ON A GENERIC INVERSE DIFFERENTIAL GALOIS PROBLEM FOR $\mathrm{GL_n}$

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Abstract. In this paper we construct a generic Picard-Vessiot extension for the general linear groups. In the case when the differential base field has finite transcendence degree over its field of constants we provide necessary and sufficient conditions for solving the inverse differential Galois problem for this groups via specialization from our generic extension.

Introduction

Le F be a differential field of characteristic zero with algebraically closed field of constants C. In this paper we give an affirmative answer, for the group $\mathrm{GL_n}(C)$, to the following

Generic Inverse Differential Galois Problem: *For a connected algebraic group G over C find a generic Picard-Vessiot extension of F with differential Galois group G.*

By *generic extension* we mean a Picard-Vessiot extension of a *generic field* that contains F and such that every Picard-Vessiot extension of F for G in the usual sense can be obtained from the generic one by specialization.

We point out that any such specialization will provide a solution to the inverse differential Galois problem in the usual sense, namely, to determine, given F and C as above, and a linear algebraic group G over C, what differential field extensions $E \supset F$ are Picard-Vessiot extensions with differential Galois group G and, in particular, whether there are any.

We use the terminology of A. Magid's book [13]. In [13] the reader may also find definitions and proofs of some results from differential Galois theory that will be recalled.

Our construction of a generic Picard-Vessiot extension of F with group $\mathrm{GL_n}(C)$ may be summarized as follows: Let $F\{Y_{ij}\}$ be the ring of differential polynomials over F in the differential indeterminates Y_{ij}, $1 \le i, j \le n$. Our *generic base field* will be the differential quotient field $F\langle Y_{ij}\rangle$ of $F\{Y_{ij}\}$. Let X_{ij}, $1 \le i, j \le n$, be algebraically independent over $F\langle Y_{ij}\rangle$. Extend the derivation on $F\langle Y_{ij}\rangle$ to the polynomial ring

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$F\langle Y_{ij}\rangle[X_{ij}]$ by letting $D(X_{ij}) = \sum_{\ell=1}^{n} Y_{i\ell}X_{\ell j}$. This derivation extends in a natural way to the quotient field $F\langle Y_{ij}\rangle(X_{ij})$ (note that $F\langle Y_{ij}\rangle(X_{ij})$ is also the function field of the group obtained from $\mathrm{GL_n}(C)$ by extending scalars from C to $F\langle Y_{ij}\rangle$). Then

Theorem 1. *The differential field extension $F\langle Y_{ij}\rangle(X_{ij}) \supset F\langle Y_{ij}\rangle$ is a generic Picard-Vessiot extension of F with differential Galois group $\mathrm{GL_n}(C)$.*

The main step in proving Theorem 1 is to show that $F\langle Y_{ij}\rangle(X_{ij}) \supset F\langle Y_{ij}\rangle$ is a no-new-constant extension. A new constant in $F\langle Y_{ij}\rangle(X_{ij})$ must be the quotient of two relatively prime Darboux polynomials, that is, two polynomials $p_1, p_2 \in F\{Y_{ij}\}[X_{ij}]$ that satisfy $D(p_i) = qp_i$ for some $q \in F\{Y_{ij}\}[X_{ij}]$. We use Gröbner basis techniques to show that the only Darboux polynomials in $F\{Y_{ij}\}[X_{ij}]$ are those of the form $\ell \det[X_{ij}]^a$, where $\ell \in F$ and $a \in \mathbb{N}$. This implies that there are no two such relatively prime Darboux polynomials and, therefore, no new constants.

To show that the extension is generic we let $E \supseteq F$ be any Picard-Vessiot extension of F with group $\mathrm{GL_n}(C)$. Then E is isomorphic to $F(X_{ij})$, the function field of the group obtained from $\mathrm{GL_n}(C)$ by extending scalars from C to F, as a $\mathrm{GL_n}(C)$-module and as an F-module, and there are $f_{ij} \in F$ such that the derivation on E is given by $D_E = \sum_{\ell=1}^{n} f_{i\ell}X_{\ell j}$. In this situation, the Picard-Vessiot extension $F(X_{ij}) \supseteq F$ is obtained from the Picard-Vessiot extension $C\langle f_{ij}\rangle(X_{ij}) \supseteq C\langle f_{ij}\rangle$ by extension of scalars from C to F. Therefore, any Picard-Vessiot extension $E \supset F$ can be obtained from $F\langle Y_{ij}\rangle(X_{ij}) \supset F\langle Y_{ij}\rangle$ via the specialization $Y_{ij} \mapsto f_{ij}$. That is, $F\langle Y_{ij}\rangle(X_{ij}) \supset F\langle Y_{ij}\rangle$ is a generic Picard-Vessiot extension of F for $\mathrm{GL_n}(C)$.

Now, suppose that F has finite transcendence degree over C say,

$$F = C(t_1, \ldots, t_m)[z_1, \ldots, z_k],$$

where the t_i are algebraically independent over C and the z_i are algebraic over $C(t_1, \ldots, t_m)$. Consider the differential field $F(X_{ij})$ with derivation D_E as above. Let $\mathcal{C}$ denote its field of constants. Let $F\{Y_{ij}\}[X_{ij}]$ be the differential ring with the derivation defined above. For $k \geq 1$ let $\mathbb{T}_k$ denote the set of monomials in the both the t_i and X_{ij} of total degree less than or equal to k. Fix a term order on the set $\mathbb{T}$ of monomials in the t_i and the X_{ij} and let $W_k(Y_{ij})$ denote the wronskian of $\mathbb{T}_k$ relative to that order (the order will only affect the wronskian by a sign). The following theorem summarizes our specialization results:

Theorem 2. *$F(X_{ij}) \supset F$ is a Picard-Vessiot extension for $\mathrm{GL_n}(C)$ if and only if all the wronskians $W_k(Y_{ij})$ map to nonzero elements in $F(X_{ij})$ via the specialization $Y_{ij} \mapsto f_{ij} \in F$.*

The above condition on the wronskians means that all the sets $\mathbb{T}_k$, for $k \geq 1$, are linearly independent over $\mathcal{C}$. This is in turn equivalent to the fact that the set of all the t_i and all the X_{ij} are algebraically independent over $\mathcal{C}$. Unfortunately, Theorem 2 gives infinitely many conditions. We do not know at present how to use these conditions to effectively construct solutions to the inverse problem, and this constitutes an interesting open problem.

A specialization as in Theorem 2, however, is known to exist by a result of C. Mitschi and M. Singer [17]. They give a constructive algebraic solution to the inverse problem for all connected linear algebraic groups (and, in particular, for $\mathrm{GL_n}(C)$) when F has finite transcendence degree over C. An interesting direction of research in connection with the previous open problem is to try to fully describe all possible solutions (isomorphic and non-isomorphic) that may arise in this situation.

The work of Mitschi and Singer in [17] makes use of the logarithmic derivative and an inductive technique developed by Kovacic [11], [12], to lift a solution to the inverse problem from G/R_u, where R_u is the unipotent radical of G, to the full group G. Using this machinery Kovacic proved that it is enough to find a solution to the inverse problem for reductive groups (observe that G/R_u is reductive). In [19], van der Put explains and partly proves the results in [17].

In the introduction of [17] the authors briefly review previous work on the inverse problem such as results of Bialynicki-Birula in [3], Kovacic [11], [12], Ramis [20], [21], Singer [24], Tretkoff and Tretkoff [26], Beukers and Heckman [2], Katz [10], Duval and Mitschi [7], Mitschi [15], [16], Duval and Loday-Richaud [6], Ulmer and Weil [27] and Singer and Ulmer [25]. A more extensive survey on the inverse problem can be found in M. Singer's [23].

The constructive algebraic solutions to the inverse differential Galois problem for connected algebraic groups that are currently available are based on Kolchin's Main Structure Theorem for Picard-Vessiot extensions (see Theorem 1.1.1 below). In particular, a corollary to this theorem (see Theorem 1.1.2) establishes that if $E \supset F$ is Picard-Vessiot and G is, for example, unipotent or solvable or $G = \mathrm{GL_n}$ or $G = \mathrm{SL_n}$, then E is isomorphic as an F-module and as a G-module to the function field of the group G_F obtained from G by extension of scalars from C to F. Therefore, to get a Picard-Vessiot extension $E \supset F$ with group G (if it exists) one can begin by taking E to be the function field of G_F and then the problem reduces to extending the derivation from F to E in such a way that $E \supset F$ is Picard-Vessiot for that derivation. In this paper we use this approach for our construction.

The idea of tackling the inverse problem by constructing generic extensions is inpired by the works of E. Noether [18] for the Galois theory of algebraic equations. Following her approach, L. Goldman in [8] introduced the notion of a *generic differential equation with group G*. Goldman explicitly constructed a generic equation with group G for some groups. However, after specializing Goldman's equation the group of the new equation obtained is a subgroup of the original group. In order to solve the inverse problem by this means, we need to keep the original group as the group of the equation after specialization. Goldman's generic equation for $\mathrm{GL_n}$ is equivalent to Magid's *general equation of order n* (Example 5.26 in [13]).

More work in the spirit of Goldman's generic equation came some years later in J. Miller's dissertation [14]. He defined the notion of hilbertian differential field and gave a sufficient condition for the generic equation with group G to specialize to an equation over such a field with group G as well. However, as pointed out by Mitschi and Singer in [17], his condition was stronger than the analogous one for algebraic equations and this made the theory especially difficult to apply for those groups that were not already known to be Galois groups.

This paper contains the results of the author's Ph.D. dissertation [9]. I wish to thank my Ph.D. advisor Andy Magid for the many valuable research meetings that we had. I am also grateful to Michael Singer for many enlightening conversations on the inverse problem.

1. Preliminaries

1.1. **Notation and some basic results from Differential Galois Theory.** We fix a differential field F with algebraically closed field of constants C. If $E \supseteq F$ is a differential field extension then the group of differential automorphisms of E over F is denoted by $G(E/F)$.

If G is a linear algebraic group over C and K is an overfield of C we denote by G_K the group obtained from G by extending scalars from C to K.

We will show that the differential field $F\langle Y_{ij}\rangle(X_{ij})$ to be defined in 1.2 is a Picard-Vessiot extension of $F\langle Y_{ij}\rangle$ with differential Galois group $\mathrm{GL_n}(C)$. Note that $F\langle Y_{ij}\rangle(X_{ij})$ is the function field of G_K with $G = \mathrm{GL_n}(C)$ and $K = F\langle Y_{ij}\rangle$. The following two results provide the rationale for choosing such function field for the generic Picard-Vessiot extension. The proofs can be found in [13] (Theorem 5.12 and Corollary 5.29 respectively).

Theorem 1.1.1 (Kolchin Structure Theorem). *Let $E \supseteq F$ be a Picard-Vessiot extension, let $G \leq G(E/F)$ be a Zariski closed subgroup and let T be the set of all f in E that satisfy a linear homogeneous differential equation over $K = E^G$. Then T is a finitely generated G-stable differential K-algebra with function field E, and if $\overline{K}$ denotes the algebraic closure of K, then there is a G-algebra isomorphism*

$$\overline{K} \otimes_K T \to \overline{K} \otimes_C C[G].$$

Note that $C[G]$ denotes the affine coordinate ring of G and that the target of the above isomorphism is the affine coordinate ring of the group $G_{\overline{K}}$ obtained from G by extension of scalars from C to $\overline{K}$.

