UNIVERSAL PROPERTIES OF $L(\mathbb{F}_\infty)$ IN SUBFACTOR THEORY.

SORIN POPA AND DIMITRI SHLYAKHTENKO

ABSTRACT. We construct a functor from the category of standard lattices, with embeddings as morphisms, into the category of inclusions $N \subset M$, with $N \cong M \cong L(\mathbb{F}_\infty)$, with commuting square embeddings as morphisms. This functor is a right inverse to the functor $\mathcal{G}$ assigning to an inclusion its standard lattice. We also prove that given a subfactor $Q \subset P$, with P an arbitrary II_1 factor, there exists a subfactor $\hat{Q} \subset \hat{P}$, with the same standard invariant as $Q \subset P$, $\hat{Q} \cong Q * L(\mathbb{F}_\infty)$, $\hat{P} \cong P * L(\mathbb{F}_\infty)$.

1. INTRODUCTION.

A standard lattice $\mathcal{G}$ is an abstraction of the system of higher relative commutants of a subfactor. That is to say, the relative commutants of an arbitrary finite-index inclusion of II_1 factors satisfy the axioms of a standard lattice (see [Pop95a]). Conversely, as was proved in [Pop95a], given any standard lattice $\mathcal{G}$, it can be realized as the system of higher relative commutants of some inclusion of II_1 factors, constructed out of $\mathcal{G}$. However, it remained an open problem, whether one can construct a "universal" II_1 factor, that would contain subfactors with any given standard lattice as higher relative commutants.

The main result of this paper shows that one can realize any arbitrary standard lattice $\mathcal{G}$ as the system of higher relative commutants of a type II_1 subfactor $P_{-1} \subset P_0$, where both P_{-1} and P_0 are isomorphic to the free group factor $L(\mathbb{F}_\infty)$. In other words, $L(\mathbb{F}_\infty)$ is such a universal type II_1 factor, whose subfactors realize all possible standard lattices.

Moreover, the association $\mathcal{G} \mapsto \mathcal{F}(\mathcal{G}) = (P_{-1} \subset P_0)$ with $P_{-1} \cong P_0 \cong L(\mathbb{F}_\infty)$ is functorial in the following sense. Let $\mathbb{G}$ denote the category of standard lattices, whose morphisms are embeddings of standard λ-lattices (see Section 2 for the exact definition of this latter notion). For a II_1 factor M, let $\mathbb{S}$ denote the category of all subfactors $P_{-1} \subset P_0$ with $P_{-1} \cong P_0 \cong L(\mathbb{F}_\infty)$, having as morphisms commuting square embeddings: a morphism $\imath : (P_{-1}^{(1)} \subset P_0^{(1)}) \to (P_{-1}^{(2)} \subset P_0^{(2)})$ is an embedding of algebras which forms the following commuting square

$$\begin{array}{ccc} P_{-1}^{(2)} & \subset & P_0^{(2)} \\ \cup\imath & & \cup\imath \\ P_{-1}^{(1)} & \subset & P_0^{(1)} \end{array}.$$

Then $\mathcal{F} : \mathbb{G} \to \mathbb{S}$ is a functor, which is a right inverse to the functor $\mathcal{G} : \mathbb{S} \to \mathbb{G}$, taking an inclusion $P_{-1} \subset P_0$ to its standard lattice $\mathcal{G}_{P_{-1} \subset P_0}$.

In fact, $L(\mathbb{F}_\infty)$ has a further universal property. The collection of standard lattices of all possible subfactors of a given factor M does not decrease when M is replaced by $M * L(\mathbb{F}_\infty)$. More precisely, if $N \subset M$ is an inclusion of II_1 factors, then there exists an inclusion $\hat{N} \subset \hat{M}$ (functorially

Date: Preliminary version; December 2, 2000.

associated to $N \subset M$), with $\hat{N} \cong N * L(\mathbb{F}_\infty)$, $\hat{M} \cong M * L(\mathbb{F}_\infty)$, and so that the standard lattices $\mathcal{G}_{\hat{N}\subset\hat{M}}$ and $\mathcal{G}_{N\subset M}$ are the same.

We note that these results are generalization of earlier results about realization of finite-depth subfactors inside free group factors [Răd94], [Dyk95], irreducible subfactors in $L(\mathbb{F}_\infty)$ [US00] and finite-depth subfactors of $M * L(\mathbb{F}_\infty)$, for M arbitrary [Shl99].

To prove the first result above, to a standard λ-lattice $\mathcal{G} = (A_{ij})$ we associate in a functorial way a λ-Markov commuting square

$$\begin{array}{ccc} \mathcal{B}_{-1} & \subset & \mathcal{B}_0 \\ \cup & & \cup \\ \mathcal{A}_{-1} & \subset & \mathcal{A}_0 \end{array},$$

with $\mathcal{A}_i$ semifinite type I with discrete center, and with the top row being the infinite amplification of the standard model inclusion $R_{-1} \subset R_0$ associated to $\mathcal{G}$. Thus $\mathcal{B}_i$ are hyperfinite type II_∞ algebras (possibly with non-trivial center). Most importantly, the commuting square satisfies $(\mathcal{B}_i' \cap \mathcal{A}_j) \simeq (A_{ij})$ in its Jones tower, thus encoding the standard lattice $\mathcal{G} = (A_{ij})$.

Similarly, to prove the second result, given an arbitrary inclusion of II_1 factors $M_{-1} \subset M_0$, we associate to it a canonical λ-Markov commuting square

$$\begin{array}{ccc} \mathcal{M}_{-1} & \subset & \mathcal{M}_0 \\ \cup & & \cup \\ \mathcal{A}_{-1} & \subset & \mathcal{A}_0 \end{array}$$

in which the bottom row is as before, but the top row is the infinite amplification of the inclusion $M_{-1} \subset M_0$. Also, we have again $(\mathcal{M}_i' \cap \mathcal{A}_j) \simeq (M_i' \cap M_j)$, in the corresponding Jones tower.

To construct such canonical commuting squares out of a given standard lattice or a subfactor, we use inductive limits of non-unital embeddings naturally associated to the duality isomorphisms in the Jones tower.

Using the above commuting squares, we construct amalgamated free product inclusions (along the lines of [Pop93],[Pop95a])

$$\mathcal{B}_{-1} *_{\mathcal{A}_{-1}} (Q \otimes \mathcal{A}_{-1}) \subset \mathcal{B}_0 *_{\mathcal{A}_0} (Q \otimes \mathcal{A}_0). \tag{1.1}$$

and

$$\mathcal{M}_{-1} *_{\mathcal{A}_{-1}} (Q \otimes \mathcal{A}_{-1}) \subset \mathcal{M}_0 *_{\mathcal{A}_0} (Q \otimes \mathcal{A}_0) \tag{1.2}$$

The crucial point is that (1.1) has the standard lattice $\mathcal{G}$ as its relative commutants, while (1.2) has the same higher relative commutants as $M_{-1} \subset M_0$.

We then identify the algebras involved in the inclusions (1.1) and (1.2). In fact, we show that $\mathcal{B}_i *_{\mathcal{A}_i} (Q \otimes \mathcal{A}_i) \cong L(\mathbb{F}_\infty)$ and $\mathcal{M}_i *_{\mathcal{A}_i} (Q \otimes \mathcal{A}_i) \cong (M_i * L(\mathbb{F}_\infty)) \otimes B(H)$, thereby proving the claimed universal properties of $L(\mathbb{F}_\infty)$.

The techniques needed for this identification come from free probability theory pioneered by Voiculescu ([VDN92]). The main observation is that the amalgamated free product algebra $\mathcal{B}_i *_{\mathcal{A}_i} (Q \otimes \mathcal{A}_i)$ is generated by $\mathcal{B}_i$ and $Q \cong L(\mathbb{F}_\infty)$; furthermore, Q has as generators an infinite semicircular system $X_1, X_2, \ldots$ [Voi90]. The position of this family relative to $\mathcal{B}_i$ is encoded in the statement that $\{X_n\}$ form an *operator-valued* semicircular system over $\mathcal{B}_i$ in the sense of [Shl98], [Shl99]. The rest of the proof involves manipulations with this semicircular system — in ways that parallel earlier random-matrix techniques of Voiculescu [Voi90],[VDN92], further developed by Radulescu [Răd92], [Răd94] and Dykema, [Dyk94], [Dyk93], [DR00].

Acknowledgement. The second-named author would like to thank the Wiley W. Manuel Courthouse in Oakland, CA, where an early part of the work was carried out while on breaks from jury duty. The authors would also like to thank MSRI and the organizers of the stimulating program on Operator algebras. Research supported in part by the NSF grant no. DMS-9801324, for S.P., and by an NSF postdoctoral fellowship, for D.S.

2. Some canonical commuting squares associated to a subfactor.

Let $M_{-1} \subset M_0$ be an inclusion of Type II_1 factors with finite Jones index. In this section we will associate to it a system of λ-Markov commuting squares of semifinite von Neumann algebras with trace preserving expectations

$$\mathcal{C}^0 = \begin{array}{ccc} \mathcal{M}_{-1} & \subset & \mathcal{M}_0 \\ \cup & & \cup \\ \mathcal{B}_{-1} & \subset & \mathcal{B}_0 \\ \cup & & \cup \\ \mathcal{A}_{-1} & \subset & \mathcal{A}_0 \end{array}$$

in which the upper commuting square is the ∞-amplification of

$$\begin{array}{ccc} M_{-1} & \subset & M_0 \\ \cup & & \cup \\ M_{-1}^{st} & \subset & M_0^{st} \end{array},$$

$M_{-1}^{st} \subset M_0^{st}$ being the standard model associated with $M_{-1} \subset M_0$, and in which $\mathcal{A}_{-1} \subset \mathcal{A}_0$ is an inclusion of type I von Neumann algebras with atomic centers and inclusion matrix given by the graph of $M_{-1} \subset M_0$. The construction of the lower commuting square

$$\begin{array}{ccc} \mathcal{B}_{-1} & \subset & \mathcal{B}_0 \\ \cup & & \cup \\ \mathcal{A}_{-1} & \subset & \mathcal{A}_0 \end{array}$$

will in fact only depend on the standard invariant $\mathcal{G} = \mathcal{G}_{M_{-1},M_0}$ of $M_{-1} \subset M_0$ and will be functorial in $\mathcal{G}$. This commuting square will completely encode $\mathcal{G}$, as it will satisfy $\mathcal{B}_i' \cap \mathcal{A}_j = \mathcal{M}_i' \cap \mathcal{A}_j \simeq M_i' \cap M_j$ in the Jones tower for $\mathcal{C}^0$, respectively for $M_{-1} \subset M_0$.

The commuting square $\mathcal{C}^0$ will be constructed as an inductive limit of non-unital Trace-preserving embeddings of the commuting squares

$$\begin{array}{ccc} M_{2n-1} & \subset & M_{2n} \\ \cup & & \cup \\ A_{-\infty,2n-1} & \subset & A_{-\infty,2n} \\ \cup & & \cup \\ A_{-1,2n-1} & \subset & A_{-1,2n} \end{array}$$

where $A_{ij} = M_i' \cap M_j,\ j,i \in \mathbb{Z}$ are the higher relative commutants in some tunnel-tower $\ldots \subset M_{-2} \overset{e_{-1}}{\subset} M_{-1} \overset{e_0}{\subset} M_0 \overset{e_1}{\subset} M_1 \overset{e_2}{\subset} M_2 \subset \ldots$ for $M_{-1} \subset M_0$ and $A_{-\infty,j} = \overline{\cup_{k\leq j} A_{kj}}$.

