Stable C^*-algebras

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Abstract

We give a survey of known results, and add some new results, about stable C^*-algebras. The main themes in the article are to characterize stable C^*-algebras, to discuss permanence properties of the class of stable C^*-algebras, and to investigate the relation between this topic and the structure problem for simple C^*-algebras.

1 Introduction

This article contains some new results as well as a survey of older results, mostly from the articles [12], [16], and [17], on stable C^*-algebras. Recall that a C^*-algebra A is stable if it is isomorphic to $A \otimes \mathcal{K}$, where $\mathcal{K}$ denotes the C^*-algebra of compact operators on a separable Hilbert space. Since $\mathcal{K} \otimes \mathcal{K} \cong \mathcal{K}$ it follows that $A \otimes \mathcal{K}$ is stable for every C^*-algebra A. If B_1 and B_2 are full hereditary sub-C^*-algebras of a C^*-algebra A, then $B_1 \otimes \mathcal{K} \cong B_2 \otimes \mathcal{K}$ by Brown's theorem, [4]; and so any pair of stable, full hereditary sub-C^*-algebras of a C^*-algebra A are isomorphic.

Our goals for studying stable C^*-algebras are to:

- Investigate a relation between stability of C^*-algebras and the problem if every simple C^*-algebra must be either stably finite or purely infinite. Corollaries 4.5 and 4.6 below can be viewed as a mild indication that the answer to this problem could be "no".

- Investigate permanence properties of stable C^*-algebras: Is the property of being stable closed under various natural C^*-operations. Stability is closed under inductive limits and crossed products by discrete groups (Corollary 2.3); stability is not stable in the sense that $M_2(A)$ can be stable without A being stable (Theorem 4.3). It remains an open problem if any extension of two stable C^*-algebras must be stable (see Section 6).

- Decide for which finite C^*-algebras absence of traces is equivalent to stability, and decide for which infinite C^*-algebras absence of unital quotients is equivalent to stability.

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2 Characterizing stable C^*-algebras

We begin this section by stating a result from [12] by Hjelmborg and the author that characterizes stable C^*-algebras. To state the result we need some notation.

In a C^*-algebra A, let $F(A)$ denote the set of positive elements a in A for which there exists e in A such that $ea = ae = a$. (Every element in $F(A)$ belongs to the Pedersen ideal of A; but the Pedersen ideal can in some cases contain positive elements not in $F(A)$ — this is the case whenever A is an algebraically simple, non-unital C^*-algebra.)

A C^*-algebra is said to be σ_{p}-unital if it contains a countable approximate unit consisting of projections. One can show that such an approximate unit always can always be taken to be increasing and dominating any fixed projection in the C^*-algebra.

Remark 2.1 (Equivalence of positive elements) Two positive elements a, b in a C^*-algebra A are said to be equivalent, written $a \sim b$, if there is an element x in A such that $x^*x = a$ and $xx^* = b$. Write $x = u(x^*x)^{1/2}$, where u is a partial isometry in A^{**}. Then uc belongs to A for every c in $\overline{aAa}$, and the map $c \mapsto ucu^*$ defines an isomorphism from $\overline{aAa}$ onto $\overline{bAb}$ which maps a to b.

For every positive element c in $\overline{aAa}$ we have $c \sim ucu^*$. Indeed, $y = uc^{1/2}$ belongs to A, $y^*y = c$, and $yy^* = ucu^*$.

Write $a \precsim b$ if a and b are positive elements in A such that $x_n^*bx_n \to a$ for some sequence $\{x_n\}$ in A. For a in A^+ and $\varepsilon > 0$ let $(a-\varepsilon)_+$ denote the positive part of the self-adjoint element $a - \varepsilon \cdot 1$ in the unitization of $\widetilde{A}$. Then $(a-\varepsilon)_+$ belongs to A, and $a \precsim b$ if and only if $(a-\varepsilon)_+ \sim b_\varepsilon$ for some b_ε in $\overline{bAb}$ for each $\varepsilon > 0$ (cf. [15, Proposition 2.4]).

Theorem 2.2 (Theorems 2.1 and 3.3 of [12]) *The following conditions are equivalent for every σ-unital C^*-algebra A:*

(i) *A is stable,*

(ii) *for every positive element a in A and for every positive $\varepsilon > 0$ there are positive elements b, c in A such that $\|a - b\| \le \varepsilon$, $b \sim c$, and $\|ac\| \le \varepsilon$,*

(iii) *for every a in $F(A)$ there is a positive element b in A such that $a \precsim b$ and $a \perp b$,*

(iv) *for every a in $F(A)$ there is a unitary element u in the unitization of A such that $a \perp uau^*$,*

(v) *there is a sequence $\{E_n\}_{n=1}^{\infty}$ of mutually orthogonal, mutually equivalent projections in the multiplier algebra $\mathcal{M}(A)$ of A such that $\sum_{n=1}^{\infty} E_n = 1$ (the sum converges in the strict topology).*

If A is further assumed to be σ_{p}-unital, then (i) – (v) *above are equivalent to:*

(vi) *for every projection p in A there is a projection q in A such that $p \sim q$ and $p \perp q$.*

Corollary 2.3 (Permanence)

(i) *If A is a σ-unital C^*-algebra and if A is the inductive limit of an inductive system of σ-unital stable C^*-algebras, then A is stable.*

(ii) *If A is stable, then so is every ideal in A and every quotient of A.*

(iii) *If A is a σ-unital, stable C^*-algebra and if a is a positive contraction in A, then $\overline{(1-a)A(1-a)}$ is stable.*

(iv) *If B is a sub-C^*-algebra of a σ-unital, stable C^*-algebra A and if B contains an approximate unit for A, then B is stable.*

(v) *If A is a σ-unital, stable C^*-algebra and if G is a countable discrete group acting on A, then $A \rtimes G$ is stable.*

Parts (i), (iii), (iv), and (v) are proved in [12] (and the proof of (i) and (iii) uses Theorem 2.2). To see that (ii) holds we may assume that $A = A_0 \otimes \mathcal{K}$ for some C^*-algebra A_0. If I is a closed two-sided ideal in $A_0 \otimes \mathcal{K}$, then $I = I_0 \otimes \mathcal{K}$ for some closed two-sided ideal I_0 of A_0, and it follows that I and A/I are stable.

It remains an open problem if any extension of two (σ-unital) stable C^*-algebras must be stable, cf. Section 6. If A is stable, then so is $A \otimes B$ for every C^*-algebra B. In the converse direction one cannot conclude that A is stable knowing that $A \otimes B$ is stable, not even in the case where $B = M_2(\mathbb{C})$, cf. Theorem 4.3.

No stable C^*-algebra can admit a bounded trace nor can it have a unital quotient. The converse does not hold in general (see Corollary 4.4), but it does hold for certain well-behaved C^*-algebras, cf. Proposition 2.7 below. Hjelmborg proved in [11] that Cuntz–Krieger algebras arising from infinite graphs are stable if and only if they admit no bounded trace and have no unital quotient.

For a particularly well-behaved class of finite C^*-algebras, absence of bounded traces is equivalent to stability (see Section 3); and absence of unital quotients is equivalent to stability for purely infinite C^*-algebras in the sense of [14] (see Section 5). A precursor to these results is given in Proposition 2.7 below. Reformulating that result slightly, we introduce the following:

Definition 2.4 A hereditary sub-C^*-algebra B of a C^*-algebra A is said to be *large in* A if for every every positive element a in A and for every $\varepsilon > 0$ there is x in A such that $\|x^*x - a\| \leq \varepsilon$ and xx^* belongs to B.

Any large hereditary sub-C^*-algebra is necessarily full, i.e., not contained in any proper ideal.

Every C^*-algebra A is large in itself.