Theorem 1.1.2. *Let $E \supseteq F$ be a Picard-Vessiot extension, let $G \leq G(E/F)$ be a Zariski closed subgroup with $E^G = F$. Let $\overline{F}$ be an algebraic closure of F, and suppose the Galois cohomology $H^1(\overline{F}/F, G(\overline{F}))$ is a singleton. Let $T(E/F)$ be the set of all f in E that satisfy a linear homogeneous differential equation over F. Then there are F- and G-isomorphisms $T(E/F) \to F[G_F]$ and $E \to F(G_F)$. In particular, this holds if G is unipotent or solvable, or if $G = \mathrm{GL_n}(C)$ or if $G = \mathrm{SL_n}$.*

We will use the following characterization of Picard-Vessiot extension given in [13] (Proposition 3.9):

Theorem 1.1.3. *Let $E \supseteq F$ be a differential field extension. Then E is a Picard-Vessiot extension if and only if:*

1. *$E = F\langle V\rangle$, where $V \subset E$ is a finite-dimensional vector space over C;*
2. *There is a group G of differential automorphisms of E with $G(V) \supseteq V$ and $E^G = F$;*
3. *$E \supset F$ has no new constants.*

In particular, if the above conditions hold and if $\{y_1, \ldots, y_n\}$ is a C-basis of V, then E is a Picard-Vessiot extension of F for the linear homogeneous differential operator

$$L(Y) = \frac{w(Y, y_1, \ldots, y_n)}{w(y_1, \ldots, y_n)}$$

where $w(-)$ denotes the wronskian determinant and $L^{-1}(0) = V$.

In our case the base field is $F\langle Y_{ij}\rangle$ and $G = \mathrm{GL_n}(C)$. We want to show that $F\langle Y_{ij}\rangle(X_{ij}) \supset F\langle Y_{ij}\rangle$ is Picard-Vessiot. We first show (Corollary 1.3.6) that the field of constants of $F\langle Y_{ij}\rangle$ is C. Then, conditions 1. and 2. in Theorem 1.1.3 are easily verified with V the C-span of the X_{ij} and $G = \mathrm{GL_n}(C)$. Therefore, our main goal henceforth will be to show that $F\langle Y_{ij}\rangle(X_{ij}) \supset F\langle Y_{ij}\rangle$ is a no-new-constant extesion.

1.2. **The differential fields $\mathbf{F\langle Y_{ij}\rangle(X_{ij})}$ and $\mathbf{F(X_{ij})}$.** Let Y_{ij}, $1 \le i, j \le n$, be differential indeterminates over F. For convenience, denote the k-th derivative $D^{(k)}(Y_{ij})$ by $Y_{ij,k}$, for $k \ge 0$, so that $D(Y_{ij,k}) = Y_{ij,k+1}$, $k \ge 0$. As usual, $D^{(0)}(Y_{ij}) = Y_{ij,0}$ represents the original element Y_{ij}. In this situation we will omit the k-subindex and write Y_{ij} instead of $Y_{ij,0}$.

Let $F\{Y_{ij}\}$ be the ring of differential polynomials in the Y_{ij} and $F\langle Y_{ij}\rangle$ its differential quotient field. By that we mean the usual quotient field endowed with the natural derivation:

$$\mathcal{D}\Big(\frac{p}{q}\Big) = \frac{D(p)q - pD(q)}{q^2}.$$

for $p, q \in F\{Y_{ij}\}$, where D is the derivation on $F\{Y_{ij}\}$.

Next let X_{ij}, $1 \le i, j \le n$, be algebraically independent over $F\langle Y_{ij}\rangle$. We consider the differential ring $R = F\{Y_{ij}\}[X_{ij}]$ with derivation extending the derivation on $F\{Y_{ij}\}$ by a formula

$$D(X_{ij}) = \sum_{\ell=1}^{n} Y_{i\ell} X_{\ell j}.$$

As above, this derivation extends to the quotient field

$$Q = F\langle Y_{ij}\rangle(X_{ij})$$

in a natural way.

Henceforth we fix the differential field Q as defined above.

Likewise we will regard the polynomial ring $F[X_{ij}]$ as a differential ring with derivation extending the derivation on F by a formula

$$D(X_{ij}) = \sum_{\ell=1}^{n} f_{i\ell} X_{\ell j}$$

with $f_{ij} \in F$. Then, as before, we extend this derivation to the quotient field $F(X_{ij})$ in a natural way.

The multinomial notation $a_\alpha \mathbf{X}^\alpha$ will be used to denote a term of the form $a_{\alpha_{11}\cdots\alpha_{k\ell}} X_{11}^{\alpha_{11}} \cdots X_{k\ell}^{\alpha_{k\ell}}$.

The ring $F[X_{ij}]$ is assumed to be ordered with the degree reverse lexicographical order (*degrevlex*). That is, the set

$$\mathbb{T}^{n^2} = \{\mathbf{X}^\beta \mid \mathbf{X} = (X_{ij}),\ \beta = (\beta_{ij}) \in \mathbb{N}^{n^2}\}$$

of the power products in the X_{ij} is ordered by $X_{11} > \cdots > X_{1n} > \cdots > X_{n1} > \cdots > X_{nn}$, and

$$\mathbf{X}^\alpha < \mathbf{X}^\beta \iff \begin{cases} \sum_{j=1}^n \sum_{i=1}^n \alpha_{ij} < \sum_{j=1}^n \sum_{i=1}^n \beta_{ij} \\ \textit{or} \\ \sum_{j=1}^n \sum_{i=1}^n \alpha_{ij} = \sum_{j=1}^n \sum_{i=1}^n \beta_{ij}, \textit{ and the first coordinates} \\ \alpha_{ij},\ \beta_{ij} \textit{ from the right which are different satisfy } \alpha_{ij} > \beta_{ij}. \end{cases}$$

Henceforth the leading power product of a polynomial in $F[X_{ij}]$ is assumed to be with respect to this order.

The above definitions as well as the *Multivariable Division Algorithm* that will be used in Remark 1.4.14, can be found in [1].

1.3. Darboux polynomials and the constants of $\mathbf{F}\langle\mathbf{Y_{ij}}\rangle(\mathbf{X_{ij}})$.

We need to show that the field of constants $\mathcal{C}$ of $Q = F\langle Y_{ij}\rangle(X_{ij})$ coincides with the field of constants C of F. In this section we will show (Corollary 1.3.4) that this can be reduced to proving that the only *Darboux polynomials* in R are, up to a scalar multiple in F, powers of $\det[X_{ij}]$.

Definition 1.3.1. *Let D be a derivation on the polynomial ring $A = k[Y_1, \ldots, Y_s]$. A polyomial $p \in A$ is called a Darboux polynomial if there is a polynomial $q \in A$ such that $D(p) = qp$. That is, p divides $D(p)$.*

Darboux polynomials correspond to generators of principal differential ideals in A. Chapter I of J.A. Weil's Thesis [28] is devoted to constants and Darboux polynomials in Differential Algebra.

The following basic proposition (proven in [28] for A as in Definition 1.3.1) characterizes new constants for the extension $Q \supset F$ in terms of Darboux polynomials:

Proposition 1.3.2. *Let $p_1, p_2 \in R = F\{Y_{ij}\}[X_{ij}]$, $p_1, p_2 \neq 0$, be relatively prime. Then $D(\frac{p_1}{p_2}) = 0$, if and only if p_1 and p_2 are Darboux polynomials. Moreover, if $q_1, q_2 \in R$ are such that $D(p_1) = q_1 p_1$ and $D(p_2) = q_2 p_2$, then $q_1 = q_2$.*

Proof. For the necessity of the condition we have

$$D\left(\frac{p_1}{p_2}\right) = \frac{D(p_1)p_2 - p_1 D(p_2)}{p_2^2} = 0,$$

thus $D(p_1)p_2 - p_1 D(p_2) = 0$, that is

$$D(p_1)p_2 = p_1 D(p_2). \tag{1}$$

Since p_1 and p_2 are relatively prime, the last equation implies that p_1 divides $D(p_1)$ and p_2 divides $D(p_2)$.

Now, let $q_1, q_2 \in R$ be such that $D(p_1) = q_1 p_1$ and $D(p_2) = q_2 p_2$, respectively. Then it follows from (1) that

$$q_1 p_1 p_2 = q_2 p_1 p_2.$$

Hence, $q_1 = q_2$.

The proof of the converse is obvious. □

Proposition 1.3.3. *Let $p = \ell \det[X_{ij}]^a$, with $\ell \in F$. Then p is a Darboux polynomial in R with $q = \frac{\ell'}{\ell} + a \sum_{i=1}^n Y_{ii}$.*

Proof. We have

$$D(p) = \ell' \det[X_{ij}]^a + \ell(a \det[X_{ij}]^{a-1} D(\det[X_{ij}])).$$

And,

$$D(\det[X_{ij}]) = \Big(\sum_{i=1}^n Y_{ii}\Big) \det[X_{ij}].$$

Thus,

$$\begin{aligned} D(p) &= \ell' \det[X_{ij}]^a + \ell(a \det[X_{ij}]^{a-1} \Big(\sum_{i=1}^n Y_{ii}\Big) \det[X_{ij}] \\ &= \Big(\frac{\ell'}{\ell} + a \sum_{i=1}^n Y_{ii}\Big)\, \ell \det[X_{ij}]^a. \end{aligned}$$

That is,

$$p = \ell \det[X_{ij}]^a$$

is a Darboux polynomial in R with

$$q = \frac{\ell'}{\ell} + a \sum_{i=1}^n Y_{ii}.$$

□

Corollary 1.3.4. *Suppose that if $p \in R$ is a Darboux polynomial then $p = \ell \det[X_{ij}]^a$, with $\ell \in F$, $a \in \mathbb{N}$ and $q = \frac{\ell'}{\ell} + a \sum_{i=1}^n Y_{ii}$. Then $F\langle Y_{ij}\rangle(X_{ij}) \supset F$ is a no-new-constant extension.*

Proof. $F\langle Y_{ij}\rangle(X_{ij})$ is the fraction field of $F\{Y_{ij}\}[X_{ij}]$. Thus, if an element $f \in F\langle Y_{ij}\rangle(X_{ij})$ satisfies $D(f) = 0$ and $f \notin F$ then, by proposition 1.3.2, $f = p_1/p_2$ with $p_1, p_2 \in R$ relatively prime Darboux polynomials. If the hypothesis is true, this contradicts the fact that p_1 and p_2 are relatively prime. □

Next we show that the constants of $F\langle Y_{ij}\rangle$ coincide with the constants of F. For simplicity, if $h(Y) \in F\{Y_{ij}\}$, we will use the notation $h'(Y)$ for $D(h(Y))$. Notice that this is not the usual meaning $h'(Y) = \sum h'_\alpha \mathbf{Y}^\alpha$.

Proposition 1.3.5. *If $h(Y) \in F\{Y_{ij}\}$ satisfies $h'(Y) = g(Y)h(Y)$ for some $g(Y) \in F\{Y_{ij}\}$ then $h(Y) \in F$. That is, there are no non-trivial Darboux polynomials in $F\{Y_{ij}\}$.*

Proof. Suppose that the hypothesis of the proposition is true. According to our notation $D^{(k)}(Y_{ij}) = Y_{ij,k}$. Consider the set of subindices $\{ij,k\}$, $i,j,k \in \mathbb{N}$, ordered with the lexicographical order. That is, $\{i_1j_1,k_1\} > \{i_2j_2,k_2\}$ if and only if the first coordinates s_1 and s_2 from the left, for $s = i,j,k$ above, which are different satisfy $s_1 > s_2$.

Let $\{mn,t\}$ be the largest subindex such that $Y_{mn,t}$ occurs in $h(Y)$.