Lemma 2.1. *For each $k \geq 0$, $n \geq 0$, let α_n^k be the map from M_{2n+k} into M_{2n+k+2} given by $\alpha_n^k(x) = \lambda^{-k-1} e_{2n+1}e_{2n+2}\cdots e_{2n+k+1}e_{2n+k+2}xe_{2n+k+2}\cdots e_{2n+1},\ x \in M_{2n+k}$. Then α_n^k are non-unital $*$-isomorphisms and they satisfy:*

1°. $\alpha_n^k(M_{2n+j-1}) = e_{2n+1}M_{2n+j+1}e_{2n+1},\ j = 0, 1, \dots, k+1$, *with* ${\alpha_n^{k+1}}_{|M_{2n+j-1}} = \alpha_n^k$ *if* $j \leq k+1$.

3°. $\alpha_n^k(x) = \sigma'(x)e_{2n+1},\ x \in M'_{2n-1} \cap M_{2n+k}$, *where σ' is the duality isomorphism on $\cup_{i,j\in\mathbb{Z}} A_{ij}$ (see e.g.* [Pop99]*).*

2°. $\alpha_n^k(A_{i,2n+j-1}) = e_{2n+1}A_{i,2n+j+1}e_{2n+1},\ j = 0, 1, \dots, k+1,\ -\infty \leq i \leq -1.$

4°. *If* Tr_n *is the rescaled trace on $\cup_k M_{2n+k}$ given by* $\mathrm{Tr}_n = \lambda^{-n}\tau$ *then* $\mathrm{Tr}_{n+1}(\alpha_n^k(x)) = \mathrm{Tr}_n(x), \forall x \in M_{2n+k}, \forall k \geq 0$.

Proof. Since $[x,\ e_{2n+k+2}] = 0$ for $x \in M_{2n+k}$ and $\lambda^{-k-1/2}e_{2n+1}\cdots e_{2n+k+2}$ is a partial isometry, α_n^k follows a *-isomorphism. For the properties $1^\circ - 4^\circ$ we have:

1°. Since $e_{2n+j+1}M_{2n+j-1}e_{2n+j+1} = M_{2n+j-1}e_{2n+j+1} = e_{2n+j+1}M_{2n+j+1}e_{2n+j+1}$ it follows that $e_{2n+1}\dots e_{2n+k+2}M_{2n+j-1}e_{2n+k+2}\dots e_{2n+1} = e_{2n+1}\dots e_{2n+j+1}M_{2n+j-1}e_{2n+j+1}\dots e_{2n+1} = e_{2n+1}\dots e_{2n+j+1}M_{2n+j+1}e_{2n+j+1}\dots e_{2n+1} = e_{2n+1}M_{2n+j+1}e_{2n+1}$.

2°. Since $e_{2n+1}, e_{2n+2}, \dots, e_{2n+k+2} \in A_{-1,2n+k+2}$ and $A_{i,2n+j-1}e_{2n+j+1} = e_{2n+j+1}A_{i,2n+j+1}e_{2n+j+1}$ for each $j = 0, 1, \dots, k+1$ and $-\infty \leq i \leq -1$, this part follows by 1°.

3°. This is trivial by the definition of σ'.

4°. Since $\tau(xe_{2n+k+2}) = \lambda\tau(x)$ for $x \in M_{2n+k}$, one gets $\tau(\alpha_n^k(x)) = \lambda\tau(x)$ so that $\mathrm{Tr}_{n+1}(\alpha_n^k(x)) = \mathrm{Tr}_n(x)$. ☐

Notation 2.2. To simplify the notations we will denote by $\mathcal{C}_n$ the system of commuting squares

$$\begin{array}{ccccccc}
M_{2n-1} & \subset & M_{2n} & \subset \dots \subset & M_{2n+k} & \subset \dots \\
\cup & & \cup & & \cup & \\
A_{-\infty,2n-1} & \subset & A_{-\infty,2n} & \subset \dots \subset & A_{-\infty,2n+k} & \subset \dots \\
\cup & & \cup & & \cup & \\
A_{-1,2n-1} & \subset & A_{-1,2n} & \subset \dots \subset & A_{-1,2n+k} & \subset \dots
\end{array}$$

with $\mathcal{C}_n^k$ denoting its truncation up to $k,\ k = 0, 1, \dots$. Thus, with these notations Lemma 2.1 states that α_n^k identifies the commuting square $(\mathcal{C}_n^k,\ \mathrm{Tr}_n)$ with the "corner" $e_{2n+1}\mathcal{C}_{n+1}^k e_{2n+1}$ of the commuting square $(\mathcal{C}_{n+1}^k,\ \mathrm{Tr}_{n+1})$ endowed with the restriction of the trace Tr_{n+1}.

Moreover, since by 2.1.1° we have ${\alpha_n^{k+1}}_{|M_{2n+j-1}} = \alpha_n^k$, for $0 \leq j \leq k+1$, with the sequence $\{\alpha_n^k(x)\}_k$ being constant from a certain point on, for each $x \in M_{2n+j}, \forall j$, we immediately get the following:

Corollary 2.3. *For each $n \geq 0$ and $x \in \cup_{j\geq 0} M_{2n+j}$ let $\alpha_n(x) \overset{\text{def}}{=} \lim_{k\to\infty} \alpha_n^k(x)$. Then we have*

1°. $\alpha_n(\mathcal{C}_n) = e_{2n+1}\mathcal{C}_{n+1}e_{2n+1}$.

2°. $\alpha_n(x) = \sigma'(x)e_{2n+1},\ x \in \cup_j A_{2n-1,2n+j} = \cup_j(M'_{2n-1} \cap M_{2n+j})$, *where σ' is the duality endomomorphism on $\cup_j A_{0j}$ that sends A_{ij} onto $A_{i+2,j+2}$, $\forall j \geq i \geq 0$ (as defined in* [Pop99]*).*

3°. $\mathrm{Tr}_{n+1} \circ \alpha_n = \mathrm{Tr}_n$ *and α_n takes the* Tr_n*-preserving expectations (= τ-preserving expectations) in $\mathcal{C}_n$ into the restrictions to $e_{2n+1}\mathcal{C}_{n+1}e_{2n+1}$ of the* Tr_{n+1}*-preserving expectations in $\mathcal{C}_{n+1}$.*

4°. *The top row of commuting squares in $\mathcal{C}_n$ is a sequence of basic constructions of the initial homogeneous λ-Markov commuting square of inclusions*

$$\mathcal{C}_n^0 = \begin{array}{ccc} M_{2n-1} & \subset & M_{2n} \\ \cup & & \cup \\ A_{-\infty,2n-1} & \subset & A_{-\infty,2n}. \end{array}$$

Moreover, $\alpha_n(1_{\mathcal{C}_n}) = \alpha_n(1_{M_{2n}}) = e_{2n+1} \in M_{2n+2}$ (the latter regarded as an algebra in $\mathcal{C}_{n+1}$) has scalar central trace in $A_{-\infty,2n+1}$ (as an algebra in $\mathcal{C}_{n+1}$), so that $\mathcal{C}_{n+1}$ is the λ^{-1}-amplification of $\mathcal{C}_n$.

Proof. α_n is well defined because for each x and k large enough one has $\alpha_n(x) = \alpha_n^k(x)$ (by 2.1.1°). Then properties 1°-3° are just reformulations of 2.1.1°–2.1.4°. The last property 4° is well known (see e.g. [Pop95b]). $\square$

Definition 2.4. We define $\mathcal{C}$ to be the system of inclusions of von Neumann algebras

$$\begin{array}{ccccccc} \mathcal{M}_{-1} & \subset & \mathcal{M}_0 & \subset \ldots \subset & \mathcal{M}_k & \subset \ldots \\ \cup & & \cup & & \cup & \\ \mathcal{B}_{-1} & \subset & \mathcal{B}_0 & \subset \ldots \subset & \mathcal{B}_k & \subset \ldots \\ \cup & & \cup & & \cup & \\ \mathcal{A}_{-1} & \subset & \mathcal{A}_0 & \subset \ldots \subset & \mathcal{A}_k & \subset \ldots \end{array}$$

obtained as the inductive limit of the sequence of non-unital trace preserving embeddings of commuting squares

$$(\mathcal{C}_0;\ \mathrm{Tr}_0) \overset{\alpha_0}{\hookrightarrow} (\mathcal{C}_1;\ \mathrm{Tr}_1) \overset{\alpha_1}{\hookrightarrow} (\mathcal{C}_2;\ \mathrm{Tr}_2) \hookrightarrow .$$

By this we mean the following:

(2.4.1). We first take the (non-unital!) algebraic inductive limit $\mathcal{M}_i^0$ of $M_i \overset{\alpha_0}{\hookrightarrow} M_{i+2} \overset{\alpha_1}{\hookrightarrow} M_{i+4} \hookrightarrow \ldots$. We note that $\mathcal{M}_{-1}^0 \subset \mathcal{M}_0^0 \subset$ ...., in a natural way.

(2.4.2). For each $n \geq 0, j \geq -1$ and $x \in M_{i+2n}$ we denote by $\tilde{\alpha}_n(x) = \ldots \alpha_{n+1} \circ \alpha_n(x)$ the image of x in $\mathcal{M}_i^0$. With this notation, one clearly has $\mathcal{M}_i^0 = \cup_n \tilde{\alpha}_n(M_{i+2n})$.

(2.4.3). On $\mathcal{M}_i^0$ one takes the C^*-norm defined by $\|x\| = \|\tilde{\alpha}_n(x)\|_{M_{i+2n}}$, if $x \in M_{i+2n}$.

(2.4.4). One defines a positive tracial functional Tr on $\mathcal{M}_i^0$ by $\mathrm{Tr}(\tilde{\alpha}_n(x)) = \mathrm{Tr}_n\, x$, if $x \in M_{i+2n}$.

(2.4.5). One defines $\mathcal{M}_i$ to be the completion of $\mathcal{M}_i^0$ in the topology of convergence in the norm $\|x\|_{2,\mathrm{Tr}} = \mathrm{Tr}(x^*x)^{1/2}$ on bounded sets (in C^*-norm) (note that $\mathcal{M}_i$ can also be defined through the GNS construction for $(\mathcal{M}_i^0,\ \mathrm{Tr})$).

(2.4.6). Note that Tr extends to a normal semifinite faithful trace on $\mathcal{M}_i$, still denoted Tr. Moreover, $\mathcal{M}_i$ this way defined clearly satisfy $\mathcal{M}_i \subset \mathcal{M}_{i+1}$ with $\mathrm{Tr}_{\mathcal{M}_{i+1}|\mathcal{M}_i} = \mathrm{Tr}_{\mathcal{M}_i}$ (the notation being self-explanatory).

(2.4.7). Define $\mathcal{B}_i$, respectively $\mathcal{A}_i,\ i \geq -1$, as the closure in the same topology of $\|\ \ \|_{2,\mathrm{Tr}}$ -convergence on bounded sets of the *-subalgebras $\overline{\bigcup_n \tilde{\alpha}_n(A_{-\infty})}$, respectively $\overline{\bigcup_n \tilde{\alpha}_n(A_{-1,i+2n})}$, of $\mathcal{M}_i^0$.

