# EXTENSIONS OF NUMBER FIELDS WITH WILD RAMIFICATION OF BOUNDED DEPTH

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ABSTRACT. We consider $p$-extensions of number fields such that the filtration of the Galois group by higher ramification groups is of prescribed finite length. We extend the basic properties of such towers (similar to the tamely ramified case but with more complications) to this more general setting; for instance, we show that these towers are "asymptotically good," (we give an explicit bound for the root discriminant). We study the difficult problem of bounding the relation-rank of the Galois groups in question; by using results of Gordeev and Wingberg, we show that they can tend to infinity when the set of ramified primes is fixed but the length of the ramification filtration becomes large. We discuss the $p$-adic representations of these Galois groups in the context of the Fontaine-Mazur and provide some examples

## Introduction

Fix a prime number $p$, a number field $K$, and a finite set $S$ of primes of $K$. Let $S_p$ be the set of all primes of $K$ of residue characteristic $p$. Inside a fixed algebraic closure $\overline{K}$ of $K$, let $K_S$ be the maximal $p$-extension of $K$ unramified outside $S$, and put $G_S = \mathrm{Gal}(K_S/K)$. The study of $G_S$ is governed by a dichotomy between the tame ($S \cap S_p = \emptyset$) and wild ($S \cap S_p \neq \emptyset$) cases.

One distinguishing feature of this dichotomoy is the following: in the tame case, every open subgroup of $G_S$ has finite abelianization; on the other hand, if $S_p \subseteq S$, then $G_S$ has a surjection onto $\mathbb{Z}_p^{r_2+1}$ (induced by the $\mathbb{Z}_p$-extensions of $K$), where $r_2$ is the number of imaginary places of $K$; (for surjections of $G_S$ to $\mathbb{Z}_p$ when $S \subset S_p$, see [M]). In fact, a conjecture of Fontaine and Mazur [FM], predicts that, in the tame case, $G_S$ has no infinite $p$-adic analytic quotients.

A second, and subtly related, distinguishing feature is the following: for $\mathfrak{p} \in S - S_p$, the filtration of $G_S$ by higher ramification groups at $\mathfrak{p}$ has length at most 2, i.e. $D^{(1)}(K_S/K, \mathfrak{p})$ vanishes, whereas in the case of wild ramification in an infinite $p$-extension, it is often the case that the higher ramification groups of all indices are non-trivial (for example in a $\mathbb{Z}_p$-extension).

In this paper, we study a generalization of tame towers, namely towers with wild ramification of bounded "depth". To be precise, let $\nu : S \to [0, \infty]$ be an arbitrary map (which will serve to limit the depth of ramification). Now define the group $G_{S,\nu}$ as the quotient of $G_S$ by the closed normal subgroup generated by higher ramification groups (conjugacy classes), in the upper numbering, $D^{(\nu_\mathfrak{p})}(K_S/K, \mathfrak{p})$ as $\mathfrak{p}$ runs over $S$. The fixed field $K_{S,\nu}$ of this subgroup, with Galois group $\mathrm{Gal}(K_{S,\nu}/K) = G_{S,\nu}$, is the compositum of all finite $p$-extensions of $K$ having vanishing ${\nu_\mathfrak{p}}^{th}$ ramification group at

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$\mathfrak{p}$ for all $\mathfrak{p} \in S$. If the image of $\nu$ is simply $\nu(S) = \{\infty\}$, then $G_{S,\nu}$ is nothing but $G_S$, and much – most notably an estimate for its relation-rank – is known about this group [Sh], [K2], [NSW].

The case of greatest interest for us is when $\nu$ is "finite", i.e. $\nu(S) \subseteq [0, \infty)$, because this ensures that the number fields in the tower $K_{S,\nu}/K$ form an "asymptotically good" family (in the sense of Tsfasman and Vladut [TV]), i.e. the root discriminant of these number fields remains bounded. The basic reason is that the exponent of a prime $\mathfrak{p}$ in the relative discriminant of $L/K$, where $L$ is contained in $K_S$, is a sum of orders of ramification groups and therefore grows linearly in $[L : K]$. Thus, the root discriminant of the fields in $K_S/K$ remains bounded. This is discussed in detail in section 5. For applications of asymptotically good families in the number field and function field settings, see [AM], [HM], and especially [TV].

We investigate the group $G_{S,\nu}$ via generators and relations. While calculating its generator-rank $d(G_{S,\nu}) = \dim_{\mathbb{F}_p} H^1(G_{S,\nu}, \mathbb{F}_p)$ reduces to a standard calculation in class field theory, estimating $r(G_{S,\nu}) = \dim_{\mathbb{F}_p} H^2(G_{S,\nu}, \mathbb{F}_p)$ presents an essential technical difficulty we have not been able to overcome. This is explained in section 6.

We are, however, able to make certain interesting observations regarding the relation-rank of $G_{S,\nu}$. First, there are non-trivial cases (i.e. where $\nu_{\mathfrak{p}} \geq 2$ for some $\mathfrak{p} \in S \cap S_p$) where $G_{S,\nu}$ is finite, and, of course, $G_{S,\nu}$ can be infinite since already $G_\emptyset$ can be infinite. However, it is not at all clear that, in general, $G_{S,\nu}$ is even finitely presentable (has finite relation-rank), even if $\nu$ is finite. In fact, using results of Wingberg and Gordeev, we construct a family of examples with infinite $\nu$ achieving $r(G_{S,\nu}) = \infty$. These examples allow us to show that for fixed $S$ and growing, but finite, $\nu$, the relation-rank can become arbitrarily large.

In the final section, further questions about the structure of $G_{S,\nu}$ are discussed, especially as regards $p$-adic representations of these groups. Fontaine and Mazur conjecture that when $S \cap S_p = \emptyset$, $G_S$ has no infinite $p$-adic analytic quotients. When $\nu$ is finite, we suspect $G_{S,\nu}$ is subject to the same conclusion, and provide some supporting evidence. A related question is whether every $p$-adic representation of $G_{S,\nu}$ is potentially semistable at $p$ (in the sense of Fontaine) when $\nu$ is finite. We may summarize our questions as follows.

**Question.** Suppose $K$ is a number field and $L/K$ is a Galois extension such that $\mathrm{Gal}(L/K)$ is an infinite $p$-adic analytic group. Does it follow that

a) the root discriminant of the fields in this tower tend to infinity?

b) for some prime $\mathfrak{P}$ of $L$, all the higher ramification groups $D^y(L/K, \mathfrak{P})$ are non-trivial (for all $0 < y < \infty$)?

c) there exists a $p$-adic representation $\rho : \mathrm{Gal}(L/K) \to GL_n(E)$ (for some $E/\mathbb{Q}_p$ finite and some $n \geq 1$) which is not potentially semistable?

The organization of the paper is as follows. The first two sections comprise a preliminary chapter of definitions and properties of ramification groups. We define the extensions of interest to us in section 3, and calculate the behavior of root discriminants in them in section 4. In sections 5 and 6, we use theorems of Sen, Gordeev, and Wingberg to explore the structure of the groups $G_{S,\nu}$. In particular, we pinpoint the difficulty in extending the method for calculating the relation rank of $G_S$ to that of $G_{S,\nu}$, and we construct a family of $G_{S,\nu}$ with infinite $\nu$ which are not finitely presentable. Section 7 investigates a possible extension of Conjecture 5a of [FM] with some supporting evidence: We give examples of $G_{S,\nu}$ which do not admit infinite analytic quotients (Corollaries 40 and 41).

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# Part 1. Preliminaries

## 1. The setup

Let $p$ be a prime number. Consider a number field $K$, a finite set $S$ of primes of $K$ equipped with an indexing function $\nu : S \to [0,\infty]$ sending $\nu$ to $\nu(\mathfrak{p}) = \nu_{\mathfrak{p}}$; we will call the pair $(S,\nu)$ an *indexed set* for $K$. If $\nu(S) \subseteq [0,\infty)$, we will call $(S,\nu)$ a *finitely indexed set*. Sometimes it will be convenient to extend $\nu$ to all places of $K$ by setting $\nu_{\mathfrak{p}} = 0$ for $\mathfrak{p} \notin S$. If $\nu(\mathfrak{p}) = 0$ or $\infty$ for all $\mathfrak{p} \in S \cap S_p$, then the pair $(S,\nu)$ is abbreviated by $S$.

We fix once and for all an algebraic closure $\overline{K}$; an "extension of $K$" will mean a subfield of this fixed algebraic closure. Let $\mathfrak{p}$ be a prime ideal of $K$. We will use the following notation.

