Classification of Simple C^*-algebras of Tracial Topological Rank Zero *

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Abstract

We give a classification theorem for unital separable simple nuclear C^*-algebras with tracial topological rank zero which satisfy the Universal Coefficient Theorem. We prove that if A and B are two such C^*-algebras and

$$(K_0(A), K_0(A)_+, [1_A], K_1(A))$$

$$\cong (K_0(B), K_0(B)_+, [1_B], K_1(B)),$$

then $A \cong B$.

1 Introduction

There has been rapid progress in recent years in the program of classification of nuclear C^*-algebras, in particular, in the case that C^*-algebras are simple and have real rank zero. For example, Elliott and Gong ([EG2]) show that simple AH C^*-algebras of real rank zero (with slow dimension growth) can be classified (up to isomorphisms) by their scaled ordered K_0 and K_1. Kirchberg and Phillips ([K] and [P1]) show that separable nuclear purely infinite simple C^*-algebras satisfying the Universal Coefficient Theorem (UCT) can also be classified by their K-theory. These two results are the highlights of the program of classification of nuclear C^*-algebras initiated by G. A. Elliott (see [Ell2]). More recent results in the case of C^*-algebras which have real rank one can be found in a paper of Elliott, Gong and Li ([EGL]). Many other mathematicians have contributed to this program. (See for example, [BEEK], [BBEK], [D1], [D2], [DE2], [DL1], [DL2], [DG], [Ell1], [Ell2], [Ell3], [Ell4], [EG1], [EE], [EG2], [EGLP], [EGS], [ER], [ES], [G1], [G2], [JS], [BK1], [BK2], [Li1], [Li2], [Ln1]-[Ln9], [Ln11]-[Ln12], [LP1], [LP2], [LP3], [LS], [LqP], [NT], [P1], [P2], [Ro1], [Ro2], [Ro3], [ER], [Su1], [Su2], [Th1], [Th2] - an incomplete list). The application of the classification program began to spread to different fields, in particular, to the study of dynamical systems. In this paper we will only consider the case that C^*-algebras are simple and have real rank zero and stable rank one.

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During the development of classification of nuclear separable purely infinite simple C^*-algebras, a special class of inductive limits of finite direct sums of even Cuntz-algebras was first classified by Rørdam ([Ro1]). Rørdam's paper kicked off a series of work on classification of nuclear purely infinite simple C^*-algebras. More general examples of purely infinite simple C^*-algebras were classified, such as simple C^*-algebras which are inductive limits of O_n tensored with $C(S^1)$ ([LP1]) and those C^*-algebras arising as crossed products of stable simple AF-algebra with a $\mathbb{Z}$-action determined by an automorphisms ([Ro2]; see also [ER] and other papers listed above). These purely infinite simple C^*-algebras exhaust all possible K-theory. Later Kirchberg and Phillips classified separable nuclear purely infinite simple C^*-algebras (satisfying the UCT) without assuming an inductive limit structure or other special forms.

Naturally, it is important to classify a class of (unital) separable nuclear stably finite simple C^*-algebras without assuming they are inductive limits of certain simple forms.

In [Ln7] and [Ln10], we introduced the notion of tracial topological rank. Recall that a simple unital C^*-algebra A is said to have tracial topological rank zero (written $TR(A)=0$), if for any $\varepsilon>0$, any finite subset $\mathcal{F}\subset A$ and any $a\in A_+\setminus\{0\}$, there exists a nonzero projection p and a finite dimensional C^*-subalgebra $B\subset A$ with $1_B=p$ such that

(1) $\|px-xp\|<\varepsilon$,

(2) $\mathrm{dist}(pxp,F)<\varepsilon$ for all $x\in\mathcal{F}$ and

(3) $1-p$ is equivalent to a projection in $\mathrm{Her}(a)=\overline{aAa}$.

Roughly speaking, if an AF-algebra could be described as C^*-algebra which can be approximated by finite dimensional C^*-subalgebras in norm, then a C^*-algebra A with $TR(A)=0$ could be described as C^*-algebras which can be approximated by finite dimensional C^*-subalgebras in "measure," or rather in trace. This also resembles a key structure of unital separable nuclear purely infinite simple C^*-algebra: Every such C^*-algebra A has the following property: for any $\varepsilon>0$ any finite subset $\mathcal{F}\subset A$, there exists a nonzero projection p and a C^*-subalgebra $B\cong O_2$ such that

(1') $\|px-xp\|<\varepsilon$ and

(2') $\mathrm{dist}(pxp,B)<\varepsilon$ for all $x\in\mathcal{F}$.

((3') Note that $1-p$ is equivalent to a projection in $Her(a)=\overline{aAa}$ since A is purely infinite simple.) This property plays a very important role in the classification theorem of Kirchberg and Phillips.

It is shown (in [Ln7]) that a unital separable simple C^*-algebra A with $TR(A)=0$ has real rank zero, stable rank one, weakly unperforated $K_0(A)$ and is quasidiagonal. Simple AH-algebras with slow dimension growth have stable rank one (see [DNNP]) and are also quasidiagonal and have weakly unperforated K_0-groups. It is proved in [EG2] that all simple AH-algebras with slow dimension growth and with real rank zero have tracial topological rank zero. It seems that the class of simple C^*-algebras with tracial topological rank zero is the right substitute for class of quasidiagonal simple C^*-algebras with real rank zero, stable rank one and weakly unperforated K_0. It was shown in [Ln9] that unital separable simple C^*-algebras which are inductive limits of type I C^*-algebras with real rank zero, stable rank one, weakly unperforated K_0 and unique tracial states have tracial topological rank zero. N. C. Phillips ([LqP]) shows certain simple crossed products arising from smooth minimal diffeomorphisms have tracial topological rank zero. In studying non-commutative shift, recently, A. Kishimoto ([Ki]) shows that the crossed product of the two sided infinite tensor product $\otimes_{\mathbb{Z}}A$ by the shifts has tracial topological rank zero.

Tracial topological rank zero (or TAF) for simple C^*-algebras was introduced for the purpose of applying the so-called uniqueness theorem originally established in [Ln6]. An immediate consequence of the uniqueness theorem and the notion of TAF is that a unital separable simple nuclear C^*-algebras of tracial topological rank zero with the same K-theory as that of UHF-algebra Q with $K_0(Q) = \mathbb{Q}$ is actually isomorphic to Q ([Ln7]). Further attempts were made to classify all separable unital simple nuclear C^*-algebras with tracial topological rank zero. For example, it was shown that if A is a separable unital simple nuclear C^*-algebras with $RR(A) = 0$ and $K_0(A) = \mathbb{Q} \oplus \ker\tau$, where τ is the unique trace (see [Ln8] and [Ln11]), then it is isomorphic to any other such C^*-algebras. This was also obtained in [DE2] in the case that $K_0(A) = \mathbb{Q}$. It is also shown in [Ln8] that A is a unital separable simple nuclear C^*-algebra with $TR(A) = 0$ such that $K_0(A)$ is a subring of $\mathbb{Q}$ (such as $\mathbb{Z}[1/2]$) and $K_1(A)$ is torsion free then it is isomorphic to a unital $A\mathbb{T}$-algebra with the same K-theory.

In this paper by combining several previous results, we give a classification theorem for unital separable simple nuclear C^*-algebras with tracial topological rank zero which satisfy the UCT. C^*-algebras in this class are classified (up to isomorphism) by their scaled ordered K_0-groups and K_1-groups.

The proof of classification theorem (5.2) as before consists of a uniqueness theorem, an existence theorem and an intertwining argument of Elliott. With the results in [Ln6] and the introduction of the notion of tracial topological rank (or TAF), the desired uniqueness theorem (5.1) was established in [Ln8]. However, the version of the (half) existence theorem (4.1) in [Ln8] (see also [DE2]) does not provide a true existence theorem for the purpose of classification of simple nuclear C^*-algebras with tracial topological rank zero. Extra conditions on the order structure of $K_0(A)$ were needed in [Ln8], [DE2] and [Ln11]. The main technical result in this paper is to recover the order information lost in 4.1.

In Section 2 we show that a nuclear separable simple C^*-algebras with tracial topological rank zero has an approximate structure which is similar to a standard inductive construction. Section 3 gives a special version of an interpolation property. In section 4 we establish the required existence theorem, by applying section 2 and 3 together with the half existence theorem (4.1 from [Ln8]) and a cutting lemma from our previous result in [Ln12]. After we establish the required existence theorem, the classification theorem can be established using the results already established in [Ln8].

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2 An Approximation Construction

Recall that a unital simple C^*-algebra A is said to have tracial topological rank zero (written $TR(A) = 0$), if for any finite subset $\mathcal{F} \subset A$ any $\varepsilon > 0$ and any nonzero element $a \in A_+$, there exists a finite dimensional C^*-subalgebra $B \subset A$ with $p = 1_B$ such that

(1) $\|px - xp\| < \varepsilon$,

(2) $\mathrm{dist}(pxp, B) < \varepsilon$ for all $x \in \mathcal{F}$ and

(3) $1-p$ is equivalent to a projection in $\overline{aAa}$.

2.1 We begin by reviewing a construction of C^*-algebras of real rank zero. This construction appeared in several places ([BBEK], [Go] and [D1]). We present a construction of Dadarlat's version. Recall that a C^*-algebra B is said to be residually finite dimensional (RFD), if B has a separating family of finite dimensional irreducible representations. If B is separable RFD C^*-algebra, then there is a sequence of finite dimensional irreducible representations $\{\pi_n\}$ of B which separates A. Let B be such a RFD C^*-algebra, $\{\pi_n\}$ be a separating sequence of finite dimensional irreducible representations and $\{x_n\}$ be a dense sequence of the unit ball in B. Suppose that the rank of π_n is $r(n)$.

