):F] = 8$ and $(a,b)_F = (a,c)_F = 0$ in the Brauer group $Br(F)$. Then $a \in \mathrm{im}(N_{F(\sqrt{b})/F}) \cap \mathrm{im}(N_{F(\sqrt{c})/F})$ and so there exist $\beta \in F(\sqrt{b})$ and $\gamma \in F(\sqrt{c})$ such that $N_{F(\sqrt{b})/F}(\beta) = N_{F(\sqrt{c})/F}(\gamma) = a$. Let $E = F(\sqrt{b},\sqrt{c})$. Then [Wd], Lemma 2.14, implies there exist $\delta \in E$ and $d \in F$ such that $N_{E/F(\sqrt{b})}(\delta) = \beta d$ and $N_{E/F(\sqrt{c})}(\delta) = \gamma d$.

Let $E' = E(\sqrt{a})$, $K = E'(\sqrt{\beta d},\sqrt{\gamma d})$, and $L = K(\sqrt{\delta})$.

Proposition 4.6. 1. *K/F is a Galois extension with* $\mathrm{Gal}(K/F) \cong D \curlywedge D$.
2. *L/F is a Galois extension with* $\mathrm{Gal}(L/F) \cong G_2$.
3. *$K \subseteq F^{(3)}$, $L \subseteq F^{\{3\}}$, $L \nsubseteq F^{(3)}$.*

Proof. We have $F(\sqrt{a},\sqrt{b},\sqrt{\beta d}) = D^{a,b}$ and $F(\sqrt{a},\sqrt{c},\sqrt{\gamma d}) = D^{a,c}$. Since $D^{a,b}$ and $D^{a,c}$ are each Galois extensions of F, it follows that $K = D^{a,b}D^{a,c}$ is a Galois extension of F. The reasoning in the proof of Proposition 4.1 lets us conclude $\mathrm{Gal}(K/F) \cong D \curlywedge D$. This proves (1).

We have $D^{a,b}$ and $D^{a,c}$ contained in $F^{(3)}$ and thus $K \subseteq F^{(3)}$. Since $L = E'(\sqrt{\beta d},\sqrt{\gamma d},\sqrt{\delta})$ and βd, γd, $\delta \in F^{(2)}$, it follows $L \subseteq F^{\{3\}}$. If $L \subseteq F^{(3)}$, then $F^{(2)}(\sqrt{\delta})/F$ is a Galois extension. This implies $N_{E/F(\sqrt{b})}(\delta) = \beta d \in (F^{(2)})^2$. But this is impossible since $D^{a,b} \nsubseteq F^{(2)}$. This proves (3).

Since $[L:K] = 2$ by (3) and $[K:F] = 32$ by (1), it follows $[L:F] = 64$. Since $\delta \in E$ and K/F is a Galois extension, in order to show L/F is a Galois extension, it is sufficient to show $\sigma(\delta)\delta \in K^2$ for all $\sigma \in \mathrm{Gal}(E/F)$. This is accomplished as in the proof of Proposition 3.6.

Now we begin to show $\mathrm{Gal}(L/F) \cong G_2$. There exist automorphisms $\sigma_a, \sigma_b, \sigma_c \in \mathrm{Gal}(L/F)$ such that σ_a fixes $\sqrt{b}, \sqrt{c}$ and $\sigma_a(\sqrt{a}) = -\sqrt{a}$, and similarly σ_b fixes $\sqrt{a}, \sqrt{c}$, but not $\sqrt{b}$, and σ_c fixes $\sqrt{a}, \sqrt{b}$, but not $\sqrt{c}$.