- $S_p$ is the set of primes of $K$ of residue characteristic $p$; $S - S_p$ means $S-(S\cap S_p)$.
- $K_{\mathfrak{p}}$ is the completion of $K$ at $\mathfrak{p}$;
- $\mathcal{O}_{\mathfrak{p}}$ the ring of integers of $K_{\mathfrak{p}}$;
- $U_{\mathfrak{p}}$ the group of units of $\mathcal{O}_{\mathfrak{p}}$;
- $U_{\mathfrak{p}}^{(i)} = \{u \in U_{\mathfrak{p}} | u-1 \in \mathfrak{p}^{\lceil i \rceil}$ for $i \geq 0$; $U_{\mathfrak{p}}^{(0)} = U_{\mathfrak{p}}$; $U_{\mathfrak{p}}^{\infty} = \{1\}$;
- $J_K$ is the group of idèles of $K$;
- $\mathcal{U}_S = \prod_{\mathfrak{p}\notin S} U_{\mathfrak{p}}$; $\mathcal{U}_{S,\nu} = \prod_{\mathfrak{p}} U_{\mathfrak{p}}^{(\nu_{\mathfrak{p}})}$; these products are extended over the infinite places $\mathfrak{p}$ as well, where, for such a place, $U_{\mathfrak{p}} = U_{\mathfrak{p}}^0 = K_{\mathfrak{p}}^{\times}$.
- $\hat{K}$ is the maximal $p$-extension of $K$, with Galois group $\mathcal{G} = \mathrm{Gal}(\hat{K}/K)$;
- $\hat{K}_{\mathfrak{p}}$ is the maximal $p$-extension of $K_{\mathfrak{p}}$ in a fixed Galois closure $\overline{K_{\mathfrak{p}}}$ of $K_{\mathfrak{p}}$, with Galois group $D_{\mathfrak{p}} = \mathrm{Gal}(\hat{K}_{\mathfrak{p}}/K_{\mathfrak{p}})$;
- $N_{S,\nu}$ is the closed normal subgroup of $\mathcal{G}$ generated by all higher ramification groups $D^{\nu_{\mathfrak{p}}}(\hat{K}/K,\mathfrak{p})$ for all $K$-primes $\mathfrak{p}$;
- $G_{S,\nu} = \mathcal{G}/N_{S,\nu} = \mathrm{Gal}(K_{S,\nu}/K)$ is the Galois group of the maximal $p$-extension of $K$ unramified outside $S$ and with ramification of depth at most $\nu_{\mathfrak{p}}$ for $\mathfrak{p} \in S$;
- $d(G) = p\text{-rk}G = \dim_{\mathbb{F}_p} G/[G,G]G^p$ is the $p$-rank of a group $G$;
- for a pro-$p$ group, $r(G) = \dim_{\mathbb{F}_p} H^2(G,\mathbb{F}_p)$ is the minimal number of relations for defining $G$ as a pro-$p$ group;
- $d_{S,\nu} = d(G_{S,\nu}), r_{S,\nu} = r(G_{S,\nu})$;
- $\mu_p$ is the group of $p$th roots of unity in $\overline{K}$;
- $\delta_{\mathfrak{p}} = 1$ if $K_{\mathfrak{p}}$ contains $\mu_p$, 0 otherwise;
- $\delta(\mathfrak{p},\nu_{\mathfrak{p}})$ is 1 if $\mu_{\mathfrak{p}} \subseteq U_{\mathfrak{p}}^{(\nu_{\mathfrak{p}})}$, and is 0 otherwise;
- $\delta_K = 1$ if $K$ contains $\mu_p$, 0 otherwise;
- $\theta_S = 0$ unless $\delta_K = 1$ and $S$ is empty, in which case, $\theta_S = 1$.

Let $S'$ be another finite set of prime ideals of $K$. For later use, we introduce the groups

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- $\Delta = \{x \in K^\times |\ (x)$ is a $p$th power in the group of fractional ideals of $K\}$
- $\Delta_S = \{x \in \Delta |\ x \in K_\mathfrak{p}^{\times p}\ \forall \mathfrak{p} \in S\}/K^{\times p}$;
- $\Delta_{S,\nu} = \{x \in \Delta |\ x \in K_\mathfrak{p}^{\times p} U_\mathfrak{p}^{(\nu_\mathfrak{p})}\ \forall \mathfrak{p} \in S\}/K^{\times p}$;
- $\Delta_S^{S'} = \{x \in K^\times |\ x \in U_\mathfrak{p} K_\mathfrak{p}^{\times p}\ \forall \mathfrak{p} \notin S', x \in K_\mathfrak{p}^{\times p}\ \forall \mathfrak{p} \in S\}/K^{\times p}$.

Using Kummer theory, one has

**Proposition 1.** *Assume $\mu_p \subseteq K$.*

- *If $S_p \subseteq S$, then $p$-rk$(\mathrm{Cl}_K^{S,+}) = p$-rk$(\Delta_S)$, where $\mathrm{Cl}_K^{S,+}$ is the $S$-class group of $K$ in the narrow sense.*
- *If $S' \cup S = S_p$, then $p$-rk$(\Delta_S^{S'}) = p$-rk$(G_{S'}^S)$, where $G_{S'}^S$ is the Galois group over $K$ of the maximal $p$-extension of $K$ unramified outside $S'$ in which $S$ splits completely.*

## 2. Higher Ramification groups

For the convenience of the reader, we collect here some definitions and standard properties of higher ramification groups. The reader is invited to consult this section only as needed, and is referred to Serre [Se1] for more details.

### 2.1. Ramification groups with lower numbering.

Consider a finite Galois extension $L/K$ of number fields and a prime $\mathfrak{P}$ of $L$ lying over $\mathfrak{p}$ of $K$.

**Definition 2.** *The decomposition group $D(L/K, \mathfrak{P}) \subseteq \mathrm{Gal}(L/K)$ of $L/K$ at $\mathfrak{P}$ is the stabilizer of $\mathfrak{P}$ and can be identified with the local Galois group $D(\mathfrak{P}/\mathfrak{p}) = \mathrm{Gal}(L_\mathfrak{P}/K_\mathfrak{p})$. For $x \geq -1$, we define the higher ramification group of index $x$ in the lower numbering $D_x(L/K, \mathfrak{P}) \subseteq D(\mathfrak{P}/\mathfrak{p})$ by*

$$D_x(L/K, \mathfrak{P}) = \{\sigma \in D(\mathfrak{P}/\mathfrak{p}) |\ v_\mathfrak{P}(\sigma(\alpha) - \alpha) \geq x + 1, \forall \alpha \in \mathcal{O}_L\}.$$

In the sequel, we will sometimes abbreviate the data $(L/K, \mathfrak{P})$ as $(\mathfrak{P}/\mathfrak{p})$, for instance $D_x(\mathfrak{P}/\mathfrak{p})$ in place of $D_x(L/K, \mathfrak{P})$, etc. Note that the ramification groups of index $-1, 0$ are the decomposition and inertia groups of $L/K$ at $\mathfrak{P}$, respectively. The higher ramification groups give a finite decreasing filtration of the decomposition group: $D_x(L/K, \mathfrak{P}) \subseteq D_y(L/K, \mathfrak{P})$, for $x \geq y$ and $D_x(L/K, \mathfrak{P}) = \{1\}$ for all large enough $x$.

### 2.2. Ramification groups with upper numbering.

Consider the map $\varphi = \varphi_{L/K,\mathfrak{P}} = \varphi_{\mathfrak{P}/\mathfrak{p}}$ from $[-1, \infty)$ to $[-1, \infty)$ defined as follows: For $-1 \leq u \leq 0$, $\varphi(u) = u$; for $u \geq 0$, let $m = \lfloor u \rfloor$ and put

$$\varphi(u) = \frac{1}{g_0}\left(g_1 + \cdots + g_m + (u - m)g_{m+1}\right),$$

where $g_i = |D_i(L/K, \mathfrak{P})|$; in particular,

$$\varphi(m) = -1 + \frac{1}{g_0}\sum_{i=0}^{m} g_i.$$

**Proposition 3.** *The function $\varphi$ is continuous, piecewise linear, strictly increasing, concave, and satisfies $\varphi(0) = 0, \varphi(x) \leq x$.*

Let $\psi = \psi_{L/K,\mathfrak{P}} = \psi_{\mathfrak{P}/\mathfrak{p}} : [-1, \infty) \to [-1, \infty)$ be the inverse of $\varphi$.