Let $A_1 = B$. Set $R(2) = 1 + k(1)r(1)$ and $A_2 = M_{R(2)}$, Define a homomorphism $h_1 : A_1 \to A_2$ by

$$h_1(a) = \mathrm{diag}(a, \pi_1(a), ..., \pi_1(a)) \ \text{ for } \ a \in A_1,$$

where π_1 repeats $k(1)$ times. Let $A_3 = M_{R(3)}(A_2)$, where $R(3) = 1 + k(2,1)r(1) + k(2,2)r(2)$, and define $h_2 : A_2 \to A_3$ by

$$h_2(a) = \mathrm{diag}(a, \tilde{\pi}_1(a), ..., \tilde{\pi}_1(a), \tilde{\pi}_2(a), ..., \tilde{\pi}_2(a))$$

for $a \in A_2$, where $\tilde{\pi}_1 = \pi_1 \otimes \mathrm{id}_{k(1)r(1)}$, $\tilde{\pi}_2 = \pi_2 \otimes \mathrm{id}_{k(1)r(1)}$ and $\tilde{\pi}_1(a)$ repeats $k(2,1)$ times and $\tilde{\pi}_2(a)$ repeats $k(2,2)$ times. Continuing this fashion, we obtain a sequence of homomorphisms $h_n : A_n \to A_{n+1}$ and set $A = \lim_n(A_n, h_n)$. If $k(n,i) \to \infty$ as $n \to \infty$ for each i, as in [Ln7], A is simple and $TR(A) = 0$. Let $\phi_{n,n+m} = h_{m+n} \circ \cdots h_n$. Then

$$\phi_{1,n}(a) = \begin{pmatrix} a & & & & \\ & \Phi_1^{(n)}(a) & & & \\ & & \Phi_2^{(n)}(a) & & \\ & & & \ddots & \\ & & & & \Phi_n^{(n)}(a) \end{pmatrix}$$

for $a \in A_1$, where $\Phi_1^{(n)}(a)$ is $s(1,n)$ many copies of $\pi_1(a)$, $\Phi_2(a)$ is $s(2,n)$ many copies of $\pi_2(a)$,..., Φ_n is $s(n,n)$ copies of $\pi_n(a)$. From the construction, $\inf_n\{s(k,n)r(k)/\sum_{i=1}^n s(i,n)r(i)\}$ is positive for each k and $\sum_{i=1}^n s(i,n)r(i)/(1+\sum_{i=1}^n s(i,n)r(i)) \to 1$.

It is clear that this construction can be made even more general (for example, A_k is a finite direct sum of matrix algebra over B with different size) but still have the similar properties. It is clear while this construction is special, it is typical. But much more is true.

In what follows we will show that every nuclear separable simple C^*-algebra A with $TR(A) = 0$ has a similar structure, or at least an approximated one. We need to replace homomorphisms by approximate multiplicative morphisms and π_k will be replaced by those whose ranges are contained in finite dimensional C^*-algebras.

Lemma 2.2 (cf. 5.3 in [Ln7]) *Let A be a separable unital nuclear simple C^*-algebra with $RR(A) = 0$. Let $\{A_n\}$ be a sequence of nuclear RFD C^*-algebras such that $A_n \subset A_{n+1}$ and A is the closure of*

$\cup_n A_n$. *Fix $\varepsilon_n > 0$ which decreases to zero, a sequence of nonzero homomorphisms h_n from A_n to a finite dimensional C^*-algebra F_n and a finite subset $\mathcal{F}_n \subset A_n$. There exists a sequence of non-zero projections $\{e_n\}$ and a sequence of monomorphisms $h_n' : F_n \to A$ with $h_n'(1_{F_n}) = e_n$ satisfying the following:*

(1) $\|e_n x - x e_n\| < \varepsilon_n$ and

(2) $\|h_n' \circ h_n(x) - e_n a e_n\| < \varepsilon_n$ for all $x \in \mathcal{F}_n$.

Proof: Fix an integer n. Let $\varepsilon > 0$ and a finite subset $\mathcal{F}_n = \{x_1, x_2, ..., x_k\} \subset A_n$ be given. Let $I = \ker h_n \subset A_n$ and B be the hereditary C^*-subalgebra of A generated by I. Let C be the closure of $A_n + B$. We will show that for any $c \in C, bc, cb \in B$ for all $b \in B$. It suffices to consider those elements $c \in A_n$. Note that B is the closure of IAI. Fix $b \in B$. For any $\varepsilon > 0$, there is a positive element $e \in I$ such that

$$\|eb - b\| < \varepsilon.$$

We also have $ce \in I \subset B$ and $ceb \in B$ for all $c \in A_n$. Therefore

$$\|cb - ceb\| < \varepsilon.$$

This implies that $cb \in B$. Similarly, $bc \in B$. Thus C is a C^*-subalgebra containing B as a (closed) ideal. Since $C/B = A_n/B \cap I = A_n/I \cong F_n$, we see that $C/B \cong F_n$. To save notation, we may write $F_n = C/B$. Let $\pi : C \to C/B$ be the quotient map. Hence we may identify h_n with $\pi|_{A_n}$.

By 5.2 in [Ln8], every projection in F_n lifts to a projection in C. Note that B has real rank zero, so it admits an approximate identity consisting of projections. Now a standard argument (see Lemma 9.8 in [Ef]) shows that C contains a finite dimensional C^*-subalgebra F_n' such that $F_n' \cong F_n$ and $\pi(F_n') = F_n$. Thus there is a monomorphism $\tilde{h}_n : F_n \to F_n'$ such that $\pi \circ \tilde{h}_n = \mathrm{id}_{F_n}$. In particular, $\tilde{h}_n \circ \pi(x) - x \in B$ for all $x \in C$.

Let $q = 1_{F_n'}$, then $(1-q)C(1-q) = (1-q)B(1-q)$. Write $F_n' = M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_k}$ and assume that $d_i \in M_{n_i}$ are minimal projections in M_{n_i}. Let $\{e_0^{(m)}\}$ be an approximate identity for $(1-q)B(1-q)$ consisting of projections. Let $\{c_i^{(m)}\}$ be an approximate identity for $d_i B d_i$ consisting of projections, $i = 1, 2, ..., k$. Write the hereditary C^*-subalgebra of C generated by M_{n_i} as $M_{n_i}(d_i C d_i)$. Put $e_i^{(m)} = diag(c_i^{(m)}, c_i^{(m)}, ..., c_i^{(m)})$ ($c_i^{(m)}$ repeats n_i times) and $E_m = e_0^{(m)} + \sum_{i=1}^k e_i^{(m)}$. Then it is clear that $\{E_m\}$ forms an approximate identity for B consisting of projections. Furthermore, by construction,

$$E_m x = x E_m$$

for each $x \in F_n'$. Since $\tilde{h}_n \circ h_n(x_i) - x_i \in B$, for sufficiently large m,

$$\|(1 - E_m)(\tilde{h}_n \circ h_n(x_i) - x_i)(1 - E_m)\| < \varepsilon/2, \quad i = 1, 2, ..., k.$$

Moreover, since $E_m(\tilde{h}_n \circ h_n(x_i)) = (\tilde{h}_n \circ h_n(x_i))E_m$,

$$\|(1 - E_m)x_i - x_i(1 - E_m)\| < \varepsilon/2, \quad i = 1, 2, ..., k,$$

and $h_n'(a) = (1 - E_m)\tilde{h}_n(a)(1 - E_m)$ ($a \in F_n$) defines a homomorphism. Let $e_n = (1 - E_m)$. Then $h_n'(1_{F_n}) = e_n$,

$$\|e_n x_i - x_i e_n\| < \varepsilon/2 \quad \text{and} \quad \|h_n' \circ h_n(x_i) - e_n x_i e_n\| < \varepsilon$$

$i = 1, 2, ..., k$. ∎

Definition 2.3 *Let A and B be C^*-algebras, let $L : A \to B$ be a contractive completely positive linear map, let $\varepsilon > 0$ and let $\mathcal{F} \subset A$ be a subset. L is said to be $\mathcal{F}$-ε-multiplicative, if*

$$\|L(xy) - L(x)L(y)\| < \varepsilon$$

for all $x, y \in \mathcal{F}$.

2.4 Let A be a separable unital nuclear simple C^*-algebra with $TR(A) = 0$. It follows from a result of B. Blackadar and E. Kirchberg ([BK1] and [BK2]) that A is an inductive limit of RFD C^*-algebras. Let $\{A_n\}$ be a sequence of nuclear RFD C^*-algebras such that $A_n \subset A_{n+1}$ and A is the closure of $\cup_n A_n$. Fix a finite subset $\mathcal{F}_1 \subset A_1$, $\delta_0 > 0$ and a homomorphism h from A_1 to a finite dimensional C^*-subalgebra F_0. Suppose that $\{x_1, ..., x_n, ...\}$ is a dense sequence of elements in the unit ball of A. For the convenience, we assume that $x_1 \in \mathcal{F}_1$. Applying Lemma 2.1, we obtain a nonzero projection $e_0 > 0$ and a homomorphism $h' : F_0 \to A$ with $e_0 = h'(1_{F_0})$ such that

(i_0) $\|e_0 a - a e_0\| < \delta_0/8$ and

(ii_0) $\|h' \circ h(a) - e_0 a e_0\| < \delta_0/32$ for all $a \in \mathcal{F}_1$. Set $H = h' \circ h$.

Since $TR((1 - e_0)A(1 - e_0)) = 0$, there is a projection $q_1' \le (1 - e_0)$ and a finite dimensional C^*-subalgebra F_1' with $1_{F_1'} = q_1'$ such that

(i_0') $\|q_1' x - x q_1'\| < \delta_0/8$,

(ii_0') $\mathrm{dist}(q_1' x q_1', F_1') < \delta_0/32$ for all $x \in \mathcal{F}_1$ and

(iii_0') $\tau(1 - q_1') < 1/4$ for all tracial states on A.

Set $q_1 = q_1' + e_0$ and $F_1 = F_1' \oplus h(F_0)$. Thus

(i) $\|q_1 x - x q_1\| < \delta_0/4$,

(ii) $\mathrm{dist}(q_1 x q_1, F_1) < \delta_0/16$ for all $x \in \mathcal{F}_1$ and

(iii) $\tau(1 - q_1) < 1/4$ for all tracial states τ on A.

Let $\mathcal{F}_2$ be the union of $\{x_2\} \cup \mathcal{F}_1$ and a set S_1 of standard generators of F_1. Since A is nuclear, there is a contractive completely positive linear map $L_1' : q_1 A q_1 \to F_1$ such that $L_1'|_{F_1} = \mathrm{id}_{F_1}$. Set $L_1(a) = L_1'(q_1 a q_1)$ for all $a \in A$. Then L_1 is a contractive completely positive linear map from A to F_1 and $L_1|_{F_1} = \mathrm{id}_{F_1}$. Note that L_1 is $\{x_1\}$-$\delta_0/2$-multiplicative. There is $\delta_2' > 0$ such that for any $\mathcal{S}_1$-δ_2'-multiplicative contractive completely positive linear map L from F_1 there is a homomorphism h from F_1 such that

$$\|L(a) - h(a)\| < \delta_0/16$$

for all $a \in \{L_1(x_1)\} \cup S_1$. Let $\delta_2 = \min\{\delta_2', \delta_0/4\}$. Since $TR(A) = 0$, there is a projection q_2 and a finite dimensional C^*-subalgebra F_2 with $1_{F_2} = q_2$ such that

(i') $\|q_2 x - x q_2\| < \delta_2/4$ for all $x \in \mathcal{F}_2$,

(ii') $\mathrm{dist}(q_2 x q_2, F_2) < \delta_2/16$ for all $x \in \mathcal{F}_2$ and

(iii') $\tau(1 - q_2) < 1/8$ for tracial states on A.