We want to choose σ_b, σ_c more carefully as follows. Choose $\sigma_b \in \mathrm{Gal}(L/F)$ such that

$$\sigma_b|_{F(\sqrt{a},\sqrt{c},\sqrt{\gamma d})} = 1, \quad \sigma_b^2|_{F(\sqrt{a},\sqrt{b},\sqrt{\beta d})} = 1, \quad \sqrt{\beta d}\,\sigma_b(\sqrt{\beta d}) = \sqrt{a}d.$$

To see this is possible, observe that $\mathrm{Gal}(F(\sqrt{a},\sqrt{b},\sqrt{\beta d})/F) \cong D$ and hence there exists σ_b such that the second and third conditions hold. Since $K = F(\sqrt{a},\sqrt{b},\sqrt{\beta d})F(\sqrt{a},\sqrt{c},\sqrt{\gamma d})$, $\mathrm{Gal}(K/F) \cong D \curlywedge D$ and $\sigma_b|_{F(\sqrt{a})} = 1$, the first condition can be arranged simultaneously with the second and third.

Similarly we can choose $\sigma_c \in \mathrm{Gal}(L/F)$ such that

$$\sigma_c|_{F(\sqrt{a},\sqrt{b},\sqrt{\beta d})} = 1, \quad \sigma_c^2|_{F(\sqrt{a},\sqrt{c},\sqrt{\gamma d})} = 1, \quad \sqrt{\gamma d}\,\sigma_c(\sqrt{\gamma d}) = \sqrt{a}d.$$

We now show $\sigma_a, \sigma_b, \sigma_c$ satisfy the relations in Definition 4.4. Since σ_a fixes each element of E and $\delta \in E$, it follows that $\sigma_a^2(\sqrt{\delta}) = \sqrt{\delta}$. Similar reasoning shows that σ_a^2 fixes E' and K elementwise and hence $\sigma_a^2 = 1$.

We have that $\sigma_b^2|_K = \sigma_c^2|_K = 1$. To show $\sigma_b^2 = \sigma_c^2 = 1$, it is enough to show $\sigma_b^2(\sqrt{\delta}) = \sqrt{\delta}$ and $\sigma_c^2(\sqrt{\delta}) = \sqrt{\delta}$. Since

$$\delta\,\sigma_b(\delta) = N_{E/F(\sqrt{c})}(\delta) = \gamma d,$$

we have $\sqrt{\delta}\,\sigma_b(\sqrt{\delta}) = (-1)^{\epsilon}\sqrt{\gamma d}$ and thus

$$\sigma_b(\sqrt{\delta})\,\sigma_b^2(\sqrt{\delta}) = (-1)^{\epsilon}\sigma_b(\sqrt{\gamma d}) = (-1)^{\epsilon}\sqrt{\gamma d}.$$

This implies $\sigma_b^2(\sqrt{\delta}) = \sqrt{\delta}$. Since $\delta\,\sigma_c(\delta) = N_{E/F(\sqrt{b})}(\delta) = \beta d$, we have $\sqrt{\delta}\,\sigma_c(\sqrt{\delta}) = (-1)^{\epsilon'}\sqrt{\beta d}$, and a similar proof gives $\sigma_c^2(\sqrt{\delta}) = \sqrt{\delta}$.

We have $[\sigma_a, \sigma_b]^2 = [\sigma_a, \sigma_c]^2 = 1$ since $\mathrm{Gal}(L/F)$ is a homomorphic image of $G_F^{\{3\}}$. We have $[\sigma_b, \sigma_c]|_K = 1$ because $\sigma_b\sigma_c = \sigma_c\sigma_b$ on each subfield $F(\sqrt{a},\sqrt{b},\sqrt{\beta d})$ and $F(\sqrt{a},\sqrt{c},\sqrt{\gamma d})$. To show $[\sigma_b, \sigma_c] = 1$, it is enough to show $\sigma_b\sigma_c(\sqrt{\delta}) = \sigma_c\sigma_b(\sqrt{\delta})$.