**Proposition 4.** *The map $\psi$ is continuous, piecewise linear, strictly increasing, convex, and satisfies $\psi(0) = 0, \psi(y) \geq y$. If $n$ is an integer, then $\psi(n)$ is an integer.*

**Definition 5.** *We define the ramification group of index $y \geq -1$ in the upper numbering by $D^y(L/K, \mathfrak{P}) = D_{\psi(y)}(L/K, \mathfrak{P})$.*

*Remark* 2.1. We note that $D_x(L/K, \mathfrak{P}) = D^{\varphi(x)}(L/K, \mathfrak{P})$, $D^y(L/K, \mathfrak{P}) \subseteq D_y(L/K, \mathfrak{P})$, and for $y \geq z$, $D^y(L/K, \mathfrak{P}) \subseteq D^z(L/K, \mathfrak{P})$. The ramification groups of $\mathfrak{P}^\sigma$, $\sigma \in \mathrm{Gal}(L/K)$, are the conjugates by $\sigma$ of the ramification groups for $\mathfrak{P}$. We let $D_x(L/K, \mathfrak{p})$ $(D^y(L/K, \mathfrak{p}))$ denote the conjugacy class in $\mathrm{Gal}(L/K)$ of $D_x(L/K, \mathfrak{P})$ $(D^y(L/K, \mathfrak{P}))$.

Now let us consider the behavior of these groups in a tower $L'/L/K$ where $L'/K$ and $L/K$ are finite Galois extensions, $\mathfrak{P}'$ is a prime of $L'$ lying over $\mathfrak{P}$ of $L$ and $\mathfrak{p}$ of $K$.

**Proposition 6.** *We have*

1. $\varphi_{\mathfrak{P}'/\mathfrak{p}} = \varphi_{\mathfrak{P}/\mathfrak{p}} \circ \varphi_{\mathfrak{P}'/\mathfrak{P}}$, *and* $\psi_{\mathfrak{P}'/\mathfrak{p}} = \psi_{\mathfrak{P}'/\mathfrak{P}} \circ \psi_{\mathfrak{P}/\mathfrak{p}}$;
2. $D_x(\mathfrak{P}'/\mathfrak{P}) = D_x(\mathfrak{P}'/\mathfrak{p}) \cap \mathrm{Gal}(L'/L)$;
3. $D^y(\mathfrak{P}/\mathfrak{p}) = D^y(\mathfrak{P}'/\mathfrak{p})D(\mathfrak{P}'/\mathfrak{P})/D(\mathfrak{P}'/\mathfrak{P})$; *in other words, $D^y(\mathfrak{P}/\mathfrak{p})$ is the image of $D^y(\mathfrak{P}'/\mathfrak{p})$ in* $\mathrm{Gal}(L/K)$.

The groups in the lower numbering behave well under taking subgroups, whereas the groups in the upper numbering behave well under taking quotients. As noted in [Se1], the restriction property allows one to define the ramification groups $D^y(L/K, \mathfrak{P})$ with upper numbering for all profinite extensions $L/K$. We now list some simple consequences of the above proposition, which we will use later.

**Proposition 7.**
1. *If $\mathfrak{P}/\mathfrak{p}$ is unramified, then $D^y(\mathfrak{P}/\mathfrak{p}) = D^y(\mathfrak{P}'/\mathfrak{P})$.*
2. *If $D^y(\mathfrak{P}'/\mathfrak{P})$ and $D^y(\mathfrak{P}/\mathfrak{p})$ are trivial, then $D^y(\mathfrak{P}'/\mathfrak{p})$ is also trivial.*
3. *If $L''$ is a Galois extension of $K$ such that $LL'' = L'$, and $D^y(\mathfrak{P}/\mathfrak{p}) = D^y(\mathfrak{P}''/\mathfrak{p}) = \{1\}$, where $\mathfrak{P}'' = \mathfrak{P}' \cap L''$, then $D^y(\mathfrak{P}'/\mathfrak{p}) = \{1\}$.*

*Proof.* 1. As $\mathfrak{P}/\mathfrak{p}$ is unramified, the lower numbering ramification groups of $\mathfrak{P}'/\mathfrak{P}$ are the same as those of $L'/K$. The map $\psi_{\mathfrak{P}'/\mathfrak{p}}$ is then the same as $\psi_{\mathfrak{P}'/\mathfrak{P}}$. Then one has:

$$\begin{aligned} D^y(\mathfrak{P}'/\mathfrak{p}) &= D_{\psi_{\mathfrak{P}'/\mathfrak{p}}(y)}(\mathfrak{P}'/\mathfrak{p}) = D_{\psi_{\mathfrak{P}'/\mathfrak{P}}(y)}(\mathfrak{P}'/\mathfrak{p}) \\ &= D_{\psi_{\mathfrak{P}'/\mathfrak{P}}(y)}(\mathfrak{P}'/\mathfrak{P}) = D^y(\mathfrak{P}'/\mathfrak{P}). \end{aligned}$$

2. The restriction property shows that $D^y(\mathfrak{P}'/\mathfrak{p}) \subseteq \mathrm{Gal}(L''/L)$. Moreover $D^y(\mathfrak{P}'/\mathfrak{p}) = D_{\psi_{\mathfrak{P}'/\mathfrak{p}}(y)}(\mathfrak{P}'/\mathfrak{p})$. Now using the transitivity of $\psi$ and other elementary properties, we have $D_{\psi_{\mathfrak{P}'/\mathfrak{p}}(y)}(\mathfrak{P}'/\mathfrak{p}) \subseteq D_{\psi_{\mathfrak{P}'/\mathfrak{P}}(y)}(\mathfrak{P}'/\mathfrak{p})$. Thus,

$$D_{\psi_{\mathfrak{P}'/\mathfrak{p}}(y)}(\mathfrak{P}'/\mathfrak{p}) \subseteq \mathrm{Gal}(L''/L) \cap D_{\psi_{\mathfrak{P}'/\mathfrak{P}}(y)}(\mathfrak{P}'/\mathfrak{p}) = D_{\psi_{\mathfrak{P}'/\mathfrak{P}}(y)}(\mathfrak{P}'/\mathfrak{P}) = D^y(\mathfrak{P}'/\mathfrak{P}).$$

The triviality of $D^y(\mathfrak{P}'/\mathfrak{P})$ allows us to conclude.

3. Follows easily from the restriction property. □

**Proposition 8.** *Suppose $L/K$ is a finite $p$-extension.*

*1) If $\mathfrak{p}$ has residue characteristic $\ell \neq p$, $D^y(\mathfrak{P}/\mathfrak{p})$ vanishes for all $y > 0$.*

*2) If $\mathfrak{p}$ has residue characteristic $p$, and $D^y(\mathfrak{P}/\mathfrak{p})$ vanishes for some $y \leq 1$, then $\mathfrak{P}/\mathfrak{p}$ is unramified.*

*Proof.* 1) Since $x = \psi_{\mathfrak{P}/\mathfrak{p}}(y) > 0$, we have $D^y(\mathfrak{P}/\mathfrak{p}) = D_x(\mathfrak{P}/\mathfrak{p}) = D_1(\mathfrak{P}/\mathfrak{p})$. Since $L/K$ is a $p$-extension, $\mathfrak{P}/\mathfrak{p}$ is at most tamely ramified and the wild inertia group $D_1(\mathfrak{P}/\mathfrak{p})$ is trivial.

2) We claim that $x := \psi_{\mathfrak{P}/\mathfrak{p}}(1) = 1$. We already know that $x$ is a positive integer. Let us show that $x \leq 1$. We have

$$1 = \varphi_{\mathfrak{P}/\mathfrak{p}}(x) = \frac{g_1 + \cdots + g_x}{g_0},$$

where $g_i = |D_i(\mathfrak{P}/\mathfrak{p})|$. As $D_0(\mathfrak{P}/\mathfrak{p})/D_1(\mathfrak{P}/\mathfrak{p})$ is trivial, $g_0 = g_1$. One obtains:

$$1 + \frac{x-1}{g_0} \leq 1,$$

and so $x = 1$. Now we see that if $y \leq 1$, then

$$D_1(\mathfrak{P}/\mathfrak{p}) = D_{\psi_{\mathfrak{P}/\mathfrak{p}}(1)}(\mathfrak{P}/\mathfrak{p}) \subseteq D_{\psi_{\mathfrak{P}/\mathfrak{p}}(y)}(\mathfrak{P}/\mathfrak{p}) = D^y(\mathfrak{P}/\mathfrak{p}) = \{1\},$$

and this implies the triviality of $D_0(\mathfrak{P}/\mathfrak{p})$ because this is a $p$-group and $D_0(\mathfrak{P}/\mathfrak{p})/D_1(\mathfrak{P}/\mathfrak{p})$ has order prime to $p$. □

*Remark* 2.2. If $\mathfrak{P}$ has residue characteristic $p$, $\psi_{\mathfrak{P}/\mathfrak{p}}(y) = y$ for $y \in [0, 1]$.