If B is a large hereditary sub-C^*-algebra of A, then for each a in $F(A)$ there is x in A such that $x^*x = a$ and xx^* belongs to B. Indeed, if e is a positive contraction in A such that $ea = ae = a$ then $z^*(e - 1/2)_+ z = a$ for some z in A (in the notation of Remark 2.1). Find x in A such that $\|x^*x - e\| < 1/2$ and xx^* belongs to B. By [13, Lemma 2.2] there is y in A such that $y^*x^*xy = (e - 1/2)_+$. Put $w = xyz$. Then ww^* belongs to B and $w^*w = a$ as desired.

The argument above also shows that for every projection p in A there is a projection q in B such that $p \sim q$ (whenever B is large in A).

Lemma 2.5 *Every full, hereditary sub-C^*-algebra of a purely infinite C^*-algebra is large.*

Proof: Suppose that B be a full, hereditary sub-C^*-algebra of a purely infinite C^*-algebra A. Let a be a positive element in A and let $\varepsilon > 0$ be given. Then a belongs to the closed two-sided ideal generated by B, hence $(a - \varepsilon/3)_+$ belongs to the algebraic ideal generated by B, and hence $(a - 2\varepsilon/3)_+$ belongs to the algebraic ideal generated by some positive element b in B. Since A is purely infinite, $(a-\varepsilon)_+ = y^*by$ for some y in A. This shows that $(a-\varepsilon)_+ = x^*x$ and $xx^* \in B$ when $x = b^{1/2}y$. □

Lemma 2.6 *Any full, stable, hereditary sub-C^*-algebra of a separable C^*-algebra is large.*

Proof: Let B be a full, stable, hereditary sub-C^*-algebra of a C^*-algebra A, let a be a positive element in A and let $\varepsilon > 0$ be given. Since $F(B)$ is dense in B^+ and since B is full in A, the algebraic ideal in A generated by $F(B)$ is dense in A. It follows that we can find b in $F(B)$ and $x_1, \dots, x_n$ in A such that

$$\|\sum_{j=1}^{n} x_j^* b x_j - a\| \leq \varepsilon.$$

It follows from Theorem 2.2 that there are mutually orthogonal and mutually equivalent positive elements $b_1 = b, b_2, \dots, b_n$ in B. Find $u_1, \dots, u_n$ in B such that $u_j^* u_j = b$ and $u_j u_j^* = b_j$, so that $u_i^* u_j = 0$ when $i \neq j$. Put $x = \sum_{j=1}^{n} u_j x_j$. Then xx^* belongs to B and $x^*x = \sum_{j=1}^{n} x_j^* u_j^* u_j x_j = \sum_{j=1}^{n} x_j^* b x_j$. □

Proposition 2.7 (Proposition 5.1 of [12]) *Let A be a σ-unital C^*-algebra. Suppose that any full, hereditary sub-C^*-algebra B of A is large if it admits no non-zero bounded trace. Then A is stable if and only if there is no non-zero bounded trace on A and A has no non-trivial unital quotient.*

We shall in Section 4 give an example of a non-stable σ_{p}-unital C^*-algebra A without bounded traces and unital quotients. It follows from Proposition 2.7 that A has a full, hereditary sub-C^*-algebra B that is not large in A and that does not have a bounded trace.

The example below is due to Ken Dykema.

Example 2.8 The full free product $\mathcal{K} * \mathcal{K}$ is not stable; hence the class of stable C^*-algebras is not closed under forming free products.

Indeed, $\mathcal{K} * \mathcal{K}$ has a unital quotient. To see this, let $\{e_{ij}\}_{i,j=1}^{\infty}$ be the standard matrix units for $\mathcal{K}$. Observe that if D is a C^*-algebra and if $f_1, f_2, \dots$ is a sequence of mutually orthogonal and equivalent projections in D, then there is an embedding $\varphi\colon \mathcal{K} \to D$ such that $\varphi(e_{jj}) = f_j$. Take the Cuntz algebra $\mathcal{O}_2$ with its two canonical generators s_1 and s_2. Since every pair of non-zero projections in $\mathcal{O}_2$ are equivalent and any non-zero projection in $\mathcal{O}_2$ has countably many mutually orthogonal non-zero sub-projections, there are embeddings $\varphi_1, \varphi_2\colon \mathcal{K} \to \mathcal{O}_2$ such that $\varphi_1(e_{11}) = s_1 s_1^*$ and $\varphi_2(e_{11}) = s_2 s_2^*$. By the universal property of free products there is a *-homomorphism $\varphi\colon \mathcal{K} * \mathcal{K} \to \mathcal{O}_2$ whose restriction to the first and the second copy of $\mathcal{K}$ is φ_1, respectively, φ_2. Accordingly, $1 = s_1s_1^* + s_2s_2^*$ belongs to the image of φ. Hence $\mathcal{K} * \mathcal{K}$ has a unital quotient.

3 Stability of finite C^*-algebras

Blackadar proved in [1] that a (simple) AF-algebra is stable if and only if it admits no bounded trace. We shall in this section pursue generalizations of this result. Let us first remark that any unital, properly infinite C^*-algebra is traceless but not stable. One will therefore expect the two properties, being stable and being traceless, to be equivalent only for finite C^*-algebras; and even here the equivalence does not hold without qualifications.

As in [1] it is convenient to consider also a third property of a C^*-algebra that the scale of its K_0-group equals the entire positive cone. The positive cone and the scale of the K_0-group of a C^*-algebra A are given by

$$K_0(A)^+ = \{[p]_0 : p \in \mathcal{P}(A \otimes \mathcal{K})\}, \qquad \mathcal{D}_0(A) = \{[p]_0 : p \in \mathcal{P}(A)\}.$$

It follows from Lemma 2.6 that $\mathcal{D}_0(A) = K_0(A)^+$ for all stable C^*-algebras.

An axiomatic description of a scaled ordered Abelian group is given in the following:

Definition 3.1 A triple (G, G^+, Σ) will be called a *scaled, ordered Abelian group* if (G, G^+) is an ordered Abelian group and Σ is an upper directed, hereditary, full subset of G^+, i.e.,

(i) $\forall x_1, x_2 \in \Sigma\ \exists x \in \Sigma : x_1 \le x,\ x_2 \le x$,

(ii) $\forall x \in G^+\ \forall y \in \Sigma : x \le y \Rightarrow x \in \Sigma$,

(iii) $\forall x \in G^+\ \exists y \in \Sigma\ \exists k \in \mathbb{N} : x \le ky$.

A (σ_{p}-unital) C^*-algebra A is said to be *finite* if it contains no infinite projections, and A is *stably finite* if $M_n(A)$ is finite for every n. (A projection is infinite if it is Murray–von Neumann equivalent to a proper subprojection of itself.) If

$$\forall p, q \in \mathcal{P}(A \otimes \mathcal{K}) : [p]_0 = [q]_0 \ \text{ in }\ K_0(A) \implies p \sim q,$$

then A is said to have *cancellation*. We have

$$\mathrm{sr}(A) = 1 \implies A \ \text{ has cancellation } \implies A \ \text{ is stably finite,}$$

for all C^*-algebras A.

Lemma 3.2 *Let A be a C^*-algebra with the cancellation property, let p be a projection in A, and let g be an element in $K_0(A)$.*

(i) *If $0 \le g \le [p]_0$, then there is a projection q in A such that $q \le p$ and $[q]_0 = g$.*

(ii) *If A is σ_{p}-unital, if $[p]_0 \le g$, and if g belongs to $\mathcal{D}_0(A)$, then there is a projection q in A such that $p \le q$ and $[q]_0 = g$.*

Proof: (i). Find projections e, f in matrix algebras over A such that $[e]_0 = g$ and $[f]_0 = [p]_0 - g$. Then $[e \oplus f]_0 = [p]_0$ and because A is assumed to have the cancellation property we conclude that $e \oplus f \sim p$. Find a rectangular matrix v over A such that $v^*v = e \oplus f$ and $vv^* = p$, and set $q = v(e \oplus 0)v^*$. Then q belongs to A, $q \le p$, and $[q]_0 = g$.