The equation $\sqrt{\delta}\,\sigma_b(\sqrt{\delta}) = (-1)^{\epsilon}\sqrt{\gamma d}$ gives

$$\sigma_c(\sqrt{\delta})\,\sigma_c\,\sigma_b(\sqrt{\delta}) = (-1)^{\epsilon}\sigma_c(\sqrt{\gamma d}),$$

and the equation $\sqrt{\delta}\,\sigma_c(\sqrt{\delta}) = (-1)^{\epsilon'}\sqrt{\beta d}$ gives

$$\sigma_b(\sqrt{\delta})\,\sigma_b\,\sigma_c(\sqrt{\delta}) = (-1)^{\epsilon'}\sigma_b(\sqrt{\beta d}).$$

Since $\sqrt{\beta d}\,\sigma_b(\sqrt{\beta d}) = \sqrt{\gamma d}\,\sigma_c(\sqrt{\gamma d})$, some routine manipulation gives $\sigma_b\sigma_c(\sqrt{\delta}) = \sigma_c\sigma_b(\sqrt{\delta})$.

We now show $[\sigma_b,\ [\sigma_a,\sigma_c]\,] = [\sigma_c,\ [\sigma_a,\sigma_b]\,]$ has order dividing 2 and commutes with $\sigma_a,\sigma_b,\sigma_c$. The two expressions are equal by Corollary 2.10. The order divides 2 by Lemma 2.4. We have $[\sigma_a,\sigma_b]$ in the center of $\mathrm{Gal}(F(\sqrt{a},\sqrt{b},\sqrt{\beta d})/F)$ by Proposition 2.6 and thus $[\sigma_c,\ [\sigma_a,\sigma_b]\,]|_{F(\sqrt{a},\sqrt{b},\sqrt{\beta d})} = 1$. Similarly, $[\sigma_a,\sigma_c]$ is in the center of $\mathrm{Gal}(F(\sqrt{a},\sqrt{c},\sqrt{\gamma d})/F)$ and thus $[\sigma_b,\ [\sigma_a,\sigma_c]\,]|_{F(\sqrt{a},\sqrt{c},\sqrt{\gamma d})} = 1$. Thus, $[\sigma_b,\ [\sigma_a,\sigma_c]\,]|_K = 1$ and so

$$[\sigma_b,\ [\sigma_a,\sigma_c]\,] \in \mathrm{Gal}(L/K) \subseteq Z(\mathrm{Gal}(L/F)).$$

Therefore $[\sigma_b,\ [\sigma_a,\sigma_c]\,]$ commutes with $\sigma_a,\sigma_b,\sigma_c$.

We have now shown that $\sigma_a,\sigma_b,\sigma_c$ satisfy the relations in Definition 4.4. Next we show $\sigma_a,\sigma_b,\sigma_c$ generate $\mathrm{Gal}(L/F)$.

Let $H = \langle \sigma_a,\sigma_b,\sigma_c\rangle$. If $H \subsetneq \mathrm{Gal}(L/F)$, then there exists a quadratic extension $F(\sqrt{e})$ of F in the fixed field of H and $F(\sqrt{e}) \subseteq F^{(3)} \cap L = K$. An easy check shows that no quadratic extension of F inside $F(\sqrt{a},\sqrt{b},\sqrt{c})$ is in the fixed field of H. Then $F(\sqrt{a},\sqrt{b},\sqrt{c},\sqrt{e}) \subseteq K$ and $(\mathbb{Z}/2\mathbb{Z})^4$ is a quotient of $\mathrm{Gal}(K/F) \cong D \curlywedge D$ by a normal subgroup N of order 2. This implies the commutator subgroup of $D \curlywedge D$ is contained in N and this is impossible since the commutator subgroup of $D \curlywedge D$ has order 4. Thus $(\mathbb{Z}/2\mathbb{Z})^4$ is not a quotient of $D \curlywedge D$ and therefore $H = \mathrm{Gal}(L/F)$. Since $\sigma_a,\sigma_b,\sigma_c$ generate $\mathrm{Gal}(L/F)$ and satisfy the relations in Definition 4.4, it follows that $\mathrm{Gal}(L/F) \cong G_2$. □