**Proposition 9.** *Suppose $L/K$ is an abelian extension. Then $D_1(\mathfrak{P}/\mathfrak{p})$ vanishes if and only if $D^1(\mathfrak{P}/\mathfrak{p})$ does.*

*Proof.* One direction is trivial because $D^1(\mathfrak{P}/\mathfrak{p}) \subseteq D_1(\mathfrak{P}/\mathfrak{p})$. We know that the image of principal units under the local reciprocity map generates $D^1(\mathfrak{P}/\mathfrak{p})$. Thus, if this group is trivial, by using the structure of the group of local units we deduce the fact that the inertia group has order prime to the residue characteristic of $\mathfrak{p}$, and then $D_1(\mathfrak{P}/\mathfrak{p})$ is trivial. □

**Proposition 10.** *Suppose $D^y(\mathfrak{P}'/\mathfrak{p}) = \{1\}$ for some $y$. Then $D^z(\mathfrak{P}'/\mathfrak{P}) = \{1\}$ for $z = \psi_{\mathfrak{P}/\mathfrak{p}}(y)$.*

*Proof.* One has:

$$\begin{aligned} D^z(\mathfrak{P}'/\mathfrak{P}) &= D_{\psi_{\mathfrak{P}'/\mathfrak{P}}(z)}(\mathfrak{P}'/\mathfrak{P}) = D_{\psi_{\mathfrak{P}'/\mathfrak{p}}(y)}(\mathfrak{P}'/\mathfrak{P}) \\ &= D_{\psi_{\mathfrak{P}'/\mathfrak{p}}(y)}(\mathfrak{P}'/\mathfrak{p}) \cap \mathrm{Gal}(L'/L) \\ &= D^y(\mathfrak{P}'/\mathfrak{p}) \cap \mathrm{Gal}(L'/L) \\ &= \{1\}. \end{aligned}$$

□

# Part 2. Wild ramification of bounded depth

## 3. Definition and simple properties of towers with bounded ramification

Let $S$ be a finite set of primes of $K$, equipped with a map $\nu : S \to [0, \infty]$ sending $\mathfrak{p}$ to $\nu_{\mathfrak{p}}$.

**Definition 11.** *Suppose $L/K$ is a finite Galois extension and $\mathfrak{p}$ is a prime of $K$. We say that the ramification of $L/K$ at $\mathfrak{p}$ is of depth at most $y$ if $D^y(\mathfrak{P}/\mathfrak{p})$ vanishes for all primes $\mathfrak{P}$ of $L$ lying over $\mathfrak{p}$.*

The ramification of $L/K$ at $\mathfrak{p}$ is of depth at most 0 (at most 1) means that $\mathfrak{p}$ is unramified (at most tamely ramified) in $L/K$.

**Definition 12.** *Let $K_{S,\nu}$ be the compositum, inside a fixed algebraic closure of $K$, of all finite $p$-extensions $L$ of $K$ unramified outside $S$ and with the property that, for every $\mathfrak{p} \in S$, the ramification of $L/K$ at $\mathfrak{p}$ is of depth at most $\nu_\mathfrak{p}$. Let $G_{S,\nu} = \mathrm{Gal}(K_{S,\nu}/K)$.*

By Proposition 7, $K_{S,\nu}$ is the maximal $p$-extension of $K$ unramified outside $S$ and ramified to depth at most $\nu_\mathfrak{p}$ for every $\mathfrak{p} \in S$. If $K_S$ is the maximal $p$-extension of $K$ unramified outside $S$, then $K_{S,\nu}$ is the fixed field corresponding to the closed normal subgroup of $G_S = \mathrm{Gal}(K_S/K)$ generated by all $D^{(\nu_\mathfrak{p})}(K_S/K, \mathfrak{p})$. If $\nu$ vanishes identically, then $K_{S,\nu}$ is simply the Hilbert $p$-class field tower of $K$. By Proposition 8, we can and do assume without loss of generality that

$$\mathfrak{p} \in S - S_p \Rightarrow \nu_\mathfrak{p} = 1, \qquad \mathfrak{p} \in S \cap S_p \Rightarrow \nu_\mathfrak{p} > 1.$$

**Definition 13.** *Suppose $L/K$ is a finite extension contained in $K_{S,\nu}$. We lift the indexed set $(S, \nu)$ of $K$ to an indexed set $(S,\nu)_L = (S_L, \nu_L)$ of $L$ by*

$$S_L = \{\mathfrak{P} \subseteq \mathcal{O}_L | \mathfrak{P} \textit{ divides } \mathfrak{p}\mathcal{O}_L \textit{ for some } \mathfrak{p} \in S\}, \qquad \nu_L(\mathfrak{P}) = \psi_{\mathfrak{P}/\mathfrak{p}}(\nu_\mathfrak{p}).$$

Let us note that for a tower $F/L/K$, the indexed sets $(S,\nu)_F$ and $(S_L, \nu_L)_F$ coincide, thanks to the transitivity of $\psi$.

Suppose $\nu$ is *finite*, and $K_1$ is the maximal abelian extension of $K$ contained in $K_{S,\nu}$; this is the field associated by class field theory to the idèle subgroup:

$$\mathcal{U}_{S,\nu} = \prod_{\mathfrak{p} \notin S} U_\mathfrak{p} \prod_{\mathfrak{p} \in S} U_\mathfrak{p}^{(\nu_\mathfrak{p})}.$$

Thus, $K_1$ is the ray class field of $K$ modulo $\mathfrak{m}_{S,\nu} = \prod_{\mathfrak{p} \in S} \mathfrak{p}^{\lceil \nu_\mathfrak{p} \rceil}$; in particular, $K_1/K$ is finite.

To simplify notation, let us write $(S_1, \nu_1)$ for $(S_{K_1}, \nu_{K_1})$. Define inductively a tower of finite abelian extensions as follows: $K_{n+1}$ is the maximal abelian extension of $K_n$ contained in $K_{S_n,\nu_n}$ and $(S_{n+1}, \nu_{n+1})$ is the lift of $(S_n, \nu_n)$ to $K_{n+1}$. Let $K_\infty = \cup_n K_n$. By maximality for each step, $K_\infty/K$ is a Galois pro-$p$-extension.

**Theorem 14.** *Suppose $(S,\nu)$ is a finitely indexed set for $K$. With notations as above, we have*

1. *$K_\infty = K_{S,\nu}$;*
2. *If $K/K_0$ is a Galois extension such that $(S,\nu)$ is $\mathrm{Gal}(K/K_0)$-stable, then $K_{S,\nu}/K_0$ is a Galois extension;*
3. *If $L/K$ is a finite Galois extension contained in $K_{S,\nu}$, then $K_{S,\nu} = L_{S_L,\nu_L}$;*
4. *The open subgroups of $G_{S,\nu}$ have finite abelianization.*

*Proof.* 1) For $\mathfrak{p} \in S$, choose a compatible system $\mathfrak{P}_n \in S_n$, lying over $\mathfrak{p}$. Recall that $\nu_n(\mathfrak{P}_n) = \psi_{\mathfrak{P}_n/\mathfrak{p}}(\nu_\mathfrak{p})$. To increase the readability of what follows, let us use the temporary notation $D[j](\mathfrak{P}/\mathfrak{p}) = D^j(\mathfrak{P}/\mathfrak{p})$ and $D\{j\}(\mathfrak{P}/\mathfrak{p}) = D_j(\mathfrak{P}/\mathfrak{p})$ for the upper and lower ramification groups respectively. Consider a positive integer $n$ such that