Since A is nuclear, there is a contractive completely positive linear map $L_2' : q_2 A q_2 \to F_2$ such that $L_2'|_{F_2} = \mathrm{id}_{F_2}$. Set $L_2(a) = L_2'(q_2 a q_2)$ for all $a \in A$. Note that L_2 is a contractive completely positive linear map from A to F_2 such that $L_2|_{F_2} = \mathrm{id}_{F_2}$ and it is $\mathcal{F}_2$-δ_2-multiplicative. Thus there is a homomorphism $h_2 : F_1 \to F_2$ such that

$$\|L_2(a) - h_2(a)\| < \delta_0/16$$

for all $a \in \{L_1(x_1)\} \cup S_1$. Since $L_2(z) \neq 0$ (by (i'),(ii'))and (iii')) if $z \in S_1$, h_2 has to be injective.

Continuing this way, we obtain a sequence of finite subsets $\mathcal{F}_0, \mathcal{F}_1, ..., \mathcal{F}_n, ...$ in the unit ball of A which is dense, a sequence of decreasing positive numbers δ_n (with $\delta_n < \delta_{n-1}/4^n$), a sequence of projections $\{q_n\} \subset A$, a sequence of finite dimensional C^*-subalgebras F_n with $1_{F_n} = q_n$ and a sequence of homomorphisms $h_{n+1} \to F_n \to F_{n+1}$ such that

(1) $\|q_n x - x q_n\| < \delta_n/4$ for $\mathcal{F}_n$

(2) $\mathrm{dist}(q_n x_i q_n, F_n) < \delta_n/16$, $i = 1, ..., n$.

(3) $\tau(1-q_n) < 1/2^n$ for all tracial states τ on A,

(4) $\mathcal{F}_{n+1} \supset S_n$, where S_n is a set of standard generators of F_n,

(5) $\|L_{n+1}(a) - h_{n+1}(a)\| < 1/16^n$ for all $a \in \{L_n(\mathcal{F}_n)\} \cup \{S_n\}$, where $L_n : A \to F_n$ is a contractive completely positive linear map such that $L_n|_{F_n} = \mathrm{id}_{F_n}$, $n = 1, 2, ...$

Let $D_{n,1}, D_{n,2}, ..., D_{n,m(n)}$ be simple summands of F_n and $\pi_{n,i} : F_n \to D_{n,i}$ be the quotient map. Let $\Psi_n : A \to (1-q_n)A(1-q_n)$ defined by $\Psi_n(a) = (1-q_n)a(1-q_n)$ for $a \in A$. Note that Ψ_n are $\mathcal{F}_n$-$\delta_n/2$-multiplicative. Define $J_n : A \to A$ by $J_n(a) = L_n(a) \oplus \Psi_n(a)$ (for $a \in A$). Note that J_n is $\mathcal{F}_n$-$\delta_n/2$-multiplicative. Set $J_{m,n} = J_n \circ J_{n-1} \circ \cdots \circ J_m$ and $h_{m,n} = h_n \circ h_{n-1} \circ \cdots h_{m+1} : F_m \to F_n$. Note also that $J_{m,n}$ is $\mathcal{F}_m$-δ_m-multiplicative. To save notation, we will also use L_n, Ψ_n, J_n, $J_{m,n}$, h_m, and $h_{m,n}$ for $L_n \otimes \mathrm{id}_{M_k}$, $\Psi_n \otimes \mathrm{id}_{M_k}$, $J_n \otimes \mathrm{id}_{M_k}$ and $J_{m,n} \otimes \mathrm{id}_{M_k}$, $h_m \otimes \mathrm{id}_{M_k}$ and $h_{m,n} \otimes \mathrm{id}_{M_k}$.

The following has been discussed in many recent papers (cf. [Ln8], [Ln12] and [DE2]). We present here for the sake of notation.

Definition 2.5 Let A be a nuclear C^*-algebra and B be a σ-unital C^*-algebra. Let $\mathcal{T}(A, B)$ be the set of those extensions of B by A whose six-term exact sequence in K-theory gives pure extensions

$$0 \to K_i(B) \to K_i(E) \to K_i(A) \to 0 \quad i = 0, 1$$

(i.e., every finitely generated subgroup of $K_i(B)$ splits). Following Rørdam, set $KL(A, B) = Ext(A, SB)/\mathcal{T}(A, SB)$. Let $\beta \in KL(A, B)$, then β gives an homomorphism from $K_0(A) \to K_0(B)$. Suppose that both A and B are stably finite. Denote by $KL(A, B)_+$ those β which induce positive homomorphisms.

Let C_n be a commutative C^*-algebra with $K_0(C_n) = \mathbb{Z}/n\mathbb{Z}$ and $K_1(C_n) = 0$. Suppose that A is a C^*-algebra. Then $K_i(A, \mathbb{Z}/k\mathbb{Z}) = K_i(A \otimes C_k)$. Let $\mathbf{P}(A)$ be the set of all projections in $M_\infty(A)$, $M_\infty(C(S^1) \otimes A)$, $M_\infty((A \otimes C_m)\tilde{})$ and $M_\infty((C(S^1) \otimes A \otimes C_m)\tilde{})$. We have the following commutative diagram ([Sc2]):

$$\begin{array}{ccccc} K_0(A) & \to & K_0(A, \mathbb{Z}/k\mathbb{Z}) & \to & K_1(A) \\ \uparrow_{\mathbf{k}} & & & & \downarrow_{\mathbf{k}} \\ K_0(A) & \leftarrow & K_1(A, \mathbb{Z}/k\mathbb{Z}) & \leftarrow & K_1(A) \end{array}$$

As in [DL1], we use the notation

$$\underline{K}(A) = \oplus_{i=0,1, n \in \mathbb{Z}_+} K_i(A; \mathbb{Z}/n\mathbb{Z}).$$

By $Hom_\Lambda(\underline{K}(A), \underline{K}(B))$ we mean all homomorphisms from $\underline{K}(A)$ to $\underline{K}(B)$ which respect the direct sum decomposition and the so-called Bockstein operations (see [DL1]). It follows from [DL1] that if A satisfies the Universal Coefficient Theorem, then $Hom_\Lambda(\underline{K}(A), \underline{K}(B)) = KL(A, B)$.

Let A and B be two C^*-algebras and $L : A \to B$ a completely positive linear map. Then L induces maps from $A \otimes C_m \to B \otimes C_m$, from $C(S^1) \otimes A \otimes C_m$ to $C(S^1) \otimes B \otimes C_m$, namely, $L \otimes \mathrm{id}$. For convenience, we will also denote the induced map by L. Given a projection $p \in \mathbf{P}(A)$, if L is $\mathcal{G}$-ε-multiplicative with sufficiently large $\mathcal{G}$ and sufficiently small ε, $L(p)$ is close to a projection. Let $L(p)'$ be that projection. Fix finite subsets of $\mathcal{P}_1 \subset \mathbf{P}(A)$. It is easy to see that $L(p)'$ and $L(q)'$ are in the same equivalence class of projections of $\mathbf{P}(A)$, if p and q are in $\mathcal{P}_1$ and are in the same equivalence class of projections of $\mathbf{P}(A)$, provided that $\mathcal{F}$ is sufficiently large and ε is sufficiently small. We use $[L](p)$ for the class of projections containing $[L](p)'$. In what follows, whenever we write $[L](p)$, we assume that $\mathcal{F}$ is sufficiently large and ε is sufficiently small so that $[L](p)$ is well-defined on $\mathcal{P}_1$. Furthermore, abusing the language, we write $[L]([p])$ as well as $[L](p)$, where $[p]$ is the equivalence class containing p. Suppose that q is in $\mathbf{P}(A)$ with $[q] = k[p]$ for some integer k, by adding sufficiently many elements (partial isometries) in $\mathcal{F}$, we can assume that $[L](q) = k[L](p)$. Suppose that G is a finitely generated group generated by $\mathcal{P}$ and $G = \mathbb{Z}^n \oplus \mathbb{Z}/k_1\mathbb{Z} \oplus \cdots \mathbb{Z}/k_m\mathbb{Z}$. Let $g_1, g_2, ..., g_n$ be free generators of $\mathbb{Z}^n$ and $t_i \in \mathbb{Z}/k_i\mathbb{Z}$ be the generator with order k_i, $i = 1, 2, ..., m$. Since every element in $K_0(C)$ (for any unital C^*-algebra C) may be written as $[p_1] - [p_2]$ for projections $p_1, p_2 \in A \otimes M_l$, for some $l > 0$, with sufficiently large $\mathcal{F}$ and sufficiently small ε, one can define $[L](g_j)$ and $[L](t_i)$. Moreover (with sufficiently large $\mathcal{F}$ and sufficiently small ε), the order of $[L](t_i)$ divides k_i. Then we can define a map $[L]|_G$ by defining $[L](\sum_i^n n_i g_i + \sum_j^m m_j t_j) = \sum_i^k n_i [L](g_i) + \sum_j^m m_j [L](t_j)$. Thus $[L]$ is a group homomorphism on G. Note, in general, $[L]|_{\mathcal{P}}$ may *not* coincide with $[L]|_G$ on $\mathcal{P}$. However, if $\mathcal{F}$ is large enough and ε is small enough, they coincide. In what follows, if $\mathcal{P}$ is given, we say $[L]|_G$ is well-defined and write $[L]|_G$ if $[L]|_{\mathcal{P}}$ is well-defined, $[L]|_G$ is well-defined and is a homomorphism and $[L]|_{\mathcal{P}} = [L]|_G$ on $\mathcal{P}$. We will also use $[L]|_{\mathcal{P}}$ for $[L]|_{[\mathcal{P}]}$ whenever it is convenient.

Definition 2.6 Let S be a compact convex set and $\mathrm{Aff}(S)$ be the space of all affine continuous functions on S. $\mathrm{Aff}(S)$ has the following order:

$$\mathrm{Aff}(S)_+ = \{f \in \mathrm{Aff}(S) : f(s) > 0 \ \ s \in S\} \cup \{0\}.$$

This is the order we use in this paper.

Let A be a simple C^*-algebra with $TR(A) = 0$ and with the tracial space $T(A)$. If $\tau \in T(A)$, we extend it to a trace on $A \otimes M_n$ on every n by using $\tau \otimes Tr$, where Tr is the standard trace on M_n. We will use τ for the extension.

Denote by $\rho_A : K_0(A) \to \mathrm{Aff}(T(A))$ the homomorphism defined by $\rho_B([p])(\tau) = \tau(p)$ for all $\tau \in T(A)$ and for projections $p \in A \otimes M_n$ $(n = 1, 2, ...)$. We will often write $\tau([p])$ for $\rho_A([p])$.

We now return to the construction in 2.4.