(ii). There is an approximate unit $\{p_n\}_{n=1}^{\infty}$ for A where each p_n is a projection dominating p. Now, $[p_n]_0 \ge g$ for some n. Indeed, take a projection q' in A such that $g = [q']_0$ and choose n such that $\|(1 - p_n)q'\| < 1$. Then $q' \precsim p_n$, and so $[p_n]_0 \ge [q']_0 = g$. Use (i) to find a projection e in A such that $e \le p_n - p$ and $[e]_0 = g - [p]_0$. The projection $q = p + e$ will then be as desired. □

Lemma 3.3 *The triple $\big(K_0(A), K_0(A)^+, \mathcal{D}_0(A)\big)$ is a scaled, ordered, Abelian group if A is a σ_{p}-unital C^*-algebra with the cancellation property.*

Conversely, if A is a stable, σ_{p}-unital C^*-algebra with the cancellation property, and if Σ is a subset of $K_0(A)^+$ for which the triple $\big(K_0(A), K_0(A)^+, \Sigma\big)$ is a scaled, ordered, Abelian group, then there is a full, σ_{p}-unital, hereditary sub-C^*-algebra B of A such that

$$\big(K_0(B), K_0(B)^+, \mathcal{D}_0(B)\big) \cong \big(K_0(A), K_0(A)^+, \Sigma\big).$$

Proof: Assume that A is a σ_{p}-unital C^*-algebra with the cancellation property. Let $\{p_n\}_{n=1}^{\infty}$ be an approximate unit for A consisting of projections. (i) in Definition 3.1 holds as we can take x to be $[p_n]_0$ for some large enough n. (ii) follows from Lemma 3.2 (i). If q is a projection in $M_k(A)$, then q is equivalent to a projection in $M_k(p_nAp_n)$ for some large enough n whence $[q]_0 \le k[p_n]_0$. Hence (iii) in Definition 3.1 holds.

To prove the second part of the lemma, use (i), (ii), and (iii) in Definition 3.1 to find $0 \le x_1 \le x_2 \le x_3 \le \dots$ in Σ such that for every g in $K_0(A)^+$ the following two conditions are satisfied:

- $g \le kx_n$ for some positive integers k and n, and

- g belongs to Σ if and only if $g \leq x_n$ for some n.

Use Lemma 3.2 (ii) to find an increasing sequence $\{q_n\}_{n=1}^{\infty}$ of projections in A such that $[q_n]_0 = x_n$. Let B be the closure of $\bigcup_{n=1}^{\infty} q_n A q_n$. Then B is a full σ_{p}-unital sub-C^*-algebra of A. By construction of B, if g is an element in $K_0(A)$, then g belongs to Σ if and only if there is a projection e in B such that $g = [e]_0$. It follows that the isomorphism $K_0(B) \to K_0(A)$ induced by the inclusion mapping $B \hookrightarrow A$ maps $\mathcal{D}_0(B)$ onto Σ. $\square$

An ordered Abelian group (G, G^+) is said to be *weakly unperforated* if $ng > 0$ implies $g > 0$ for every g in G and for every positive integer n. (Other texts have assigned other meanings to the term weak unperforation.)

Proposition 3.4 *Let A be a σ_{p}-unital C^*-algebra with the cancellation property, and consider the following three conditions:*

(i) *A is stable,*

(ii) *$\mathcal{D}_0(A) = K_0(A)^+$,*

(iii) *A admits no bounded trace.*

Then

$$\text{(i)} \iff \text{(ii)} \implies \text{(iii)},$$

and (iii) $\Rightarrow$ (ii) *if A is exact, $K_0(A)$ is weakly unperforated, and every ideal in A is σ_{p}-unital.*

Proof: The implication (i) $\Rightarrow$ (ii) holds for all C^*-algebras (as noted above). The assumption that A is σ_{p}-unital implies that every non-zero, densely defined trace τ on A induces a non-zero state $\widehat{\tau}$ on $K_0(A)$, and

$$\|\tau\| \geq \sup\{\widehat{\tau}(g) : g \in \mathcal{D}_0(A)\} = \sup\{\widehat{\tau}(g) : g \in K_0(A)^+\} = \infty,$$

when (ii) holds. Therefore (ii) $\Rightarrow$ (iii).

(ii) $\Rightarrow$ (i): Assume that (ii) holds. Let p be a projection in A. Then $2[p]_0$ belongs to $\mathcal{D}_0(A)$, and so it follows from Lemma 3.2 (ii) that there is a projection q in A with $p \leq q$ and $[q]_0 = 2[p]_0$. Using again that A has the cancellation property we find that $q - p \sim p$. It now follows from Theorem 2.2 that A is stable.

(iii) $\Rightarrow$ (ii): Assume next that $K_0(A)$ is weakly unperforated, each ideal in A is σ_{p}-unital, A is exact, and that (iii) holds. Take g in $K_0(A)^+$ and find a projection p in $A \otimes \mathcal{K}$

such that $g = [p]_0$. Let $I \otimes \mathcal{K}$ be the closed two-sided ideal in $A \otimes \mathcal{K}$ generated by p, and take an increasing approximate unit $\{p_n\}_{n=1}^{\infty}$ of projections for I. Let T be the compact set of traces τ on I such that $\tau(p) = 1$. Then

$$\sup_{n \in \mathbb{N}} \tau(p_n) = \infty$$

for every τ in T (otherwise τ would extend to a bounded trace on I and in turns to a bounded trace on A).

Each projection q in I (or in $I \otimes \mathcal{K}$) defines a continuous affine function $\widehat{q}\colon T \to \mathbb{R}$, and $\{\widehat{p}_n\}$ is an increasing sequence of functions tending pointwise to infinity. Since T is compact we have $\widehat{p}_n > 1$ for some n. In other words, $\tau(p) < \tau(p_n)$ for all τ in T. We infer that $f([p]_0) < f([p_n]_0)$ for all states f on $(K_0(I), K_0(I)^+)$ with $f([p]_0) = 1$. Indeed, each such state f lifts to a quasitrace τ on I (by [3]) and each quasitrace on an exact C^*-algebra is a trace (by Haagerup's theorem in [10]). By Goodearl–Handelman's extension theorem (see [9]), $k[p]_0 < k[p_n]_0$ in $K_0(I)$ (and hence in $K_0(A)$) for some natural number k. Since $K_0(A)$ is weakly unperforated we can conclude that $[p]_0 < [p_n]_0$. This entails that $g = [p]_0$ belongs to $\mathcal{D}_0(A)$ using Lemma 3.3 and Definition 3.1 (ii). $\square$

The three conditions of Proposition 3.4 are equivalent for all separable, exact, real rank zero C^*-algebras with the cancellation property and with weakly unperforated K_0-group. This is a lot to ask for, but many commonly encountered C^*-algebras satisfy these properties. For example, all AF-algebras, and more generally, all AH-algebras of real rank zero and of slow dimension growth have these properties (see [8] and [2]).

Stability of a finite C^*-algebra can also be expressed in terms of properties of its multiplier algebra as in the proposition below from [17]. Recall that a unital C^*-algebra is *properly infinite* if it contains two mutually orthogonal projections p, q such that $1 \sim p \sim q$.