$K_n \subseteq K_{S,\nu}$ (this is the case for $n=1$). We calculate:

$$\begin{aligned}
D[\nu_{\mathfrak{p}}](\mathfrak{P}_{n+1}/\mathfrak{p}) \cap \mathrm{Gal}(K_{n+1}/K_n) &= D\{\psi_{\mathfrak{P}_{n+1}/\mathfrak{p}}(\nu_{\mathfrak{p}})\}(\mathfrak{P}_{n+1}/\mathfrak{p}) \cap \mathrm{Gal}(K_{n+1}/K_n) \\
&= D\{\psi_{\mathfrak{P}_{n+1}/\mathfrak{p}}(\nu_{\mathfrak{p}})\}(\mathfrak{P}_{n+1}/\mathfrak{P}_n) \\
&= D\{\psi_{\mathfrak{P}_{n+1}/\mathfrak{P}_n}(\psi_{\mathfrak{P}_n/\mathfrak{p}}(\nu_{\mathfrak{p}}))\}(\mathfrak{P}_{n+1}/\mathfrak{P}_n) \\
&= D[\psi_{\mathfrak{P}_n/\mathfrak{p}}(\nu_{\mathfrak{p}})](\mathfrak{P}_{n+1}/\mathfrak{P}_n) \\
&= \{1\}.
\end{aligned}$$

But, by the restriction property of higher ramification groups, and the fact that $D[\nu_{\mathfrak{p}}](\mathfrak{P}_n/\mathfrak{p})$ vanishes, $D[\nu_{\mathfrak{p}}](\mathfrak{P}_{n+1}/\mathfrak{p}) \subseteq \mathrm{Gal}(K_{n+1}/K_n)$, hence is trivial. By induction, $K_\infty \subseteq K_{S,\nu}$.

To show the reverse inclusion, it suffices to show that if $L$ is a finite abelian $p$-extension of $K_n$ (for an arbitrary $n$) which is contained in $K_{S,\nu}$, then $L \subseteq K_{n+1}$. If $\mathfrak{P}$ is a prime of $L$ dividing $\mathfrak{P}_n$, we have:

$$\begin{aligned}
D[\nu_n(\nu_{\mathfrak{p}})](\mathfrak{P}/\mathfrak{P}_n) &= D[\psi_{\mathfrak{P}_n/\mathfrak{p}}](\mathfrak{P}/\mathfrak{P}_n) \\
&= D\{\psi_{\mathfrak{P}/\mathfrak{P}_n} \circ \psi_{\mathfrak{P}_n/\mathfrak{p}}(\nu_{\mathfrak{p}})\}(\mathfrak{P}/\mathfrak{P}_n) \\
&= D\{\psi_{\mathfrak{P}/\mathfrak{p}}(\nu_{\mathfrak{p}})\}(\mathfrak{P}/\mathfrak{p}) \cap \mathrm{Gal}(L/K_n) \\
&= D[\nu_{\mathfrak{p}}](\mathfrak{P}/\mathfrak{p}) \\
&= \{1\}.
\end{aligned}$$

By the maximality of $K_{n+1}/K_n$, we then have $L \subseteq K_{n+1}$, completing the proof of 1.

2) is clear.

3) Proposition 10 gives $K_{S,\nu} \subseteq L_{S_L,\nu_L}$. We show the reverse inclusion. Let $(L_n)_n$ be the sequence of maximal abelian extensions giving $\cup_n L_n = L_{S_L,\nu_L}$. Put $L' = L_n$ for an arbitrary fixed $n$. By 2), $L'/K$ is Galois. One gets:

$$\begin{aligned}
\{1\} = D[\nu_L(\mathfrak{P})](\mathfrak{P}'/\mathfrak{P}) &= D\{\psi_{\mathfrak{P}'/\mathfrak{P}}(\nu_L(\mathfrak{P}))\}(\mathfrak{P}'/\mathfrak{P}) \\
&= D\{\psi_{\mathfrak{P}'/\mathfrak{p}}(\nu_{\mathfrak{p}})\}(\mathfrak{P}'/\mathfrak{P}) \\
&= D\{\psi_{\mathfrak{P}'/\mathfrak{p}}(\nu_{\mathfrak{p}})\}(\mathfrak{P}'/\mathfrak{p}) \cap \mathrm{Gal}(L'/L) \\
&= D[\nu_{\mathfrak{p}}](\mathfrak{P}'/\mathfrak{p}) \cap \mathrm{Gal}(L'/L).
\end{aligned}$$

Moreover, $D[\nu_{\mathfrak{p}}](\mathfrak{P}/\mathfrak{p}) = \{1\}$, and so $D[\nu_{\mathfrak{p}}](\mathfrak{P}'/\mathfrak{p}) \subseteq \mathrm{Gal}(L'/L)$. In conclusion $D[\nu_{\mathfrak{p}}](\mathfrak{P}'/\mathfrak{p}) = \{1\}$, and $L_n = L' \subseteq K_{S,\nu}$.

4) If $H$ is an open subgroup of $G_{S,\nu}$, its fixed field $L$ is a finite extension of $K$ and the abelianization of $H = G_{S_L,\nu_L}$ is isomorphic to a ray class group, hence is finite. □

*Remark* 3.1. 1. A compact topological group whose open subgroups have finite abelianization is sometimes called "FAB."

2. The naive lift of $\nu$ from $K$ to $L$ namely $\mathfrak{P} \mapsto \nu_{\mathfrak{p}}$ (for a prime $\mathfrak{P}$ of $L$ lying over $\mathfrak{p}$ of $K$) would lead to examples where $L_{S_L,\nu_L} \neq K_{S,\nu}$. Here is a simple example for $p=2$: let $K = \mathbb{Q}$, $L = \mathbb{Q}(\sqrt{3})$ and $F = K(\sqrt{2})$. We let $\mathfrak{P}, \mathfrak{p}$ be the unique primes of $F, L$ (respectively) dividing 2. One checks that $D^{(3)}(\mathfrak{P}/2)$ vanishes but that $D^{(3)}(\mathfrak{P}/\mathfrak{p})$ does not.
3. We should remark that the towers (of function fields) constructed in Perret [P] are based on this naive lift of $(S,\nu)$. They are contained in $K_{S,\nu}$ by Proposition 7), and thanks to Hasse-Arf, have a slightly lower root discriminant than the tower $K_{S,\nu}/K$, cf. Remark 4.1. The Galois group of Perret's tower, however, being less natural, is probably very difficult to study. For example, Neiderreiter

and Xing [NX] have shown that the relation-rank estimate conjectured by Perret is not correct, at least for function fields over $\mathbb{F}_2$.

4. By the above theorem, the fields $K_n$ are simply the fields fixed by the "commutator series" of $G_{S,\nu}$. Also, $K_\emptyset$ = the maximal unramified $p$-extension of $K$, is contained in $K_{S,\nu}$, for an arbitrary $(S,\nu)$.

We conclude this section with a calculation of the generator rank of $G_{S,\nu}$. By the Burnside Basis theorem, this reduces to calculating the $p$-rank of its maximal abelian quotient, which can be expressed, by class field theory, to the $p$-rank of an appropriate ray class group. The reduction to abelian extensions also allows us to assume, without loss of generality, that $\nu$ takes integer values (thanks to Hasse-Arf). We also note that if $\min_{\mathfrak{p}\in S}\nu_\mathfrak{p}$ is large enough, then $p\text{-rk}(G_{S,\nu}) = p\text{-rk}(G_S)$.

**Theorem 15.** *The generator-rank $d_{S,\nu}$ of $G_{S,\nu}$ is the $p$-rank of the ray class group $\mathrm{Cl}_{K,\mathfrak{m}}$ where $\mathfrak{m} = \prod_{\mathfrak{p}\in S}\mathfrak{p}^{\lceil \nu_\mathfrak{p}\rceil}$. It satisfies*

$$\begin{aligned} d_{S,\nu} &= p\text{-rk}\Delta_{S,\nu} + p\text{-rk}\frac{\mathcal{U}_\emptyset}{\mathcal{U}_{S,\nu}\mathcal{U}_\emptyset^p} - p\text{-rk}E_K \\ &= p\text{-rk}\Delta_{S,\nu} + \sum_{\mathfrak{p}\in S-S_p}\delta_\mathfrak{p} + \sum_{\mathfrak{p}\in S\cap S_p} p\text{-rk}\frac{U_\mathfrak{p}^{(1)}}{U_\mathfrak{p}^{(\nu_\mathfrak{p})}} - p\text{-rk}E_K \end{aligned}$$

*Proof.* The main observation, as mentioned above, is that the fixed field $K_1$ of (the closure of) the commutator subgroup $[G_{S,\nu}, G_{S,\nu}]$ is the field associated by class field theory to the idèle subgroup:

$$\mathcal{U}_{S,\nu} = \prod_{\mathfrak{p}\notin S} U_\mathfrak{p} \prod_{\mathfrak{p}\in S} U_\mathfrak{p}^{(\nu_\mathfrak{p})}.$$