Let $h(Y) = \sum_\alpha a_\alpha Y_{11}^{\alpha_{11}} \cdots Y_{mn,t}^{\alpha_{mn,t}}$. Then

$$\begin{aligned} h'(Y) &= \sum_\alpha a'_\alpha Y_{11}^{\alpha_{11}} \cdots Y_{mn,t}^{\alpha_{mn,t}} + \sum_\alpha a_\alpha \alpha_{11} Y_{11}^{\alpha_{11}-1} Y_{11,1}^{\alpha_{11,1}+1} \cdots Y_{mn,t}^{\alpha_{mn,t}} \\ &\qquad + \cdots + \sum_\alpha a_\alpha \alpha_{mn,t} Y_{11}^{\alpha_{11}} \cdots Y_{mn,t}^{\alpha_{mn,t}-1} Y_{mn,t+1} \\ &= h_1(Y_{11}, \cdots, Y_{mn,t}) + \Big(\sum_\alpha a_\alpha \alpha_{mn,t} Y_{11}^{\alpha_{11}} \cdots Y_{mn,t}^{\alpha_{mn,t}-1} \Big) Y_{mn,t+1} \\ &= g(Y)h(Y). \end{aligned}$$

For $Y_{mn,t+1} = D(Y_{mn,t})$ we have $\{mn,t+1\} > \{mn,t\}$, thus it does not occur in $h(Y)$ by the choice of $\{mn,t\}$. Also, it does not occur in $h_1(Y_{11}, \cdots, Y_{mn,t})$. Thus the above equation implies that $Y_{mn,t+1}$ must occur in $g(Y)$. Let $g_{t+1}(Y)$ be its coefficient in $g(Y)$ and let

$$h_2(Y) = \sum_\alpha a_\alpha \alpha_{mn,t} Y_{11}^{\alpha_{11}} \cdots Y_{mn,t}^{\alpha_{mn,t}-1}.$$

Then we have

$$h(Y) g_{t+1}(Y) Y_{mn,t+1} = h_2(Y) Y_{mn,t+1}$$

or

$$h(Y) g_{t+1}(Y) = h_2(Y).$$

But the total degree of $h_2(Y)$ is strictly less than the total degree of $h(Y)$. This forces $h(Y) \in F$. □

Corollary 1.3.6. *The field of constants of $F\langle Y_{ij}\rangle$ coincides with C, the field of constants of F.*

Proof. This is a consequence of Propositions 1.3.2 and 1.3.5. □

1.4. **Darboux polynomials in $\mathbf{R} = \mathbf{F}\{\mathbf{Y_{ij}}\}[\mathbf{X_{ij}}]$.** Proposition 1.3.3 shows that scalar multiples of $\det[X_{ij}]$ and its powers are Darboux polynomials in R. Corollary 1.3.4 implies that if these are the only Darboux polynomials in R then we are done since consequently there will be no new constants in Q. In this section we will show that that is the case, namely, the only Darboux polynomials in R are those of the form $p = \ell \det[X_{ij}]^a$, with $\ell \in F$, $a \in \mathbb{N}$ and $q = \frac{\ell'}{\ell} + a \sum_{i=1}^n Y_{ii}$.

Remarks. 1.4.1 (Derivative of a power product in the X_{ij}). *Let*

$$\mathbf{X}^\alpha = X_{11}^{\alpha_{11}} \cdots X_{1n}^{\alpha_{1n}} \cdots X_{n1}^{\alpha_{n1}} \cdots X_{nn}^{\alpha_{nn}},$$

then

$$\begin{aligned} D(\mathbf{X}^\alpha) = &\Big(\sum_{i=1}^n\sum_{j=1}^n \alpha_{ij}Y_{ii}\Big)\mathbf{X}^\alpha \\ &+\sum_{i=1}^n\sum_{j=1}^n\Big(\sum_{\ell>i}\alpha_{ij}Y_{i\ell}X_{11}^{a_{11}}\cdots X_{ij}^{\alpha_{ij}-1}\cdots X_{\ell j}^{\alpha_{\ell j}+1}\cdots X_{nn}^{\alpha_{nn}} \\ &\qquad+\sum_{\ell<i}\alpha_{ij}Y_{i\ell}X_{11}^{a_{11}}\cdots X_{\ell j}^{\alpha_{\ell j}+1}\cdots X_{ij}^{\alpha_{ij}-1}\cdots X_{nn}^{\alpha_{nn}}\Big). \end{aligned}$$

1.4.2. *Given $\mathbf{X}^\alpha$ as in 1.4.1, we need to know which power products in the X_{ij} contain in their derivatives a Y-multiple of $\mathbf{X}^\alpha$. That is, we need to find the power products $\mathbf{X}^\beta$ such that $D(\mathbf{X}^\beta)$ contains an expression of the form $Y_{rt}\mathbf{X}^\alpha$. By Remark 1.4.1 such power products are*

$$\mathbf{X}^{\alpha_{rs,t}} = \begin{cases} X_{11}^{\alpha_{11}}\cdots X_{rs}^{\alpha_{rs}+1}\cdots X_{ts}^{\alpha_{ts}-1}\cdots X_{nn}^{\alpha_{nn}} & \text{if } r<t \\ X_{11}^{\alpha_{11}}\cdots X_{ts}^{\alpha_{ts}-1}\cdots X_{rs}^{\alpha_{rs}+1}\cdots X_{nn}^{\alpha_{nn}} & \text{if } r>t \end{cases}$$

for $1\le r,s\le n$, $t\ne r$, and $\mathbf{X}^\alpha$ itself.

1.4.3. *Let $p\in R$. Since $D(X_{ij})=\sum_{\ell=1}^n Y_{i\ell}X_{\ell j}$, then the total degree of p with respect to the X_{ij} does not change after differentiation. Therefore, if $D(p)=qp$ then $q\in F\{Y_{ij}\}$.*

Proposition 1.4.4. *Let $p\in R$. Write it as $p=\sum_\alpha p_\alpha(Y)\mathbf{X}^\alpha$, with $p_\alpha(Y)\in F\{Y_{ij}\}$. Then for any α with $p_\alpha(Y)\ne 0$, the coefficient of $\mathbf{X}^\alpha$ in $D(p)$ is*

$$p'_\alpha(Y)+p_\alpha(Y)\sum_{i=1}^n\sum_{j=1}^n\alpha_{ij}Y_{ii}+\sum_{i=1}^n\sum_{j=1}^n(\alpha_{ij}+1)\sum_{\ell\ne i}p_{\alpha_{ij,\ell}}(Y)Y_{i\ell},$$

where $\alpha_{ij,\ell}$ is the exponent vector of the power product

$$\mathbf{X}^{\alpha_{ij,\ell}} = \begin{cases} X_{11}^{\alpha_{11}}\cdots X_{ij}^{\alpha_{ij}+1}\cdots X_{\ell j}^{\alpha_{\ell j}-1}\cdots X_{nn}^{\alpha_{nn}} & \text{if } i<\ell \\ X_{11}^{\alpha_{11}}\cdots X_{\ell j}^{\alpha_{\ell j}-1}\cdots X_{ij}^{\alpha_{ij}+1}\cdots X_{nn}^{\alpha_{nn}} & \text{if } \ell>i \end{cases}$$

as in Remark 1.4.2.

Proof. This is a direct consequence of Remarks 1.4.1 and 1.4.2. □

Proposition 1.4.5. *Let $p\in R$ and suppose that $D(p)=qp$, for some $q\in F\{Y_{ij}\}$. Then $p\in F[X_{ij}]$.*

Proof. Let $p=\sum_\alpha p_\alpha(Y)\mathbf{X}^\alpha$. Then

$$\begin{aligned} D(p) &= \sum_\alpha p'_\alpha(Y)\mathbf{X}^\alpha+p_\alpha(Y)D(\mathbf{X}^\alpha) \\ &= qp \\ &= \sum_\alpha q(Y)p_\alpha(Y)\mathbf{X}^\alpha. \end{aligned}$$

By Proposition 1.4.4, for each α with $p_\alpha(Y) \neq 0$ the corresponding coefficient of $\mathbf{X}^\alpha$ in $D(p)$ is

$$\begin{aligned} D(p)_\alpha = p'_\alpha(Y) \quad &+ \quad p_\alpha(Y) \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij} Y_{ii} \\ &+ \quad \sum_{i=1}^{n}\sum_{j=1}^{n} (\alpha_{ij}+1) \sum_{\ell \neq i} p_{\alpha_{ij,\ell}}(Y) Y_{i\ell}. \end{aligned}$$

Since $D(p) = qp$, it must be $D(p)_\alpha = q(Y)p_\alpha(Y)$ or, equivalently,

$$\begin{aligned} q(Y)p_\alpha(Y) = p'_\alpha(Y) \quad &+ \quad p_\alpha(Y) \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij} Y_{ii} \\ &+ \quad \sum_{i=1}^{n}\sum_{j=1}^{n} (\alpha_{ij}+1) \sum_{\ell \neq i} p_{\alpha_{ij,\ell}}(Y) Y_{i\ell}. \end{aligned}$$

This means that for each α, the coefficient $p_\alpha(Y)$ of $\mathbf{X}^\alpha$ in p divides the expression

$$p'_\alpha(Y) + \sum_{i=1}^{n}\sum_{j=1}^{n} (\alpha_{ij}+1) \sum_{\ell \neq i} p_{\alpha_{ij,\ell}}(Y) Y_{i\ell}.$$

Thus, for each α, there is $u_\alpha(Y)$ such that

$$p_\alpha(Y)u_\alpha(Y) = p'_\alpha(Y) + \sum_{i=1}^{n}\sum_{j=1}^{n} (\alpha_{ij}+1) \sum_{\ell \neq i} p_{\alpha_{ij,\ell}}(Y) Y_{i\ell}.$$

As in the proof of Proposition 1.3.5, order the triples $\{ij, k\}$, $i, j, k \in \mathbb{N}$, with the lexicographical order. Let $\{mn, t\}$ be the largest subindex such that $Y_{mn,t}$ occurs in p. We have $D(Y_{mn,t}) = Y_{mn,t+1}$ and $\{mn, t+1\} > \{mn, t\}$.