Lemma 2.7 *Let $\mathcal{P} \subset M_k(A)$ be a finite subset of projections. Assume that $\mathcal{P} \subset M_k(A_1)$ and $\mathcal{F}_1$ is sufficiently large and δ_0 is sufficiently small. With notation in 2.4, we may assume that $[L_{1+n} \circ J_{1,n}]|_{\mathcal{P}}$ and $[L_{n+1} \circ J_{1,n}]|_G$ are well defined, where G is the subgroup determined by $\mathcal{P}$, and*

$$\lim_{n\to\infty} \tau([L_{1+n} \circ J_{1,n}]([p])) = \tau([p])$$

and the convergent is uniform on $T(A)$. Furthermore,

$$|\tau(h_{1+n} \circ \cdots \circ h_1([p])) - \tau(h_n \circ \cdots \circ h_1([p])| < 1/2^{n+1}$$

for all $\tau \in T(A)$ and

$$\lim_{n\to\infty} \tau(h_{1+n}\circ\cdots h_1([p])\geq (1-\sum_{k=1}^{\infty}2^{k+1})\tau(h_1([p]))>0.$$

for $p \in \mathcal{P}$ and for all tracial states on A (and the convergent is uniform on $T(A)$).

Proof: Choose a finite subset $\mathcal{F}_1 \subset A_1$ and $\delta_0 > 0$ so that $[L]|_{\mathcal{P}}$ and $[L]|_G$ are well defined provided that L is a $\mathcal{F}_1$-δ_0-multiplicative contractive completely positive linear map. Without loss of generality, we may assume that $\delta_0 < 1/2$. This implies that $[p] = [J_{1,n}(p)]$ for $p \in \mathcal{P}$. In particular $\tau([p]) = \tau([J_{1,n}(p)])$ for $p \in \mathcal{P}$. The first limit formula together with the uniformness of the convergence follows from the fact that $\tau(1-q_n) < 1/2^{n+1}$ for all tracial states τ and Theorem 6.8 in [Ln10] (see also 3.4 and 3.6 in [Ln7]) that A has the fundamental comparison property of Blackadar. Second formula also follows from the the same fact that $\tau(1-q_k) < 1/2^{k+1}$, $k = 1, 2, \ldots.$ ∎

2.8 Fix a finite subset $\mathcal{P}$ of projections of A. Let N be a sufficiently large integer. Then

$$\begin{aligned}
[L_{N+1}\circ J_N] &= [L_{N+1}\circ L_N]\oplus[L_{N+1}\circ\Psi_N]\\
&= [h_{N+1}\circ(\mathrm{diag}(\pi_{N,1},\pi_{N,2},...,\pi_{N,m(N)}))\circ[L_N]\oplus[L_{N+1}\circ\Psi_N]
\end{aligned}$$

on $\mathcal{P}$, and

$$\begin{aligned}
[L_{N+2}\circ J_{N,N+1}] &= [L_{N+2}\circ L_{N+1}\circ J_N]\oplus[L_{N+2}]\circ[\Psi_{N+1}\circ J_N]\\
&= [L_{N+2}\circ L_{N+1}\circ L_N]\oplus[L_{N+2}\circ L_{N+1}\circ\Psi_N]\oplus[L_{N+2}]\circ[\Psi_{N+1}\circ J_N]\\
&= [h_{N,N+2}\circ(\mathrm{diag}(\pi_{N,1},\pi_{N,2},...,\pi_{N,m(N)}))\circ[L_N]\\
&\qquad\oplus[L_{N+2}\circ L_{N+1}\circ\Psi_N]\oplus[L_{N+2}]\circ[\Psi_{N+1}\circ J_N].
\end{aligned}$$

Moreover,

$$\begin{aligned}
[L_{N+n}\circ J_{N,N+n-1}] &= [h_{N,N+n}\circ(\mathrm{diag}(\pi_{N,1},\pi_{N,2},...,\pi_{N,m(N)}))\circ[L_N]\\
&\quad\oplus[L_{N+n}\circ\Psi_{N+n-1}\circ J_{N,N+n-2}]\oplus[L_{N+n}\circ L_{N+n-1}\circ\Psi_{N+n-2}\circ J_{N,N+n-3}]\\
&\qquad\oplus\cdots\oplus[L_{N+n}\circ\cdots L_{N+1}\circ\Psi_N].
\end{aligned}$$

on $\mathcal{P}$. Denote by $\tilde{\psi}_{(N,i)} = \pi_{N,i}\circ[L_N]$, $\tilde{\psi}_{(N+1,i)} = \pi_{N+1,i}\circ L_{N+1}\circ\Psi_N, ..., \tilde{\psi}_{(N+n,i)} = \pi_{N+n-1,i}\circ L_{N+n-1}\circ \Psi_{N+n-2}$. Let $c_{(N+n,i,m)}(\tau) = \tau(h_{N+n,N+n+m}\circ\tilde{\psi}_{N,i}(1_A)) > 0$. Moreover, we have computed (in 2.7) that $c_{(N+n,i,m)}(\tau)\to c_{(N+n,i)}(\tau) > 0$ uniformly (as $m\to\infty$) on $T(A)$.

Rearranging $\{\tilde{\psi}_{N+m,i}\}$ $(m = 1, 2, ...)$ as ψ_j, $j = 1, 2, ...$ Set $s(k) = \sum_{l=1}^k m(N+l-1)$. Suppose that the image of ψ_j is isomorphic to $M_{r(j)}$. So the rank of the image of ψ_j is $r(j)$.

2.9 Identify $K_0(M_{r(j)})$ with $\mathbb{Z}$. Let g_j be the element in $K_0(A)$ corresponding to the minimal projection in $\psi_j(A)$. Fix $p \in \mathcal{P}$. Set $z = tr\circ[\psi_j]([p])$, where tr is the normalized trace on matrix algebras. Then $(r(j)z)g_j \in K_0(A)$. If $\psi_j = \tilde{\psi}_{N+m,i}$, define $g_j^{(n)} = [h_{N+m,N+m+n}](g_j)$ and $a_j^{(n)} = c_{(N+m,i,n)}$. Then $\tau(g_j^{(n)}) = a_j^{(n)}(\tau))/r(j)$.

We have the following lemma

Lemma 2.10 *For any $p \in \mathcal{P}$,*

$$\tau([p]) = \lim_{k\to\infty} \sum_{j=1}^{s(k)} a_j^{(k)} tr \circ [\psi_j]([p]) \text{ uniformly on } T(A),$$

where tr denotes the normalized standard trace on matrix algebras. Furthermore, $a_j^{(k)} > 0$ and $a_j^{(k)} \to a_j > 0$ uniformly on $T(A)$ (as $k \to \infty$),$j = 1, 2, ..., s(k)$.

Proof: The expression of τ and the uniformness of the convergence follow from 2.7 and the fact that $\tau([\Psi_N + n](1)) = \tau(1 - q_{N+n}) \to 0$. The assertion that $a_j > 0$ follows from the fact that each $c_{(m,i,n)} > 0$. ∎

Corollary 2.11 *Let $G_0 = G \cap K_0(A)$ and let $\tilde{\rho} : G \to l^\infty(\mathbb{Q})$ be defined by*

$$g \mapsto (tr \circ [\psi_1](g), ..., tr \circ [\psi_n], ...).$$

Then, $\ker\tilde{\rho} \subset \ker\tau$.

Proof: Clearly from 2.10, if $tr \circ [\psi_j](g) = 0$ for every j, then $\tau(g) = 0$. ∎

2.12 Summery: The above construction (roughly) says, if $\varepsilon > 0$, $\delta > 0$, finite subset of projections in $A_N \otimes M_k$ (for some $k > 0$) and finite subset $\mathcal{F} \subset A_N$ are given, we may write ($a \in \mathcal{F}$)

$$a \approx_\varepsilon \begin{pmatrix} \Psi_n(a) & & & & \\ & \Phi_1^{(n)}(a) & & & \\ & & \Phi_2^{(n)}(a) & & \\ & & & \ddots & \\ & & & & \Phi_{s(n)}^{(n)}(a) \end{pmatrix},$$

where $\Phi_1^{(n)} = h_{N,N+n} \circ \psi_1, \Phi_{s(k)}^{(n)} = h_{N+n-1,N+n} \circ \psi_{s(n)}$. For any tracial state τ, we have $\tau(\Phi_j^{(n)})(a) = a_j^n(\tau){\cdot}tr(\psi_1(a))$. It is important that $a_j^{(n)}$ converges to some positive element $a_j \in \mathrm{Aff}(T(A))$ uniformly on $T(A)$ and

$$|\tau([p]) - \sum_{j=1}^{s(n)} a_j^{(n)} tr([\psi_j](p)| < \delta$$

for all $\tau \in T(A)$ and for $p \in \mathcal{P}$.

This concludes this section.

3 A Technical Lemma

3.1 Let S be a compact convex set. Let r be a positive integer or $r = \infty$ and $\mathbb{D} \subset \mathrm{Aff}(S)$ be a subgroup. Set $l_r^\infty(\mathbb{D})$ be set of r-tuple of elements in $\mathbb{D}$ (or the set of bounded sequences in $\mathbb{D}$).

If $f = \{f_n\} \in l_r^\infty(\mathbb{D})$, we write

$$\|f\|_\infty = \sup\{\|f_n\| : n = 1, 2, ..., \}.$$

The following lemma will not be used in this paper. But it makes the next lemma much more transparent.

Let $\{x_{ij}\}$ be a $r \times \infty$ matrix, $a_j^{(n)} > 0$ such that $a_j^{(n)} \to a_j > 0$, $j = 1, 2, ...$ Suppose that

$$(x_{ij})_{r\times\infty}(a_j^{(n)})_{\infty\times 1} \to z = (z_j)_{1\times\infty}.$$

We want to find finitely many non-negative $b_1, ..., b_n$ for some integer $n > 0$ such that

$$(x_{ij})_{r\times n}(b_j)_{n\times 1} = (z_j)_{n\times 1}.$$

The following lemma states that much more than this is true. Note that both constant δ and integer K are important in the lemma.

Lemma 3.2 *Let S be a compact convex set and $\mathrm{Aff}(S)$ be the space of all affine continuous functions on S. Let $\mathbb{D}$ be a dense ordered subgroup of $\mathrm{Aff}(S)$. Let $\{x_{ij}\}_{0<i\le r, 0<j<\infty}$ be a $r\times\infty$ matrix having rank r and with each $x_{ij} \in \mathbb{Q}$, and let $\{a_j^{(n)}\}$ be sequences of positive elements in $\mathbb{D}$ such that $a_j^{(n)} \to a_j(>0)$ uniformly on S as $n \to \infty$. For each n,*

$$(x_{ij})_{r\times n} v_n = y_n,$$

where $v_n = (a_j^{(n)})_{n\times 1}$ is an $n \times 1$ column vector and $y_n = (b_i^{(n)}) \in \mathbb{D}^r$ is an $r \times 1$ column vector.