We recall that we defined in section 1, certain number groups $\Delta_{S,\nu}$. We leave to the reader the verification of the exactness of the sequence:

$$1 \longrightarrow \Delta_{S,\nu} \longrightarrow \Delta_\emptyset \longrightarrow \frac{\mathcal{U}_\emptyset}{\mathcal{U}_\emptyset^p\mathcal{U}_{S,\nu}} \longrightarrow \frac{J_K}{\mathcal{U}_{S,\nu}J_K^p} \longrightarrow \frac{J_K}{K^\times\mathcal{U}_\emptyset J_K^p} \simeq \mathrm{Cl}_K/\mathrm{Cl}_K^p \longrightarrow 1.$$

Moreover, it is easy to establish that $p\text{-rk}\Delta_K = p\text{-rk}\mathrm{Cl}_K + p\text{-rk}E_K$, using the obvious map from $\Delta$ to the ideal classes killed by $p$. Putting all of this together, we obtain the desired formula. □

**Corollary 16.** *The genertor rank $d_S$ of $G_S$ satisfies:*

$$d_S = p\text{-rk}\Delta_S + \sum_{\mathfrak{p}\in S-S_p}\delta_\mathfrak{p} + \sum_{\mathfrak{p}\in S} p\text{-rk}U_\mathfrak{p}^{(1)} - p\text{-rk}E_K.$$

*Proof.* For $\mathfrak{p}\in S$, take $\nu_\mathfrak{p} = \infty$. This is the well-known formula of Shafarevich [Sh]. □

## 4. Behavior of root discriminant

We now suppose we have a finitely indexed set $(S,\nu)$ of $K$. We will give an upper bound for the root discriminant of the number fields in the tower $K_{S,\nu}/K$.

**Definition 17.** *Let $K$ be a number field of degree $n$, and discriminant $\mathrm{disc}_K$. The root discriminant $\mathrm{rd}_K$ of $K$ is defined by*

$$\mathrm{rd}_K = |\mathrm{disc}_K|^{1/n}.$$

**Theorem 18.** *In the $p$-extension $K_{S,\nu}/K$, the root discriminant is bounded. More precisely, if $L$ is a finite extension of $K$ contained in $K_{S,\nu}$, then*

$$\mathrm{rd}_L \leq \mathrm{rd}_K \prod_{\mathfrak{p}\in S-S_{\mathfrak{p}}} (N_{K/\mathbb{Q}}\mathfrak{p})^{1/[K:\mathbb{Q}]} \prod_{\mathfrak{p}\in S\cap S_{\mathfrak{p}}} (N_{K/\mathbb{Q}}\mathfrak{p})^{(\nu_{\mathfrak{p}}+1)/[K:\mathbb{Q}]}.$$

*Proof.* It suffices to consider $K \subseteq L \subseteq K_{S,\nu}$ such that $L/K$ is Galois. Fix one of the $r_{\mathfrak{p}}$ primes $\mathfrak{P}$ of $L$ dividing $\mathfrak{p} \in S$; since $L/K$ is Galois, the ramification index $e_{\mathfrak{P}/\mathfrak{p}} = e_{\mathfrak{p}}$ and residue degree $f_{\mathfrak{P}/\mathfrak{p}} = f_{\mathfrak{p}}$ depend only on $\mathfrak{p}$. Let $\varphi = \varphi_{\mathfrak{P}/\mathfrak{p}}$, $\psi = \psi_{\mathfrak{P}/\mathfrak{p}}$.

By the restriction property, $D^{\nu_{\mathfrak{p}}}(\mathfrak{P}/\mathfrak{p})$ is trivial for all $\mathfrak{p}$ in $S$; moreover, $L/K$ is unramified outside $S$. By definition,

$$D^{\nu_{\mathfrak{p}}}(\mathfrak{P}/\mathfrak{p}) = D_{\psi(\nu_{\mathfrak{p}})}(\mathfrak{P}/\mathfrak{p}) = \{1\}.$$

Let $n = \psi(\nu_{\mathfrak{p}})$ and $m = \lfloor n \rfloor$. The valuation $v_{\mathfrak{P}}(\mathfrak{d}_{L/K})$ at $\mathfrak{P}$ of the different of $L/K$ is given by Hilbert's formula [Se1]:

$$v_{\mathfrak{P}}(\mathfrak{d}_{L/K}) = g_0 + g_1 + \cdots + g_{m-1} - m,$$

where $g_j = |D_j(\mathfrak{P}/\mathfrak{p})|$; note that $g_j = 1$ for $j \geq m$. In particular, for $\mathfrak{p} \in S - S_p$ the ramification is tame, so we have

$$v_{\mathfrak{P}}(\mathfrak{d}_{L/K}) = e_{\mathfrak{p}} - 1.$$

From the definitions of $\psi, \varphi$, one has:

$$\nu_{\mathfrak{p}} = \varphi(n) = \frac{g_1 + \cdots + g_m + (n-m)}{g_0}.$$

Moreover,

$$\begin{aligned} v_{\mathfrak{P}}(\mathfrak{d}_{L/K}) &= g_0 + g_1 + \cdots + g_m - (m+1) \\ &= g_0(\nu_{\mathfrak{p}} + 1) - (m+1) - (n-m) \\ &= g_0(\nu_{\mathfrak{p}} + 1) - (n+1) \\ &\leq (e_{\mathfrak{p}} - 1)(\nu_{\mathfrak{p}} + 1), \end{aligned}$$

since $g_0 = e_{\mathfrak{P}/\mathfrak{p}}$ and $\nu_{\mathfrak{p}} = \varphi(n) \leq n$.

Putting the local data together and recalling that $e_{\mathfrak{p}} f_{\mathfrak{p}} r_{\mathfrak{p}} = [L:K]$, we have

$$\begin{aligned} N_{L/K}\mathfrak{d}_{L/K}^{1/[L:K]} &\leq \prod_{\mathfrak{p}\in S-S_p} \mathfrak{p}^{(e_{\mathfrak{p}}-1)f_{\mathfrak{p}}r_{\mathfrak{p}}/[L:K]} \prod_{\mathfrak{p}\in S\cap S_p} \mathfrak{p}^{(\nu_{\mathfrak{p}}+1)(e_{\mathfrak{p}}-1)f_{\mathfrak{p}}r_{\mathfrak{p}}/[L:K]} \\ &\leq \prod_{\mathfrak{p}\in S-S_p} \mathfrak{p}^{(1-1/e_{\mathfrak{p}})} \prod_{\mathfrak{p}\in S\cap S_p} \mathfrak{p}^{(\nu_{\mathfrak{p}}+1)(1-1/e_{\mathfrak{p}})}. \end{aligned}$$

We are now done, since

$$\mathrm{rd}_L = \mathrm{rd}_K N_{L/\mathbb{Q}}(\mathfrak{d}_{L/K})^{1/[L:\mathbb{Q}]} = \mathrm{rd}_K N_{K/\mathbb{Q}}(N_{L/K}(\mathfrak{d}_{L/K})^{1/[L:K]})^{1/[K:\mathbb{Q}]}.$$

□

*Remark* 4.1. For the analogue of Perret's tower in the number field case, one has the following root discriminant bound:

$$\mathrm{rd}_L \leq \mathrm{rd}_K \prod_{\mathfrak{p}\in S} (N_{K/\mathbb{Q}}\mathfrak{p})^{\lceil \nu_{\mathfrak{p}} \rceil/[K:\mathbb{Q}]}.$$

The details, which are essentially in [P], are left to the reader.

## 5. The relation-rank of $G_S$ and $G_{S,\nu}$

We put

$$r_S = \dim_{\mathbb{F}_p} H^2(G_S, \mathbb{F}_p), \qquad r_{S,\nu} = \dim_{\mathbb{F}_p} H^2(G_{S,\nu}, \mathbb{F}_p),$$

for the relation-rank of $G_S$ and $G_{S,\nu}$, respectively. An estimate for $r_S$ was given by Shafarevich [Sh] in 1963. It turns out that estimating $r_{S,\nu}$ is a more delicate problem than estimating $r_S$. In fact, using results of Gordeev and Wingberg, we can find examples, with $\nu_{\mathfrak{p}}$ finite for some $\mathfrak{p}$ and infinite for others, where $G_{S,\nu}$ is not finitely presentable! The main problem, then, is

*Question* 1. a) Do we have $r_{S,\nu} < \infty$ for finitely indexed sets $(S,\nu)$?

b) If so, give an explicit upper bound $r_{S,\nu} \leq f(S,\nu)$.