Now, for each α such that $Y_{mn,t}$ occurs in $p_\alpha(Y)$ we have that $Y_{mn,t+1}$ will occur in $p'_\alpha(Y)$ but not in $p_\alpha(Y)$ or in

$$\sum_{i=1}^{n}\sum_{j=1}^{n} (\alpha_{ij}+1) \sum_{\ell \neq i} p_{\alpha_{ij,\ell}}(Y) Y_{i\ell}$$

by the choice of $\{mn, t\}$. Therefore, it must occur in $p_\alpha(Y)u_\alpha(Y)$. Let

$$p_\alpha(Y) = \sum a_\beta Y_{11}^{\beta_{11}} Y_{12}^{\beta_{12}} \cdots Y_{mn,t}^{\beta_{mn,t}}$$

then

$$\begin{aligned} p'_\alpha(Y) = \sum \; & a'_\beta Y_{11}^{\beta_{11}} \cdots Y_{mn,t}^{\beta_{mn,t}} \\ & + \sum a_\beta \; \beta_{11} \; Y_{11}^{\beta_{11}-1} Y_{11,1}^{\beta_{11,1}+1} \cdots Y_{mn,t}^{\beta_{mn,t}} + \ldots \\ & \quad + \sum a_\beta \; \beta_{mn,t} \; Y_{11}^{\beta_{11}} \cdots Y_{mn,t}^{\beta_{mn,t}-1} Y_{mn,t+1}. \end{aligned}$$

So $Y_{mn,t+1}$ occurs in $p'_\alpha(Y)$ only in

$$\begin{aligned}\sum a_\beta\ \beta_{mn,t}\ Y_{11}^{\beta_{11}}\cdots Y_{mn,t}^{\beta_{mn,t}-1}Y_{mn,t+1} \\ = \Big(\sum a_\beta\ \beta_{mn,t}\ Y_{11}^{\beta_{11}}\cdots Y_{mn,t}^{\beta_{mn,t}-1}\Big)Y_{mn,t+1} \\ = v(Y)Y_{mn,t+1}.\end{aligned}$$

Since $Y_{mn,t+1}$ occurs in $p_\alpha(Y)u_\alpha(Y)$ and not in $p_\alpha(Y)$ it must occur in $u_\alpha(Y)$. Let $u_{\alpha,t+1}(Y)$ be the coefficient of $Y_{mn,t+1}$ in $u_\alpha(Y)$. Then it has to be

$$p_\alpha(Y)u_{\alpha,t+1}(Y)Y_{mn,t+1} = v(Y)Y_{mn,t+1}.$$

The above equation implies that $p_\alpha(Y)$ divides $v(Y)$. But this is impossible since the total degree of $v(Y)$ is strictly less than the total degree of $p_\alpha(Y)$. This contradiction yields the result. □

Lemma 1.4.6. *Let $p \in F[X_{ij}]$ and suppose that there is $q \in F\{Y_{ij}\}$ such that $D(p) = qp$. Then q is a linear polynomial in the Y_{ij}. If $\beta = (\beta_{ij})$ is sucht that $\mathbf{X}^\beta$ occurs in p, then for $1 \le i \le n$ the coefficient of Y_{ii} in q is $\sum_{j=1}^n \beta_{ij}$. In particular, the sums $\sum_{j=1}^n \beta_{ij}$, for $1 \le i \le n$, are independent of the choice of $\mathbf{X}^\beta$.*

Proof. We have $p = \sum a_\beta \mathbf{X}^\beta$, with $a_\beta \in F$.

Thus,

$$\begin{aligned} D(p) &= \sum a'_\beta \mathbf{X}^\beta + a_\beta D(\mathbf{X}^\beta) \\ &= qp \\ &= \sum q(Y) a_\beta \mathbf{X}^\beta. \end{aligned}$$

By Proposition 1.4.4, the coefficient of $\mathbf{X}^\beta$ in $D(p)$ is

$$a'_\beta + a_\beta \sum_{i=1}^n\sum_{j=1}^n \beta_{ij}Y_{ii} + \sum_{i=1}^n\sum_{j=1}^n(\beta_{ij}+1)\sum_{\ell\neq i} a_{\beta_{ij,\ell}}Y_{i\ell}.$$

Hence, it must be

$$q(Y)a_\beta = a'_\beta + a_\beta\Big(\sum_{i=1}^n\sum_{j=1}^n \beta_{ij}Y_{ii} + \sum_{i=1}^n\sum_{j=1}^n(\beta_{ij}+1)\sum_{\ell\neq i} a_{\beta_{ij,\ell}}Y_{i\ell}\Big).$$

From this,

$$q(Y) = \frac{a'_\beta}{a_\beta} + \sum_{i=1}^n\sum_{j=1}^n \beta_{ij}Y_{ii} + \sum_{i=1}^n\sum_{j=1}^n(\beta_{ij}+1)\sum_{\ell\neq i} \frac{a_{\beta_{ij,\ell}}}{a_\beta}Y_{i\ell}.$$

The coefficient of Y_{ii} in the above expression is $\sum_{j=1}^n \beta_{ij}$, for $1 \le i \le n$. Since this expression for q is valid for any index β, the "in particular" part follows immediately. □

Corollary 1.4.7. *Let p be as in Lemma 1.4.6. Let $\mathbf{X}^\alpha$ be the leading power product of p. Let $\mathbf{X}^\beta$ be any power product with non-zero coefficient in p. Then $\sum_{j=1}^n \beta_{ij} = \sum_{j=1}^n \alpha_{ij}$, for $1 \le i \le n$. Thus p is homogeneous of degree $\sum_{j=1}^n\sum_{i=1}^n \alpha_{ij}$.*

Proof. This is an immediate consequence of the "in particular" part in Lemma 1.4.6. $\square$

Corollary 1.4.8. *Let $p \in F[X_{ij}]$ and suppose that $D(p) = qp$, for some $q \in F\{Y_{ij}\}$. Let $\mathbf{X}^\alpha$ be the leading power product of p, and let $\ell \in F$ be its coefficient. Then*

$$q = \frac{\ell'}{\ell} + \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} Y_{ii}.$$

Proof. By Proposition 1.4.4 and since $D(p) = qp$, the coefficient of $\mathbf{X}^\alpha$ in $D(p)$ is

$$\ell q = \ell' + \ell \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} Y_{ii} + \sum_{i=1}^{n} \sum_{j=1}^{n} (\alpha_{ij} + 1) \sum_{\ell \neq i} p_{\alpha_{ij,\ell}} Y_{i\ell} \Big). \tag{1}$$

The $p_{\alpha_{ij,k}}$ are the coefficients of the power products $\mathbf{X}^{\alpha_{ij,k}}$ in$p$, with $\alpha_{ij,k} \neq \alpha$, such that $D(\mathbf{X}^{\alpha_{ij,k}})$ contains an expression of the form $Y_{st}\mathbf{X}^\alpha$. By Remark 1.4.2, these power products are

$$\mathbf{X}^{\alpha_{rs,t}} = \begin{cases} X_{11}^{\alpha_{11}} \cdots X_{rs}^{\alpha_{rs}+1} \cdots X_{ts}^{\alpha_{ts}-1} \cdots X_{nn}^{\alpha_{nn}} & \text{if} \quad r < t \\ X_{11}^{\alpha_{11}} \cdots X_{ts}^{\alpha_{ts}-1} \cdots X_{rs}^{\alpha_{rs}+1} \cdots X_{nn}^{\alpha_{nn}} & \text{if} \quad r > t \end{cases},$$

all of which violate Corollary 1.4.7 for $i = r$ and $i = t$. Therefore it must be $p_{\alpha_{ij,k}} = 0$, for all $1 \le i, j \le n$; $k \neq i$. But now, substituting back in (1), we see that

$$\ell q = \ell' + \ell \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} Y_{ii}.$$

Hence,

$$q = \frac{\ell'}{\ell} + \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} Y_{ii}.$$

$\square$

Our next step in order to show that the Darboux polynomials $p \in R$ have the desired form will be to show that such a p *is not reduced with respect to* $\det[X_{ij}]$. For that we will show that the leading power product of p is a power of the leading power product of $\det[X_{ij}]$. First, we have

Lemma 1.4.9. *Let $p \in F[X_{ij}]$ be such that $D(p) = qp, q \in F\{Y_{ij}\}$. Let $\mathbf{X}^\alpha$ be its leading power product. Then $\alpha_{ij} = 0$ for $j \neq n - i + 1$ and $\alpha_{i,n-i+1} > 0$, $1 \le i \le n$. That is, $\mathbf{X}^\alpha = X_{1n}^{\alpha_{1n}} X_{2,n-1}^{\alpha_{2,n-1}} \cdots X_{n1}^{\alpha_{n1}}$.*

Proof. To prove that $\alpha_{ij} = 0$ for $j \neq n - i + 1$ we first show that $\alpha_{ij} = 0$ for $j > n - k + 1$, $i \ge k$, $2 \le k \le n$. Indeed, for $k = 2$ we have $j > n - 1$, so $j = n$ and

$$D(\mathbf{X}^\alpha) = \alpha_{nn} \sum_{k=1}^{n-1} Y_{nk} X_{11}^{\alpha_{11}} \cdots X_{kn}^{\alpha_{kn}+1} \cdots X_{nn}^{\alpha_{nn}-1} + \dots$$

Since q has no Y_{ij} with $i \neq j$, each term in $D(\mathbf{X}^\alpha)$ containing such a Y_{ij} must be cancelled. In particular we need to cancel the terms containing

$$Y_{nj}X_{11}^{\alpha_{11}}\cdots X_{jn}^{\alpha_{jn}+1}\cdots X_{nn}^{\alpha_{nn}-1}$$

for $1 \leq j \leq n-1$ above. For that we can only use the derivatives of power products of the form

$$\mathbf{X}^{\alpha_{nl,j}} = X_{11}^{\alpha_{11}}\cdots X_{j\ell}^{\alpha_{j\ell}-1}\cdots X_{jn}^{\alpha_{jn}+1}\cdots X_{n1}^{\alpha_{n1}}\cdots X_{n\ell}^{\alpha_{n\ell}+1}\cdots X_{nn}^{\alpha_{nn}-1}, \qquad \ell < n.$$

But these are all strictly greater than $\mathbf{X}^\alpha$ (the leading power product of p), and they may not occur in p. As a consequence, it has to be $\alpha_{nn} = 0$. Now let $k > 2$ be such that $\alpha_{in} = 0$ for $i \geq k$. Then

$$\mathbf{X}^\alpha = X_{11}^{\alpha_{11}}\cdots X_{k-1,n}^{\alpha_{k-1,n}}\cdots X_{k,n-1}^{\alpha_{k,n-1}}X_{k+1,1}^{\alpha_{k+1,1}}\cdots X_{k+1,n-1}^{\alpha_{k+1,n-1}}\cdots X_{n,n-1}^{\alpha_{n,n-1}}$$

and

$$\begin{aligned} D(\mathbf{X}^\alpha) = \alpha_{k-1,n}\Big(&\sum_{i<k-1} Y_{k-1,i}X_{11}^{\alpha_{11}}\cdots X_{in}^{\alpha_{in}+1}\cdots X_{k-1,n}^{\alpha_{k-1,n}-1}\cdots X_{n,n-1}^{\alpha_{n,n-1}} \\ + &\sum_{i>k-1} Y_{k-1,i}X_{11}^{\alpha_{11}}\cdots X_{k-1,n}^{\alpha_{k-1,n}-1}\cdots X_{in}^{\alpha_{in}+1}\cdots X_{n,n-1}^{\alpha_{n,n-1}}\Big) + \dots \end{aligned}$$

Likewise, we need to cancel all the terms in $D(\mathbf{X}^\alpha)$ that contain $Y_{k-1,i}$, for $i \neq k-1$. In particular, we need to cancel

$$Y_{k-1,i}X_{11}^{\alpha_{11}}\cdots X_{in}^{\alpha_{in}+1}\cdots X_{k-1,n}^{\alpha_{k-1,n}-1}\cdots X_{n,n-1}^{\alpha_{n,n-1}},$$

for $i < k-1$. For that we can only use the power products of the form

$$\mathbf{X}^{\alpha_{k-1,\ell,i}} = X_{11}^{\alpha_{11}}\cdots X_{i\ell}^{\alpha_{i\ell}-1}\cdots X_{in}^{\alpha_{in}+1}\cdots X_{k-1,\ell}^{\alpha_{k-1,\ell}+1}\cdots X_{k-1,n}^{\alpha_{k-1,n}-1}\cdots X_{n,n-1}^{\alpha_{n,n-1}},$$

for $i < k-1$.

But all of them are strictly greater than $\mathbf{X}^\alpha$ and cannot occur in p. Thus, it has to be $\alpha_{k-1,n} = 0$. Since this argument is valid for any $k > 2$, it follows that $\alpha_{kn} = 0$, for $2 \leq k \leq n$. This makes the statement that $\alpha_{ij} = 0$ for $j > n-k+1$, $i \geq k$, true for $k = 2$.