Theorem 4.7. *The following statements are equivalent.*

1. *There is a Galois extension K/F such that* $\mathrm{Gal}(K/F) \cong D \curlywedge D$.
2. *There is a Galois extension L/F such that* $\mathrm{Gal}(L/F) \cong G_2$.
3. *There exist $a,b,c \in F$ such that $[F(\sqrt{a},\sqrt{b},\sqrt{c}) : F] = 8$ and $(a,b)_F = (a,c)_F = 0$.*
4. *There exist $a,b,c \in \dot{F}\backslash\dot{F}^2$ such that a,b,c are independent mod squares and $b,c \in D_F(\langle 1,-a\rangle)$.*
5. *There exists $a \in \dot{F}\backslash\dot{F}^2$ such that either $|D_F(\langle 1,-a\rangle)| \geq 8$ or both $|D_F(\langle 1,-a\rangle)| = 4$ and $a \notin D_F(\langle 1,1\rangle)$.*

Proof. (1) and (3) are equivalent by Proposition 4.1. The construction in Proposition 4.6 shows (3) implies (2). (2) implies (1) since $D \curlywedge D$ is a quotient of G_2 (as the construction in Proposition 4.6 shows). It is straightforward to check that (3)-(5) are equivalent using the observation $a \in D_F(\langle 1,-a\rangle)$ if and only if $a \in D_F(\langle 1,1\rangle)$. □

In the following proposition, DD denotes the central product of two copies of the dihedral group D. That is, if g denotes the nontrivial element in the center of D, then $DD \cong (D \times D)/\{1,(g,g)\}$. Thus DD

is obtained from the direct product $D \times D$ by identifying the centers of the two copies of D.

Proposition 4.8. *Suppose* $\mathrm{Gal}(L/F) \cong G_2$. *Let* $K = F^{(3)} \cap L$, *so that* $\mathrm{Gal}(K/F) \cong D \wr D$. *Then there is a unique quadratic extension* $F(\sqrt{a})$ *of* F *such that* $\mathrm{Gal}(K/F(\sqrt{a}))$ *is an abelian group of order* 16. *In addition the following hold.*

1. *There is a unique subgroup* N *of* G_2 *of order* 2 *such that* $G_2/N \cong D \wr D$.
2. $\mathrm{Gal}(L/F(\sqrt{a})) \cong DD$.
3. $\mathrm{Gal}(L/F(\sqrt{a}))$ *is the unique subgroup of* G_2 *isomorphic to* DD.
4. *The element* $-a$ *is nonrigid.*

Proof. Using Proposition 4.2, let $F(\sqrt{a})$ be the quadratic extension of F that corresponds to the unique abelian subgroup of $\mathrm{Gal}(K/F)$ of order 16. Any extension K' of F that lies in L with $\mathrm{Gal}(K'/F) \cong D \wr D$ must lie in $F^{(3)}$. Thus K is uniquely determined, so this proves (1).

In the notation of Proposition 4.6, let $D_1 = \langle \sigma_b, [\sigma_a, \sigma_c] \rangle$, let $D_2 = \langle \sigma_c, [\sigma_a, \sigma_b] \rangle$ and let $H = \langle \sigma_b, \sigma_c, [\sigma_a, \sigma_c], [\sigma_a, \sigma_b] \rangle$. Note that $D_1 \cong D_2 \cong D$ since σ_b and $[\sigma_a, \sigma_c]$ have order 2 and $[\sigma_b,\ [\sigma_a, \sigma_c]\,]$ is central of order 2. A similar argument holds for D_2. We have $|H| = 32$, by observing the representation of elements of G_2 at the end of the proof of Lemma 4.5. We show $\mathrm{Gal}(L/F(\sqrt{a})) \cong D_1 D_2 \cong DD$. The groups D_1 and D_2 commute elementwise by Definition 4.4 and Lemma 4.5. Therefore the injections $D_1 \mapsto H$ and $D_2 \mapsto H$ induce a surjective homomorphism $D_1 \times D_2 \mapsto H$. The element