Proposition 3.5 *Let A be a C^*-algebra and let $\mathcal{M}(A)$ denote its multiplier algebra.*

(i) *If A is stable, then $\mathcal{M}(A)$ is properly infinite.*

(ii) *If A is σ-unital,* $\mathrm{sr}(A) = 1$, *and A is not stable, then $\mathcal{M}(A)$ is not properly infinite.*

(iii) *If A is σ-unital, simple,* $\mathrm{sr}(A) = 1$, *and A is not stable, then $\mathcal{M}(A)$ is finite.*

Part (i) is standard and follows from the fact that $\mathcal{M}(A) \otimes \mathcal{M}(\mathcal{K})$ (maximal tensor product) maps into $\mathcal{M}(A \otimes \mathcal{K})$. Part (i) and (ii) say that for σ-unital C^*-algebras A of stable rank one, A is stable if and only if $\mathcal{M}(A)$ is properly infinite.

If A is a unital, properly infinite C^*-algebra, then $\mathcal{M}(A) = A$, and hence $\mathcal{M}(A)$ is properly infinite. On the other hand, A is not stable. We can therefore not in general deduce that A is stable knowing that $\mathcal{M}(A)$ is properly infinite.

4 Stability is not a stable property

One often refers to a property of C^*-algebras as being stable if it is preserved by passing from A to $M_n(A)$ and vice versa for each n. Being stable is not a stable property in this sense, as shown by the author in [16] using techniques of Villadsen from [18].

We first state a result that limits how exotic this behavior can be:

Proposition 4.1 (Proposition 2.1 of [16]) *Let A be a σ-unital C^*-algebra. If $M_n(A)$ is stable for some integer n, then $M_k(A)$ is stable for all $k \geq n$.*

The proof uses Theorem 2.2.

Let us indicate at the level of scaled, ordered Abelian groups why there should exists a non-stable C^*-algebra A such that $M_2(A)$ is stable:

Example 4.2 (Example 3.4 of [16]) Let $\mathbb{Z}_2$ denote the group $\mathbb{Z}/2\mathbb{Z}$, and let $\mathbb{Z}_2^{(\infty)}$ denote the group of all sequences $t = (t_j)_{j=1}^{\infty}$, with $t_j \in \mathbb{Z}_2$ and where $t_j \neq 0$ for at most finitely many j. For each $t \in \mathbb{Z}_2^{(\infty)}$ let $d(t)$ be the number of elements in the set $\{j \in \mathbb{N} \mid t_j \neq 0\}$. Set

$$G = \mathbb{Z} \oplus \mathbb{Z}_2^{(\infty)}, \qquad G^+ = \{(k,t) \mid d(t) \leq k\}, \qquad \Sigma = \{(k,t) \mid d(t) = k\}.$$

Then (G, G^+, Σ) is a scaled, ordered Abelian group, cf. Definition 3.1. To see this, let $e_j \in \mathbb{Z}_2^{(\infty)}$ be the generator of the jth copy of $\mathbb{Z}_2$ and set $g_j = (1, e_j) \in G^+$. Then

$$\Sigma = \bigcup_{j=1}^{\infty} \{x \in G^+ \mid x \leq g_1 + g_2 + \cdots + g_j\},$$

and in this picture it is easy to see that Σ satisfies the axioms of Definition 3.1.

The element $(2, e_1)$ belongs to G^+ but not to Σ, and so $\Sigma \neq G^+$.

If A is a C^*-algebra whose scaled ordered K_0-group is isomorphic to (G, G^+, Σ), then the scaled ordered K_0-group of $M_2(A)$ is isomorphic to $(G, G^+, \Sigma \hat{+} \Sigma)$, where $\Sigma \hat{+} \Sigma$ is the set of elements x in G^+ for which there exist y_1, y_2 in Σ such that $x \leq y_1 + y_2$. In the given example, $\Sigma \hat{+} \Sigma = G^+$, because if $g = (k,t)$ belongs to G^+, then

$$g \leq g + g = (2k, 0) = 2(g_1 + g_2 + \cdots + g_k).$$

If we can find a σ_{p}-unital C^*-algebra A with the cancellation property such that the scaled ordered K_0-group of A is isomorphic to (G, G^+, Σ), then A will be non-stable and $M_2(A)$ will be stable by Proposition 3.4. The C^*-algebra found in Theorem 4.3 below, corresponding to $n = 2$, has the property that a *subgroup* of its K_0-group is isomorphic to (G, G^+, Σ).

For the formulation of the next result, recall that an AH-algebra is a C^*-algebra that is the inductive limit of a sequence of C^*-algebras of the form $p(C(X) \otimes \mathcal{K})p$, where X is a (not necessarily connected) compact Hausdorff space and p is a projection in $C(X) \otimes \mathcal{K}$.

Theorem 4.3 (Theorem 5.3 and Corollary 4.2 of [16])

(i) *For each natural number n there is a simple, separable, σ_{p}-unital AH-algebra A of stable rank one such that $M_n(A)$ is stable but $M_{n-1}(A)$ is not stable.*

(ii) *For each natural number n there is continuous field C^*-algebra $A = (A_x)_{x \in X}$, where X is a compact Hausdorff space and where each fiber A_x is isomorphic to $\mathcal{K}$, such that $M_{n-1}(A)$ is not stable and $M_n(A)$ is stable.*

We indicate here the proof of part (ii) in the case where $n = 2$. As mentioned above, the proof follows ideas of Villadsen.

Let $Y = \mathbb{RP}^2$ be the real projective plane and recall that its cohomology (over $\mathbb{Z}$) is given as:

$$H^0(Y; \mathbb{Z}) \cong \mathbb{Z}, \qquad H^1(Y; \mathbb{Z}) = 0, \qquad H^2(Y; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}.$$

We have

$$C(Y) = \{f \in C(\mathbb{D}) : f(z) = f(-z) \ \text{ for all } \ z \in \mathbb{T}\}.$$

Let ξ_0 be a complex line bundle over Y with non-trivial Euler class $\mathrm{e}(\xi_0)$ in $H^2(Y; \mathbb{Z})$. This line bundle corresponds to the projection p in $M_2(C(Y))$ given by

$$p(re^{it}) = \begin{pmatrix} r & e^{it}\sqrt{r(1-r)} \\ e^{-it}\sqrt{r(1-r)} & 1-r \end{pmatrix}, \qquad r \in [0,1], \ \ t \in [0, 2\pi].$$

Also, $\xi_0 \oplus \xi_0 \cong \theta_2$, the trivial 2-dimensional complex bundle over Y.

Put $X = \prod_{n=1}^{\infty} Y$ and let $\pi_n \colon X \to Y$ be the coordinate map onto the nth copy of Y. Put $\xi_n = \pi_n^*(\xi_0)$, so that each ξ_n is a complex line bundle over X. We have $\xi_n \oplus \xi_n = \pi_n^*(\xi_0 \oplus \xi_0) \cong \theta_2$ for every n. An application of Künneth's theorem shows that $\xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_n$ has non-trivial Euler class for every n. It follows that there for no n is a complex bundle η such that $\xi_1 \oplus \eta \cong \xi_2 \oplus \cdots \oplus \xi_n$ since that would entail

$$\theta_2 \oplus \eta \cong \xi_1 \oplus \xi_1 \oplus \eta \cong \xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_n.$$

This cannot be because $\theta_2 \oplus \eta$ has trivial Euler class, whereas $\xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_n$ was constructed to have non-trivial Euler class.

Choose mutually orthogonal projections $p_1, p_2, \dots$ in $C(X) \otimes \mathcal{K}$ such that p_n corresponds to the line bundle ξ_n, and let e be a projection in $C(X) \otimes \mathcal{K}$ corresponding to the trivial bundle θ_1. Then

(a) $p_n \oplus p_n \sim e \oplus e$ for all n, and

(b) p_1 is not equivalent to a sub-projection of $p_2 + p_3 + \cdots + p_n$ for any n.