Now assume that k is such that $\alpha_{ij} = 0$ for $j > n-k+1, i \geq k$. So

$$\mathbf{X}^\alpha = X_{11}^{\alpha_{11}}\cdots X_{1n}^{\alpha_{1n}}\cdots X_{k,n-k+1}^{\alpha_{k,n-k+1}}X_{k+1,1}^{\alpha_{k+1,1}}\cdots X_{k+1,n-k+1}^{\alpha_{k+1,n-k+1}}\cdots X_{n,n-k+1}^{\alpha_{n,n-k+1}}$$

and for $i > k$

$$\alpha_{i,n-k+1}Y_{ij}X_{11}^{\alpha_{11}}\cdots X_{1n}^{\alpha_{1n}}\cdots X_{k,n-k+1}^{\alpha_{k,n-k+1}+1}\cdots X_{i,n-k+1}^{\alpha_{i,n-k+1}-1}\cdots X_{n,n-k+1}^{\alpha_{n,n-k+1}}$$

occurs in $D(\mathbf{X}^\alpha)$. Thus we need to cancel it. For that we can only use the derivatives of power products of the form

$$\mathbf{X}^{\alpha_{ij,k}} = X_{11}^{\alpha_{11}} \cdots X_{kj}^{\alpha_{kj}-1} \cdots X_{k,n-k+1}^{\alpha_{k,n-k+1}+1} \cdots X_{ij}{}^{\alpha_{ij}+1} \cdots X_{i,n-k+1}^{\alpha_{i,n-k+1}-1} \cdots X_{n,n-k+1}^{\alpha_{n,n-k+1}}$$

with $j < n-k+1$ since $\alpha_{kj} = 0$ for all $j > n-k+1$ by hypothesis. But all such power products are strictly greater than $\mathbf{X}^\alpha$ and therefore they cannot occur in p. This forces $\alpha_{i,n-k+1} = 0$ for $i > k$. We can repeat this process until $k = n$ and get $\alpha_{ij} = 0$ for all $j > n-k+1$, $i \geq k$, $2 \leq k \leq n$, that is,

$$\mathbf{X}^\alpha = X_{11}^{\alpha_{11}} \cdots X_{1n}^{\alpha_{1n}} X_{21}^{\alpha_{21}} \cdots X_{2,n-1}^{\alpha_{2,n-1}} X_{31}^{\alpha_{31}} \cdots X_{n-1,2}^{\alpha_{n-1,2}} X_{n1}^{\alpha_{n1}}.$$

Now we show that $\alpha_{ij} = 0$ for $j < n-k+1$, $1 \leq k \leq n-1$, $i \leq k$. The process is analogous to what we just did. First we show that $\alpha_{i1} = 0$ for $i < n$. Indeed, for each i we have for $\ell > i$ that

$$\alpha_{i1} Y_{i\ell} X_{11}^{\alpha_{11}} \cdots X_{i1}^{\alpha_{i1}-1} \cdots X_{\ell 1}^{\alpha_{\ell 1}+1} \cdots X_{n1}^{\alpha_{n1}}$$

occurs in $D(\mathbf{X}^\alpha)$. So, in order to cancel it, we need to use the derivatives of power products of the form

$$\mathbf{X}^{\alpha_{ij,\ell}} = X_{11}^{\alpha_{11}} \cdots X_{i1}^{\alpha_{i1}-1} \cdots X_{ij}{}^{\alpha_{ij}+1} \cdots X_{\ell 1}^{\alpha_{\ell 1}+1} \cdots X_{\ell j}^{\alpha_{\ell j}-1} \cdots X_{n1}^{\alpha_{n1}}$$

with $j > 1$, all of which are strictly greater than $\mathbf{X}^\alpha$ if $\ell < n$, and for $\ell = n$ we cannot simply have one of those since $\alpha_{nj} = 0$ for $j \neq 1$. Thus such power products cannot occur in p and it has to be $\alpha_{i1} = 0$ for $i < n$.

Let $k \leq n-1$ be such that $\alpha_{ij} = 0$ for $j < n-k+1$, $i \leq k$. We have

$$\mathbf{X}^\alpha = X_{1,n-k+1}^{\alpha_{1,n-k+1}} \cdots X_{1n}^{\alpha_{1n}} \cdots X_{k,n-k+1}^{\alpha_{k,n-k+1}} \cdots X_{n1}^{\alpha_{n1}}$$

and for all $i < k$, $\ell > i$, we have that

$$\alpha_{i,n-k+1} Y_{i\ell} X_{1,n-k+1}^{\alpha_{1,n-k+1}} \cdots X_{i,n-k+1}^{\alpha_{i,n-k+1}-1} \cdots X_{\ell,n-k+1}^{\alpha_{\ell,n-k+1}+1} \cdots X_{n1}^{\alpha_{n1}}$$

occurs in $D(\mathbf{X}^\alpha)$ and in order to cancel it we only have the derivatives of power products of the form

$$\mathbf{X}^{\alpha_{ij,\ell}} = X_{1,n-k+1}^{\alpha_{1,n-k+1}} \cdots X_{i,n-k+1}^{\alpha_{i,n-k+1}-1} \cdots X_{ij}{}^{\alpha_{ij}+1} \cdots X_{\ell,n-k+1}^{\alpha_{\ell,n-k+1}+1} \cdots X_{\ell j}^{\alpha_{\ell j}-1} \cdots X_{n1}^{\alpha_{n1}}$$

with $j > n-k+1$ since $\alpha_{ij} = 0$ for $i \leq k$, $j < n-k+1$.

For $\ell < k$, all these power products are strictly greater than $\mathbf{X}^\alpha$ and therefore they cannot occur in p. For $\ell \geq k$ we cannot simply have such power products since for $\ell \geq k$, $\alpha_{\ell j} = 0$ if $j > n-k+1$. Thus it has to be $\alpha_{i,n-k+1} = 0$ for $i \leq k-1$.

We can repeat this process until $k = n-1$ and get $\alpha_{ij} = 0$, $j < n-k+1$, $i \leq k$, $1 \leq k \leq n-1$. This completes the proof of the first part of the lemma.

To prove that $\alpha_{i,n-i+1} \neq 0$, for all $1 \leq i \leq n$, suppose that there is i such that $\alpha_{i,n-i+1} = 0$ and let $j \neq i$ be such that $\alpha_{j,n-j+1} \neq 0$. Then $D(\mathbf{X}^\alpha)$ will contain

$$\alpha_{j,n-j+1} Y_{ji} X_{1n}^{\alpha_{1n}} \cdots X_{j,n-j+1}^{\alpha_{j,n-j+1}-1} \cdots X_{i,n-j+1} \cdots X_{n1}^{\alpha_{n1}} + \dots \qquad \text{if } i > j$$

or

$$\alpha_{j,n-j+1} Y_{ji} X_{1n}^{\alpha_{1n}} \cdots X_{i,n-j+1} \cdots X_{j,n-j+1}^{\alpha_{j,n-j+1}-1} \cdots X_{n1}^{\alpha_{n1}} + \dots \qquad \text{if } i < j.$$

As noted above, since q does not contain any Y_{ij} with $i \neq j$, we need to cancel the terms in $D(p)$ involving either of the above. But that is impossible since $\alpha_{ij} = 0$ for all j and by Corollary 1.4.7 all the power products

$$X_{11}^{\beta_{11}} \cdots X_{ij}{}^{\beta_{ij}} \cdots X_{nn}^{\beta_{nn}}$$

in p must have $\beta_{ij} = 0$ for $j = 1, \dots, n$. In particular, we cannot have in p power products of the form $\mathbf{X}^{\alpha_{j,n-j+1,i}}$ as in Remark 1.4.2 ☐

Next we show that the exponents α_{st} of the X_{st} in $\mathbf{X}^\alpha$, the leading power product of p, are all equal:

Lemma 1.4.10. *Let $p \in F[X_{ij}]$ be such that $D(p) = qp$, $q \in F\{Y_{ij}\}$. Let*

$$\mathbf{X}^\alpha = X_{1n}^{\alpha_{1n}} X_{2,n-1}^{\alpha_{2,n-1}} \cdots X_{n1}^{\alpha_{n1}}$$

be its leading power product. Then $\alpha_{i,n-i+1} = \alpha_{1n}$, for $i > 1$, that is, if $a = \alpha_{1n}$, then

$$\mathbf{X}^\alpha = (X_{1n} X_{2,n-1} \cdots X_{n1})^a.$$

Proof. Let ℓ be the coefficient of $\mathbf{X}^\alpha$ in p. We have

$$\begin{aligned}
&D(\ell X_{1n}^{\alpha_{1n}} X_{2,n-1}^{\alpha_{2,n-1}} \cdots X_{n1}^{\alpha_{n1}}) = \\
&\quad\Big(\sum_{i=1}^{n} \alpha_{i,n-i+1} \ell Y_{ii}\Big) X_{1n}^{\alpha_{1n}} X_{2,n-1}^{\alpha_{2,n-1}} \cdots X_{n1}^{\alpha_{n1}} \\
&\qquad + \alpha_{1n} \ell \sum_{k \neq 1} Y_{1k} X_{1n}^{\alpha_{1n}-1} \cdots X_{k,n-k+1}^{\alpha_{k,n-k+1}} \cdots X_{kn} \cdots X_{n1}^{\alpha_{n1}} \\
&+ \ell \sum_{1<i} \alpha_{i,n-i+1} \sum_{k>i} Y_{ij} X_{1n}^{\alpha_{1n}} \cdots X_{i,n-i+1}^{\alpha_{i,n-i+1}-1} \cdots X_{k,n-k+1}^{\alpha_{k,n-k+1}} \cdots X_{k,n-i+1} \cdots X_{n1}^{\alpha_{n1}} \\
&+ \ell \sum_{1<i} \alpha_{i,n-i+1} \sum_{k>i} Y_{ij} X_{1n}^{\alpha_{1n}} \cdots X_{k,n-i+1} \cdots X_{k,n-k+1}^{\alpha_{k,n-k+1}} \cdots X_{i,n-i+1}^{\alpha_{i,n-i+1}-1} \cdots X_{n1}^{\alpha_{n1}} \\
&+ \ell' X_{1n}^{\alpha_{1n}} X_{2,n-1}^{\alpha_{2,n-1}} \cdots X_{n1}^{\alpha_{n1}}.
\end{aligned}$$

In order to cancel

$$\alpha_{1n} \ell Y_{1k} X_{1n}^{\alpha_{1n}-1} \cdots X_{k,n-k+1}^{\alpha_{k,n-k+1}} \cdots X_{kn} \cdots X_{n1}^{\alpha_{n1}}, \qquad k \neq 1,$$

above, we can only use the derivatives of the power product

$$\begin{aligned}
&\mathbf{X}^{\alpha_{1,n-k+1,k}} = \\
&\qquad X_{1,n-k+1} \cdots X_{1n}^{\alpha_{1n}-1} \cdots X_{k,n-k+1}^{\alpha_{k,n-k+1}-1} \cdots X_{kn} \cdots X_{n1}^{\alpha_{n1}},
\end{aligned}$$

since for $j \neq n-k+1$ we have $\alpha_{kj} = 0$.