$$([\sigma_b,\ [\sigma_a, \sigma_c]\,],\ [\sigma_c,\ [\sigma_a, \sigma_b]\,])$$

lies in the kernel of the map $D_1 \times D_2 \mapsto H$. Since $[\sigma_b,\ [\sigma_a, \sigma_c]\,]$ and $[\sigma_c,\ [\sigma_a, \sigma_b]\,]$ are the nontrivial elements in the center of D_1 and D_2, it follows that H is isomorphic to the central product $D_1 D_2 \cong DD$. It is clear that $H \subseteq \mathrm{Gal}(L/F(\sqrt{a}))$ and therefore $\mathrm{Gal}(L/F(\sqrt{a})) = H \cong DD$. This proves (2).

Suppose E is a quadratic extension of F such that $\mathrm{Gal}(L/E) \cong DD$. Then there is a subfield M of L such that $[L : M] = 2$ and $\mathrm{Gal}(M/E) \cong (\mathbb{Z}/2\mathbb{Z})^4$. (Note that $D \times D/(\{1, g\} \times \{1, g\}) \cong (\mathbb{Z}/2\mathbb{Z})^4$.) Thus $M = E(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_4})$, where $\alpha_i \in E$. The Galois closure of $E(\sqrt{\alpha_i})/F$ lies in $F^{(3)}$ for each i and hence the Galois closure of M/F lies in $F^{(3)}$. Thus M lies in $F^{(3)}$ and it follows $M = K$. Thus $\mathrm{Gal}(M/F) = \mathrm{Gal}(K/F) \cong D \wr D$. Since $D \wr D$ contains a unique abelian subgroup of order 16, it follows $E = F(\sqrt{a})$ and this proves (3).

We have from before that $(a, b)_F = (a, c)_F = 0$ and this implies $-a$ is not rigid by Lemma 2.1. $\square$

5. Main Theorems

Theorem 5.1. *F is not a rigid field if and only if either G_1 or G_2 occurs as a Galois group over F.*

Proof. First assume that F is not a rigid field. Then Lemma 2.1 implies there exists $a \in \dot{F}\backslash F^2$ such that $|D_F(\langle 1, -a\rangle)/\dot{F}^2| \geq 4$. Then either the statement in Theorem 3.7(4) holds or the statement in Theorem 4.7(5) holds. Therefore, either G_1 or G_2 occurs as a Galois group over F.

Now assume that either G_1 or G_2 occurs as a Galois group over F. Then by Theorems 3.7 and 4.7, there exists an element $a \in \dot{F}\backslash F^2$ such that $D_F(\langle 1, -a\rangle)$ contains at least four square classes. Then F is nonrigid by Lemma 2.1. □

Theorem 5.2. *Let $a \in \dot{F}\backslash(F^2 \cup -F^2)$. Then the element $-a$ is not rigid in F if and only if at least one of the following two statements holds.*

1. *An imbedding $F \subseteq F(\sqrt{a}) \subseteq E \subseteq L$ exists such that $\mathrm{Gal}(E/F) \cong C$ and $\mathrm{Gal}(L/F) \cong G_1$.*
2. *An imbedding $F \subseteq F(\sqrt{a}) \subseteq L$ exists such that $\mathrm{Gal}(L/F(\sqrt{a})) \cong DD$ and $\mathrm{Gal}(L/F) \cong G_2$.*

Proof. This follows from Proposition 3.8, Proposition 4.8, the construction in Proposition 3.6 and the construction in Proposition 4.6. □

Corollary 5.3. *Assume F is nonrigid. Let v be a valuation on F satisfying $1+M \subseteq \dot{F}^2$ and $B(F) = \dot{F}^2U$. Then the following statements are equivalent.*