Below, we will first indicate briefly why the proof of Shafarevich does not easily generalize to our case, and then we will give examples with $r_{S,\nu} = \infty$ (where $(S,\nu)$ is not finitely indexed), which leads to a family of examples showing that $r_{S,\nu}$ can tend to infinity for finitely indexed sets $S,\nu$, where $S$ is fixed, but $\max_{\mathfrak{p}\in S} \nu_{\mathfrak{p}}$ tends to infinity.

5.1. **Shafarevich's relation-rank bound.** The fundamental artithmetic result about $G_S$ is an estimate for its relation rank due to Shafarevich [Sh].

**Theorem 19.** *We have*

$$r_S - d_S \leq p\text{-rk} E_K - \delta_K + \theta_S - \sum_{\mathfrak{p}\in S_p} [K_{\mathfrak{p}} : \mathbb{Q}_p].$$

We note in passing that when $S_p \subseteq S$, we in fact know the relation rank exactly: $r_S - d_S = -(r_2+1)$ where $r_2$ is the number of imaginary places of $K$ (cf. [H] and, for $p=2$, [Sch]).

We will sketch the proof of Theorem 19 given by Koch (cf. [K1], [K2]), indicating a difficulty one encounters in attempting to generalize it by replacing $S$ with $(S,\nu)$.

Recall the group of $(S,\nu)$-unit idèles is

$$\mathcal{U}_{S,\nu} = \prod_{\mathfrak{p}} U_{\mathfrak{p}}^{(\nu_{\mathfrak{p}})},$$

where we have extended $\nu$ to all places by setting $\nu_{\mathfrak{p}} = 0$ for $\mathfrak{p} \notin S$.

The local reciprocity map $\rho_{\mathfrak{p}}$ for a local Galois extension $L_{\mathfrak{P}}/K_{\mathfrak{p}}$ is defined on $L_{\mathfrak{P}} \cap K_{\mathfrak{p}}^{ab}$, i.e in $D(\mathfrak{P}/\mathfrak{p})/[D(\mathfrak{P}/\mathfrak{p}), D(\mathfrak{P}/\mathfrak{p})]$. Moreover, by the restriction property of higher ramification groups, we have

$$D^{\nu_{\mathfrak{p}}}(L_{\mathfrak{P}} \cap K_{\mathfrak{p}}^{ab}/K_{\mathfrak{p}}) = \frac{D^{\nu_{\mathfrak{p}}}(\mathfrak{P}/\mathfrak{p})}{D^{\nu_{\mathfrak{p}}}(\mathfrak{P}/\mathfrak{p}) \cap [D(\mathfrak{P}/\mathfrak{p}), D(\mathfrak{P}/\mathfrak{p})]}.$$

Also, $\rho_{\mathfrak{p}}(U_{\mathfrak{p}}^{(\nu_{\mathfrak{p}})}) = D^{\nu_{\mathfrak{p}}}(L_{\mathfrak{P}} \cap K_{\mathfrak{p}}^{ab}/K_{\mathfrak{p}})$.

Since $D(\mathfrak{P}/\mathfrak{p})/D^0(\mathfrak{P}/\mathfrak{p})$ is cyclic, we have

$$[D(\mathfrak{P}/\mathfrak{p}), D(\mathfrak{P}/\mathfrak{p})] = [D(\mathfrak{P}/\mathfrak{p}), D^0(\mathfrak{P}/\mathfrak{p})].$$

This seemingly minor simplification will turn out to be quite important, as we will see in a moment. Recall that $K_S$ is the maximal $p$-extension of $K$ unramified outside $S$, with Galois group $G_S = \mathrm{Gal}(K_S/K)$; it will be convenient here to think of $K_S$ as $K_{S,\nu}$ with $\nu_{\mathfrak{p}} = \infty$ for $\mathfrak{p} \in S$.

Let $\mathcal{G}$ be the Galois group over $K$ of its maximal $p$-extension $\hat{K}$. We let $D_\mathfrak{p}$ be the Galois group of the maximal $p$-extension $\hat{K}_\mathfrak{p}$ of $K_\mathfrak{p}$. Let $N_{S,\nu}$ be the closed normal subgroup of $\mathcal{G}$ generated by all higher ramification groups $D_\mathfrak{p}^{\nu_\mathfrak{p}}$, for $\mathfrak{p} \in S$.

**Definition 20.** *Let* $B_{S,\nu} = \ker\left(H^2(G_{S,\nu}, \mathbb{F}_p) \to H^2(\mathcal{G}, \mathbb{F}_p)\right)$.

The exact sequence

$$1 \longrightarrow N_{S,\nu} \longrightarrow \mathcal{G} \longrightarrow G_{S,\nu} \longrightarrow 1$$

gives, via Hochschild-Serre, the exact sequence

$$\{1\} \longrightarrow B_{S,\nu}^* \longrightarrow \frac{N_{S,\nu}}{N_{S,\nu}^p[\mathcal{G}, N_{S,\nu}]} \xrightarrow{\quad\alpha\quad} \frac{\mathcal{G}}{\mathcal{G}^p[\mathcal{G},\mathcal{G}]},$$

allowing us to identify $\ker(\alpha)$ with the dual of $B_{S,\nu}$.

**Proposition 21.** *1) There is a natural injection*

$$H^2(\mathcal{G}, \mathbb{F}_p) \hookrightarrow \oplus_{\mathfrak{p}\notin S} H^2(D_\mathfrak{p}, \mathbb{F}_p) \oplus_{\mathfrak{p}\in S} H^2(D_\mathfrak{p}, \mathbb{F}_p).$$

*Moreover if $K$ contains $\mu_p$, we can omit an arbitrary place in the sum.*

*2) If $K_\mathfrak{p}$ does not contain $\mu_p$, $H^2(D_\mathfrak{p}, \mathbb{F}_p)$ is trivial.*

*3) If $K_\mathfrak{p}$ contains $\mu_p$, $D_\mathfrak{p}$ is a group with only one relation. So $p$-rk$H^2(D_\mathfrak{p}, \mathbb{F}_p) = 1$*

*4) In the composite map*

$$H^2(G_{S,\nu}, \mathbb{F}_p) \longrightarrow H^2(\mathcal{G}, \mathbb{F}_p) \longrightarrow \oplus_{\mathfrak{p}\notin S} H^2(D_\mathfrak{p}, \mathbb{F}_p) \oplus_{\mathfrak{p}\in S} H^2(D_\mathfrak{p}, \mathbb{F}_p),$$

*the image of $H^2(G_{S,\nu}, \mathbb{F}_p)$ in $\oplus_{\mathfrak{p}\notin S} H^2(D_\mathfrak{p}, \mathbb{F}_p)$ is trivial.*

*Proof.* This is proved in [K2]; for part 4), Koch treats only $G_S$ but the same argument applies to $G_{S,\nu}$. See also [Se2], [Sh], [K1]) □

We have therefore proved

**Theorem 22.** *For a finite set $S$ of places of $K$ and an arbitrary indexing function $\nu$ of $S$, we have*

$$r_{S,\nu} = p\text{-rk}H^2(G_{S,\nu}, \mathbb{F}_p) \le \sum_{\mathfrak{p}\in S} \delta_\mathfrak{p} - \delta_K + \theta_S + p\text{-rk}B_{S,\nu}.$$

Thus, in order to complete the proof of theorem 19, (or to answer Question 1), it suffices to bound $p$-rk$B_{S,\nu}$ from above. Let us attempt to follow the proof given by Shafarevich and Koch for bounding $B_S$ by relating it to the kernel of the natural map

$$\eta : \mathcal{U}_{S,\nu}/\mathcal{U}_{S,\nu}^p \longrightarrow J_K/K^\times J_K^p.$$

Let us observe, first of all, that $\ker(\eta)$ is under control.