Let $a_{\alpha_{1,n-k+1,k}}$ be the coefficient of $X^{\alpha_{1,n-k+1,k}}$ in p. Then

$$a_{\alpha_{1,n-k+1,k}} = -\ell\alpha_{1n} \tag{2}$$

On the other hand, in order to cancel

$$\alpha_{k,n-k+1}\ell Y_{k1}X_{1,n-k+1}\cdots X_{1n}^{\alpha_{1n}}\cdots X_{k,n-k+1}^{\alpha_{k,n-k+1}-1}\cdots X_{n1}^{\alpha_{n1}}, \qquad k \neq 1$$

above, the only power product that we can use is, again,

$$\begin{aligned}\mathbf{X}^{\alpha_{kn,1}} &= X_{1,n-k+1}\cdots X_{1n}^{\alpha_{1n}-1}\cdots X_{k,n-k+1}^{\alpha_{k,n-k+1}-1}\cdots X_{kn}\cdots X_{n1}^{\alpha_{n1}}\\ &= X^{\alpha_{1,n-k+1,k}},\end{aligned}$$

since $\alpha_{1j} = 0$ for $j \neq n$. Thus it must be

$$a_{\alpha_{1,n-k+1,k}} = -\ell\alpha_{k,n-k+1} \tag{3}$$

as well.

From (2) and (3) it follows that, for $k \neq 1$, $\alpha_{1n} = \alpha_{k,n-k+1}$. □

As a consequence of the above results we obtain the following expression for q:

Corollary 1.4.11. *Let $p \in F[X_{ij}]$ and suppose that $D(p) = qp$, $q \in F\{Y_{ij}\}$. Let $\mathbf{X}^\alpha$ be the leading power product of p. Let $a \in \mathbb{N}$ be such that*

$$\mathbf{X}^\alpha = (X_{1n}X_{2,n-1}\cdots X_{n1})^a$$

and let $\ell \in F$ be the coefficient of $\mathbf{X}^\alpha$ in p. Then

$$q = \frac{\ell'}{\ell} + a\sum_{i=1}^{n} Y_{ii}.$$

Proof. This is a consequence of Corollary 1.4.8 and Lemma 1.4.10. □

Corollary 1.4.12. *Let p be as in Corollary 1.4.11. Then p is homogeneous of degree na.*

Proof. This is a consequence of Corollary 1.4.6 and Lemma 1.4.10 □

Lemma 1.4.10 implies that p is not reduced with respect to $\det[X_{ij}]$. Since this is a key point in the proof of our main result we restate it as the following

Theorem 1.4.13. *Let $p \in F[X_{ij}]$ be such that $D(p) = qp$, $q \in F\{Y_{ij}\}$. Let $\mathbf{X}^\alpha$ be its leading power product. Then*

$$\mathbf{X}^\alpha = (X_{1n}X_{2,n-1}\cdots X_{n1})^a = \mathrm{lp}(\det[X_{ij}])^a.$$

Thus p is not reduced with respect to $\det[X_{ij}]$.

Note. If f is a polynomial, $\mathrm{lp}(f)$ denotes its leading power product with respect to a given order.

Proof. This is just a restatement of Lemma 1.4.10. □

Remark 1.4.14. *Let $p_1, p_2 \in F[X_{ij}]$ be two polynomials such that $lp(p_1) = X^\alpha = lp(p_2)$. Then we can write $p_1 = f\,p_2 + r$ where $f \in F$ and r is reduced with respect to p_2. Indeed, since $lp(p_1) = lp(p_2)$, we have that $lp(p_2)$ divides $lp(p_1)$. So p_1 is not reduced with respect to p_2 . We may apply the Multivariable Division Algorithm to p_1 and p_2, to get $f, r \in F[X_{ij}]$, such that $p_1 = f\,p_2 + r$, with r reduced with respect to p_2 and $lp(p_1) = lp(f)lp(p_2)$. The last equation implies that $lp(f) = 1$. Hence, $f \in F$.*

We are now ready to prove our main result on the form of the Darboux polynomials in R:

Theorem 1.4.15. *Let $p \in F[X_{ij}]$ and $q \in F\{Y_{ij}\}$ be polynomials in R that satisfy the Darboux condition $D(p) = qp$. Then there is $a \in \mathbb{N}$ and $\ell \in F$ such that*

$$p = \ell\ \det[X_{ij}]^a$$

and

$$q = \frac{\ell'}{\ell} + a\sum_{i=1}^{n} Y_{ii}.$$

Proof. Let $q_1 = \sum_{i=1}^n a\,Y_{ii}$, so that,

$$D(\det[X_{ij}]^a) = q_1 \det[X_{ij}]^a = (q - \frac{\ell'}{\ell}) \det[X_{ij}]^a.$$

By Remark 1.4.14 we can write $p = \ell \det[X_{ij}]^a + r$, with r reduced with respect to $\det[X_{ij}]^a$. Now,

$$\begin{aligned} D(p) &= D(\ell \det[X_{ij}]^a) + D(r) \\ &= \ell' \det[X_{ij}]^a + \ell(q - \frac{\ell'}{\ell}) \det[X_{ij}]^a + D(r) \\ &= \ell' \det[X_{ij}]^a + q\ell \det[X_{ij}]^a - \ell' \det[X_{ij}]^a + D(r) \\ &= q\ell \det[X_{ij}]^a + D(r). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} D(p) &= qp \\ &= q\ell \det[X_{ij}]^a + qr. \end{aligned}$$

Therefore, it has to be $D(r) = qr$. But r is reduced with respect to $\det[X_{ij}]^a$. It follows, by Theorem 1.4.13, that $r = 0$. The statement about the form of q is just the content of Corollary 1.4.11. ☐

Now, for $1 \le i, j \le n$, let $D_{E(ij)} \in \mathrm{Lie}(\mathrm{GL_n}(C))$ be the derivation given by multiplication by the matrix $E(ij)$, with 1 in position (i,j) and zero elsewhere. The set $\{D_{E(ij)} | 1 \le i, j \le n\}$ is a basis for $\mathrm{Lie}(\mathrm{GL_n}(C))$ with respect to which the derivation in Theorem 1.4.15 is expressed. We show that the result does not depend on the basis chosen on $\mathrm{Lie}(\mathrm{GL_n}(C))$:

Theorem 1.4.16. *Let $\mathcal{D}_{st}$, $1 \le s, t \le n$, be any basis of* $\mathrm{Lie}(\mathrm{GL_n}(C))$. *Define a derivation in the ring $R = F\{Y_{ij}\}[X_{ij}]$ by $\mathcal{D} = \sum Y_{st}\mathcal{D}_{st}$. Let p and q be polynomials in R that satisfy the Darboux condition $\mathcal{D}(p) = qp$. Then there is $a \in \mathbb{N}$ and $\ell \in F$ such that $p = \ell \det[X_{ij}]^a$ and $q = \frac{\ell'}{\ell} + a\sum_{i=1}^n Y_{ii}$.*

Proof. Since $\{D_{E(ij)} | 1 \leq i, j \leq n\}$ is a basis of $\mathrm{Lie}(\mathrm{GL_n}(C))$ we have

$$\mathcal{D}_{st} = \sum c_{st,ij} D_{E(ij)},$$

with $c_{st,ij} \in C$. Thus,

$$\begin{aligned}
\mathcal{D} &= \sum_{s,t} Y_{st} \mathcal{D}_{st} \\
&= \sum_{s,t} Y_{st} \sum_{i,j} c_{st,ij} D_{E(ij)} \\
&= \sum_{i,j} \sum_{s,t} c_{st,ij} Y_{st} D_{E(ij)} \\
&= \sum_{i,j} Z_{ij} D_{E(ij)},
\end{aligned}$$

where $Z_{ij} = \sum_{s,t} c_{st,ij} Y_{st}$. Now, $[c_{st,ij}]$ is a matrix of change of basis so it is invertible. Also the $c_{st,ij}$ are contants for D, thus the map $Z_{ij,k} \to Y_{ij,k}$ is a differential bijection. In other words, the differential rings

$$R = F\{Y_{ij}\}[X_{ij}], D$$

and

$$R' = F\{Z_{ij}\}[X_{ij}], \mathcal{D}$$

are isomorphic and therefore we can apply Theorem 1.4.15 to R'. □

2. A Generic Picard-Vessiot Extension for $\mathrm{GL_n}(C)$

In this section we prove the statement about the generic Picard-Vessiot extension for F with differential Galois group $\mathrm{GL_n}(C)$. We provide some specialization properties in the case when F has finite transcendence degree over C.

2.1. The generic extension.

Theorem 2.1.1. *$F\langle Y_{ij}\rangle(X_{ij}) \supset F\langle Y_{ij}\rangle$ is a generic Picard-Vessiot extension with differential Galois group* $\mathrm{GL_n}(C)$.

Proof. First we need to show that $F\langle Y_{ij}\rangle(X_{ij}) \supset F\langle Y_{ij}\rangle$ is a Picard-Vessiot extension with differential Galois group $\mathrm{GL_n}(C)$. We will use the characterization of Theorem 1.1.3. We have

1. $F\langle Y_{ij}\rangle(X_{ij}) = F\langle Y_{ij}\rangle\langle V\rangle$, where $V \subset F\langle Y_{ij}\rangle(X_{ij})$ is the finite dimensional vector space over C spanned by the X_{ij}.
2. The group $G = \mathrm{GL_n}(C)$ acts as a group of differential automorphisms of $F\langle Y_{ij}\rangle(X_{ij})$ with $G(V) \subseteq V$ and $F\langle Y_{ij}\rangle(X_{ij})^G = F\langle Y_{ij}\rangle$. This follows from the fact that $F\langle Y_{ij}\rangle(X_{ij})$ is the function field of $\mathrm{GL_n}(C)_{F\langle Y_{ij}\rangle}$.
3. $F\langle Y_{ij}\rangle(X_{ij}) \supseteq F\langle Y_{ij}\rangle$ has no new constants. This is a consequence of Proposition 1.3.2, Corollary 1.3.4 and Theorem 1.4.15.

Now, suppose that $E \supseteq F$ is a Picard-Vessiot extension of F with differential Galois group $\mathrm{GL_n}(C)$. By Theorems 1.1.1 and 1.1.2, we have that in this situation E is isomorphic to $F(X_{ij})$ (the function field of $\mathrm{GL_n}(C)_F$) as a $\mathrm{GL_n}(C)$-module and as an F-module. Any $\mathrm{GL_n}(C)$ equivariant derivation D_E on $F(X_{ij})$ extends the derivation on F in such a way that

$$D_E(X_{ij}) = \sum_{\ell=1}^{n} f_{i\ell} X_{\ell j}$$

with $f_{ij} \in F$. Since $E \supset F$ is a Picard-Vessiot extension for $\mathrm{GL_n}(C)$, then so is $C\langle f_{ij}\rangle(X_{ij}) \supset C\langle f_{ij}\rangle$, the derivation on $C\langle f_{ij}\rangle(X_{ij})$ being the corresponding restriction of D_E. From this Picard-Vessiot extension one can retrieve $F(X_{ij}) \supset F$ by extension of scalars from C to F. In this way, any Picard-Vessiot extension $E \supset F$ with differential Galois group $\mathrm{GL_n}(C)$ can be obtained from $F\langle Y_{ij}\rangle(X_{ij}) \supset F\langle Y_{ij}\rangle$ via the specialization $Y_{ij} \mapsto f_{ij}$. This means that $F\langle Y_{ij}\rangle(X_{ij}) \supset F\langle Y_{ij}\rangle$ is a generic Picard-Vessiot extension of F for $\mathrm{GL_n}(C)$. □

2.2. **Specializing to a Picard-Vessiot extension of F.** In this section we give necessary and sufficient conditions for a specialization $Y_{ij} \mapsto f_{ij}$, $f_{ij} \in F$, with $C\langle f_{ij}\rangle(X_{ij}) \supset C\langle f_{ij}\rangle$ a Picard-Vessiot extension, to exist. We restrict ourselves to the case when F has finite transcendence degree over C.