1. *v is not 2-divisible.*
2. *There exists an element $a \in \dot{F}\backslash \pm F^2$ such that the two imbedding problems in Theorem 5.2 have no solutions for both $F(\sqrt{a})$ and $F(\sqrt{-a})$.*

Proof. If v is not 2-divisible, then Corollary 2.3 implies there exists an element $a \in \dot{F}\backslash \pm F^2$ such that a is double rigid. Now Theorem 5.2 implies that both statements in Theorem 5.2 fail for both $F(\sqrt{a})$ and $F(\sqrt{-a})$. Conversely, if both statements in Theorem 5.2 fail for both $F(\sqrt{a})$ and $F(\sqrt{-a})$, then a and $-a$ are rigid. Thus v is not 2-divisible by Corollary 2.3. □

6. The case $|\dot{F}/(\dot{F})^2| = 4$

Proposition 6.1. *Assume $|\dot{F}/(\dot{F})^2| = 4$.*

1. *The group G_1 occurs as a Galois group over F if and only if F is nonrigid.*
2. *If F is nonreal, then G_1 occurs as a Galois group over F if and only if $Br_2(F) = 0$ (i.e., $(a,b)_F = 0$ for all $a, b \in F$).*
3. *If F is formally real, then G_1 occurs as a Galois group over F if and only if F is uniquely ordered, which in this case is equivalent to F not being Pythagorean.*

Proof. (1). If G_1 occurs as a Galois group over F, then F is nonrigid by Theorem 3.7. Now assume F is nonrigid. Then Lemma 2.1 implies there exists $a \in \dot{F} \backslash F^2$ such that $|D_F(\langle 1, -a\rangle)/\dot{F}^2| \geq 4$. Thus $D_F(\langle 1, -a\rangle) = \dot{F}$. There exists $b \in F$ such that $\dot{F} = \{1, a, b, ab\}\dot{F}^2$ and so statement (3) of Theorem 3.7 holds and G_1 occurs as a Galois group over F.

(2). Let $F = F^2 \cup aF^2 \cup bF^2 \cup abF^2$. Then $[F(\sqrt{a}, \sqrt{b}) : F] = 4$. If $Br_2(F) = 0$, then $(a,a)_F = (a,b)_F = 0$. Thus G_1 occurs as a Galois group over F by Theorem 3.7.

Now assume G_1 occurs as a Galois group over F. Then there exist $a, b \in F$ such that $[F(\sqrt{a}, \sqrt{b}) : F] = 4$ and $(a,a)_F = (a,b)_F = 0$. It follows that $F = F^2 \cup aF^2 \cup bF^2 \cup abF^2$ and thus $(a, \dot{F})_F = 0$. Suppose first that either $-1 \in F^2$ or $a \in -F^2$. Then $(-1, \dot{F})_F = 0$. This implies $(b,b)_F = (-1,b)_F = 0$. Then $(b, \dot{F})_F = 0$ (since $(b,a)_F = 0$ and $(b,b)_F = 0$) and so $(ab, \dot{F})_F = 0$. Therefore $Br_2(F) = 0$.

Now assume $-1 \notin F^2$ and $a \notin -F^2$. Then we can assume $b = -1$ (since we also have $(a, ab)_F = 0$) and so $F = F^2 \cup -F^2 \cup aF^2 \cup -aF^2$. It is clear that $F^2 \cup aF^2 \subseteq D_F(\langle 1, a\rangle)$ and we have $F^2 \cup aF^2 \subseteq D_F(\langle 1, 1\rangle)$ since $(-1, a)_F = 0$. Since $F^2 \cup aF^2$ is not an ordering of F, we must have either $F^2 \cup aF^2 \subsetneq D_F(\langle 1, a\rangle)$ or $F^2 \cup aF^2 \subsetneq D_F(\langle 1, 1\rangle)$. In the first case we have $(-a, \dot{F})_F = 0$ and in the second case we have $(-1, \dot{F})_F = 0$. Each case implies $Br_2(F) = 0$.