Suppose that $y_n \to z$ for some $z = (z_j)_{r\times 1} \in \mathbb{D}^r$ uniformly (in $\mathrm{Aff}(S)^r$ norm).

Then there is $\delta > 0$ and a positive integer $K > 0$ satisfying the following:

For some sufficiently large n, there is $u = (c_j)_{n\times 1} \in (1/K)(\mathbb{D}_+^n)$ (where $c_j(\tau) > 0$ for all $\tau \in S$ or $c_j = 0$) such that

$$(x_{ij})_{r\times n} u = z'$$

if $z' \in \mathbb{D}_+^r$ and $\|z - z'\|_\infty < \delta$.

Proof: To save notation, without loss of generality, we may assume that $(x_{ij})_{r\times r}$ has rank r. Set $A_n = (x_{ij})_{r\times n}$ $(n \ge r)$. Then there exists an invertible matrix $B \in M_r(\mathbb{Q})$ (which does not depend on n) such that $BA_n = C_n$, where $C_n = (c_{ij})_{r\times n}$, $c_{ii} = 1$ for $i = 1, 2, ..., r$, and $c_{ij} = 0$ if $i \ne j$, $j = 1, 2, ..., r$, and $c_{ij} \in \mathbb{Q}$. Since $B \in M_r(\mathbb{Q})$, there is a positive integer $K > 0$ such that $K \cdot B \in M_r(\mathbb{Z})$. Moreover, $K \cdot C_n \in M_{r\times n}(\mathbb{Z})$.

Let I_r be the $r \times r$ identity matrix. We may write

$$C_n = (I_r, D_n'),$$

where D_n' is a $r \times (n-r)$ matrix. Note that $K \cdot D_n' \in M_{r\times(n-r)}(\mathbb{Z})$. Thus we have

$$C_n v_n = By_n \quad \text{and} \quad I_r v_n' = By_n - D_n v_n,$$

where $v_n' = (a_1^{(n)}, a_2^{(n)}, ..., a_r^{(n)})$ (as a column) and $D_n = (0, D_n')$ is a $r \times n$ matrix. Note that for any $n \times 1$ column vector v with the form $(t_1, t_2, ..., t_r, a_{r+1}^{(n)}, a_{r+2}^{(n)}, ..., a_n^{(n)})$, $D_n v = D_n v_n$. Since $a_j^{(n)} \to a_j > 0$ uniformly on S and S is compact, there is $N_1 > 0$ such that

$$a_j^{(n)}(\tau) \ge \inf\{a_j(\tau)/2 : \tau \in S\} > 0 \quad)s \in S)$$

for all $n \geq N_1$ and $j = 1, 2, ..., r$. Let $0 < \varepsilon < min\{\inf\{a_j(\tau)\}/8 : \tau \in S, j = 1, 2, ..., r\}$. There is $N_2 > 0$ such that

$$\|By_n - Bz\|_\infty < \varepsilon/4$$

if $n \geq N_2$. There is $\delta > 0$ depending only on B such that, if $\|z - z'\|_\infty < \delta$,

$$\|Bz - Bz'\|_\infty < \varepsilon/4$$

(B is determined by $\{x_{ij}\}$). Therefore

$$\|By_n - Bz'\|_\infty < \varepsilon/2$$

for all $n \geq N_2$. Set $N = max\{N_1, N_2\}$. Let

$$u' = Bz' - D_n v_n,$$

where $u' = (c_1, c_2, .., c_r)$ (column vector). Since $I_r v_n' = v_n'$ and $I_r u' = u'$, we have

$$\|u' - v_n'\|_\infty = \|(Bz' - D_n v_n) - (By_n - D_n v_n)\| < \varepsilon$$

if $n \geq N$. Therefore $c_j > 0$ for $j = 1, 2, ..., r$. Set $u = (c_1, ..., c_r, a_{r+1}^{(n)}, a_{r+2}^{(n)}, ..., a_n^{(n)})$. Then

$$I_r u' = Bz' - D_n u$$

($n \geq N$). Since $I_r u' = Bz' - D_n u$, $Bz' \in (1/K)\mathbb{D}^r$ and $K \cdot D_n \in M_{r\times(n-r)}(\mathbb{Z})$

$$(c_1, c_2, ..., c_r) = u' = I_r u' \in (1/K)(\mathbb{D}^r)_+.$$

Since $D_n = C_n - I_r$, we have

$$C_n u = Bz'.$$

Finally, since B is invertible, we have

$$A_n u = z'.$$

∎

3.3 Let G be a group and $A = (a_{ij})_{m\times k} \in M_{m\times k}(\mathbb{Z})$. Viewing $y = (g_1, ..., g_k) \in G^k$ as a column, one defines $Ay = (\sum_{j=1}^k a_{1j}g_j, \sum_{j=1}^k a_{2j}g_j, ..., \sum_{j=1}^k a_{mj}g_j)$. Thus A maps G^k to G^m. If $A \in M_{m\times k}(\mathbb{Z})$ and $B \in M_{r\times m}(\mathbb{Z})$, then $B(Ay) = (BA)(y)$. Note that if $B \in M_{r\times m}(\mathbb{Q})$ but $BA \in M_{r\times k}(\mathbb{Z})$ does not imply $B(Ay) = (BA)(y)$ in general, since G might have torsion.

Lemma 3.4 *Let S be a compact convex set and $\mathrm{Aff}(S)$ be the set of all affine continuous functions on S. Let $\mathbb{D}$ be a dense ordered subgroup of $\mathrm{Aff}(S)$ and G be an ordered group with the order determined by a surjective homomorphism $\rho : G \to \mathbb{D}$, i.e.,*

$$G_+ = \{g \in G : \rho(g)(\tau) > 0\} \cup \{0\}.$$

Let $\{x_{ij}\}_{0<i\leq r, 0<j<\infty}$ be a $r \times \infty$ matrix having rank r and with each $x_{ij} \in \mathbb{Q}_+$, $r(j) \in \mathbb{N}$ such that $r(j)x_{ij} \in \mathbb{Z}_+$ for all i, j, $g_j^{(n)} \in G$ such that $\rho(g_j^{(n)}) = a_j^{(n)}(\tau)/r(j)$ ($\tau \in S$), where $\{a_j^{(n)}\}$ is a sequence of positive elements in $\mathbb{D}$ such that $a_j^{(n)} \to a_j(> 0)$ uniformly on S as $n \to \infty$. For each n, ($s(n) \geq n$)

$$(r(j)x_{ij})_{r\times s(n)}\tilde{v}_n = \tilde{y}_n,$$

where $\tilde{v}_n = (g_j^{(n)})_{s(n)\times 1}$ *is a* $s(n)\times 1$ *column and* $\tilde{y}_n = (\tilde{b}_j^{(n)}) \in G^r$ *is a* $r\times 1$ *column vector. Set* $b_i^{(n)} = \rho(\tilde{b}_j^{(n)})$ *and* $y_n = (b_i^{(n)})$. *Suppose that* $y_n \to z$ *on* S *uniformly on* S *for some* $z = (z_j)_{r\times 1} \in \mathbb{D}^r$ *(in* $\mathrm{Aff}(S)^r$ *norm).*

Then there is $\delta > 0$ *and a positive integer* $K > 0$ *satisfying the following:*

For some sufficiently large n, *there is* $u = (\tilde{c}_j)_{s(n)\times 1} \in G_+^{s(n)}$ *(*$\rho(\tilde{c}_j) > 0$ *or* $\rho(\tilde{c}_j) = 0$ *) such that*

$$(r(j)x_{ij})_{r\times s(n)}\tilde{u} = \tilde{z}'$$

if $\tilde{z}' \in G^r$ *and there is* $\tilde{z}'' \in G^r$ *such that* $K^3\tilde{z}'' = \tilde{z}'$ *and* $\|z - Mz'\|_\infty < \delta$, *where* $\tilde{z}' = (\tilde{z}_1', ..., \tilde{z}_r')$ *such that* $z_j' = \rho(\tilde{z}_j')$, $j = 1, 2, ..., r$ *and* M *is a positive integer.*

Proof: The proof is a repetition of the proof of 3.2 with some modification. The important difference is that G may have torsion. We proceed as in the proof of 3.2. To save notation, without loss of generality, we may assume that $(x_{ij})_{r\times r}$ has rank r. Set $A_n' = (r(j)x_{ij})_{r\times s(n)} \in M_{r\times s(n)}(\mathbb{Z})$ $(s(n) \geq r)$. Then there exists an invertible matrix $B' \in M_r(\mathbb{Q})$ (which does not depend on n) such that $B'A_n' = C_n$, where $C = (c_{ij})_{r\times s(n)}$, $c_{ii} = 1$ for $i = 1, 2, ..., r$, and $c_{ij} = 0$ if $i \neq j$, $j = 1, 2, ..., r$, and $c_{ij} \in \mathbb{Q}$. Since $B' \in M_r(\mathbb{Q})$, there is a positive integer $K > 0$ such that $K\cdot B' \in M_r(\mathbb{Z})$. Moreover, $K\cdot C_n \in M_{r\times s(n)}(\mathbb{Z})$. We may also assume that $K\cdot (B')^{-1} \in M_r(\mathbb{Z})$.

Let I_r be the $r\times r$ identity matrix. We may write

$$C_n = (I_r, D_n'),$$

where D_n' is a $r\times (s(n)-r)$ matrix. Note that $K\cdot D_n' \in M_{r\times(s(n)-r)}(\mathbb{Z})$. Since $\mathbb{D}$ is dense in $\mathrm{Aff}(S)$, there are $\xi_n \in G^{s(n)}$ such that $\xi_n = (\tilde{d}_j^{(n)})_{s(n)\times 1}$ and

$$\|K^3\rho(\tilde{d}_j^{(n)}) - a_j^{(n)}/Mr(j)\| < \frac{1}{s(n)^2\cdot 2^n}, \quad j = 1, 2, ..., s(n), n = 1, 2, ...$$

Let $\tilde{w}_n = K^3\xi_n$ and $\rho(\tilde{d}_j^{(n)}) = d_j^{(n)}$. Then $K^3 d_j^{(n)} \to K^3 d_j = a_j/Mr(j) > 0$ uniformly on S. Set $\tilde{y}_n' = A_n'\tilde{w}_n = K^3A_n'\xi_n$. Note that $A_n'\xi_n \in G^r$ $(A_n' \in M_{r\times s(n)}(\mathbb{Z}))$. Let $\rho^{(s(n))} : G^{s(n)} \to \mathbb{D}^{s(n)}$ be defined by $\rho^{(s(n))}(g_1, ..., g_{s(n)}) = (\rho(g_1), ..., \rho(g_{s(n)}))$ for $g \in G$. Let $y_n' = \rho^{(s(n))}(\tilde{y}_n')$. Then, from the construction above, $y_n' \to z/M$ uniformly on S.