Our goal is to find $f_{ij} \in F$ such that the specialization (homomorphism) from $C\{Y_{ij}\}$ to F given by $Y_{ij} \mapsto f_{ij}$ is such that $C\langle f_{ij}\rangle(X_{ij}) \supset C\langle f_{ij}\rangle$, with derivation given by $D(X_{ij}) = \sum_{\ell=1}^{n} f_{i\ell} X_{\ell j}$, has no new constants. We have:

Theorem 2.2.1. *Let $F = C(t_1, \dots, t_m)[z_1, \dots, z_k]$ where the t_i are algebraically independent over C and the z_i are algebraic over $C(t_1, \dots, t_m)$. Assume that the derivation on F has field of constants C and that it extends to $F(X_{ij})$ so that $D(f \otimes X_{ij}) = D(f) \otimes X_{ij} + f \otimes \sum_{\ell=1}^{n} f_{i\ell} X_{\ell j}$ on $F \otimes C[X_{ij}]$. Let $\mathcal{C}$ be the field of constants of $F(X_{ij})$. Then $\mathcal{C} = C$ if and only if the set of all the t_i and all the X_{ij} are algebraically independent over $\mathcal{C}$.*

Proof. (Sufficiency) Suppose that $\mathcal{C}$ properly contains C. Let r be the transcendence degree of $\mathcal{C}$ over C. Since C is algebraically closed, r has to be at least one.

We have the tower of fields

$$C \subset \mathcal{C} \subset \mathcal{C}(X_{ij}) \subset F(X_{ij})$$

where the transcendence degree of $\mathcal{C} \subset \mathcal{C}(X_{ij})$ is n^2 and the transcendence degree of $C \subset F(X_{ij})$ is $n^2 + m$. Since $r \geq 1$ the transcendence degree ℓ of $\mathcal{C}(X_{ij}) \subset F(X_{ij})$ has to be $\ell < m$ and therefore there is an algebraic relation among the t_i over $\mathcal{C}(X_{ij})$. Let $g(X_{ij}), f_i(X_{ij}) \in \mathcal{C}[X_{ij}]$, $g(X_{ij}) \not\equiv 0$, be such that

$$t^{\delta_s} + \frac{f_{s-1}(X_{ij})}{g(X_{ij})} t^{\delta_{s-1}} + \cdots + \frac{f_0(X_{ij})}{g(X_{ij})} = 0.$$

Then

$$g(X_{ij}) t^{\delta_s} + f_{s-1}(X_{ij}) t^{\delta_{s-1}} + \cdots + f_0(X_{ij}) = 0.$$

Since the $f_i(X_{ij})$ and $g(X_{ij})$ are polynomials in the X_{ij} with coefficients in $\mathcal{C}$, the last equation gives an algebraic relation among the t_i and the X_{ij} over $\mathcal{C}$.

For the necessity we only need to point out that by construction the set of all the t_i and all the X_{ij} are algebraically independent over C. $\square$

Now to check whether the set of all the t_i and all the X_{ij} are algebraically independent over $\mathcal{C}$, we let $\mathbb{T}_k$, $k \geq 1$, denote the set of monomials in both the t_i and the X_{ij} of total degree less than or equal to k. Then the set of all the t_i and all the X_{ij} are algebraically independent over $\mathcal{C}$ if and only if, for each k, the set $\mathbb{T}_k$ is linearly independent over $\mathcal{C}$.

Fix a term order on the set $\mathbb{T}$ of all monomials in both the t_i and the X_{ij} and let W_k denote the wronskian of the set $\mathbb{T}_k$ relative to that order. Then the above condition is equivalent to the fact that $W_k \neq 0$ for $k \geq 1$. Now go back to $C\{Y_{ij}\}[X_{ij}]$ and extend scalars from C to F. Let $W_k(Y_{ij})$ be the Wronskian of $\mathbb{T}_k$ in $F \otimes C\{Y_{ij}\}[X_{ij}]$.

Then, the condition of Theorem 2.2.1 for finding a specialization $Y_{ij} \mapsto f_{ij}$ so that $C\langle f_{ij}\rangle(X_{ij}) \supset C\langle f_{ij}\rangle$ has no new constants can be expressed as follows:

Theorem 2.2.2. *There is a specialization of the Y_{ij} with no new constants if and only if there are $f_{ij} \in F$ such that all the wronskians $W_k(Y_{ij})$, $k \geq 1$, map to non-zero elements under $Y_{ij} \mapsto f_{ij}$.*

3. Results for connected linear algebraic groups

We do not know at present whether generic Picard-Vessiot extensions exist for arbitrary connected linear algebraic groups. However, the proofs of the specialization theorems in 2.2 can be easily generalized for such groups.

3.1. **Specialization results.** We point out that the proofs of Theorems 2.2.1 and 2.2.2 do not make any special use of the fact that $G = \mathrm{GL_n}(C)$.

Let $F = C(t_1, \ldots, t_m)[z_1, \ldots, z_k]$ where the t_i are algebraically independent over C and the z_i are algebraic over $C(t_1, \ldots, t_m)$.

Let $Y_1, \ldots, Y_n$ be differential indeterminates over F and $X_1, \ldots, X_n$ algebraically independent over $F\langle Y_i\rangle$.

We consider the group G to be a connected linear algebraic group over C with function field $C(G) = C(X_i)$.

If $\{D_1, \ldots, D_n\}$ is a basis for Lie(G), let $D_Y = \sum_{i=1}^n Y_i D_i$ be a G-equivariant derivation on $F\langle Y_i\rangle(X_i)$ and $D = \sum_{i=1}^n f_i D_i$, $f_i \in F$, a specialization of D_Y to a G-equivariant derivation on $F(X_i)$. Let $\mathcal{C}$ be the field of constants of $F(X_i)$ for this derivation. We have,

Theorem 3.1.1. *Let F, C and $\mathcal{C}$ be as above. Then $\mathcal{C} = C$ if and only if the set of all the t_i and the X_i are algebraically independent over $\mathcal{C}$.*

Now, fix an order in the set $\mathbb{T}$ of monomials in both the t_i and the X_i and let $W_k(Y_i)$ be the wronskian (with respect to this order) of the monomials in both the t_i and the X_i of degree less than or equal to k computed in $F \otimes C\{Y_i\}[X_i]$. Then,

Theorem 3.1.2. *There is a specialization of the Y_i with no new constants if and only if there are $f_i \in F$ such that all the wronskians $W_k(Y_i)$, $k \geq 1$, map to non-zero elements under $Y_i \mapsto f_i$.*

For the proofs of Theorems 3.1.1 and 3.1.2 we only need to replace the X_{ij} with X_i , the Y_{ij} with Y_i and n^2 with n in the proofs of Theorems 2.2.1 and 2.2.2. □

Observe that the proofs of Theorems 3.1.1 and 3.1.2 do not use the fact that $C(X_i)$ is the function field of G. However, this hypothesis is used in the following theorem to show that $F(X_i) \supset F$ is a Picard-Vessiot extension with group G.

Under the hypothesis and notation of Theorems 3.1.1 and 3.1.2 we have:

Theorem 3.1.3. *$F(X_i) \supset F$ is a Picard-Vessiot extension with Galois group G if and only if the set of all the t_i and all the X_i are algebraically independent over the field of constants $\mathcal{C}$ of $F(X_i)$.*

Proof. First assume that $F(X_i) \supset F$ is a Picard-Vessiot extension. Then the field of constants $\mathcal{C}$ of $F(X_i)$ coincides with C. So we can apply Theorem 3.1.1 and get the result.

Conversely, if the set of all the t_i and all the X_i are algebraically independent over $\mathcal{C}$, by Theorem 3.1.1, $F(X_i) \supset F$ is a no new constant extension. On the other hand, $F(X_i)$ is obtained from $C(X_i)$ by the extension of scalars:

$$\begin{aligned} F(X_i) &= q.f.(F \otimes_C C(X_i)) \\ &= q.f.(F \otimes_C C[G]) \end{aligned}$$

where $C[G]$ is the coordinate ring of G and G acts on $F \otimes_C C[G]$ fixing F. So, $G \subseteq G(F(X_i)/F)$. Counting dimensions we get that $G = G(F(X_i)/F)$ since $C(X_i) = C(G)$, the function field fo G. Finally, $F(X_i) = F\langle V\rangle$, where V is the finite-dimensional vector space over C spanned by the X_i. By Theorem 1.1.3, $F(X_i) \supset F$ is a Picard-Vessiot extension. □

Applying Theorems 3.1.2 and 3.1.3 we also obtain:

Theorem 3.1.4. *There is a specialization of the Y_i such that $F(X_i) \supset F$ is a Picard-Vessiot extension if and only if there are $f_i \in F$ such that all the $W_k(Y_i)$, $k \geq 1$, map to non-zero elements via $Y_i \mapsto f_i$.*

4. Computing new constants

Let C be an algebraically closed field with trivial derivation. Let $F = C(t_1, \ldots, t_m)[z_1, \ldots, z_k]$ where the t_i are algebraically independent over C and the z_i are algebraic over $C(t_1, \ldots, t_m)$. Assume that the derivation on F has field of constants C and that it extends to $F(X_{ij})$ so that

$$D(f \otimes X_{ij}) = D(f) \otimes X_{ij} + f \otimes \sum_{\ell=1}^{n} f_{i\ell} X_{\ell j}$$

on $F \otimes C[X_{ij}]$, for certain $f_{ij} \in F$. By Theorem 2.2.1, if there is an algebraic relation among the set of all the t_i and all the X_{ij} over the field of constants $\mathcal{C}$ of $F(X_{ij})$ then $\mathcal{C}$ properly contains C.

In this section we will produce a new constant from such an algebraic relation. In order to simplify the computations we will assume $F = C$. So, in particular, the coefficients f_{ij} in the derivation above are constant. In this situation, since the transcendence degree of F over C is zero, if $\mathcal{C} \supsetneqq C$, the condition of Theorem 2.2.1 means that the X_{ij} are algebraically dependent over $\mathcal{C}$.

We will restrict ourselves to the case $n = 2$ and use a particular linear dependence relation.

Extend the derivation on F to $F(X_{11}, X_{12}, X_{21}, X_{22})$ by letting

$$D(X_{ij}) = \sum_{\ell=1}^{2} f_{i\ell} X_{\ell j},$$

where the f_{ij} are such that the wronskian $W_1 = w(X_{11}, X_{12}, X_{21}, X_{22}) = 0$. That is, the X_{ij} are linearly dependent over $\mathcal{C}$. Furthermore, assume that the linear relation among the X_{ij} is such that there are $\beta_{12}, \beta_{21}, \beta_{22} \in \mathcal{C}$ with

$$X_{11} = \beta_{12} X_{12} + \beta_{21} X_{21} + \beta_{22} X_{22} \tag{1}$$

and that X_{12}, X_{21} and X_{22} are linearly independent. In order to simplify the computations we will also assume that $\det[f_{ij}] = 0$.

We want to find $a, b, c \in F$ such that $p = aX_{12} + bX_{21} + cX_{22}$ is a Darboux polynomial in $F[X_{ij}]$, that is $D(aX_{12} + bX_{21} + cX_{22}) = q(aX_{12} + bX_{21} + cX_{22})$ for certain $q \in F$.