Put $q_n = p_1 + \cdots + p_n$ and set

$$A = \overline{\bigcup_{n=1}^{\infty} q_n \big(C(X) \otimes \mathcal{K}\big) q_n}.$$

With $\rho_x \colon A \to \mathcal{K}$ the restriction to A of the evaluation mapping $C(X) \otimes \mathcal{K} \to \mathcal{K}$ at x, A gets the structure of a continuous field C^*-algebra with base space X and with each fiber isomorphic to $\mathcal{K}$.

By (b) above, there is no projection q in A such that $q \sim p_1$ and $q \perp p_1$, and it follows from (a) that $M_2(A)$ is stable. $\square$

We can now conclude that there are non-stable C^*-algebras that do not have bounded traces or unital quotients:

Corollary 4.4 *There is a non-stable, non-unital, separable, nuclear, simple, σ_{p}-unital C^*-algebra A that admits no bounded traces.*

Proof: Take A as in Theorem 4.3 (i) corresponding to $n = 2$. Then A is non-stable, separable, nuclear, simple and σ_{p}-unital. Since $M_2(A)$ is stable, A is not unital, nor can it have a bounded trace. $\square$

It is an important open problem if every simple C^*-algebra is either stably finite or purely infinite. The existence of a simple C^*-algebra with a non-zero finite and an infinite projection would provide a negative answer to this problem, but we do not know if there is such a simple C^*-algebra. A unital, finite C^*-algebra A such that $M_2(A)$ is infinite is known (see [6]) but there is no example of a simple C^*-algebra A with this behavior.

Cuntz has shown that every infinite projection in a simple C^*-algebra is properly infinite. It would therefore be natural to attempt to apply some algebraic argument to solve the problem about simple C^*-algebras by showing that if A is a unital, finite (possibly non-simple) C^*-algebra, then $M_2(A)$ cannot be properly infinite. This conclusion, however, is false:

Corollary 4.5 *For each natural number n there is a unital C^*-algebra B such that $M_n(B)$ is properly infinite, but $M_k(B)$ is finite for $k < n$.*

Proof: Take A to be the C^*-algebra constructed in Theorem 4.3 (i). Let $B = \mathcal{M}(A)$ be the multiplier algebra of A. Then $M_k(B) \cong \mathcal{M}(M_k(A))$. We can now apply Proposition 3.5 to conclude that B is as desired. □

The C^*-algebra B constructed in Corollary 4.5 is not separable, not simple, and not nuclear. Whereas I do not know how to construct a simple (or a nuclear) example, it is easy to make B separable: Take two isometries s_1, s_2 in $M_n(B)$ such that $s_1 s_1^* \perp s_2 s_2^*$. Let $s_k(i,j) \in B$ be the matrix entries for s_k, $k = 1, 2$, and let B_0 be the separable sub-C^*-algebra of B generated by the $2n^2$ elements $s_k(i,j)$. Then s_1, s_2 belong to $M_n(B_0)$, and this makes $M_n(B_0)$ properly infinite. Being a sub-C^*-algebra of the finite C^*-algebra $M_k(B)$, $M_k(B_0)$ is finite when $k < n$.

We can rephrase Corollary 4.5 as follows: There is a unital, properly infinite C^*-algebra A such that $(1-e)A(1-e)$ is finite for some projection $e \neq 1$ in A, and e can be chosen to have size $1/n$. The next corollary says that the example can be sharpened in that e can be chosen to have infinitesimal size.

Corollary 4.6 *There is a properly infinite, unital C^*-algebra A and an embedding $\varphi\colon \mathcal{K} \to A$ such that the corner C^*-algebra $(1-\varphi(e))A(1-\varphi(e))$ is finite whenever e is a non-zero projection in $\mathcal{K}$.*

Proof: By Corollary 4.5 there is for each natural number n a unital C^*-algebra B_n such that $M_n(B_n)$ is properly infinite and $M_{n-1}(B_n)$ is finite. Put

$$A = \prod_{n=1}^{\infty} M_n(B_n) \Big/ \sum_{n=1}^{\infty} M_n(B_n),$$

where $\prod_{n=1}^{\infty} M_n(B_n)$ is the C^*-algebra of all bounded sequences $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in M_n(B_n)$, and $\sum_{n=1}^{\infty} M_n(B_n)$ is the ideal of those $\{x_n\}_{n=1}^{\infty}$ for which $\|x_n\| \to 0$. Let $\pi\colon \prod_{n=1}^{\infty} M_n(B_n) \to A$ denote the quotient mapping.

Since each $M_n(B_n)$ is properly infinite, $\prod_{n=1}^\infty M_n(B_n)$ and hence A are properly infinite.

Let $\{e_{ij}\}_{i,j=1}^\infty$ be a set of matrix units for the compact operators $\mathcal{K}$. For n in $\mathbb{N}$ and for $1 \le i,j \le n$, let $g_{ij}^{(n)} \in M_n(\mathbb{C}) \subseteq M_n(B_n)$ be the (i,j)th standard matrix unit (wrt. the natural embedding of $M_n(\mathbb{C})$ into $M_n(B_n)$ defined by the unit of B_n). Set $g_{ij}^{(n)} = 0$ if i or j is greater than n. Put

$$g_{ij} = (g_{ij}^{(1)}, g_{ij}^{(2)}, g_{ij}^{(3)}, \dots), \qquad f_{ij} = \pi(g_{ij}).$$

Then $\{f_{ij}\}_{i,j=1}^\infty$ are matrix units for $\mathcal{K}$, and so there is a *-homomorphism $\varphi\colon \mathcal{K} \to A$ given by $\varphi(e_{ij}) = f_{ij}$. We proceed to check that $(1-\varphi(e))A(1-\varphi(e))$ is finite for all non-zero projections e in $\mathcal{K}$. It suffices to consider the case $e = e_{11}$.

Suppose, to reach a contradiction, that $(1-\varphi(e_{11}))A(1-\varphi(e_{11}))$ is infinite and take a non-unitary isometry s in that algebra. Lift s to an element $x = (x_1, x_2, \dots)$ in $\prod_{n=1}^\infty M_n(B_n)$. Upon replacing each x_n by $(1-g_{11}^{(n)})x_n(1-g_{11}^{(n)})$ we may assume that each $x_n = (1-g_{11}^{(n)})x_n(1-g_{11}^{(n)})$. Since

$$\pi(1-g_{11}^{(1)}, 1-g_{11}^{(2)}, \dots) = 1-\varphi(e_{11}) = \pi(x_1^*x_1, x_2^*x_2, \dots),$$

we conclude that $\|x_n^*x_n - (1-g_{11}^{(n)})\| \to 0$, and so $x_n^*x_n$ is invertible (in the corner algebra $(1-g_{11}^{(n)})M_n(B_n)(1-g_{11}^{(n)})$) for all sufficiently large n. As $(1-g_{11}^{(n)})M_n(B_n)(1-g_{11}^{(n)}) \cong M_{n-1}(B_n)$ and this C^*-algebra is finite, we can further conclude that x_n is invertible for all large enough n. But then s is invertible, a contradiction. $\square$

By an argument similar to the one outlined below Corollary 4.5, the C^*-algebra A in Corollary 4.6 can be taken to be separable. One cannot take A to be simple: any simple, unital C^*-algebra that admits an embedding of $\mathcal{K}$ is properly infinite (cf. [7]); and there are embeddings $\mathcal{K} \hookrightarrow (1-e)\mathcal{K}(1-e) \hookrightarrow (1-\varphi(e))A(1-\varphi(e))$.