(2.4.8). We note that the trace Tr on $\mathcal{M}_i$ restricts to semifinite traces on both $\mathcal{B}_i$ and $\mathcal{A}_i$, for each $i \geq -1$.

(2.4.9). If for each n we choose an inclusion $\mathcal{Q}_n \subset \mathcal{P}_n$ between two of the algebras in the commuting square $\mathcal{C}_n$, but so that for each n the algebras are chosen at the same "spot", and if we denote by $\mathcal{E}^{\mathcal{P}}_{\mathcal{Q}}$ the unique Tr-preserving expectation of the inductive limit $\mathcal{P} \stackrel{\text{def}}{=} \overline{\bigcup_n \tilde{\alpha}_n(P_n)}$ onto the inductive limit $\mathcal{Q} \stackrel{\text{def}}{=} \overline{\bigcup_n \tilde{\alpha}_n(Q_n)}$ then by Corollary 2.3 we have $\mathcal{E}^{\mathcal{P}}_{\mathcal{Q}}(\tilde{\alpha}_n(x)) = \tilde{\alpha}_n(E^{P_n}_{Q_n}(x))$, for $x \in P_n$.

In particular, by 2.3.3°, the properties (2.4.8) and (2.4.9) above show that the system of inclusions $\mathcal{C}$, endowed with the corresponding Tr-preserving expectations between its algebras, is a system of commuting squares.

We now examine more closely the main properties of $\mathcal{C}$.

Lemma 2.5. *If for each $n \geq 0$ we let 1_n be the identity in $\mathcal{C}_n$, i.e. $1_n = 1_{M_{2n-1}} = 1_{A_{-1,2n-1}} = 1_{M_{2n+k}} = 1_{A_{-1,2n+k}}, \forall k \geq 0$, and define $p_n = \tilde{\alpha}_n(1_n)$ then we have:*

1°. *p_n belong to $\mathcal{A}_{-1}$, $\operatorname{Tr} p_n = \lambda^{-n}, \forall n$, and $p_0 \leq p_1 \leq p_2 \leq \ldots$ with $p_n \nearrow 1_{\mathcal{A}_{-1}} (= 1_{\mathcal{C}})$.*

2°. *For each n, $p_n \mathcal{C} p_n$ is naturally isomorphic to $\mathcal{C}_n$, via $\tilde{\alpha}_n$ (as commuting squares of trace preserving expectations).*

3°. *p_n has scalar central trace in $p_{n+1}\mathcal{B}_{-1}p_{n+1}, \forall n \geq 0$.*

4°. *For each $j \geq i \geq -1$ and $x \in A_{ij}$ there exists a unique element $\alpha(x)$ in $\cup_k \mathcal{M}_k$ such that $[\alpha(x),\ p_n] = 0, \forall n, \alpha(x)p_n = \tilde{\alpha}_n(\sigma'^n(x))$, where σ' is the duality isomorphism as in 2.3.3°. Moreover, α is a $*$-isomorphism and $\alpha(A_{ij}) = \mathcal{M}'_i \cap \mathcal{M}_j = \mathcal{M}'_i \cap \mathcal{A}_j, \forall j \geq i \geq -1$.*

5°. *$\alpha(e_j)$ belongs to $\mathcal{A}_j, \forall j \geq 1$ and $\alpha(e_{n+1})$ implements the Tr-preserving conditional expectation of $\mathcal{M}_n$ onto $\mathcal{M}_{n-1}, \forall n \geq 0$.*

Proof. 1° is clear by the definitions, and so is the equality $p_n \mathcal{C} p_n = \tilde{\alpha}_n(\mathcal{C}_n)$ of condition 2°. Then p_n have scalar central trace in $p_{n+1}\mathcal{B}_{-1}p_{n+1}$ because e_{2n+1} has scalar central trace in $A_{-\infty,2n+1}$ (see e.g. [Pop95b]).

The first part in 4° is clear by property 2° in Corollary 2.3. Then the equality $\alpha(A_{ij}) = \mathcal{M}'_i \cap \mathcal{M}_j$ is immediate by the definitions of α, $\mathcal{M}_i, \mathcal{M}_j$. Also, by the way it is defined, $\alpha(A_{ij})$ is clearly contained in $\mathcal{A}_j$, so $\alpha(A_{ij}) = \mathcal{M}'_i \cap \mathcal{A}_j$. (Note that in 5.2.1° we prove a stronger statement about these higher relative commutants, with the above argument being given in details).

Since e_j lies in $A_{-1,j}$ for $j \geq 1$, it follows by 4° that $\alpha(e_j)$ lies in $\mathcal{A}_j$. Since e_{2k+1} implements the expectation of M_{2k} onto M_{2k-1}, it follows that $\tilde{\alpha}_k(e_{2k+1})$ implements the conditional expectation of $p_k \mathcal{M}_0 p_k$ onto $p_k \mathcal{M}_{-1} p_k$. Since $p_k \nearrow 1$ and $\alpha(e_1)p_k = \tilde{\alpha}_k(e_{2k+1})$, we get the last part of 5° as well. $\square$

The next lemma clarifies the structure of the inclusions $\mathcal{A}_{-1} \subset \mathcal{A}_0 \subset \ldots$. To state it, let us denote by $\Gamma = \Gamma_{M_{-1},M_0} = (a_{kl})_{k\in K, l\in L}$ the standard graph of $M_{-1} \subset M_0$ (or, equivalently, of $\mathcal{G} = \mathcal{G}_{M_{-1},M_0}$), which describes the sequence of inclusions $A_{-1,-1} \subset A_{-1,0} \subset A_{-1,1} \subset \ldots$. Thus, if $* \in K$ denotes the *initial vertex* of Γ and $K_n = (\Gamma\Gamma^t)^n(\{*\}), L_n = (\Gamma\Gamma^t)^n\Gamma(\{*\})$, then $K = \cup_n K_n, L_n = \cup_n L_n$ with the sets K_n, L_n having the following significance:

The set of simple summands of $\mathcal{Z}(A_{-1,2n-1})$ (respectively $\mathcal{Z}(A_{-1,2n})$) naturally identifies with the set K_n (respectively L_n), with the inclusion $K_n \subset K_{n+1}$ (respectively $L_n \subset L_{n+1}$) corresponding to the embedding of $\mathcal{Z}(A_{-1,2n-1})$ into $\mathcal{Z}(A_{-1,2n+1})$ (respectively of $\mathcal{Z}(A_{-1,2n})$ into $\mathcal{Z}(A_{-1,2n+2})$) given by the applications

$$\mathcal{Z}(A_{-1,j}) \ni z \mapsto z' \in Z(A_{-1,j+2}),$$

with z' the unique element in $Z(A_{-1,j+2})$ such that $ze_{j+2} = z'e_{j+2}$.

Moreover, the inclusion graphs of $A_{-1,2n-1} \subset A_{-1,2n}$ (respectively $A_{-1,2n} \subset A_{-1,2n+1}$) are given by ${}_{K_n}\Gamma$ (respectively ${}_{L_n}\Gamma^t$).

Also, there exists a unique vector $\vec{s} = (s_k)_{k\in K}$ such that $s_* = 1$, $\Gamma\Gamma^t\vec{s} = \lambda^{-1}\vec{s}$ and such that if $\vec{t} = (t_l)_{l\in L} = \lambda\Gamma^t\vec{s}$ then $(\lambda^n s_k)_{k\in K_n}$ resp. $(\lambda^n t_l)_{l\in L_n}$ give the traces of the minimal projections in $A_{-1,2n-1}$ resp. $A_{-1,2n}$.

With these notations at hand we have:

Lemma 2.6. *$\mathcal{A}_{-1} \subset \mathcal{A}_0 \subset \dots$ are inclusions of atomic von Neumann algebras. More precisely, for each $n \geq 0$ the reduced sequence of inclusions $p_n(\mathcal{A}_{-1} \subset \mathcal{A}_0 \subset \dots)p_n$ is isomorphic via $\tilde{\alpha}_n$ to the sequence of inclusions $(A_{-1,2n-1} \subset A_{-1,2n} \subset \dots)$, with the trace Tr on the former corresponding to the trace Tr_n on the latter. Moreover, if one identifies the set of factor summands of $\mathcal{A}_{-1}$ which contain non-zero parts of the projection p_n with the set of factor summands of $A_{-1,2n-1}$, i.e., with K_n, via the identification of $p_n\mathcal{A}_{-1}p_n$ with $A_{-1,2n-1}$, then the inclusion matrix for $\mathcal{A}_{-1} \subset \mathcal{A}_0$ is given by Γ, while the trace Tr is given on the minimal projections of $\mathcal{A}_{-1}$ by the eigenvector $\vec{s} = (s_k)_{k\in K}$ and on the minimal projections of $\mathcal{A}_0$ by the vector $\vec{t}$. Similarly, the inclusion graph for $\mathcal{A}_i \subset \mathcal{A}_{i+1}$ is given by Γ if i is odd and by Γ^t if i is even, with the trace vector for the minimal projections of $\mathcal{A}_{2k-1}$ (resp. $\mathcal{A}_{2k}$) being given by $\lambda^k\vec{s}$ (resp. $\lambda^k\vec{t}$).*

Proof. We have already noted in 2.5 that the non-unital isomorphism $\tilde{\alpha}_n$ takes the sequence of inclusions $(A_{-1,-1} \subset A_{-1,0} \subset A_{-1,1} \subset \dots)$ onto the sequence of inclusions $p_n(\mathcal{A}_{-1} \subset \mathcal{A}_0 \subset \dots)p_n$, with $\mathrm{Tr} \circ \tilde{\alpha}_n = \mathrm{Tr}_n$. Since A_{ij} are all atomic and $p_n \nearrow 1$, it follows that $\mathcal{A}_k$ are all atomic.

From the above and the discussion preceding 2.6, the last part follows now trivially. □

Lemma 2.7. *The sequence of inclusions $\mathcal{A}_{-1} \subset \mathcal{A}_0 \overset{\alpha(e_1)}{\subset} \mathcal{A}_1 \overset{\alpha(e_2)}{\subset} \mathcal{A}_2 \dots$ is a Jones tower of λ-Markov inclusions.*

Proof. By 2.5.5° $\alpha(e_{n+1})$ belongs to $\mathcal{A}_{n+1}$ and by commuting squares with $\mathcal{M}_{n-1} \subset \mathcal{M}_n$, it implements the Tr-preserving expectation of $\mathcal{A}_n$ onto $\mathcal{A}_{n-1}$.

By the definitions, we see that $p_n\mathcal{A}_1p_n$ is contained in $\overline{\mathrm{sp}}(p_{n+1}\mathcal{A}_0p_{n+1})\alpha(e_1)(p_{n+1}\mathcal{A}_0p_{n+1})$. Since $p_n \nearrow 1$, this shows that $\overline{\mathrm{sp}}\mathcal{A}_0\alpha(e_1)\mathcal{A}_0 = \mathcal{A}_1$.

But by 2.6 the traces of the minimal projections in $\mathcal{A}_{-1} \subset \mathcal{A}_0$ satisfy the conditions in [Jon83]. Thus, the basic construction $\mathcal{A}_{-1} \subset \mathcal{A}_0 \overset{\alpha(e_{\mathcal{A}_{-1}})}{\subset} \langle \mathcal{A}_0, e_{\mathcal{A}_{-1}} \rangle$ has a λ-Markov trace that extends Tr.