We have,

$$\begin{aligned}
&D(a\,X_{12} + b\,X_{21} + c\,X_{22}) \\
&\quad = \; a(f_{11}X_{12} + f_{12}X_{22}) + b(f_{21}X_{11} + f_{22}X_{21}) + c(f_{21}X_{12} + f_{22}X_{22}) \\
&\quad = \; bf_{21}X_{11} + (af_{11} + cf_{21})X_{12} + bf_{22}X_{21} + (af_{12} + cf_{22})X_{22} \\
&\quad = \; bf_{21}(\beta_{12}X_{12} + \beta_{21}X_{21} + \beta_{22}X_{22}) + (af_{11} + cf_{21})X_{12} + bf_{22}X_{21} \\
&\qquad\qquad + (af_{12} + cf_{22})X_{22} \\
&\quad = \; (af_{11} + bf_{21}\beta_{12} + cf_{21})X_{12} + b(f_{22} + f_{21}\beta_{12})X_{21} \\
&\qquad\qquad + (af_{12} + bf_{21}\beta_{22} + cf_{22})X_{22} \\
&\quad = \; qaX_{12} + qbX_{21} + qcX_{22}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&[a(f_{11} - q) + bf_{21}\beta_{12} + cf_{21}]X_{12} + b(f_{22} + f_{21}\beta_{12} - q)X_{21} \\
&\qquad\qquad + (af_{12} + bf_{21}\beta_{22} + c(f_{22} - q)X_{22} = 0.
\end{aligned} \tag{2}$$

Since we are assuming that X_{12}, X_{21} and X_{22} are linearly independent their coefficients in (2) must be equal to zero. So we have the following homogeneous linear system in a, b, c:

$$\begin{array}{rcrcrcl}
(f_{11} - q)\,a & + & f_{21}\beta_{12}\,b & + & f_{21}\,c & = & 0 \\
 & & (f_{22} + f_{21}\beta_{12} - q)\,b & & & = & 0 \\
f_{12}\,a & + & f_{21}\beta_{22}\,b & + & (f_{22} - q)\,c & = & 0
\end{array}$$

In order for the above system to have non-trivial solutions we need that

$$\det\begin{bmatrix} f_{11}-q & f_{21}\beta_{12} & f_{21} \\ 0 & f_{22}+f_{21}\beta_{12}-q & 0 \\ f_{12} & f_{21}\beta_{22} & f_{22}-q \end{bmatrix} = 0.$$

But,

$$\begin{aligned} &\det\begin{bmatrix} f_{11}-q & f_{21}\beta_{12} & f_{21} \\ 0 & f_{22}+f_{21}\beta_{12}-q & 0 \\ f_{12} & f_{21}\beta_{22} & f_{22}-q \end{bmatrix} \\ &\quad = (f_{22}+f_{21}\beta_{12}-q)\det\begin{bmatrix} f_{11}-q & f_{21} \\ f_{12} & f_{22}-q \end{bmatrix} \\ &= (f_{22}+f_{21}\beta_{12}-q)\Big(\det[f_{ij}] - \Big(\sum_{i=1}^{2} f_{ii}\Big)q + q^2\Big) \\ &= 0. \end{aligned}$$

This gives either

$$f_{22}+f_{21}\beta_{12}-q = 0 \tag{3}$$

or

$$\det[f_{ij}] - \Big(\sum_{i=1}^{2} f_{ii}\Big)q + q^2 = 0. \tag{4}$$

From (3)-(4) we get

$$q = f_{22}+f_{21}\beta_{12} \tag{5}$$

or

$$q = \frac{\sum_{i=1}^{2} f_{ii} \pm \sqrt{\big(\sum_{i=1}^{2} f_{ii}\big)^2 - 4\det[f_{ij}]}}{2} \tag{6}$$

Since we are assuming that $\det[f_{ij}] = 0$, (6) becomes:

$$q = \begin{cases} \sum_{i=1}^{2} f_{ii}, & \text{or} \\ 0 \end{cases} \tag{7}$$

Choose $q = \sum_{i=1}^{2} f_{ii}$ and assume that $q \neq 0$, $q \neq f_{22}+f_{21}\beta_{12}$. Then the second equation in the system implies that $b = 0$ and the system becomes:

$$\begin{aligned} -f_{22}\,a + f_{21}\,c &= 0 \\ f_{12}\,a - f_{11}\,c &= 0 \end{aligned}$$

If $f_{22} \neq 0$ then the above system has the general solution

$$a = \frac{f_{21}}{f_{22}}\,c, \text{ where } c \in \mathcal{C}\,.$$

In particular, if we take $c = 1$ then $p = \frac{f_{21}}{f_{22}} X_{12} + X_{22}$ satisfies

$$D\Big(\frac{f_{21}}{f_{22}} X_{12} + X_{22}\Big) = \Big(\sum_{i=1}^{2} f_{ii}\Big)\Big(\frac{f_{21}}{f_{22}} X_{12} + X_{22}\Big).$$

On the other hand we also have that

$$D(\det[X_{ij}]) = \Big(\sum_{i=1}^{2} f_{ii}\Big) \det[X_{ij}].$$

Let

$$\theta = \frac{\frac{f_{21}}{f_{22}} X_{12} + X_{22}}{\det[X_{ij}]}.$$

We have,

$$\begin{aligned} & D(\theta) \\ = \;& D\Big(\frac{\frac{f_{21}}{f_{22}} X_{12} + X_{22}}{\det[X_{ij}]}\Big) \\ = \;& \frac{D\big(\frac{f_{21}}{f_{22}} X_{12} + X_{22}\big) \det[X_{ij}] - \big(\frac{f_{21}}{f_{22}} X_{12} + X_{22}\big) D(\det[X_{ij}])}{\det[X_{ij}]^2} \\ = \;& \frac{\big(\sum_{i=1}^{2} f_{ii}\big)\big(\frac{f_{21}}{f_{22}} X_{12} + X_{22}\big) \det[X_{ij}] - \big(\frac{f_{21}}{f_{22}} X_{12} + X_{22}\big)\big(\sum_{i=1}^{2} f_{ii}\big) \det[X_{ij}]}{\det[X_{ij}]^2} \\ = \;& 0. \end{aligned}$$

That is, θ *is a new constant in* $F(X_{ij})$.

Now we show that under the restrictions that we imposed on the f_{ij} it is possible to find a non-zero f_{22}.

Since we have a linear dependence relation among the X_{ij}, the wronskian W_1 must be equal to zero. This Wronskian can be expressed, up to a sign, as the following product of determinants:

$$W_1 = \begin{vmatrix} 1 & 0 & 0 & 1 \\ f_{11} & f_{12} & f_{21} & f_{22} \\ A & B & E & F \\ C & D & G & H \end{vmatrix} \begin{vmatrix} X_{11} & X_{12} & 0 & 0 \\ X_{21} & X_{22} & 0 & 0 \\ 0 & 0 & X_{11} & X_{12} \\ 0 & 0 & X_{21} & X_{22} \end{vmatrix} = M(f_{ij}) \det[X_{ij}]^2,$$

where

$$
\begin{aligned}
A &= f_{11}' + f_{11}^2 + f_{12}f_{21}, \\
B &= f_{12}' + f_{11}f_{12} + f_{12}f_{22} \\
C &= f_{11}A + f_{21}B + A' \\
&= 3f_{11}f_{11}' + 2f_{11}f_{12}f_{21} + 2f_{12}'f_{21} + f_{11}'' + f_{12}f_{21}' + f_{11}^3, \\
D &= f_{12}A + f_{22}B + B' \\
&= 2f_{11}'f_{12} + f_{11}^2f_{12} + f_{12}^2f_{21} + f_{21}f_{22}^2 + 2f_{12}'f_{22} + f_{11}f_{12}' \\
&\qquad + f_{12}f_{22}' + f_{12}'' + f_{11}f_{12}f_{22}, \\
E &= f_{21}' + f_{21}f_{11} + f_{22}f_{21}, \\
F &= f_{22}' + f_{12}f_{21} + f_{22}^2, \\
G &= f_{11}E + f_{21}F + E' \\
&= 2f_{21}'f_{11} + f_{21}f_{11}^2 + f_{22}f_{21}f_{11} + 2f_{22}'f_{21} + f_{12}f_{21}^2 \\
&\qquad + f_{22}^2f_{21} + f_{21}'' + f_{21}f_{11}' + f_{22}f_{21}', \\
H &= f_{22}F + f_{12}E + F' \\
&= f_{21}f_{11}f_{12} + 2f_{22}f_{21}f_{12} + +3f_{22}f_{22}' + 2f_{12}f_{21}' + f_{12}'f_{21} \\
&\qquad + f_{22}^3 + f_{22}''.
\end{aligned}
$$

and

$$
M(f_{ij}) = \begin{vmatrix} 1 & 0 & 0 & 1 \\ f_{11} & f_{12} & f_{21} & f_{22} \\ A & B & E & F \\ C & D & G & H \end{vmatrix}.
$$

We have after simplifying using the hypothesis that $\det[f_{ij}] = 0$,

$$
\begin{aligned}
M(f_{ij}) = {} & (f_{22} - f_{11})(f_{12}'f_{21}'' - f_{21}'f_{12}'') + (f_{22}' - f_{11}')(f_{12}''f_{21} - f_{12}f_{21}'') \\
& - f_{12}'f_{21}'(f_{11} - f_{22})^2 - f_{12}f_{21}(f_{11}' - f_{22}') \\
& + f_{12}f_{21}'(f_{11}f_{11}' + f_{22}f_{22}' - f_{11}'f_{22} - f_{11}f_{22}' + f_{22}'' - f_{11}'' + f_{12}f_{21}' - f_{12}'f_{21}) \\
& + f_{12}'f_{21}(f_{11}f_{11}' + f_{22}f_{22}' - f_{11}'f_{22} - f_{11}f_{22}' + f_{11}'' - f_{22}'' + f_{12}'f_{21} - f_{12}f_{21}').
\end{aligned}
$$

Getting the above expression for $M(f_{ij})$ took long and involved computations. We first computed the determinant directly and then we checked the result using Dogson's method [5], [22].

The wronskian $W_1 = 0$ if and only if $M(f_{ij}) = 0$. Now, observe that if $f_{12} = 0$ then $f_{12}' = 0$ which implies that $B = 0$ and $D = 0$ as well. Therefore $M(f_{ij}) = 0$. So, if we let $M(Y_{ij})$ be the differential polynomial in the Y_{ij} whose specialization to the f_{ij} is $M(f_{ij})$ then $M(Y_{ij})$ is in the differential ideal

$$
\begin{aligned}
\mathcal{I} &= \{\det[Y_{ij}], Y_{12}\} \\
&= \{Y_{11}Y_{22} - Y_{12}Y_{21},\ Y_{12}\} \\
&= \{Y_{11}Y_{22},\ Y_{12}\}
\end{aligned}
$$

of $C\{Y_{11}, Y_{12}, Y_{21}, Y_{22}\}$. It is easy to see that Y_{22} is not in $\mathcal{I}$. Indeed, suppose that

$$Y_{22} = p\, Y_{11} Y_{22} + q\, Y_{12} + r, \tag{8}$$

where $p,\, q \in C\{Y_{11}, Y_{12}, Y_{21}, Y_{22}\}$,

$$r = \sum_{i,j} \left[p_i\, (Y_{11} Y_{22})^{(i)} + q_j\, Y_{12}^{(j)} \right]$$

with $p_i, q_j \in C\{Y_{11}, Y_{12}, Y_{21}, Y_{22}\}$.

Now, consider the map

$$\psi : C\{Y_{11}, Y_{21}, Y_{22}\} \longrightarrow C[Y_{11}, Y_{21}, Y_{22}]$$

given by $\psi(Y_{22}) = Y_{22}$ and $\psi(Y_{ij}) = 0$ for $i,\, j \neq 2$. Let $\overline{p} = \psi(p)$, $\overline{q} = \psi(q)$, $\overline{r} = \psi(r)$. We have that $\overline{r} = 0$ and (8) becomes

$$Y_{22} = 0.$$

which is impossible. □

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