(3). Since F is formally real and $|\dot{F}/(\dot{F})^2| = 4$, there is an ordering such that a is positive in this ordering and $F = F^2 \cup -F^2 \cup aF^2 \cup -aF^2$. If F is not Pythagorean then $a \in D_F(\langle 1, 1\rangle)$, and thus F is uniquely ordered since $F^2 \cup aF^2$ is contained in the positive cone of any ordering of F. If F is Pythagorean, then one checks that $-1, a \notin D_F(\langle 1, -a\rangle)$ and so $F^2 \cup -aF^2$ is also the positive cone of an ordering. Thus F is not uniquely ordered. The result now follows from Theorem 3.7. □

Proposition 6.2. *Assume $|\dot{F}/(\dot{F})^2| = 4$, F is formally real and not Pythagorean. Then $[F^{(3)} : F] = 16$ with $\mathrm{Gal}(F^{(3)}/F) \cong D \leftthreetimes C$, and $[F^{\{3\}} : F] = 32$ with $\mathrm{Gal}(F^{\{3\}}/F) \cong G_1$.*

Proof. We have $F = F^2 \cup -F^2 \cup aF^2 \cup -aF^2$ and $D_F(\langle 1,1\rangle) = F^2 \cup aF^2$ where $F^2 \cup aF^2$ is the positive cone of the unique ordering on F. Let $E = F(\sqrt{-1})$. Then the square class exact sequence ([La], p. 202) applied to E/F gives $|\dot{E}/(\dot{E})^2| = 4$ since $D_F(\langle 1,1\rangle)$ contains exactly two square classes. Since $F^{(2)} = E(\sqrt{a})$, the square class exact sequence applied to $F^{(2)}/E$ gives $|\dot{F}^{(2)}/(\dot{F}^{(2)})^2| \leq \frac{1}{2}|\dot{E}/(\dot{E})^2| = 8$. Therefore $[F^{\{3\}} : F^{(2)}] = |\dot{F}^{(2)}/(\dot{F}^{(2)})^2| \leq 8$ and so $[F^{\{3\}} : F] \leq 32$. Since G_1 occurs as a Galois group for some extension in $F^{\{3\}}$, it follows $[F^{\{3\}} : F] = 32$ and $\mathrm{Gal}(F^{\{3\}}/F) \cong G_1$. Since $D \wr C$ occurs as a Galois group over F and $F^{(3)} \subsetneq F^{\{3\}}$, it follows $\mathrm{Gal}(F^{(3)}/F) \cong D \wr C$. □

Proposition 6.3. *Assume $|\dot{F}/(\dot{F})^2| = 4$, F is nonreal and $Br_2(F) = 0$ (so that G_1 occurs as a Galois group over F). Then $[F^{(3)} : F] = 32$ and $[F^{\{3\}} : F] = 128$.*

Proof. We have $F^{(2)} = F(\sqrt{a}, \sqrt{b})$ for some $a, b \in F$. Let $E = F(\sqrt{a})$. The square class exact sequence applied to E/F gives $|\dot{E}/(\dot{E})^2| = \frac{1}{2}|\dot{F}/(\dot{F})^2| = 8$, since $u(F) = 2$ and so binary quadratic forms over F are universal. Since F is nonreal, it is known that $u(E) = 2$ also and thus the square class exact sequence applied to $F^{(2)}/E$ gives $|\dot{F}^{(2)}/(\dot{F}^{(2)})^2| = \frac{1}{2} \cdot 8^2 = 32$. Thus $[F^{\{3\}} : F^{(2)}] = |\dot{F}^{(2)}/(\dot{F}^{(2)})^2| = 32$ and so $[F^{\{3\}} : F] = 128$.