Altogether, this shows that $\mathcal{A}_0 \ni x \mapsto x \in \mathcal{A}_0$ and $e_{\mathcal{A}_{-1}} \mapsto \alpha(e_1)$ extends to a trace preserving isomorphism of $\mathcal{A}_{-1} \subset \mathcal{A}_0 \overset{\alpha(e_1)}{\subset} \mathcal{A}_1$ onto $\mathcal{A}_{-1} \subset \mathcal{A}_/ \overset{\alpha(e_{\mathcal{A}_{-1}})}{\subset} \langle \mathcal{A}_0, e_{\mathcal{A}_{-1}} \rangle$. □

Let us summarize all the properties of the commuting square $\mathcal{C}$ emphasized this far. To state it, recall the following

Definition 2.8. Let $\mathcal{Q}_i, \mathcal{P}_i, i = -1, 0$ be arbitrary semifinite von Neumann algebras with inclusions

$$\begin{array}{ccc} \mathcal{P}_{-1} & \subset & \mathcal{P}_0 \\ \cup & & \cup \\ \mathcal{Q}_{-1} & \subset & \mathcal{Q}_0 \end{array}$$

such that the normal semifinite faithful trace Tr on $\mathcal{P}_0$ is semifinite on each of the smaller algebras and such that the corresponding Tr-preserving expectations make the above into a commuting square with both row inclusions of finite index $\leq \lambda^{-1}$. Let $\mathcal{P}_{-1} \subset \mathcal{P}_0 \overset{e}{\subset} \mathcal{P}_1 = \langle \mathcal{P}_0, e \rangle$ be the basic construction for $\mathcal{P}_{-1} \subset \mathcal{P}_0$. If there exists a trace on $\mathcal{P}_1$, Tr, which extends the trace Tr on $\mathcal{P}_0$ such that $\mathrm{Tr}(xey) = \lambda \,\mathrm{Tr}(xy), \forall x, y \in \mathcal{P}_0$ and such that $\mathcal{Q}_0$ contains an orthonormal basis for $\mathcal{P}_{-1} \subset \mathcal{P}_0$, then the above commuting square will be called a *λ-Markov commuting square*.

Note that λ-Markov commuting squares are non-degenerate, in the sense of [Pop94] and that each of the row inclusions is λ-Markov, as defined in [Pop95b]. Note also that if one has such a commuting square then the above projection e also implements the basic construction for $\mathcal{Q}_{-1} \subset \mathcal{Q}_0$. Moreover, the resulting system of inclusions

$$\begin{array}{ccc} \mathcal{P}_0 & \subset & \mathcal{P}_1 \\ \cup & & \cup \\ \mathcal{Q}_0 & \subset & \mathcal{Q}_1 \end{array}$$

where Q_1 is the algebra generated by $\mathcal{Q}_0$ and e, is itself a λ-Markov commuting square (with respect to the Tr-preserving expectations). Thus, one can iterate the basic construction and obtain from the initial λ-Markov commuting square a whole Jones tower of λ-Markov commuting squares.

Theorem 2.9. 1°. *The commuting squares in the initial inclusion*

$$\mathcal{C}^0 = \begin{array}{ccc} \mathcal{M}_{-1} & \subset & \mathcal{M}_0 \\ \cup & & \cup \\ \mathcal{B}_{-1} & \subset & \mathcal{B}_0 \\ \cup & & \cup \\ \mathcal{A}_{-1} & \subset & \mathcal{A}_0 \end{array}$$

of $\mathcal{C}$, with its Tr*-preserving expectations, are all λ-Markov.*

2°. *$\mathcal{C}$ is obtained by iterating the basic construction for $\mathcal{C}^0$.*

3°. *The commuting square*

$$\begin{array}{ccc} \mathcal{M}_{-1} & \subset & \mathcal{M}_0 \\ \cup & & \cup \\ \mathcal{B}_{-1} & \subset & \mathcal{B}_0 \end{array}$$

is isomorphic to the ∞-amplification of the commuting square

$$\begin{array}{ccc} M_{-1} & \subset & M_0 \\ \cup & & \cup \\ A_{-\infty,-1} & \subset & A_{-\infty,0} \end{array}$$

i.e., it is obtained by tensoring the latter by $B(\ell^2(\mathbb{N}))$.

4°. *$\mathcal{A}_{-1} \subset \mathcal{A}_0$ is an inclusion of infinite type* I *von Neumann algebras with discrete centers, having inclusion graph given by $\Gamma = \Gamma_{M_{-1},M_0}$ and the trace* Tr *is given on the minimal projections of $\mathcal{A}_{-1}$ by $\vec{s}$ and on the minimal projections of $\mathcal{A}_0$ by $\vec{t} = \lambda\Gamma^t\vec{s}$.*

5°. *$\mathcal{M}_i' \cap \mathcal{M}_j = \mathcal{M}_i' \cap \mathcal{A}_j = \mathcal{B}_i' \cap \mathcal{A}_j$ and α gives a natural isomorphism from $\mathcal{G}_{M_{-1},M_0} = (M_i' \cap M_j)_{j \geq i \geq -1}$ onto $(\mathcal{M}_i' \cap \mathcal{A}_j)_{j \geq i \geq -1} = (\mathcal{B}_i' \cap \mathcal{A}_j)_{j \geq i \geq -1}$.*

The last result in this section describes the functoriality properties of the commuting squares appearing in $\mathcal{C}^0$. To state it, recall from [Pop95a], [Pop99] that given two standard λ-lattices $\mathcal{G}^0 = (A^0_{ij})_{j \geq i \geq -1}, \mathcal{G} = (A_{ij})_{j \geq i \geq -1}$, an *embedding* of $\mathcal{G}^0$ into $\mathcal{G}$ is a trace preserving isomorphism

$\imath$ from $\cup_n A^0_{-1,n}$ into $\cup_n A_{-1,n}$ such that $\imath(A^0_{ij}) = A_{ij}, \forall j \geq i \geq -1$ and such that $\imath$ takes the Jones λ-sequence of projections $\{e^0_n\}_{n\geq 1}$ of $\mathcal{G}^0$ into a Jones sequence of projections for $\mathcal{G}$, satisfying the smoothness condition

$$E_{A_{01}}(\imath(e^0_1)) = \imath(E_{A^0_{01}}(e^0_1)) \tag{2.10.1}$$

Theorem 2.10. 1°. *Denote by $\mathcal{C}_{M_{-1},M_0}$ the object consisting of the commuting square*

$$\begin{array}{ccc} \mathcal{M}_{-1} & \subset & \mathcal{M}_0 \\ \cup & & \cup \\ \mathcal{A}_{-1} & \subset & \mathcal{A}_0 \end{array}$$

together with the fixed projection $p_0 \in \mathcal{A}_{-1}$ is canonically associated with $M_{-1} \subset M_0$.

2°. *The object $\mathcal{C}^{st}_{\mathcal{G}}$ consisting of the commuting square*

$$\begin{array}{ccc} \mathcal{B}_{-1} & \subset & \mathcal{B}_0 \\ \cup & & \cup \\ \mathcal{A}_{-1} & \subset & \mathcal{A}_0 \end{array}$$

together with the fixed projection $p_0 \in \mathcal{A}_{-1}$, is canonically associated with the standard λ-lattice $\mathcal{G}$ and it is functorial in $\mathcal{G}$: If $\mathcal{G}_0 \subset \mathcal{G}$ is a standard λ-lattice embedded in $\mathcal{G}$ then $\mathcal{C}^{st}_{\mathcal{G}_0}$ is naturally non-degenerately embedded[1] *in $\mathcal{C}^{st}_{\mathcal{G}}$ with commuting squares and with the corresponding projections p_0 coinciding.*

Proof. 1°. This part is clear by the construction of

$$\begin{array}{ccc} \mathcal{B}_{-1} & \subset & \mathcal{B}_0 \\ \cup & & \cup \\ \mathcal{A}_{-1} & \subset & \mathcal{A}_0 \end{array}$$

as the inductive limit of the canonical commuting squares

$$\begin{array}{ccc} M_{2n-1} & \subset & M_{2n} \\ \cup & & \cup \\ A_{-1,2n-1} & \subset & A_{-1,2n} \end{array}$$

via embeddings which are canonical as well (being defined by using only the Jones projections in the tower $e_1, e_2, \dots$). Also, $p_0 = \tilde{\alpha}_0(1)$ so that the position of p_0 inside $\mathcal{A}_{-1}$ is canonical as well.

2°. The fact that $\mathcal{C}^{st}_{\mathcal{G}}$ is canonically associated with $\mathcal{G}$ follows by first noticing that the extended standard lattice $\tilde{\mathcal{G}} = (A_{ij})_{i,j\in\mathbb{Z}}$, associated with $\mathcal{G}$ as in [Pop99], is canonically constructed from $\mathcal{G}$ by repeated basic constructions starting from the inclusion $A_{0,\infty} \subset A_{-1,\infty}$ (see the second paragraph in the proof of 2.2 in [Pop99]). In particular, the sequence of inclusions $A_{-\infty,-1} \subset A_{-\infty,0} \subset \dots$, with all the system of inclusions of higher relative commutants into it, follow canonical. From this, an argument similar to the one in part 1° ends the proof.

If $\mathcal{G}_0 \subset \mathcal{G}$ in an embedding of standard λ-lattices with the same Jones projections then by the definition of the embeddings in the inductive limits of 2.4, which only depends on the Jones projections, it follows that the inductive limit algebras involved in $\mathcal{C}^{st}_{\mathcal{G}_0}$ are naturally embedded into the corresponding algebras of $\mathcal{C}^{st}_{\mathcal{G}}$, with commuting squares. To see that the embedding of

[1] This means that all the sides of the commuting "cube" arising from the inclusion of the two commuting squares are all non-degenerate commuting squares, with respect to the trace-preserving conditional expectations.

the two commuting squares is non-degenerate note first that the embedding $\mathcal{G}^0 \subset \mathcal{G}$ implements a natural embedding between the corresponding extended standard lattices $tilde\mathcal{G}^0$, $\tilde{\mathcal{G}}$ (thus, with commuting squares!). This fact in turn is an immediate consequence of the definitions, taking into account the smoothness condition (2.10.1). □

3. Amalgamated free products over type I algebras.

The main result of this section is the following theorem:

Theorem 3.1. *Let B be a von Neumann algebra, and $\mathcal{A} \subset B$ be a subalgebra. Let $E : B \to \mathcal{A}$ be a normal faithful conditional expectation. Assume that there exists a normal faithful semifinite trace Tr on $\mathcal{A}$, so that $\mathrm{Tr} \circ E$ is a trace on B. Assume lastly that $\mathcal{A}$ is type I and has discrete center.*
Let

$$M = (B, E) *_{\mathcal{A}} (\mathcal{A} \otimes L(\mathbb{F}_\infty), \mathrm{id} \otimes \tau).$$

If B is a of type II_∞ and $p \in B$ is a projection, $\mathrm{Tr}(p) = 1$, so that there is a system of matrix units $\{e_{ij}\} \subset B$ with $e_{11} = p$, $\sum e_{ii} = 1$, then

$$M \cong [(pBp, \mathrm{Tr}(p\cdot)) * (L(\mathbb{F}_\infty), \tau)] \otimes B(H);$$

The proof of the theorem will consist of a sequence of lemmas. The notation and assumptions of the first paragraph of the theorem remain fixed throughout this section.