We have

$$C_n\tilde{w}_n = K^3C_n\xi_n = K^3B'A_n'\xi_n (= B'\tilde{y}_n') \quad \text{and} \quad I_r\tilde{v}_n' = B'\tilde{y}_n' - D_n\tilde{w}_n,$$

where $\tilde{v}_n' = (K^3\tilde{d}_1^{(n)}, ..., K^3\tilde{d}_r^{(n)})$ (as a column) and $D_n = (0, D_n')$ is a $r\times s(n)$ matrix. Note that for any $s(n)\times 1$ column vectors $\tilde{v}$ and v with the form $(s_1, ..., s_r, K^3\tilde{d}_{r+1}^{(n)}, K^3\tilde{d}_{r+2}^{(n)}, ..., K^3\tilde{d}_{s(n)}^{(n)})$ and $(t_1, t_2, ..., t_r, K^3d_{r+1}^{(n)}, K^3d_{r+2}^{(n)}, ..., K^3d_{s(n)}^{(n)})$, $D_n\tilde{v} = D_n\tilde{w}_n$ and $D_nv = D_nw_n$, where $\rho^{(s(n))}(\tilde{v}) = v$ and $\rho^{(s(n))}(\tilde{w}_n) = w_n$.

Since $d_j^{(n)} \to d_j > 0$ uniformly on S, there is an $N_1 > 0$ such that

$$d_j^{(n)} \geq \inf\{d_j/2(\tau) : \tau \in S\} > 0$$

for all $n \geq N_1$ and $j = 1, 2, ..., r$. Let $0 < \varepsilon < min\{\inf\{d_j(\tau) : \tau \in S\}/8K^3 : j = 1, 2, ..., r\}$. There is $N_2 > 0$ such that

$$\|B'y_n - B'z\|_\infty < \varepsilon/2$$

if $n \geq N_2$. Thus there is $N_3 > 0$ such that

$$\|B'(y'_n) - B'(z/M)\|_\infty < \varepsilon/2$$

if $n \geq N_3$ ($y'_n \to z$ uniformly).

There is $\delta > 0$ depending only on B' such that, if $\|z - Mz'\|_\infty < \delta$,

$$\|B'y'_n - B'z'\|_\infty < \varepsilon/2.$$

if $n \geq N_3$. Now let $\tilde{z}$ and $\tilde{z}''$ be as described in the lemma. Set $N = max\{N_1, N_2, N_3\}$. Let

$$B'\tilde{z}' - D_n\tilde{w}_n = K^3B\tilde{z}'' - K^3D_n\xi_n = K^2u',$$

where $u' = (\tilde{c}_1, \tilde{c}_2, .., \tilde{c}_r) \in G^r$ (column). Set $u'' = K^2u'$. Let $\rho^{(r)}(u') = (c_1, c_2, ..., c_r) \in \mathbb{D}^r$ We may write

$$I_ru'' = B'\tilde{z}' - D_n\tilde{w}_n.$$

Since $I_r\tilde{v}'_n = \tilde{v}'_n$ and $I_ru' = u'$, we have

$$\|\rho^{(r)}(u'') - \rho^{(r)}(\tilde{v}'_n)\|_\infty = \|\rho^{(r)}(B'\tilde{z}' - D_n\tilde{w}_n) - \rho^{(r)}(B'\tilde{y}'_n - D_n\tilde{w}_n)\|_\infty < \varepsilon$$

if $n \geq N$. Since $\rho^{(r)}(\tilde{v}'_n) = (K^3d_1^{(n)}, K^3d_2^{(n)}, ..., K^3d_r^{(n)})$, therefore $c_j(\tau) > 0$ for all $\tau \in S$ and $j = 1, 2, ..., r$. Set

$$\tilde{u} = (K^2\tilde{c}_1, ..., K^2\tilde{c}_r, K^3\tilde{d}_{r+1}^{(n)}, K^3\tilde{d}_{r+2}^{(n)}, ..., K^3\tilde{d}_{s(n)}^{(n)}) \quad \text{and} \quad \bar{u} = (\tilde{c}_1, ..., \tilde{c}_r, K\tilde{d}_{r+1}^{(n)}, ..., K\tilde{d}_{s(n)}^{(n)}).$$

Then $\bar{u} \in G_+^{s(n)}$ and

$$I_ru'' = K^3B'\tilde{z}'' - K^2D_n\bar{u} = B'\tilde{z}' - D_n\tilde{u}$$

$(n \geq N)$. Since $C_n = I_r + D_n$, we have

$$C_n\tilde{u} = K^2C_n\bar{u} = K^2\bar{u} + K^2D_n\bar{u} = I_r\tilde{u} + D_n\tilde{u}$$

$$= I_ru'' + D_n\tilde{w}_n = K^3B'\tilde{z}'' = B'\tilde{z}'.$$

We have $K(B')^{-1}, KB', KB' \in M_r(\mathbb{Z})$. Therefore

$$(B')^{-1}C_n\tilde{u} = K(B')^{-1}(KC_n)\bar{u} = K^2A_n\bar{u} = A_n\tilde{u}$$

and

$$(B')^{-1}B'\tilde{z} = (K \cdot (B')^{-1})(K \cdot (B'))(K\tilde{z}'') = (K^2I_r)(K\tilde{z}'') = \tilde{z}'.$$

Hence

$$A_n\tilde{u} = \tilde{z}'.$$

∎

4 An Existence Theorem

The following theorem was proved in [Ln7]. A more general version was obtained by M. Dadarlat and S. Eilers ([DE2]).

Theorem 4.1 (Theorem 5.9 in [Ln8]) *Let A be a separable C^*-algebra satisfying UCT such that A is the closure of an increasing sequence $\{A_n\}$ of RFD C^*-algebras and B be a unital nuclear separable C^*-algebra. Then, for any $\alpha \in Hom_\Lambda(\underline{K}(A), \underline{K}(B))$, there exist two sequences of completely positive contractions $\phi_n^{(i)} : A \to B \otimes \mathcal{K}$ ($i = 1, 2$) satisfying the following:*

(1) $\|\phi_n^{(i)}(ab) - \phi_n^{(i)}(a)\phi_n^{(i)}(b)\| \to 0$ as $n \to \infty$,

(2) for each n, the images of $\phi_n^{(2)}$ are contained in a finite dimensional C^*-subalgebra of $B \otimes \mathcal{K}$ and for any finite subset $\mathcal{P} \subset \mathbf{P}(A)$, $[\phi_n^{(2)}]|_{\mathcal{P}}$ and $[\phi_n^{(2)}]|_G$ are well defined for all large n, where G is the subgroup generated by $\mathcal{P}$,

(3) for each finite subset of $\mathcal{P} \subset \mathbf{P}(A)$, there exists $m > 0$ such that

$$[\phi_n^{(1)}]|_G = \alpha + [\phi_n^{(2)}]|_G$$

for all $n \geq m$.

(4) For each n, we may assume that $\phi_n^{(2)}$ is a homomorphism on A_n.

(The condition that B is nuclear can be replaced by the condition that each A_n is nuclear).

It is shown in [Ln7] that if A is a simple nuclear separable C^*-algebras with $TR(A) = 0$, then A is strong NF. It follows a result from B. Blackadar and E. Kirchberg ([BK1] and [BK2]) that every strong NF C^*-algebra is an inductive limit of residually finite dimensional (RFD) C^*-algebras. Thus the above theorem can be applied to simple nuclear separable C^*-algebras with $TR(A) = 0$ which satisfies the UCT.

However the term $\phi_n^{(2)}$ prevents us to apply the theorem directly.

The following is proved in [Ln12]:

Lemma 4.2 (Lemma 1.8 in [Ln12]) *Let A be a unital simple AH-algebra with slow dimension growth and with real rank zero. Let G_0 be a finitely generated subgroup of $K_0(A)$ with decomposition $G_0 = G_{00} \oplus G_{01}$, where $G_{00} \subset \ker\rho_A$ and G_{01} is a finite generated free group such that $(\rho_A)|_{G_{01}}$ is injective. Suppose that $\mathcal{P} \subset \underline{K}(A)$ is a finite subset which generates a subgroup G such that $G \cap K_0(A) \supset G_0$.*

Then, for any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$, any $1 > r > 0$, and any integer K, there is an $\mathcal{F}$-ε-multiplicative map $L : A \to A$ satisfying the following:

(1) $[L]|_{\mathcal{P}}$ and $[L]|_G$ are well-defined and $[L]|_G$ is positive on G,

(2) $[L]|_{G\cap\ker\rho_A} = \mathrm{id}|_{G\cap\ker\rho_A}$, $[L]|_{G\cap K_0(A,\mathbb{Z}/k\mathbb{Z})} = \mathrm{id}|_{G\cap K_0(A,\mathbb{Z}/k\mathbb{Z})}$, $[L]|_{G\cap K_1(A)} = \mathrm{id}|_{G\cap K_1(A)}$ and $[L]|_{G\cap K_1(A,\mathbb{Z}/k\mathbb{Z})} = \mathrm{id}|_{G\cap K_1(A,\mathbb{Z}/k\mathbb{Z})}$ for those k with $G \cap K_i(A, \mathbb{Z}/k\mathbb{Z}) \neq \emptyset$ $(i = 0, 1)$,

(3) $\rho_A \circ [L](g) \leq r\rho_A(g)$ for all $g \in G \cap K_0(A)$,

(4) Let $g_1, g_2, .., g_l$ be positive generators of G_{01}. Then, there are $f_1, ..., f_l \in K_0(A)_+$ such that

$$g_i - [L](g_i) = Kf_i, \ \ i = 1, 2, ..., l.$$

Combining the above two results and 3.4, we prove the existence theorem 4.3.

To establish the following theorem, we first apply 4.1 to obtain Φ_1. Then applying 4.2 to obtain L with a small r. We then try to construct h with its range contained in a finite dimensional C^*-subalgebra, and hope that $L \circ \Phi_1 \oplus h$ will be the right map. Given 4.1, the strategy is to construct h which is a homomorphism on a finite dimensional C^*-subalgebra. So it is at least possible to have finite dimensional range and does not effect most other KK-theoretical information. While this strategy sounds, the technical difficulties remain to be overcome. Immediately, one would need to extend some certain homomorphism from a subgroup of $K_0(C)$ (C is a finite dimensional C^*-algebra) to a homomorphism on $K_0(C)$. This surely requires some divisibility of the target group. The integer K in Lemma 4.2 comes to take care of that problem. However, the homomorphism on the group also has to be positive and the extension is also need to be positive. To make things worse, Φ_1 does not even preserve the order. Furthermore, one should also be careful when maps with finite dimensional ranges are used. They may not give trivial maps on (the part of) $K_0(A, \mathbb{Z}/k\mathbb{Z})$. Lemma 3.4 together with a careful and extensive use of the integer K in 4.2 will deal with these difficulties.