An argument in the proof of Proposition 3.6 shows that $\mathrm{Gal}(F^{(3)}/F)$ is generated by any two automorphisms σ_a, σ_b that satisfy $\sigma_a(\sqrt{a}) = -\sqrt{a}$, $\sigma_a(\sqrt{b}) = \sqrt{b}$, $\sigma_b(\sqrt{a}) = \sqrt{a}$, $\sigma_b(\sqrt{b}) = -\sqrt{b}$. Then Corollary 2.8(2) implies $|\mathrm{Gal}(F^{(3)}/F)| \leq 32$. One can check that $C^a C^b \subseteq F^{(3)}$ and $D^{a,b} \subseteq F^{(3)}$. Since $[C^a C^b : F] = 16$, $\mathrm{Gal}(C^a C^b/F)$ is abelian and $\mathrm{Gal}(D^{a,b}/F)$ is nonabelian, it follows $[F^{(3)} : F] = 32$. □

Generators and relations for $\mathrm{Gal}(F^{\{3\}}/F)$ and $\mathrm{Gal}(F^{(3)}/F)$ can be found using Proposition 2.7 and Corollary 2.8.

We now briefly consider the case when $|\dot{F}/(\dot{F})^2| = 4$ and F is a rigid field. Then $F^{(3)} = F^{\{3\}}$ by [LS], although when $|\dot{F}/(\dot{F})^2| = 4$, this is easily obtained by direct calculation. In the proposition below, $Q_8 \wr C$ denotes the pullback of the system of nontrivial homomorphisms $Q_8 \mapsto \mathbb{Z}/2\mathbb{Z}$ and $C \mapsto \mathbb{Z}/2\mathbb{Z}$, where Q_8 is the quaternion group of order 8. The group $Q_8 \wr C$ is also the pullback of the system $D \mapsto \mathbb{Z}/2\mathbb{Z}$ and $C \mapsto \mathbb{Z}/2\mathbb{Z}$, where the kernel of $D \mapsto \mathbb{Z}/2\mathbb{Z}$ is the cyclic group of order 4. See [GSS] for more details.

Proposition 6.4. *Assume $|\dot{F}/(\dot{F})^2| = 4$ and F is a rigid field.*

1. *If F is formally real, then $[F^{\{3\}} : F] = 8$ and $\mathrm{Gal}(F^{\{3\}}/F) \cong D$.*
2. *If F is nonreal, then $[F^{\{3\}} : F] = 16$.*

(a) *If* $-1 \in F^2$, *then* $\mathrm{Gal}(F^{\{3\}}/F) \cong C \times C$.
(b) *If* $-1 \notin F^2$, *then* $\mathrm{Gal}(F^{\{3\}}/F) \cong Q_8 \leftthreetimes C$.

Proof. (1). We have F is Pythagorean by Proposition 6.1 and $F^{(2)} = F(\sqrt{-1}, \sqrt{a})$ for some $a \in F$. Two applications of the square class exact sequence give $|F(\sqrt{-1})/F(\sqrt{-1})^2| = 2$ and $|\dot{F^{(2)}}/(\dot{F^{(2)}})^2| = 2$. Thus $[F^{\{3\}} : F^{(2)}] = 2$ and $[F^{\{3\}} : F] = 8$. We have $\mathrm{Gal}(F^{\{3\}}/F) \cong D$ since $D^{a,-a}$ is a dihedral extension of F.

(2). When F is nonreal, two applications of the square class exact sequence give $|\dot{F^{(2)}}/(\dot{F^{(2)}})^2| = 4$ and thus $[F^{\{3\}} : F] = 16$. Part (a) follows since each quadratic extension of F can be imbedded in a cyclic quartic extension of F. One way to show (b) is to see that C^{-1} and $D^{a,-a}$ are extensions of F and that $\mathrm{Gal}(D^{a,-a}/F(\sqrt{-1})) \cong C$. Then $\mathrm{Gal}(F^{\{3\}}/F) \cong Q_8 \leftthreetimes C$. □

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