It is convenient to omit the mention of the specific conditional expectations in expressions for reduced amalgamated free products. It will always be clear from the context what conditional expectations are understood. Moreover, note that all of the conditional expectations in this paper are trace-preserving.

Lemma 3.2. *M is a factor iff the centers $Z(\mathcal{A}) \cap Z(B)$ have trivial intersection.*

Proof. By [Pop93], the relative commutant of $L(\mathbb{F}_\infty)$ inside M is equal to $\mathcal{A}$. It follows that $Z(M) \subset \mathcal{A}$, hence $Z(M) \subset Z(\mathcal{A})$. Since $\mathcal{A} \subset B$, also $Z(M) \subset Z(\mathcal{A}) \cap Z(B)$. The other inclusion is trivial. □

Let Q be a von Neumann algebra with a semifinite normal trace Tr, and let $\eta_i : Q \to Q$ be normal completely positive maps. Assume that each η_i is self-adjoint, i.e., $\mathrm{Tr}(\eta_i(x)y) = \mathrm{Tr}(x\eta_i(y))$ for all x, y trace-class in Q.

Define $\Phi(Q, \eta_1, \eta_2, \dots, \eta_n)$, $n = 1, 2, \dots$ or $+\infty$, to be the von Neumann algebra generated by Q and the Q-semicircular family $X_1, X_2, \dots, X_n$, so that: (i) X_i are free with amalgamation over Q and (ii) each X_i has covariance η_i. Denote by E_Q the canonical conditional expectation from $\Phi(Q, \eta_1, \eta_2, \dots, \eta_n)$ onto Q. By [Shl99], $\mathrm{Tr} \circ E_Q$ is a trace on $\Phi(Q, \eta_1, \dots, \eta_n)$. Moreover, $E_Q(X_i q X_j) = \delta_{ij}\eta(q)$, for all $q \in Q$. Recall [Shl98] that X_i satisfy the inequality

$$\|X_i\| \le 2\|\eta_i(1)\|^{1/2}.$$

Recall [Shl98] that if $q_i, r_i \in Q$ are elements, X is Q-semicircular of covariance η, then

$$Y_i = q_i X r_i + r_i^* X q_i^*$$

is again Q-semicircular, of covariance

$$q \mapsto q_i\eta(r_ixr_i^*)q_i^* + r_i^*\eta(q_i^*xq_i)r_i$$
$$+q_i\eta(r_ixq_i)r_i + r_i^*\eta(q_i^*xr_i^*)q_i^*.$$

In addition, $\{Y_i\}$ are free with amalgamation over Q iff $E_Q(Y_iqY_j) = 0$ for all $q \in Q$ and $i \neq j$.

Lemma 3.3. *$M \cong \Phi(B, E, E, E, \dots)$ (infinite number of copies).*

Proof. By [Shl99],

$$\begin{aligned} \Phi(B, E, E, \dots) &\cong (B, E) *_{\mathcal{A}} \Phi(\mathcal{A}, \mathrm{id}, \mathrm{id}, \mathrm{id}, \dots) \\ &\cong (B, E) *_{\mathcal{A}} (\mathcal{A} \otimes \Phi(\mathbb{C}, \mathrm{id}, \mathrm{id}, \dots)) \\ &\cong (B, E) *_{\mathcal{A}} (\mathcal{A} \otimes L(\mathbb{F}_\infty)) \\ &= M. \end{aligned}$$

□

We need a slight modification of the construction Φ, which works for semifinite completely positive maps, like $\mathrm{Tr} : B \to B$.

Lemma 3.4. *Let $\eta_i : Q \to Q$, $\mu_i : Q \to Q$ be normal self-adjoint c.p. maps. Assume that for each i, there exist (possibly unbounded) operators x_i affiliated with Q, with (possibly unbounded) inverses, so that*

$$\mu_i(q) = x_i^*\eta_i(x_iqx_i^*)x_i, \quad \forall q \in Q.$$

Then $\Phi(Q, \eta_1, \eta_2, \dots) \cong \Phi(Q, \mu_1, \mu_2, \dots)$ in a way that preserves Q and E_Q. (The equation means that μ_i can be obtained as the closure of the densely defined operator $q \mapsto x_i^*\eta_i(x_iqx_i^*)x_i$).

Proof. By definition,

$$\Phi(Q, \eta_1, \eta_2, \dots) = W^*(Q, X_1, X_2, \dots),$$

where X_i are Q-semicircular, of covariance η_i. We claim that $x_i^*X_ix_i \in \Phi(Q, \eta_1, \eta_2, \dots)$ (a priori, it may not be defined, since x_i may be unbounded). It is sufficient, by passing to the polar decomposition $x_i = u_ib_i$, $u_i \in Q$ unitary, to consider only the case that x_i are self-adjoint. Denote by x_i^t the value of the cut-off function $\{x \mapsto x\}|_{[-t,t]}$applied to x_i. Let $Y_t = x_i^tX_ix_i^t$. Then Y_t is again Q-semicircular, of covariance

$$\eta_i^t(q) = x_i^t\eta_i(x_i^tqx_i^t)x_i^t.$$

In particular,

$$\|Y_t\| \leq 2\|x_i^t\eta_i(x_i^tx_i^t)x_i^t\|^{1/2}.$$

Since $x_t^tx_i^t \leq x_i^2$ we get that

$$\begin{aligned} x_i^t\eta_i(x_i^tx_i^t)x_i^t &\leq x_i^t\eta(x_i^2)x_i^t \\ &= \chi_{[-t,t]}(x_i)x_i\eta(x_i^2)x_i\chi_{[-t,t]}(x_i) \\ &\leq x_i\eta(x_i)^2x_i = \mu(1). \end{aligned}$$

Hence we that

$$\|Y_t\| \leq 2\|\mu(1)\|^{1/2}.$$

Note that $Y_t = \chi_{[-t,t]}(x_i)x_i^s X_i x_i^s \chi_{[-t,t]}(x_i)$ if $t < s$. Hence Y_t are bounded, and moreover $\chi_{[-r,r]}(x_i)Y_t\chi_{[-r,r]}(x_i)$ does not depend on t once $t > r$. It follows that also the weak limit of Y_t exists and is bounded. We denote the limit by $x_i X_i x_i$. It is clear that $x_i X_i x_i$ is Q-semicircular of covariance $q \mapsto x_i\eta(x_i q x_i)x_i$. Note that $X_i \in W^*(Q, x_i X_i x_i)$ (one simply applies the same construction, starting with $x_i X_i x_i$ and using x_i^{-1} in the place of x_i).

Now,

$$\begin{aligned}\Phi(Q,\eta_1,\eta_2,\dots) &= W^*(Q, x_1X_1x_1, x_2X_2x_2,\dots)\\ &\cong \Phi(Q,\mu_1,\mu_2,\dots),\end{aligned}$$

since $x_i X_i x_i$ has covariance $q \mapsto x_i\eta(x_i q x_i)x_i = \mu_i(x_i)$. □

Definition 3.5. $\Phi(Q, \mathrm{Tr}, \mathrm{Tr}, \dots) = \Phi(Q, \eta, \eta, \dots)$, where η is any normal c.p. map from Q to Q, so that $\eta(q) = x^* \mathrm{Tr}(xqx^*)x$ for some $x \in Q$, having a (possibly unbounded) inverse.

It is not hard to see, from Lemma 3.4, that this definition does not depend on the choice of η. Moreover, if the trace Tr is actually *finite,* then this coincides with the previous definition of $\Phi(Q, \mathrm{Tr})$.

Remark 3.6. The "unbounded semicircular element" of Radulescu [Răd92] (see also [DR00]) is precisely the "operator" one would get if in the construction of $\Phi(Q, \mathrm{Tr})$ one were to use a semifinite trace, but completely ignore the fact that $\mathrm{Tr}(1)$ is infinite. If $\eta(\cdot) = x\,\mathrm{Tr}(x\cdot x)x$ is as above, and X is Q-semicircular of covariance η, then Radulescu's element would correspond to the operator $x^{-1}Xx^{-1}$, which does not make sense as an operator, because Tr is not a normal s.a. map from Q to itself. Note that, as used in Radulescu's work, the finite compressions $\chi_{[-t,t]}(x)x^{-1}Xx^{-1}\chi_{[-t,t]}(x)$ do make sense as operators in $\Phi(Q, \mathrm{Tr})$. In particular, $\Phi(Q, \mathrm{Tr})$ is exactly the algebra $Q * \mathcal{S}X$ described in [DR00].

Lemma 3.7. *Let* $N = \Phi(Q, \eta_1, \eta_2, \dots, \mu_1, \mu_2, \dots)$. *Denote by* $E_\eta : N \to \Phi(Q, \eta_1, \eta_2, \dots) = N_\eta$ *and* $E_\mu : N \to \Phi(Q, \mu_1, \mu_2, \dots) = N_\mu$ *the canonical conditional expectations. Then*

$$N \cong (N_\eta, E_Q) *_Q (N_\mu, E_Q) \cong \Phi(N_\eta, \mu_1 \circ E_\eta, \mu_2 \circ E_\eta, \dots)$$

in a way that preserves N_η*,* Q *and* E_η*,* E_Q*.*

Proof. By definition,

$$N = W^*(Q, X_1, X_2, \dots, Y_1, Y_2, \dots),$$

where X_i and Y_i are free over Q, and X_i is η_i-semicircular over Q, Y_i is μ_i-semicircular over Q. The claimed decomposition as an amalgamated free product follows. The second isomorphism follows from the fact that Y_i, being free from $W^*(Q, X_1, X_2, \dots) = N_\eta$ over Q, is $\mu_i \circ E_\eta$-semicircular over N_η (see [Shl98]). □

Lemma 3.8. *Assume that* Q *is a factor of type* II_∞*, and* $\eta_1, \eta_2, \dots$ *are normal self-adjoint c.p. maps from Q to itself. Assume that* $\eta_i \neq 0$ *for all* i*, and that for each* i*, there exist subalgebras* $\mathcal{A}_i$*, each of type I with discrete center, so that*

$$\eta_i = E^Q_{\mathcal{A}_i}$$

Then $\Phi(Q, \eta_1, \eta_1, \dots, \eta_2, \eta_2, \dots) \cong \Phi(Q, \mathrm{Tr}, \mathrm{Tr}, \mathrm{Tr}, \dots)$ *(each* η_i *is repeated an infinite number of times), in a way that maps* Q *to* Q*, and preserves* E_Q*.*

Proof. Since $\Phi(Q, \eta_1, \dots, \eta_2, \dots) = \Phi(Q, \eta_1, \eta_1, \dots) *_Q \Phi(Q, \eta_2, \eta_2, \dots) *_Q \dots$, it is sufficient to prove the result in the case that all η_i are the same. We can then clearly assume that all $\mathcal{A}_i = \mathcal{A}$, and $\eta_i = E_{\mathcal{A}}$.