Here is the existence theorem:

Theorem 4.3 *Let A be a unital separable simple nuclear C^*-algebra with $TR(A) = 0$ which satisfies the UCT. Suppose that B is a unital simple AH-algebra with slow dimension growth, real rank zero and*

$$(K_0(B), K_0(B)_+, [1_B], K_1(B)) \cong (K_1(A), K_1(A)_+, [1_A], K_1(A)).$$

Let $\alpha \in KK(A, B)_+$ which carries the above isomorphism. Then, for any finite subset $\mathcal{P} \subset \mathbf{P}(A)$, there is a sequence of contractive completely positive linear map $H_n : A \to B$ such that

(i) $\|H_n(ab) - H_n(a)H_n(b)\| \to 0$ for all $a, b \in A$ as $n \to \infty$ and

(ii) $[H_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$ for all sufficiently large n.

Proof: To save the notation, we may assume

$$(K_0(B), K_0(B)_+, [1_B], K_1(B)) = (K_1(A), K_1(A)_+, [1_A], K_1(A)).$$

So we will identify these two 4-tuples. Let $\mathbb{D} = \rho(K_0(A))$. Note that $\mathbb{D}$ is a dense ordered subgroup of $\mathrm{Aff}(T(A))$.

Fix $\mathcal{P}$ and let $\mathcal{P}_0 \subset \mathcal{P}$ be so that $\mathcal{P}_0$ generates a (finitely generated) subgroup G_0 so that $G_0 = G(\mathcal{P}) \cap K_0(A)$, where $G(\mathcal{P})$ is the subgroup generated by $\mathcal{P}$. We may assume that $\mathcal{P} \subset \mathbf{P}(A_1)$. (Here we identify $\mathcal{P}$ with $j(\mathcal{P})$, where $j : A_1 \to A$ is the embedding.) We may assume that $\{p_1, ..., p_l\} = \mathcal{P}_0$, where $p_i \in M_k(A_1)$ are projections (for some $k > 0$). Let $\mathcal{F}_0$ be a finite subset of A_1 and $\delta_0 > 0$ so that any contractive completely positive linear map L from A_1 which is $\mathcal{F}_0$-δ_0-multiplicative well defines $[L]|_{\mathcal{P}}$ and $[L]|_{G(\mathcal{P})}$. Let k_0 be a positive integer such that

$$G(\mathcal{P}) \cap K_i(A, \mathbb{Z}/k\mathbb{Z}) = \emptyset$$

for all $k \geq k_0$, $i = 0, 1$.

Step (I): (Construct Φ_n and fix H)

It follows from 4.1 that there is a sequence of contractive completely positive linear map $\Phi_n :$ $A \to B \otimes \mathcal{K}$ such that

$$\|\Phi_n(ab) - \Phi_n(a)\Phi_n(b)\| \to 0$$

for all $a, b \in A$ as $n \to \infty$, there is a sequence of contractive completely positive linear maps $\{\Phi_n^{(0)}\}$ from A to B so that their images contained in a finite dimensional C^*-subalgebras and $\Phi_n^{(0)}|_{A_n}$ is a homomorphism, and

$$[\Phi_n]|_G = \alpha|_G + [\Phi_n^{(0)}]|_G \quad \text{and} \quad [\Phi_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}} + [\Phi_n^{(0)}]|_{\mathcal{P}}$$

for all n. Fix $\Phi_1^{(0)}$. Let $H : A_1 \to A$ be a homomorphism such that $H = h' \circ \Phi_1^{(0)}$, where h' is a monomorphism from the image of $\Phi_1^{(0)}$ to A. The existence of such h' is given by 2.1.

For any finite subset $\mathcal{F} \subset A$ and $\eta > 0$ ($\mathcal{F}_0 \subset \mathcal{F}$ and $\eta < \delta_0/2$), to save notation, we may assume that Φ_1 is $\mathcal{F}$-η-multiplicative. Fix this H, $\mathcal{F}$ and $\eta/4$ (with $N = 1$), let $q_n, \Psi_n, L_n, J_n, J_n, J_{m,n}, \psi_n, \delta_n$ (with $\sum_{n=1}^{\infty} \delta_n < \delta_0/2$) be as constructed in 2.4. (The importance of H will become clear later.)

Step (II): (Fix M, K and $\delta > 0$) Let $\tilde{\rho} : G(\mathcal{P}) \cap K_0(A) \to l^{\infty}(\mathbb{Q})$ be defined by

$$\tilde{\rho}([p_i]) = (tr(\psi_1(p_i)), ..., tr(\psi_m(p_i)), ...) \quad \text{for } i = 1, ..., l.$$

Note that $\tilde{\rho}(G_0)$ is a finitely generated torsion free group. Consider $\{\tilde{\rho}([p_1]), \tilde{\rho}([p_2]), ..., \tilde{\rho}([p_l])\}$. Suppose that the $\mathbb{Q}$-linear span of the above l elements (in $l^{\infty}(\mathbb{Q})$) has rank r. So a subset of $\{\tilde{\rho}([p_1]), \tilde{\rho}([p_2]), ..., \tilde{\rho}([p_l])\}$ with r elements has $\mathbb{Q}$- rank r. Without loss of generality, we may assume that $\{\tilde{\rho}([p_1]), \tilde{\rho}([p_2]), ..., \tilde{\rho}([p_r])\}$ has rank r and its $\mathbb{Q}$-span includes all $\tilde{\rho}([p_i]), i = 1, ..., l$. There is an integer $M > 0$ such that for any $g \in \tilde{\rho}(G_0)$, Mg is in the subgroup of $\tilde{G}_0$ generated by $\tilde{\rho}([p_1]), \tilde{\rho}([p_2]), ..., \tilde{\rho}([p_r])$. Let $x_{ij} = tr(\tilde{\psi}_j([p_i]))$, $i = 1, 2, ..., r$ and $j = 1, 2, ...$ So we may assume that $(x_{ij})_{r\times r}$ has $\mathbb{Q}$-rank r. Let g_j, $g_j^{(n)}$, $a_j^{(n)}$ and a_j be as in 2.9. Let $z_i = \rho_A([p_i]) \in \mathbb{D}$ and $z = (z_1, ..., z_r)$. We keep the notation in 2.4, 2.8 and 2.9. We note that $a_j^{(n)} \in \mathbb{D}_+ \setminus \{0\}$, $\lim_{n\to\infty} a_j^{(n)} = a_j > 0$ uniformly on S, $\sum_{j=1}^{n} a_j^{(n)})x_{ij} \to z_i$ (or $(r(j)x_{ij})v_n \to z$) uniformly on S for $i = 1, ..., r$ as $n \to \infty$ ($v_n = (\rho_A(g_1), \rho_A(g_2), ..., \rho_A(g_{s(n)})))$, by 2.10. So Lemma 3.4 can be applied. Fix $\delta > 0$ and integer $K > 0$ given by Lemma 3.4. We also note that since H was given before we construct q_n so summands of H appears in $\{\psi_j\}$. Therefore, if $g \in \ker\tilde{\rho}$, then $g \in \ker\tau$ and $g \in \ker([\Phi_1^{(0)}] \oplus [\Phi_1^{(0)}] \oplus \cdots \oplus [\Phi_1^{(0)}])$ (for any finitely many copies of $\Phi_1^{(0)}$).

Step (III). (Define Ψ_1)

Let $\tilde{\Phi}_1^{(0)}$ be a direct sum of $MK^3(k_0+1)! - 1$ copies of $\Phi_1^{(0)}$. Note that $\tilde{\Phi}_1^{(0)}$ is a homomorphism on A_1. Set $\Psi_1 = \Phi_1 \oplus \tilde{\Phi}_1^{(0)}$. If F is a finite dimensional C^*-algebra then one has the following commutative diagram:

$$\begin{array}{ccccc} K_0(F) & \longrightarrow & K_0(F, \mathbb{Z}/k\mathbb{Z}) & \longrightarrow & K_1(F) \\ \uparrow & & & & \downarrow \\ K_0(F) & \longleftarrow & K_1(F, \mathbb{Z}/k\mathbb{Z}) & \longleftarrow & K_1(F) \end{array}$$

where $K_0(F, \mathbb{Z}/k\mathbb{Z}) = K_0(F)/kK_0(F)$, $K_1(F) = 0$, $K_1(F, \mathbb{Z}/k\mathbb{Z}) = 0$. Since $\Phi_1^{(0)}$ factors through a finite dimensional C^*-subalgebra, it is easy to check that

$$[\Phi_1^{(0)}]|_{K_1(A)\cap G} = 0, \quad \text{and} \quad [\Phi_1^{(0)}]|_{K_1(A, \mathbb{Z}/k\mathbb{Z})\cap G} = 0.$$

Moreover,

$$(k_0)![\Phi_1^{(0)}]|_{K_0(A, \mathbb{Z}/k\mathbb{Z})\cap G} = 0 \ \ (k \le k_0).$$

Therefore

$$[\Psi_1]|_{K_1(A)\cap G} = \alpha|_{K_1(A)\cap G}, \ \ [\Psi_1]|_{K_1(A, \mathbb{Z}/k\mathbb{Z})\cap G} = \alpha|_{K_1(A, \mathbb{Z}/k\mathbb{Z})\cap G},$$

and $[\Psi_1]|_{K_0(A,\mathbb{Z}/k\mathbb{Z})\cap G} = \alpha|_{K_0(A,\mathbb{Z}/k\mathbb{Z})\cap G}$. Without loss of generality, we may assume that $\Psi_1(1_A)$ is a projection in $M_R(B)$.

Step (IV) (Cut Ψ_1 according to M, K, k_0 and δ)

We may assume that there are projections $\bar{p}_1, ..., \bar{p}_l$ in $M_k(B)$ (for some $k > 0$) such that

$$\|\bar{p}_i - \Psi_1(p_i)\| < \eta < 1/4, \quad i = 1, ..., l.$$

So $[\bar{p}_i] = [\Psi_1(p_i)]$, $i = 1, ..., l$. There are projections $\bar{q}_i \le \bar{p}_i$ such that $[\bar{q}_i] = K(k_0+1)![\Phi_1^{(0)}(p_i)]$. Set $\bar{e}_i = \bar{p}_i - \bar{q}_i$. Note that $[\bar{e}_i] = \alpha([p_i])$, $i = 1, 2, ..., l$. Set $\mathcal{P}_1 = \{\mathcal{P} \cup [\tilde{\Psi}_1(\mathcal{P})]\} \cup \mathcal{P}_0' \cup \alpha(\mathcal{P})$, where $\mathcal{P}_0' = \{[\bar{q}_i], \bar{e}_i, [\bar{p}_i], [\Phi_1^{(0)}([p_i]) : i = 1, ..., l\}$. Set $G_1 = G(\mathcal{P}_1)$. We write $G_0 = G_{00} \oplus G_{01}$, where $G_{00} = G_0 \cap \ker\tau$ and $G_{01} \cong \tau(G_0)$. Let $d_1, ..., d_t$ be positive and generate G_{01}.