5 Stability of infinite C^*-algebras

A (simple or non-simple) C^*-algebra A is said to be *purely infinite* if it has no Abelian quotient and if for every pair of positive elements a, b in A, such that b belongs to the closed two-sided ideal generated by a, there is a sequence $\{x_n\}$ of elements in A with $x_n^*ax_n \to b$ (see [14]). This notion was introduced by Cuntz for simple C^*-algebras, and he defined, in agreement with the definition above, a *simple* C^*-algebra to be purely infinite if each of its non-zero hereditary sub-C^*-algebras contain an infinite projection.

There are nice characterizations of stability for purely infinite C^*-algebras, and conversely, one can characterize pure infiniteness in terms of stability.

We look first at the case of simple C^*-algebras. Here we have the following classical result of of S. Zhang from [19] (that also can be derived from Theorem 2.2 using that every purely infinite, simple, σ-unital C^*-algebra has an (increasing) approximate unit consisting of projections, and that for any pair of non-zero projections p, q in such a C^*-algebra one has $p \precsim q$):

Proposition 5.1 (Zhang's Dichotomy) *A σ-unital, purely infinite, simple C^*-algebra is either unital or stable.*

The result below is an observation of Kirchberg and it is a special case of Proposition 5.4 below for which we include a proof.

Proposition 5.2 *A simple C^*-algebra A is purely infinite if and only if every non-zero hereditary sub-C^*-algebra of A contains a (non-zero) stable sub-C^*-algebra.*

Purely infinite C^*-algebras (simple and non-simple alike) have no traces. The proposition below, proved in [14, Theorem 4.24] and which is an easy consequence of Proposition 2.7, extends Zhang's Dichotomy. There are (non-simple) purely infinite C^*-algebras that are neither stable nor unital. Take for example $C_0(\mathbb{R}) \otimes \mathcal{O}_2$.

Proposition 5.3 *A (possibly non-simple) purely infinite, σ-unital C^*-algebra is stable if and only if it has no unital quotients.*

George Elliott suggested that the following result holds:

Proposition 5.4 *Let A be a (possibly non-simple) separable C^*-algebra A. Then the following three conditions are equivalent:*

(i) *A is purely infinite,*

(ii) *every non-zero hereditary sub-C^*-algebra of A contains a full, stable, hereditary sub-C^*-algebra,*

(iii) *every non-zero hereditary sub-C^*-algebra of A contains a full, stable (not necessarily hereditary) sub-C^*-algebra.*

Proof: (i) $\Rightarrow$ (ii): Let B be a non-zero hereditary sub-C^*-algebra of A. Take a countable dense subset X of the unit ball of B^+ and put

$$Y = \{(b - 1/n)_+ : b \in X, \quad n \in \mathbb{N}\},$$

cf. Remark 2.1. Let $Y = \{b_1, b_2, \dots\}$ be an enumeration of Y. We proceed to find mutually orthogonal positive elements $c_1, c_2, \dots$ in B such that $c_j \sim b_j$ (cf. Remark 2.1) and $B \cap \{c_1, \dots, c_n\}^\perp$ is full in B for every n. The set $\{c_n, c_{n+1}, \dots\}$ will then be full in B for every natural number n. We construct the sequence $\{c_n\}_{n=1}^\infty$ by induction and to do so it suffices to justify the first step, i.e., to find c_1.

By construction, $b_1 = (b - \varepsilon)_+$ for some $\varepsilon > 0$ and some positive contraction b in B. The element b is properly infinite because A is purely infinite and we can therefore find x, y in $\overline{bAb}$ with $x^*x = y^*y = (b - \varepsilon/2)_+$ and $xx^* \perp yy^*$ (see [14, Lemma 3.2]). Let $x = u|x|$ be the polar decomposition for x as in Remark 2.1. There is a positive contraction f in the hereditary sub-C^*-algebra generated by $x^*x = (b - \varepsilon/2)_+$ such that $fb_1 = b_1 f = b_1$. Put $c_1 = ub_1u^*$, put $e = ufu^*$, and let I be the closed two-sided ideal in B generated by $B \cap \{c_1\}^\perp$. Then $c_1 \sim b_1$, cf. Remark 2.1, and it remains to show that $I = B$. Because yy^* belongs to $B \cap \{c_1\}^\perp$ we conclude that $y^*y = (b - \varepsilon/2)_+$ belongs to I. It follows that f and hence e belong to I. By construction, $ec_1 = c_1 e = c_1$ and so $(1 - e)a(1 - e)$ belongs to $B \cap \{c_1\}^\perp$ for all a in B. Now, each element a in B belongs to the ideal generated by $\{eaa^*e, (1 - e)aa^*(1 - e)\}$ and hence to I. This proves that $I = B$.

Let D and D_n be hereditary sub-C^*-algebras of B generated by $c_1, c_2, \dots$, respectively, by $c_1, \dots, c_n$. Then $D_1 \subseteq D_2 \subseteq \cdots$ and $D = \overline{\bigcup_{n=1}^\infty D_n}$. Since D contains $c_1, c_2, \dots$, the closed two-sided ideal of B generated by D contains $b_1, b_2, \dots$, and this set generates B. Therefore D is full in B. We must also show that D is stable. This follows by an application of Theorem 2.2, but it can be seen more easily by first noting that D is purely infinite, being a hereditary sub-C^*-algebra of A, and D has no unital quotient. Indeed, assume that J is a proper ideal in D and that D/J is unital. The unit of D/J will then belong to $D_n/(J \cap D_n)$ for some sufficiently large n. In that case c_k belongs to J for all $k > n$; but $c_{n+1}, c_{n+2}, \dots$ is full in D (by construction of b_n and c_n), and hence $J = D$, a contradiction. Proposition 5.3 now yields that D is stable.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i): Suppose that (iii) holds. Take a positive element a in A and find a full, stable sub-C^*-algebra D of $\overline{aAa}$. Let $\varepsilon > 0$ be given. Being separable and stable, D contains a sequence of mutually orthogonal and equivalent elements $c_1, c_2, \dots$ so that a belongs to the ideal generated c_1. (To see this, write $D = D_0 \otimes \mathcal{K}$, take a strictly positive element c in D_0 and put $c_j = c \otimes e_{jj}$.) Let u_j, $j \geq 2$, be partial isometries in A^{**} implementing the equivalence between c_1 and c_j so that $u_j^* c_j u_j = c_1$, cf. Remark 2.1, and such that $u_i u_i^* \perp c_j$

when $i \neq j$. Find n and elements $x_1, \dots, x_n$ in D such that $(a-\varepsilon)_+ = \sum_{j=1}^n x_j^* c_1 x_j$. Put

$$e = \sum_{j=1}^{n} c_j, \qquad f = \sum_{j=n+1}^{2n} c_j, \qquad x = \sum_{j=1}^{n} u_j x_j, \qquad y = \sum_{j=1}^{n} u_{n+j} x_j.$$

Then e and f are mutually orthogonal positive elements in $\overline{aAa}$ and $x^*ex = y^*fy = (a-\varepsilon)_+$. This shows that a is properly infinite, cf. [14, Proposition 3.3], and since a was arbitrary we conclude that A is purely infinite. $\square$

6 The extension problem

One motivation for characterizing stability of C^*-algebras is to study the following well-known open problem:

Question 6.1 *Given a short exact sequence of (separable) C^*-algebras*

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0.$$

Does it follow that A is stable if I and B are known to be stable?

The answer to Question 6.1 is not even known for split extensions. The following special case of Question 6.1 also remains open:

Question 6.2 *Given two stable closed two-sided ideals I and J in a (separable) C^*-algebra A. Does it follow that their sum $I + J$ is stable?*

That an affirmative answer to Question 6.2 will follow from an affirmative answer to Question 6.1 is seen by considering the short exact sequence:

$$0 \longrightarrow I \longrightarrow I + J \longrightarrow (I + J)/I \longrightarrow 0,$$

and noting that $(I + J)/I \cong J/(I \cap J)$ is stable being (isomorphic) to a quotient of the stable C^*-algebra J.