Let $q_1, q_2, \dots$ be the minimal central projections of $\mathcal{A}$, $\sum q_i = 1$. Then $\mathcal{A} = \sum q_i \mathcal{A} q_i$, and each $q_i \mathcal{A} q_i$ is a type I factor; let $n_i \in \mathbb{N} \cup \{+\infty\}$ be the rank (square root of the dimension) of $q_i \mathcal{A} q_i$. Let e^i_{st}, $1 \le s, t \le n_i$ be a system of matrix units for $q_i \mathcal{A} q_i$; that is,

$$\begin{aligned} e^i_{st} s^j_{t's'} &= \delta_{ij}\delta_{tt'} e^i_{ss'} \\ e^i_{st} &= (e^i_{ts})^* \\ q_i &= \sum_{1 \le s \le n_i} e^i_{ss}. \end{aligned}$$

Claim 3.9. Let P be an algebra of type II_∞, $\hat{P} \subset P$ be a unital subalgebra of type II_∞, so that $\hat{P}$ is a factor; and $p \in \hat{P}$ be a projection of finite trace. Let $\nu : Q \to Q$ be given by

$$\nu(q) = p \operatorname{Tr}(pqp) p, \quad q \in P.$$

Then $\Phi(P, \mathrm{Tr}, \mathrm{Tr}, \dots) \cong \Phi(P, \nu, \nu, \dots)$ in a way that preserves P and E_P.

Proof. Choose matrix units $f_{ij} \in \hat{P}$, so that $f_{11} = p$, $\sum f_{ii} = 1$, $f_{ij} f_{j'i'} = \delta_{jj'} f_{ii'}$, $f^*_{ij} = f_{ji}$. Let $x = \sum_i \frac{1}{2^i} f_{ii}$, and let $\mu(p) = x \operatorname{Tr}(xpx) x$, $p \in P$ be a completely-positive map from P to itself. Then $\Phi(P, \mathrm{Tr}, \mathrm{Tr}, \dots) \cong \Phi(P, \mu, \mu, \dots)$. Let X_i be a P-semicircular family of covariance μ; thus $\Phi(P, \mathrm{Tr}, \mathrm{Tr}, \dots) = W^*(P, X_1, X_2, \dots)$. Let $X^k_{ij} = \Re f_{1i} X_k f_{j1}$, $i \le j$, $Y^k_{ij} = \Im f_{1i} X_k f_{j1}$, $i < j$. Then $\Phi(P, \mathrm{Tr}, \mathrm{Tr}, \dots)$ is generated by P and $\{X^k_{ij}\}_{k, i \le j} \cup \{Y^k_{ij}\}_{k, i<j}$. A straightforward computation shows that $E(X^k_{ij} p X^{k'}_{i'j'}) = \text{const} \times \delta_{ii'}\delta_{jj'}\delta_{kk'} p \operatorname{Tr}(pqp)$, $E(Y^k_{ij} p Y^{k'}_{i'j'}) = \text{const} \times \delta_{ii'}\delta_{jj'}\delta_{kk'} p \operatorname{Tr}(pqp)$, and $E(X^k_{ij} p Y^{k'}_{i'j'}) = 0$. Hence, upon proper rescaling, $\{X^k_{ij}\}_{k, i \le j} \cup \{Y^k_{ij}\}_{k, i<j}$ form a P-semicircular family of covariance ν. Hence $\Phi(P, \mathrm{Tr}, \mathrm{Tr}, \dots) \cong \Phi(P, \nu, \nu, \dots)$, as claimed. ∎

Claim 3.10. $\Phi(Q, \mathrm{Tr}, \mathrm{Tr}, \dots) \cong \Phi(Q, \eta, \eta, \dots, \mathrm{Tr}, \mathrm{Tr}, \mathrm{Tr}, \dots)$, in a way that preserves Q and E_Q.

Proof. Let $p_i = e^i_{11} \in Q$. Let $\nu_i(q) = p_i \operatorname{Tr}(p_i q p_i) p_i$. We first notice that, in view of Claim 3.9,

$$\begin{aligned} \Phi(Q, \mathrm{Tr}, \mathrm{Tr}, \dots) &\cong (\Phi(Q, \mathrm{Tr}, \mathrm{Tr}, \dots) *_Q (\Phi(Q, \mathrm{Tr}, \mathrm{Tr}, \dots) *_Q \dots \\ &\cong (\Phi(Q, \mathrm{Tr}, \mathrm{Tr}, \dots, \nu_1, \nu_1, \dots) *_Q (\Phi(Q, \mathrm{Tr}, \mathrm{Tr}, \dots, \nu_2, \nu_2, \dots) *_Q \dots \\ &\cong \Phi(Q, \mathrm{Tr}, \mathrm{Tr}, \dots, \nu_1, \nu_1, \dots, \nu_2, \nu_2, \dots, \dots) \\ &\cong \Phi(Q, \nu_1, \nu_1, \dots, \nu_2, \nu_2, \dots, \dots) *_Q \Phi(Q, \mathrm{Tr}, \mathrm{Tr}, \dots). \end{aligned}$$

Let X^i_j be Q-semicircular variables, free with amalgamation over Q, and so that the covariance of X^i_j is ν_i. Note that $X^i_j = e^i_{11} X^k_j e^i_{11}$. Let

$$Y_j = \sum_i \sum_{1 \le s \le n_i} e_{s1} X^i_j e_{1s} \cdot \frac{1}{\operatorname{Tr}(e^s_{11})^{1/2}}$$

This sum converges in weakly, since X is diagonal relative to the orthogonal family of projections $\{e^i_{ss}\}$, and

$$\|e_{s1}X^i_j e_{1s}\cdot\frac{1}{\mathrm{Tr}(e^s_{11})^{1/2}}\|\leq 2\|\nu_i(1)\|^{1/2}\cdot\frac{1}{\mathrm{Tr}(e^s_{11})^{1/2}}=2.$$

It is not hard to see that $\{Y_j\}$ form a Q-semicircular family of covariance $E_{\mathcal{A}}=\eta$. Moreover, $\Phi(Q,\nu_1,\nu_1,\dots,\nu_2,\nu_2,\dots,\dots)$ is generated by Q and $\{Y_i\}_i$, since X^i_j is, up to a constant, $e^i_{11}Y_j e^i_{11}$. Hence $\Phi(Q,\nu_1,\nu_1,\dots,\nu_2,\nu_2,\dots,\dots)\cong\Phi(Q,\eta,\eta,\dots)$. Thus

$$\begin{aligned}\Phi(Q,\mathrm{Tr},\mathrm{Tr},\dots) &\cong \Phi(Q,\nu_1,\nu_1,\dots,\nu_2,\nu_2,\dots,\dots)*_Q\Phi(Q,\mathrm{Tr},\mathrm{Tr},\dots)\\ &\cong \Phi(Q,\eta,\eta,\dots)*_Q\Phi(Q,\mathrm{Tr},\mathrm{Tr},\dots)\\ &\cong \Phi(Q,\eta,\eta,\dots,\mathrm{Tr},\mathrm{Tr},\dots).\end{aligned}$$

□

We now finish the proof of the lemma. By Lemma 3.7, we get that

$$\begin{aligned}\Phi(Q,\mathrm{Tr},\mathrm{Tr},\dots) &\cong \Phi(Q,\eta,\eta,\dots,\mathrm{Tr},\mathrm{Tr},\dots)\\ &\cong \Phi(\Phi(Q,\eta,\eta,\dots),\mathrm{Tr},\mathrm{Tr},\dots).\end{aligned}$$

Noticing that $P=\Phi(Q,\eta,\eta,\dots)$ contains a II_∞ factor $\hat{P}=Q$, and setting $p=e^1_{11}\in Q$, $\nu(x)=p\,\mathrm{Tr}(pxp)p$, $x\in\Phi(Q,\eta,\eta,\dots)$, we get

$$\begin{aligned}\Phi(\Phi(Q,\eta,\eta),\mathrm{Tr},\mathrm{Tr},\dots) &\cong \Phi(\Phi(Q,\eta,\eta,\dots),\nu,\nu,\dots)\\ &\cong \Phi(Q,\eta,\eta,\dots,\nu|_Q,\nu|_Q,\dots),\end{aligned}$$

the last isomorphism because $\nu=E_Q^{\Phi(Q,\eta,\eta,\dots)}\circ\nu\circ E_Q^{\Phi(Q,\eta,\eta,\dots)}$, since $p\in Q$ (see Lemma 3.7).

Now, $\Phi(Q,\eta,\eta,\dots,\nu|_Q,\nu|_Q,\dots)$ is generated by Q and a Q-semicircular system $X_1,X_2,\dots,Y_1,Y_2,\dots$, where $\{X_i,Y_i\}_i$ are free with amalgamation over Q, X_i has covariance η and Y_i has covariance ν. Note that X_i commutes with $\mathcal{A}$ (containing $p=e^1_{11}$), and $Y_i=pY_ip$, because of the form of ν. In particular, $X_i=\sum_{i,1\leq s\leq n_i}e^i_{ss}X_ie^i_{ss}$. It follows that $\Phi(Q,\eta,\eta,\dots,\nu|_Q,\nu|_Q,\dots)$ is generated by:

$$\{q_1X_iq_1\}_i,\ \{(1-q_1)X_i(1-q_1)\}_i,\ \{e^1_{11}Y_ie^1_{11}\}_i,\ Q.$$

Furthermore, $\Phi(Q,\eta,\eta,\dots)$ is generated by:

$$\{q_1X_iq_1\}_i,\ \{(1-q_1)X_i(1-q_1)\}_i,\ Q$$

Note that the three families $\{q_1X_iq_1\}_i$, $\{(1-q_1)X_i(1-q_1)\}_i$, $\{pY_ip\}_i$ are free with amalgamation over Q; this is because for all $q\in Q$,

$$E_Q(q_1X_iq_1q(1-q_1)X_j(1-q_1))=\delta_{ij}q_1E_{\mathcal{A}}(q_1q(1-q_1))(1-q_1)=0,$$

since q_1 is a central projection in $\mathcal{A}$.