By applying 4.2, for any $\varepsilon > 0$, any finite subset $\mathcal{G} \subset M_R(B)$, any $0 < r < \delta < 1$, there is a $\mathcal{G}$-ε-multiplicative map $L : M_R(B) \to M_R(B)$ satisfying the following:

(1) $[L]|_{\mathcal{P}_1}$ and $[L]|_{G_1}$ are well-defined and $[L]|_{G_1}$ is positive on G_1,

(2) $[L]|_{G_1\cap\ker\rho_B} = \mathrm{id}|_{G_1\cap\ker\rho_B}$, $[L]|_{G_1\cap K_0(B,\mathbb{Z}/k\mathbb{Z})} = \mathrm{id}|_{G_1\cap K_0(B,\mathbb{Z}/k\mathbb{Z})}$, $[L]|_{G_1\cap K_1(B)} = \mathrm{id}|_{G\cap K_1(B)}$ and
$[L]|_{G_1\cap K_1(B,\mathbb{Z}/k\mathbb{Z})} = \mathrm{id}|_{G_1\cap K_1(B,\mathbb{Z}/k\mathbb{Z})}$ for those k with $G_1 \cap K_i(B,\mathbb{Z}/k\mathbb{Z}) \ne \emptyset$ $(i = 0, 1)$,

(3) $\rho_B \circ [L](g) \le r\rho_B(g)$ for all $g \in G \cap K_0(B)$,

(4) There are $f_1, ..., f_l \in K_0(A)_+$ such that

$$\alpha(d_i) - [L](\alpha(d_i)) = MK^3(k_0+1)!f_i, \ \ i = 1, 2, ..., t.$$

(Note that we may identify α with identity map.) We choose r so that

$$R \cdot r\rho_B([\Phi_1]([p_i])) \le (\delta/2MK^3(k_0+1)!)\rho(\alpha([p_i])).$$

This is possible since $\tau(\alpha([p_i]) = \tau([p_i]) > 0$ for all i (A and B are simple) and $T(A)$ is compact. Therefore

$$\tau \circ [L] \circ [\Psi_1]([p_i]) \le (\delta/2)\tau(\alpha([p_i])) \quad \text{and} \quad \alpha([p_i]) - [L \circ \Psi_1]([p_i]) > 0$$

for all $\tau \in T(A)$, $i = 1, ..., l$.

Let $[p_i] = \sum_j^t m_j^{(i)} d_j + s$, where $m_j^{(i)} \in \mathbb{Z}$ and $s \in G_{00}$. Then by (2) above, $\alpha(s) - [L] \circ \alpha(s) = 0$. Therefore

$$\begin{aligned} \alpha([p_i]) - [L \circ \Psi_1])([p_i]) &= \alpha(\sum_j m_j^{(i)} d_j) - [L \circ \alpha](\sum_j m_j^{(i)} d_j) - MK^3(k_0+1)![L \circ \Phi_1^{(0)}]([p_i]) \\ &= MK^3(k_0+1)!(\sum_j m_j^{(i)} f_j - [L] \circ [\Phi_1^{(0)}]([p_i])) = MK^3(k_0+1)!f_j' \end{aligned}$$

for some $f_j' \in K_0(B)$, $j = 1, 2, ..., l$.

Since $K_0(B)$ is weakly unperforated, $f_i' > 0$ for $i = 1, 2, ..., l$. Set $\beta : G(\mathcal{P}) \cap K_0(A) \to K_0(B)$ by $\beta([p_i]) = K^3(k_0+1)!f_i'$, $i = 1, 2, ..., l$.

Step (V): (Construct h) Let $\tilde{z}_i' = \beta([p_i])$ and $\tilde{z}' = (\tilde{z}_1', \tilde{z}_2', ..., \tilde{z}_r')$ (as a column). Let $\tilde{z}'' = ((k_0+1)!f_1', ..., (k_0+1)!f_r')$. Then $K^3\tilde{z}'' = \tilde{z}'$. Set $z_i' = \rho(\tilde{z}_i')$, $z' = (z_1', ..., z_r')$, Then

$$\|Mz' - z\|_\infty < \delta \qquad (z = (z_1, ..., z_r))$$

It follows from 3.4 that there is $\tilde{u} = (u_1, ..., u_{s(n)}) \in (K_0(B)^{s(n)})_+$ (for some $s(n) \ge r$) such that

$$(r(j)x_{ij})\tilde{u} = \tilde{z}'.$$

Let $D = \psi_1(A) \oplus \psi_2(A) \oplus \cdots \oplus \psi_{s(n)}(A)$. Define a homomorphism $h_0 : D \to M_k(B)$ (for some $k > 0$) such that $[h_0](e_j) = u_j$, $j = 1, ..., s(n)$, where e_j is a mininal projection in $\psi_j(A)$. Such h_0 exists since B has stable rank one and real rank zero. Define $\pi : l^\infty(\mathbb{Q}) \to l^\infty_{s(n)}(\mathbb{Q})$ by

$$\{x_k\} \mapsto (x_1, ..., x_{s(n)}).$$

So

$$[h_0](\pi \circ \tilde{\rho}([p_i])) = \beta([p_i]), \quad i = 1, ..., r.$$

Since $\ker\tilde{\rho} \subset \ker\tau \circ \alpha \cap \ker H$, if $x \in \ker\tilde{\rho}$, then $[L \circ \Psi_1](x) = [L] \circ \alpha(x)$. Recall that we identify $K_0(A)$ with $K_0(B)$. By (2) above, since $\alpha(x) \in \ker\rho_B$, $[L] \circ \alpha(x) = \alpha(x)$. Hence

$$\alpha(x) - [L \circ \Psi_1](x) = 0.$$

Therefore we may view that $\alpha - [L \circ \Psi_1]$ gives a homomorphism on $\tilde{\rho}(G_0)$. Also Mg is in the subgroup generated by $\tilde{\rho}([p_1]), ..., \tilde{\rho}([p_r])$ for any $g \in \tilde{\rho}(G_0)$. Combining these two facts, we obtain

$$M[h_0](\pi \circ \tilde{\rho}([p_i])) = M\beta([p_i]), \;\; i = 1, ..., r, ..., l.$$

Set $h_0' = h_0 \oplus \cdots h_0$, M copies of h_0. Then

$$[h_0'](\pi \circ \tilde{\rho}([p_i])) = \alpha([p_i]) - [L \circ \Psi_1]([p_i]), \;\; i = 1, ..., r, ..., l.$$

Set $h = h_0' \circ (\psi_1 \oplus \psi_2 \oplus \cdots \oplus \psi_{s(n)})$. Then h is also $\mathcal{F}$-η-multiplicative.

Moreover,

$$[h_0'] \circ [\psi_1 \oplus \psi_2 \oplus \cdots \oplus \psi_{s(n)}]([p_i]) = [h_0'] \circ (\pi \circ \tilde{\rho}([p_i])) \;\; i = 1, 2, ..., l.$$

We also have

$$[h]|_{\mathcal{P} \cap K_1(A)} = 0 \quad \text{and} \quad [h]|_{\mathcal{P}_0} = \alpha|_{\mathcal{P}_0} - [L] \circ [\Psi_1]])|_{\mathcal{P}_0}.$$

Note that, since $[h]([p_i]) = MK^3(k_0+1)!f_i'$, $i = 1, ..., r$. $[h](g) = MK^3(k_0+1)!g'$ for any $g \in G_0$ and some $g' \in K_0(B)$. Note since D is finite dimensional, we have the following (exact) commutative diagram.

$$\begin{array}{ccccc} K_0(D) & \longrightarrow & K_0(D, \mathbb{Z}/k\mathbb{Z}) & \longrightarrow & K_1(D) \\ \uparrow & & & & \downarrow \\ K_0(D) & \longleftarrow & K_1(D, \mathbb{Z}/k\mathbb{Z}) & \longleftarrow & K_1(D) \end{array}$$

Again, $K_1(D) = 0$, $K_1(D, \mathbb{Z}/k\mathbb{Z}) = \{0\}$ and $K_0(D, \mathbb{Z}/k\mathbb{Z}) = K_0(D)/kK_0(D)$ for all k. Since h factors through D, and $[h](g) = (k_0+1)!MK^3g''$, we conclude that

$$[h]|_{G \cap K_1(A, \mathbb{Z}/k\mathbb{Z})} = 0 \quad \text{and} \quad [h]|_{G \cap K_0(A, \mathbb{Z}/k\mathbb{Z})} = 0 \;\; \text{for} \;\; k \le k_0.$$

This implies that

$$[h]|_{\mathcal{P}} = \alpha|_{\mathcal{P}} - [L] \circ [\Psi_1]|_{\mathcal{P}}.$$

Now define $H_1 = L \circ \Psi_1 \oplus h$. Then H_1 is $\mathcal{F}$-η-multiplicative and

$$[H_1]|_{\mathcal{P}} = [h]|_{\mathcal{P}} + [L] \circ [\Psi_1]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}.$$

Finally, since $[H_1(1_A)] = [1_B]$, by conjegating a unitray in $(\widetilde{B \otimes \mathcal{K}})$, we may assume that $H_1(1_A) = 1_B$. ∎

5 The Main Theorem

We will use the following uniqueness theorem:

Theorem 5.1 (Theorem 2.3 in [Ln8]) *Let A be a separable unital nuclear simple C^*-algebra with $TR(A) = 0$ satisfying the UCT. Then, for any $\varepsilon > 0$, and any finite subset $\mathcal{F} \subset A$, there exist $\delta > 0$, a finite subset $\mathcal{P} \subset P(A)$ and a finite subset $\mathcal{G} \subset A$ satisfying the following: for any unital C^*-algebra B of real rank zero and stable rank one with weakly unperforated $K_0(B)$, and any two $\mathcal{G}$-δ-multiplicative morphisms $L_1, L_2 : A \to B$ with*

$$[L_1]|_{\mathcal{P}} = [L_2]|_{\mathcal{P}},$$

there exists a unitary $U \in B$ such that

$$ad(U) \circ L_1 \approx_\varepsilon L_2 \text{ on } \mathcal{F}.$$

Theorem 5.2 *Let A and B be two unital separable simple nuclear C^*-algebras with $TR(A) = TR(B) = 0$ which satisfy the UCT. Then $A \cong B$ if and only if*

$$(K_0(A), K_0(A)_+, [1_A], K_1(A)) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B)).$$

Proof: The "only if" part is obvious and known. We need to prove the "if " part only. Using the terminology in [Ln8], Theorem 4.3 implies that both A and B are pre-classifiable. Then, by 5.1, Theorem 3.7 in [Ln8] says that $A \cong B$ if

$$(K_0(A), K_0(A)_+, [1_A], K_1(A)) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B)).$$

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