Given a partially ordered set $(P, \leq)$. An element x in P is called *maximal* if $x \leq y$ implies $x = y$ for all y in X. An element x is called a *greatest element* if $y \leq x$ for every y in X. A greatest element is also a maximal element (but not conversely); a partially ordered set can have at most one greatest element, but it can have several maximal elements.

Proposition 6.3 *Every separable C^*-algebra has a maximal stable ideal (i.e., a stable ideal not properly contained in any other stable ideal).*

Proof: Use Zorn's Lemma to choose a maximal totally ordered family $\{I_i\}_{i\in\mathbb{I}}$ of stable ideals in A (counting 0 as a stable ideal) and set $I = \overline{\bigcup_{i\in\mathbb{I}} I_i}$. Then I is an ideal in A and I is not properly contained in any stable ideal in A by maximality of the set $\{I_i\}_{i\in\mathbb{I}}$. It follows from Corollary 2.3 (i) that I is stable. □

Question 6.4 *Let A be a (separable) C^*-algebra.*

(i) *Does A have a greatest stable ideal (i.e., a stable ideal that contains all other stable ideals)?*

(ii) *Does A have a stable ideal I such that A/I has no non-zero stable ideals?*

An ideal I satisfying Question 6.4 (ii) will necessarily be a greatest stable ideal in A. Indeed, if J is another stable ideal in A, then $(I+J)/I$ — which is isomorphic to $J/(I\cap J)$ — is a stable ideal in A/I, and so $(I+J)/I$ is zero. This entails that $J \subseteq I$.

Question 6.4 (ii) asks in other words if every C^*-algebra A has a (unique) greatest stable ideal I and if this ideal has the additional property that A/I has no non-zero stable ideals.

It follows from Proposition 6.3 (and its proof) that any stable ideal of a separable C^*-algebra is contained in a maximal stable ideal. We can therefore rephrase Question 6.2 (i) as follows: Does every (separable) C^*-algebra have a *unique* maximal stable ideal?

For separable C^*-algebras, Question 6.4 (i) is equivalent to Question 6.2; and Question 6.4 (ii) is equivalent to Question 6.1. Let us here show why Question 6.2 implies Question 6.4 (i):

Let A be a separable C^*-algebra, and let $\{I_i\}_{i\in\mathbb{I}}$ be the collection of all stable ideals in A (including 0). If Question 6.2 holds then $I_{i_1}+I_{i_2}$ belongs to this collection for all $i_1, i_2 \in \mathbb{I}$. It follows that $I = \overline{\bigcup_{i\in\mathbb{I}} I_i}$ is an ideal in A, and every stable ideal in A is contained in I. Corollary 2.3 (i) shows that I is stable.

Consider the continuous field C^*-algebra $A = (A_x)_{x\in X}$ constructed in Theorem 4.3 (ii). Each open subset U of X defines an ideal $A_U = (A_x)_{x\in U}$ of A consisting of those section $a = (a_x)$ in A such that $a_x = 0$ whenever $x \notin U$; and every ideal in A is of this form. In the given case, each fiber A_x is isomorphic to $\mathcal{K}$ and hence is stable, but no ideal A_U is stable — roughly because each non-empty open subset U of X contains an open cylinder set:

$$V_1 \times V_2 \times \cdots \times V_n \times Y \times Y \times \cdots \subseteq U, \qquad V_j \subseteq Y.$$

We give in Propositions 6.7 and 6.11 below a partial positive answer to Question 6.1.

Lemma 6.5 *Let A be a C^*-algebra and let I be a closed two-sided ideal in A. If I and A/I have no (non-trivial) unital quotients, then neither has A.*

Proof: Suppose, to reach a contradiction, that J is a proper closed two-sided ideal in A such that A/J is unital. Then $A/(I+J)$ is a unital quotient of A/I and therefore $I+J=A$. Hence

$$\frac{I}{I\cap J} \cong \frac{I+J}{J} = \frac{A}{J},$$

so that $I/(I\cap J)$ is unital. This entails that $I\cap J = I$. It follows that $I \subseteq J$ and consequently $J = A$, a contradiction. $\square$

Lemma 6.6 *Let A be a C^*-algebra, let I be a closed two-sided ideal in A, and assume that neither I nor A/I have (non-trivial) unital quotients. Then for each a in A, the C^*-algebra $\overline{(1-a)I(1-a^*)}$ is full in I and has no (non-trivial) unital quotients.*

Proof: Let $\widetilde{A}$ denote the unitization of A. Let J be the closed two-sided ideal in $\widetilde{A}$ generated by $1-a$, let J_0 be the closed two-sided ideal in A generated by $\overline{(1-a)A(1-a^*)}$, and let I_0 be the closed two-sided ideal in I generated by $\overline{(1-a)I(1-a^*)}$. Then $J_0 = J\cap A$ and $I_0 = J\cap I = J_0\cap I$. Let $\pi\colon \widetilde{A}\to \widetilde{A}/J$ be the quotient mapping. Then $\pi(a)=\pi(1)$, and so $\pi(A)$ is unital. The kernel of the restriction of π to A is equal to J_0. Hence A/J_0 is unital. By Lemma 6.5 and the assumption that I and A/I have no unital quotients we conclude that $J_0 = A$. It follows that $I_0 = I$ so that $\overline{(1-a)I(1-a^*)}$ is full in I.

Assume next, to reach a contradiction, that L_0 is a proper ideal in $\overline{(1-a)I(1-a^*)}$ such that $\overline{(1-a)I(1-a^*)}/L_0$ is unital. Let L be the closed two-sided ideal in I generated by L_0 so that $L_0 = \overline{(1-a)I(1-a^*)}\cap L$. Let $\pi\colon A\to A/L$ be the quotient mapping. Find e in $\overline{(1-a)I(1-a^*)}$ such that $\pi(e)$ is the unit for $\overline{(1-a)I(1-a^*)}/L_0$, and put $y = e+a-ea$. Then y belongs to A and

$$(1-y)I(1-y^*) = (1-e)(1-a)I(1-a^*)(1-e^*) \subseteq L,$$

in contradiction with the first part of the lemma saying that $\overline{(1-y)I(1-y^*)}$ is full in I. $\square$

Proposition 6.7 *Let I be a stable, closed, two-sided ideal in a separable C^*-algebra A, and suppose that A/I is stable. Then the following three conditions are equivalent:*

(i) *A is stable,*

(ii) *for each positive contraction a in A, the hereditary sub-C^*-algebra $\overline{(1-a)I(1-a)}$ is large in I (cf. Definition 2.4),*

(iii) *$\overline{(1-a)I(1-a)}$ is stable for each positive contraction a in A.*

Proof: (i) $\Rightarrow$ (iii). If A is stable, then so is $\overline{(1-a)A(1-a)}$ by Corollary 2.3 (iii). Hence $\overline{(1-a)I(1-a)}$ is stable by Corollary 2.3 (ii) being an ideal in a stable C^*-algebra.

(iii) $\Rightarrow$ (ii) follows from Lemmas 2.6 and 6.6.

(ii) $\Rightarrow$ (i). Suppose that (ii) holds. To show that A is stable we use Theorem 2.2 and find to each a in $F(A)$ a positive element a_1 in A such that $a \perp a_1$ and $a \precsim a_1$ (cf. Remark 2.1). Let $\pi \colon A \to A/I$ denote the quotient mapping.