Next, since X_i commutes with $\mathcal{A}$, we get that

$$\begin{aligned} q_1 X_i q_1 &= \sum_{1\leq s\leq n_1} e^1_{ss} X_i e^1_{ss} \\ &= \sum_{1\leq s\leq n_1} e^1_{s1} e^1_{1s} X_i e^1_{s1} e^1_{1s} \\ &= \sum_{1\leq s\leq n_1} e^1_{s1} p X_i p e^1_{1s}. \end{aligned}$$

It follows that $\Phi(Q,\eta,\eta,\dots,\nu|_Q,\nu|_Q,\dots)$ is generated by:

$$\{pX_ip\}_i,\ \{(1-q_1)X_i(1-q_1)\}_i,\ \{pY_ip\}_i,\ Q,$$

and the families $\{pX_ip\}_i$, $\{(1-q_1)X_i(1-q_1)\}_i$ $\{pY_ip\}$ are free with amalgamation over Q. Moreover, $\Phi(Q,\eta,\eta,\dots)$ is generated by

$$\{pX_ip\}_i,\ \{(1-q_1)X_i(1-q_1)\}_i,\ Q.$$

Now, $\{pX_ip\}_i$ are free with amalgamation over Q, and pX_ip is Q-semicircular with covariance

$$q \mapsto E_Q(pX_ipqpX_ip) = pE_{\mathcal{A}}(pqp)p = \text{const} \times p\,\mathrm{Tr}(pqp)p = \text{const} \times \nu(q).$$

It follows that $\{pX_ip\}_i$ (upon rescaling by some non-zero constant) form an infinite Q-semicircular family of covariance $\nu|_Q$. Hence, by renumbering, we can joint $\{pX_ip\}_i \cup \{pY_ip\}_i$ into a single semicircular family of covariance ν. It follows that the algebras

$$W^*(\{pX_ip\}_i, \{(1-q_1)X_i(1-q_1)\}_i, \{pY_ip\}_i, Q)$$

and

$$W^*(\{pX_ip\}_i,\ \{(1-q_1)X_i(1-q_1)\}_i, Q)$$

are isomorphic to each other, in a way that maps Q to Q, and preserves E_Q. But we saw before that the first of these algebras is $\Phi(Q,\eta,\eta,\dots,\nu|_Q,\nu|_Q,\dots)$, while the second is $\Phi(Q,\eta,\eta,\dots)$. $\square$

Lemma 3.11. *If B is a of type II_∞ and $p\in B$ is a projection, $\mathrm{Tr}(p)=1$, so that there is a system of matrix units $\{e_{ij}\}\subset B$ with $e_{11}=p$, $\sum e_{ii}=1$, then*

$$\Phi(B,\mathrm{Tr}) \cong [(pBp,\mathrm{Tr}(p\cdot)) * (L(\mathbb{F}_\infty),\tau)] \otimes B(H);$$

In particular, $\Phi(B,\mathrm{Tr},\mathrm{Tr},\dots)$ is a II_∞ factor.

Proof. Let $p_i = e_{ii}$ be orthogonal projections in B, $\mathrm{Tr}(p_i)=1$, $\sum p_i = 1$. Let $x=\sum \frac{1}{2^n}p_n$, $\eta : B\to B$ be given by $\eta(b) = x\,\mathrm{Tr}(xbx)x$. Then $\Phi(B,\eta,\eta,\dots)\cong\Phi(B,\mathrm{Tr},\mathrm{Tr},\dots)$, by definition. Hence $\Phi(B,\mathrm{Tr},\mathrm{Tr},\dots)\cong W^*(B,X_1,X_2,\dots)$, where X_i are B-semicircular, each of covariance η. Then

$$p_1\Phi(B,\mathrm{Tr},\mathrm{Tr},\dots)p_1 \cong W^*(p_1Bp_1,\{X^r_{ij}\}_{r,i,j}),$$

where $X^r_{ij} = e_{1i}X_re_{j1}$. It is not hard to see that $\{X^r_{ii}\}\cup\{\Re X^r_{ij} : i>j\}\cup\{\Im X^r_{ij} : i>j\}$ are free over p_1Bp_1 and are again a p_1Bp_1-semicircular family, each having covariance $\frac{1}{2^{i+j}}\cdot\mathrm{Tr}(p_1\cdot p_1)$. Denoting $p=p_1$ and $\tau(\cdot)=\mathrm{Tr}(p\cdot p)$, we get (see [Shl99])

$$p\Phi(B,\mathrm{Tr},\mathrm{Tr},\dots)p \cong \Phi(pBp,\tau,\tau,\dots)\cong (B,\tau)*L(\mathbb{F}_\infty).$$

$\square$

The following corollary, together with Lemma 3.11, implies Theorem 3.1.

Corollary 3.12. *Let M be as in Theorem 3.1. Then $M \cong \Phi(B, \mathrm{Tr}, \mathrm{Tr}, \mathrm{Tr}, \dots)$.*

Proof. Let $F : \Phi(B, E, E, \dots) \to \mathcal{A}$ denote the composition of $E : B \to \mathcal{A}$ and $E_B : \Phi(B, E, E, \dots) \to B$. Let $N = \Phi(B, \mathrm{Tr}, \mathrm{Tr}, \dots)$, and denote by $G : N \to \mathcal{A}$ the composition of $E_B : N \to B$ and $E : B \to A$. Note that F, G and E all satisfy the hypothesis of Lemma 3.8; moreover, by 3.11, N is a factor. We have

$$\begin{aligned}
M &\cong \Phi(B, E, E, \dots) \\
&\cong \Phi(\Phi(B, E, E, \dots), F, F, F, \dots) \\
&\cong \Phi(M, F, F, \dots) \\
&\cong \Phi(M, \mathrm{Tr}, \mathrm{Tr}, \dots) \\
&\cong \Phi(\Phi(B, E, E, \dots), \mathrm{Tr}, \mathrm{Tr}, \dots) \\
&\cong \Phi(B, E, E, E, \dots, \mathrm{Tr}, \mathrm{Tr}, \dots) \\
&\cong \Phi(B, \mathrm{Tr}, \mathrm{Tr}, \dots, E, E, \dots) \\
&\cong \Phi(\Phi(B, \mathrm{Tr}, \mathrm{Tr}, \dots), G, G, \dots) \\
&\cong \Phi(N, G, G, \dots) \\
&\cong \Phi(N, \mathrm{Tr}, \mathrm{Tr}, \dots) \\
&\cong \Phi(\Phi(B, \mathrm{Tr}, \mathrm{Tr}, \dots), \mathrm{Tr}, \mathrm{Tr}, \dots) \\
&\cong \Phi(B, \mathrm{Tr}, \mathrm{Tr}, \dots).
\end{aligned}$$

This completes the proof. $\square$

4. CONSTRUCTIONS OF SUBFACTORS.

Let us begin by recording the following general result (which is well-known for semifinite inclusions with trace-preserving conditional expectations).

Proposition 4.1. *Let*

$$\begin{array}{ccc}
\mathcal{P}_{-1} & \overset{\mathcal{F}_0}{\subset} & \mathcal{P}_0 \\
\cup \mathcal{E}_{-1} & & \cup \mathcal{E}_0 \\
\mathcal{Q}_{-1} & \overset{\mathcal{F}_{-1}}{\subset} & \mathcal{Q}_0
\end{array} \tag{4.1.1}$$

be a commuting square, and assume that $\mathcal{E}_i$ are faithful normal conditional expectations. Let Q be a diffuse finite von Neumann algebra with a normal finite faithful trace τ, and set

$$\mathcal{M}_i = ((\mathcal{Q}_i \otimes Q), \mathrm{id} \otimes \tau) *_{\mathcal{Q}_i} (\mathcal{P}_i, \mathcal{E}_i).$$

Then

$$\begin{array}{ccc} \mathcal{M}_{-1} & \overset{\hat{\mathcal{F}}}{\subset} & \mathcal{M}_0 \\ \cup & & \cup \\ \mathcal{P}_{-1} & \overset{\mathcal{F}_0}{\subset} & \mathcal{P}_0 \\ \cup\mathcal{E}_{-1} & & \cup\mathcal{E}_0 \\ \mathcal{Q}_{-1} & \overset{\mathcal{F}_{-1}}{\subset} & \mathcal{Q}_0 \end{array} \tag{4.1.2}$$

forms a commuting diagram of inclusions of von Neumann algebras. Moreover, $\mathcal{M}'_{-1} \cap \mathcal{M}_0 = \mathcal{P}'_{-1} \cap \mathcal{Q}_0$.

(b) Assume that (4.1.1) forms a commuting square, and $\mathcal{F}_i$ *are finite-index conditional expectations. Assume also that (4.1.1) is non-degenerate, i.e., any ONB* $\{m_i\}$ *for the inclusion* $\mathcal{Q}_{-1} \overset{\mathcal{F}_1}{\subset} \mathcal{Q}_0$ *forms an ONB for the inclusion* $\mathcal{P}_{-1} \overset{\mathcal{F}_1}{\subset} \mathcal{P}_0$ *(equivalently,* $\overline{\mathrm{sp}}(\mathcal{Q}_0\mathcal{P}_{-1}) = \mathcal{P}_0$*).*

Then all the commuting squares in (4.1.2) are non-degenerate. In particular, the index of $\mathcal{M}_{-1} \overset{\hat{\mathcal{F}}}{\subset} \mathcal{M}_0$ *is given by* $\sum m_j m_j^*$.

(c) Assume that

$$\begin{array}{ccccccc} \mathcal{P}_{-1}^{-1} & \subset & \mathcal{P}_0^{-1} & & \mathcal{P}_{-1}^0 & \subset & \mathcal{P}_0^0 \\ \cup & & \cup & \subset & \cup & & \cup \\ \mathcal{Q}_{-1}^{-1} & \subset & \mathcal{Q}_0^{-1} & & \mathcal{Q}_{-1}^0 & \subset & \mathcal{Q}_0^0 \end{array}$$

is a non-degenerate inclusion of non-degenerate commuting squares (non-degeneracy here means that all of the 6 *commuting squares obtained by combining the inclusions of* $\mathcal{P}_i^j$ *and* $\mathcal{Q}_i^j$*, are non-degenerate). Then*

$$\begin{array}{ccc} \mathcal{M}_{-1}^0 & \subset & \mathcal{M}_0^0 \\ \cup & & \cup \\ \mathcal{M}_{-1}^{-1} & \subset & \mathcal{M}_0^{-1} \end{array}$$

is again a non-degenerate commuting square.

Proof. (a) Note that the algebra generated by $\mathcal{P}_{-1}$ and Q inside $\mathcal{M}_0$ is isomorphic to $\mathcal{M}_1$; this is because Q and $\mathcal{P}_{-1}$ are free with amalgamation over $\mathcal{Q}_{-1}$, and the conditional expectations involved in the amalgamated free products are faithful.

(b) By the nondegeneracy and commuting square condition, an orthonormal basis $\{m_i\}$ for $\mathcal{Q}_{-1} \subset \mathcal{Q}_0$ "pulls out" to become an orthonormal basis for $\mathcal{M}_{-1} \subset \mathcal{M}_0$.

(c) By arguing as in part (a), we get the vertical inclusions in

$$\begin{array}{ccc} \mathcal{M}_{-1}^0 & \subset & \mathcal{M}_0^0 \\ \cup & & \cup \\ \mathcal{M}_{-1}^{-1} & \subset & \mathcal{M}_0^{-1} \end{array}.$$

Using the commuting square conditions and non-degeneracy, we see that an ONB for $\mathcal{M}_{-1}^{-1} \subset \mathcal{M}_0^{-1}$ (coming from an ONB for $\mathcal{Q}_{-1}^{-1} \subset \mathcal{Q}_0^{-1}$) is an ONB for $\mathcal{M}_{-1}^0 \subset \mathcal{M}_0^0$. $\square$

We now come to the functorial construction of subfactors in $L(\mathbb{F}_\infty)$. We denote by $\mathbb{G}$ the category whose objects are standard lattices. The morphisms in this category are by definition embeddings of standard lattices with the same index (i.e., embeddings of λ-lattices with the same λ), satisfying the smoothness condition (2.10.1).