There is a positive contractions e in $F(A)$ such that $ea = a = ae$. Set $f = \pi(e)$. Since A/I is stable and f belongs to $F(A/I)$ there is f' in $F(A/I)$ with $f \sim f'$ and $f \perp f'$ (by Theorem 2.2). Because $f' = (1-f)f'(1-f)$ we get

$$f' \in (1-f)A/I(1-f) = \pi(\overline{(1-e)A(1-e)}),$$

and we can therefore find a positive contraction e' in $\overline{(1-e)A(1-e)}$ such that $\pi(e') = f'$. Since $\pi(e') \sim \pi(e)$ there is a positive element c in I such that $(e-1/3)_+ \precsim e' \oplus c$, cf. [14, Lemma 4.2]. It follows that $(e-2/3)_+ \precsim (e'-\delta)_+ \oplus (c-\delta)_+$ for some $\delta > 0$, cf. [15, Proposition 2.4]. Put $c_0 = (c-\delta)_+ \in F(I)$ and $e_0' = (e'-\delta)_+ \in F(A)$. Then $a \precsim (e-2/3)_+ \precsim e_0' \oplus c_0$. Let g be a positive contraction in A such that $ge_0' = e_0'g = e_0'$. By assumption (and by the remarks below Definition 2.4) there is a positive element c_1 in $\overline{(1-e-g)I(1-e-g)}$ such that $c_0 \sim c_1$. Now, a, e_0', and c_1 are mutually orthogonal, positive elements in A, and

$$a \precsim e_0' \oplus c_0 \precsim e_0' \oplus c_1 \precsim e_0' + c_1.$$

We can therefore take a_1 to be $e_0' + c_1$. $\square$

Lemma 6.8 *Let A be a C^*-algebra, let I be a stable, closed two-sided ideal in A such that the quotient A/I does not have (non-trivial) unital quotients. Let a be a positive contraction in A. Then $\overline{(1-a)I(1-a)}$ admits no non-zero bounded trace.*

Proof: Assume to reach a contradiction that τ is a bounded (positive) trace on the hereditary sub-C^*-algebra $\overline{(1-a)I(1-a)}$. This hereditary sub-C^*-algebra is full in I by Lemma 6.6. We can therefore extend τ to an unbounded (because I is stable) densely defined trace τ on I. Now, I is an ideal in the unitization $\widetilde{A}$ of A, and we can extend τ to a lower semi-continuous trace function $\bar{\tau}\colon \widetilde{A}^+ \to [0,\infty]$. Let J be the closed two-sided ideal in $\widetilde{A}$ generated by all positive elements b in $\widetilde{A}$ with $\bar{\tau}(b) < \infty$. A positive element b in $\widetilde{A}$ will then belong to J if and only if $\bar{\tau}((b-\varepsilon)_+) < \infty$ for all $\varepsilon > 0$.

Now, I is contained in J because τ is densely defined on I. Since τ is not bounded on I we cannot have $\tau(1) < \infty$; thus $J \neq \widetilde{A}$. The assumption that τ is bounded on $\overline{(1-a)I(1-a)}$ leads to $\tau(1-a) < \infty$, and hence $1-a$ belongs to J.

Let $\psi\colon \widetilde{A} \to \widetilde{A}/J$ and $\pi\colon \widetilde{A}/I \to \widetilde{A}/J$ be the quotient mappings. Then $\psi(1) = \psi(a)$ because $1-a$ belongs to J, and it follows that $\psi(A)$ is unital. Since $\pi(A/I) = \psi(A)$, A/I has a unital quotient contrary to our assumptions. □

To state Proposition 6.11 in general terms the following definition is convenient.

Definition 6.9 A C^*-algebra I is called *regular* if every full, hereditary sub-C^*-algebra of I, that has no unital quotients and no bounded traces, is stable

It follows from Corollary 4.4 that not all C^*-algebras are regular. On the other hand, many C^*-algebras are regular:

Lemma 6.10 *A C^*-algebra I is regular*

(i) *if I is an exact C^*-algebra with the cancellation property,* $\mathrm{RR}(I) = 0$*, and $K_0(I)$ is weakly unperforated,* or

(ii) *if I is purely infinite.*

Proof: (i). Let I_0 be a full, hereditary sub-C^*-algebra of I. Then I_0 is σ_{p}-unital because I has real rank zero. The cancellation property, exactness, and having weakly unperforated K_0-group are all properties that pass to full hereditary sub-C^*-algebras, so I_0 has these properties. Proposition 3.4 therefore yields that I_0 is stable if I_0 has no bounded trace.

(ii). Every hereditary sub-C^*-algebra of a purely infinite C^*-algebra is again purely infinite ([14, Proposition 4.17]) and hence is stable if it has no unital quotient, cf. Proposition 5.3. □

Proposition 6.11 *Let*

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

be a short exact sequence of separable C^*-algebras and suppose that I is regular. Then A is stable if and only if I and B are stable.

All AF-algebras, and more generally all AH-algebras of real rank zero and of slow dimension growth, are regular (see the comments below Proposition 3.4). In particular, for every extension $0 \to \mathcal{K} \to A \to B \to 0$ of separable C^*-algebras one has that A is stable if and only if B is stable, a fact that implicitly is contained in the BDF-paper [5].

Proof: If A is stable, then so are I and A/I (by Corollary 2.3 (ii)). Assume now that I and A/I are stable and that I is regular. As $\overline{(1-a)I(1-a)}$ is a full hereditary sub-C^*-algebra of I that has no unital quotient (by Lemma 6.6) and no bounded traces (by Lemma 6.8) for every positive contraction a in A, the assumption that I is regular implies that $\overline{(1-a)I(1-a)}$ is stable. Proposition 6.7 then yields that A is stable. $\square$

The answer to Question 6.1 could be no in general. Suppose that there is a stable C^*-algebra I and an embedding $\varphi\colon \mathcal{K} \to \mathcal{M}(I)/I$ such that $(1-\varphi(e))\mathcal{M}(I)/I(1-\varphi(e))$ is finite (or not properly infinite) for some one-dimensional projection e in $\mathcal{K}$ and such that $\varphi(e)$ admits a lift to a projection f in $\mathcal{M}(A)$. With $\pi\colon \mathcal{M}(I) \to \mathcal{M}(I)/I$ the quotient mapping, set $A = \pi^{-1}(\varphi(\mathcal{K}))$ and notice that f belongs to A. We get a short exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow \mathcal{K} \longrightarrow 0,$$

and A is not stable. Indeed, $\mathcal{M}((1-f)I(1-f)) \cong (1-f)\mathcal{M}(I)(1-f)$ and

$$\pi\big((1-f)\mathcal{M}(I)(1-f)\big) = (1-\varphi(e))\frac{\mathcal{M}(I)}{I}(1-\varphi(e)),$$

which shows that $\mathcal{M}((1-f)I(1-f))$ is not properly infinite, whence $(1-f)I(1-f)$ is not stable, and hence A is not stable, cf. Proposition 6.7.

It is not known if there is such a C^*-algebra I, but Corollary 4.6 and its proof carry some evidence in favor of its existence. Indeed, let A_n be as in Theorem 4.3 for each n in $\mathbb{N}$, and set $I = \sum_{n=1}^{\infty} M_n(A_n)$. Then I is stable and $\mathcal{M}(I) = \prod_{n=1}^{\infty} \mathcal{M}(M_n(A_n))$. The proof of Corollary 4.6 gives an embedding

$$\varphi\colon \mathcal{K} \to \mathcal{M}(I)/\sum_{n=1}^{\infty} \mathcal{M}(M_n(A_n)) = D$$

such that $(1-\varphi(e))D(1-\varphi(e))$ is finite whenever e is a non-zero projection in $\mathcal{K}$, and e lifts to a projection in $\mathcal{M}(I)$. However, φ does not lift to an embedding of $\mathcal{K}$ into $\mathcal{M}(I)/I$.

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