Theorem 4.2. *Let $\mathbb{G}$ be the category of standard lattices, with (index-preserving) embeddings as morphisms. Let $\mathbb{S} = \mathbb{S}(L(\mathbb{F}_\infty))$ be the category of subfactors $(P_{-1} \subset P_0)$, $P_0 = L(\mathbb{F}_\infty)$, $P_{-1} \cong P_0$ of $L(\mathbb{F}_\infty)$ with morphisms $\imath : (P_{-1} \subset P_0) \to (Q_{-1} \subset Q_0)$ given by commuting square inclusions*

$$\begin{array}{ccc} Q_{-1} & \subset & Q_0 \\ \cup & & \cup \\ P_{-1} & \subset & P_0 \end{array}.$$

Denote by $\mathcal{G}$ the functor $\mathcal{G} : \mathbb{S} \to \mathbb{G}$, assigning to an inclusion its standard lattice,

$$\mathcal{G}(P_{-1} \subset P_0) = \mathcal{G}_{\mathcal{P}_{-1} \subset P_0}.$$

Then there exists a functor $\mathcal{F} : \mathbb{G} \to \mathbb{S}$, which is a right inverse for $\mathcal{G}$:

$$\mathcal{G} \circ \mathcal{F} = \mathrm{id}\,.$$

Proof. Let $\mathcal{G}_1, \mathcal{G}_2 \in \mathbb{G}$ be standard λ-lattices and assume $\mathcal{G}_1 \subset \mathcal{G}_2$. Associated to them, consider the inclusion of λ-Markov squares

$$\begin{array}{ccc} \mathcal{B}_{-1}^{(1)} & \subset & \mathcal{B}_0^{(1)} \\ \cup & & \cup \\ \mathcal{A}_{-1}^{(1)} & \subset & \mathcal{A}_0^{(1)} \end{array} \subset \begin{array}{ccc} \mathcal{B}_{-1}^{(2)} & \subset & \mathcal{B}_0^{(2)} \\ \cup & & \cup \\ \mathcal{A}_{-1}^{(2)} & \subset & \mathcal{A}_0^{(2)} \end{array}$$

as in Theorems 2.9 and 2.10. Note that $\mathcal{A}_{-1,0}^{(1),(2)}$ are type I with discrete center. Moreover, $\mathcal{B}_{-1,0}^{(1),(2)}$ are hyperfinite and each is isomorphic to an algebra of the form $R \otimes B(H)$, where R is hyperfinite type II_1 (possibly with non-trivial center). Let $Q = L(\mathbb{F}_\infty)$. Denote by $\mathcal{F}(\mathcal{G})$ the compression of the inclusion

$$(Q \otimes \mathcal{A}_{-1}^{(1)}) *_{\mathcal{A}_{-1}^{(1)}} \mathcal{B}_{-1}^{(1)} \subset (Q \otimes \mathcal{A}_0^{(1)}) *_{\mathcal{A}_0^{(1)}} \mathcal{B}_0^{(1)}$$

to the canonical trace 1 projection in $\mathcal{A}_{-1}^{(1)}$ (denoted by p_0 in Theorem 2.9). By Proposition 4.1, we get that the standard lattice of this inclusion is $\mathcal{G}_1$, i.e., $\mathcal{G} \circ \mathcal{F} = \mathcal{G}$. Also, by Proposition 4.1, the inclusions

$$\begin{array}{ccc} (Q \otimes \mathcal{A}_{-1}^{(2)}) *_{\mathcal{A}_{-1}^{(2)}} \mathcal{B}_{-1}^{(2)} & \subset & (Q \otimes \mathcal{A}_0^{(2)}) *_{\mathcal{A}_0^{(2)}} \mathcal{B}_0^{(2)} \\ \cup & & \cup \\ (Q \otimes \mathcal{A}_{-1}^{(1)}) *_{\mathcal{A}_{-1}^{(1)}} \mathcal{B}_{-1}^{(1)} & \subset & (Q \otimes \mathcal{A}_0^{(1)}) *_{\mathcal{A}_0^{(1)}} \mathcal{B}_0^{(1)} \end{array}$$

form a commuting square, which means that $\mathcal{F}$ is a functor between the categories $\mathbb{G}$ and $\mathbb{S}$. Lastly, Theorem 3.1 implies that each of the algebras involved is isomorphic to an algebra of the form $(R * L(\mathbb{F}_\infty)) \otimes B(H) \cong L(\mathbb{F}) \otimes B(H)$, where R is a type II_1 hyperfinite algebra (the last isomorphism follows from the results of Ken Dykema, [Dyk93]). It follows that the compressed inclusion $\mathcal{F}(\mathcal{G}_i)$ consists of algebras isomorphic to $L(\mathbb{F}_\infty)$. □

Corollary 4.3. *Let $\mathcal{G}$ be any standard lattice. Then there exists an inclusion $P_{-1} \subset P_0$, having $\mathcal{G}$ as its system of higher relative commutants, and so that $P_{-1} \cong P_0 \cong L(\mathbb{F}_\infty)$.*

Theorem 4.4. *Let $M_{-1} \subset M_0$ be an inclusion of II_1 factors with finite index. Then there exists an inclusion $\hat{M}_{-1} \subset \hat{M}_0$, with $\hat{M}_{-1} \cong M_{-1} * L(\mathbb{F}_\infty)$, $\hat{M}_0 \cong M_0 * L(\mathbb{F}_\infty)$, so that $\hat{M}_{-1} \subset \hat{M}_0$ has the same index and the same standard lattice of higher relative commutants as $M_{-1} \subset M_0$.*

Proof. By Theorem 2.9, there exists a non-degenerate commuting square

$$\begin{array}{ccc} M_{-1}\otimes B(H) & \subset & M_0\otimes B(H) \\ \cup & & \cup \\ \mathcal{A}_{-1} & \subset & \mathcal{A}_0 \end{array}$$

with $\mathcal{A}_{-1}$ and $\mathcal{A}_0$ type I with discrete centers. Letting $\mathcal{M}_{-1} = (M_{-1}\otimes B(H)) *_{\mathcal{A}_{-1}} (L(\mathbb{F}_\infty \otimes \mathcal{A}_{-1})$, $\mathcal{M}_0 = (M_0 \otimes B(H)) *_{\mathcal{A}_0} (L(\mathbb{F}_\infty \otimes \mathcal{A}_0)$, we see that $\mathcal{M}_{-1} \subset \mathcal{M}_0$ has the same higher relative commutants as $M_{-1} \subset M_0$. Compressing by a finite projection and noticing that in view of Theorem 3.1, $\mathcal{M}_{-1} \cong (M_{-1} * L(\mathbb{F}_\infty)) \otimes B(H)$, $\mathcal{M}_0 \cong (M_0 * L(\mathbb{F}_\infty)) \otimes B(H)$, gives the result. $\square$

5. Further constructions of subfactors of $L(\mathbb{F}_\infty)$

Returning to the setting and notations of Section 2, let us note that one can associate to a given standard λ-lattice $\mathcal{G} = (A_{ij})_{j\geq i\geq -1}$ a canonical commuting square of type I algebras as follows:

Notation 5.1. Let $(\mathcal{A}^0_{-1} \subset \mathcal{A}^0_0) \overset{\text{def}}{=} \overline{\bigcup_n \tilde{\alpha}_n(A_{-2,2n-1} \subset A_{-2,2n})}$. Note that Tr is semifinite on the algebras $\mathcal{A}^0_i$, $i = 0, -1$. Note also that if we re-denote $(\mathcal{A}_{-1} \subset \mathcal{A}_0)$ as $(\mathcal{A}^{-1}_{-1} \subset \mathcal{A}^{-1}_0)$ then one has the following inclusions with commuting squares (for the Tr-preserving expectations):

$$\begin{array}{ccc} \mathcal{B}_{-1} & \subset & \mathcal{B}_0 \\ \cup & & \cup \\ \mathcal{A}^0_{-1} & \subset & \mathcal{A}^0_0 \\ \cup & & \cup \\ \mathcal{A}^{-1}_{-1} & \subset & \mathcal{A}^{-1}_0 \end{array} \tag{5.1.1}$$

We denote by $\mathcal{C}_{\mathcal{G}}$ the pair consisting of the commuting square

$$\begin{array}{ccc} \mathcal{A}^0_{-1} & \subset & \mathcal{A}^0_0 \\ \cup & & \cup \\ \mathcal{A}^{-1}_{-1} & \subset & \mathcal{A}^{-1}_0 \end{array} \tag{5.1.2}$$

and the projection $p_0 \in \mathcal{A}^{-1}_{-1} (= \mathcal{A}_{-1})$.

Theorem 5.2. 1°. *The commuting square (5.1.2) is λ-Markov and if*

$$\begin{array}{ccccccc} \mathcal{A}^0_{-1} & \subset & \mathcal{A}^0_0 & \subset & \mathcal{A}^0_1 & \subset & \dots \\ \cup & & \cup & & \cup & & \\ \mathcal{A}^{-1}_{-1} & \subset & \mathcal{A}^{-1}_0 & \subset & \mathcal{A}^{-1}_1 & \subset & \dots \end{array}$$

is its basic construction, then

$$(\mathcal{A}^{0\prime}_i \cap \mathcal{A}^{-1}_j)_{j\geq i\geq -1} \simeq (A_{ij})_{j\geq i\geq -1}.$$

2°. *$\mathcal{C}_{\mathcal{G}}$ is canonically associated with $\mathcal{G}$ and $\mathcal{G} \mapsto \mathcal{C}_{\mathcal{G}}$ is a functor from the category of standard λ-lattices to the category of λ-Markov commuting squares of type* I *algebras with discrete centers.*

Proof. 1°. Since the commuting square formed by the top and bottom rows of (5.1.1) is λ-Markov (cf. 2.8), the commuting square (5.1.2) follows λ-Markov as well.

Moreover, since $A'_{-2,i} \cap A_{-1,j} = A_{ij}$, it follows that $A'_{-2,i+2n} \cap A_{-1,j+2n} = \sigma'^n(A_{ij})$ so that $\tilde{\alpha}_n(A_{-2,i+2n})' \cap \tilde{\alpha}_n(A_{-1,j+2n}) = \tilde{\alpha}_n(\sigma'^n(A_{ij}))$ which gives $p_n\mathcal{A}^0_i p'_n \cap p_n\mathcal{A}^{-1}_j p_n = \alpha(A_{ij})p_n$. Since $p_n \nearrow 1$, this proves 1°.

Part 2° is trivial, by the definition of $\mathcal{C}_{\mathcal{G}}$, since the construction of $\mathcal{A}^0_{-1} \subset \mathcal{A}^0_0$ only depends on the Jones projections in $\mathcal{G}$. Also, the commuting square conditions involved in the embedding $\mathcal{G}_0 \subset \mathcal{G}$ and the definition of the inductive limit, shows that $\mathcal{C}_{\mathcal{G}_0}$ sits inside $\mathcal{C}_{\mathcal{G}}$ with non-degenerate commuting squares. □

One can then prove (utilizing essentially the same techniques as Radulescu in [Răd94] or Dykema in [Dyk95]), the following:

Theorem 5.3. *Let $\mathcal{A} \subset \mathcal{C}$ be a trace-preserving inclusion of type I von Neumann algebras, each having discrete center. Then*

$$(\mathcal{A} \otimes L(\mathbb{F}_\infty)) *_{\mathcal{A}} \mathcal{C} \cong L(\mathbb{F}_\infty) \otimes B(H).$$

Combining the two theorems together and compressing to the canonical projection $p_0 \in \mathcal{A}_{-1}$ gives an alternative proof of Theorem 4.2.

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DEPARTMENT OF MATHEMATICS, UCLA, LOS ANGELES, CA 90095, USA
E-mail address: `popa@math.ucla.edu`
E-mail address: `shlyakht@math.ucla.edu`