**Lemma 23.** *We have*

$$p\text{-rk}\ker(\eta) \le p\text{-rk}E_K + d_{S,\nu} + \sum_{\substack{\mathfrak{p}\in S\cap S_p \\ \nu_\mathfrak{p}\neq\infty}} [K_\mathfrak{p} : \mathbb{Q}_p] + \delta(\mathfrak{p}, \nu_\mathfrak{p}) - \sum_{\mathfrak{p}\in S - S_p} \delta_\mathfrak{p} - \sum_{\mathfrak{p}\in S\cap S_p} p\text{-rk}\frac{U_\mathfrak{p}^{(1)}}{U_\mathfrak{p}^{(\nu_\mathfrak{p})}},$$

*where $\delta(\mathfrak{p}, \nu_\mathfrak{p})$ is 1 if $\mu_p \subseteq U_\mathfrak{p}^{(\nu_\mathfrak{p})}$ and is 0 otherwise.*

*Proof.* There is an exact sequence

$$\{1\} \longrightarrow \frac{\mathcal{U}_{S,\nu} \cap U_\emptyset^p}{U_{S,\nu}^p} \longrightarrow \ker(\eta) \xrightarrow{\phi} \Delta_{S,\nu} \longrightarrow \{1\}.$$

where $\phi$ is defined as follows. We have

$$\ker(\eta) = \frac{\mathcal{U}_{S,\nu} \cap K^\times J_K^p}{\mathcal{U}_{S,\nu} \cap \mathcal{U}_\emptyset^p}.$$

Let $u \in \mathcal{U}_{S,\nu} \cap K^\times J_K^p$; write $u = x.j^p$ with $x \in K^\times$, $j \in J_K$. We let $\phi(u)$ be the image of $x$ in $\Delta_{S,\nu}$. The kernel of $\phi$ is easily seen to be $(\mathcal{U}_{S,\nu} \cap U_\emptyset^p)/U_{S,\nu}^p$, which is trivial when $\nu(S) = \{0, \infty\}$, and whose $p$-rank in general is bounded by

$$\begin{aligned} p\text{-rk}\ker(\phi) &= \sum_{\mathfrak{p} \in S \cap S_p} p\text{-rk} U_\mathfrak{p}^{(\nu_\mathfrak{p})} \cap U_\mathfrak{p}{}^p \\ &\leq \sum_{\mathfrak{p} \in S \cap S_p} p\text{-rk} U_\mathfrak{p}^{(\nu_\mathfrak{p})} = \sum_{\substack{\mathfrak{p} \in S \cap S_p \\ \nu_\mathfrak{p} \neq \infty}} [K_\mathfrak{p} : \mathbb{Q}_p] + \delta(\mathfrak{p}, \nu_\mathfrak{p}). \end{aligned}$$

We now only need apply the formula of Theorem 15. □

Now, there are natural surjections

$$\mathcal{U}_{S,\nu} \xrightarrow{\prod_\mathfrak{p} \rho_\mathfrak{p}} \prod_\mathfrak{p} \frac{D_\mathfrak{p}^{\nu_\mathfrak{p}}}{[D_\mathfrak{p}, D_\mathfrak{p}] \cap D_\mathfrak{p}^{\nu_\mathfrak{p}}} \xleftarrow{\beta} \prod_\mathfrak{p} \frac{D_\mathfrak{p}^{\nu_\mathfrak{p}}}{[D_\mathfrak{p}, D_\mathfrak{p}^{\nu_\mathfrak{p}}]} \longrightarrow \frac{N_{S,\nu}}{[\mathcal{G}, N_{S,\nu}]}. \tag{1}$$

Note that $\beta$ is "pointing the wrong way." In the classical case (where $\nu_\mathfrak{p}$ is either 0 or $\infty$), $\beta$ is an isomorphism ($[D_\mathfrak{p}, D_\mathfrak{p}] = [D_\mathfrak{p}, D_\mathfrak{p}^0]$), so we can define a surjective global map

$$\mathcal{U}_S \xrightarrow{\rho} N_S/N_S^p[\mathcal{G}, N_S]).$$

We then have a commutative diagram:

$$\begin{tikzcd} B_S^* \arrow[r, hook] & \dfrac{N_S}{N_S^p[\mathcal{G}, N_S] \cap N_S} \arrow[r, "\alpha"] & \dfrac{\mathcal{G}}{\mathcal{G}^p[\mathcal{G}, \mathcal{G}]} \\ \ker(\eta) \arrow[u, "\exists \rho''"] \arrow[r, hook] & \dfrac{\mathcal{U}_S}{\mathcal{U}_S^p} \arrow[u, "\rho"] \arrow[r, "\eta"] & \dfrac{J_K}{K^\times J_K^p} \arrow[u, "\rho'"] \end{tikzcd}$$

where $\rho$ is as defined above. As $\rho'$ is injective, there is a natural surjection $\rho'' : \ker(\eta) \longrightarrow B_{S,\nu}^*$. Putting this together with Theorem 22 and Lemma 23, we have proved theorem 19.

To summarize, in the general case, what is lacking is a bound for the dimension of $B_{S,\nu}$, for which it would suffice to have a natural map $\rho^? : \mathcal{U}_{S,\nu}/\mathcal{U}_{S,\nu}^p \longrightarrow N_{S,\nu}/N_{S,\nu}^p[\mathcal{G}, N_{S,\nu}]$ with finite cokernel. The classical approach sketched above does not work because the map $\beta$ (an isomorphism in the classical case) has infinite cokernel in general, by the following remarkable theorem of Gordeev [G].

**Theorem 24** ([G]). *Let $K_\mathfrak{p}$ be a finite extension of $\mathbb{Q}_p$. Let $\hat{K}_\mathfrak{p}$ be the maximal $p$-extension of $K_\mathfrak{p}$ and put $D_\mathfrak{p} = \mathrm{Gal}(\hat{K}_\mathfrak{p}/K_\mathfrak{p})$. For $\nu_\mathfrak{p} > 1$,*

$$r(D_\mathfrak{p}/D_\mathfrak{p}^{\nu_\mathfrak{p}}) = \dim_{\mathbb{F}_p} H^2(D_\mathfrak{p}/D_\mathfrak{p}^{\nu_\mathfrak{p}}) = \infty.$$

**Corollary 25.** *The map $\beta$ of (1) has infinite cokernel. Indeed, with hypotheses as in the theorem,*

$$\dim_{\mathbb{F}_p} \frac{D_{\mathfrak{p}}^{\nu_{\mathfrak{p}}}}{[D_{\mathfrak{p}}, D_{\mathfrak{p}}^{\nu_{\mathfrak{p}}}](D_{\mathfrak{p}}^{\nu_{\mathfrak{p}}})^p} = \infty, \qquad \dim_{\mathbb{F}_p} \frac{D_{\mathfrak{p}}^{\nu_{\mathfrak{p}}}}{[D_{\mathfrak{p}}, D_{\mathfrak{p}}] \cap D_{\mathfrak{p}}^{\nu_{\mathfrak{p}}}(D_{\mathfrak{p}}^{\nu_{\mathfrak{p}}})^p} \leq [K_{\mathfrak{p}} : \mathbb{Q}_p] + 1.$$

*Proof.* Let us take the exact sequence

$$1 \longrightarrow D_{\mathfrak{p}}^{\nu_{\mathfrak{p}}} \longrightarrow D_{\mathfrak{p}} \longrightarrow D_{\mathfrak{p}}/D_{\mathfrak{p}}^{\nu_{\mathfrak{p}}} \longrightarrow 1.$$

Applying the Hochschild-Serre sequence, we get

$$\left(\ker\left(H^2(D_{\mathfrak{p}}/D_{\mathfrak{p}}^{\nu_{\mathfrak{p}}}, \mathbb{F}_p) \to H^2(D_{\mathfrak{p}}, \mathbb{F}_p)\right)\right)^* \hookrightarrow \frac{D_{\mathfrak{p}}^{\nu_{\mathfrak{p}}}}{[D_{\mathfrak{p}}, D_{\mathfrak{p}}^{\nu_{\mathfrak{p}}}](D_{\mathfrak{p}}^{\nu_{\mathfrak{p}}})^p} \longrightarrow \cdots$$

But $D_{\mathfrak{p}}$ is a $p$-group with $p\text{-rk}H^2(D_{\mathfrak{p}}, \mathbb{F}_p) = 0$ or 1. Moreover by Theorem 24, $D_{\mathfrak{p}}/D_{\mathfrak{p}}^{\nu_{\mathfrak{p}}}$ has infinite relation-rank. Then the $p$-rank of $\frac{D_{\mathfrak{p}}^{\nu_{\mathfrak{p}}}}{[D_{\mathfrak{p}}, D_{\mathfrak{p}}^{\nu_{\mathfrak{p}}}](D_{\mathfrak{p}}^{\nu_{\mathfrak{p}}})^p}$ is infinite. The quotient $\frac{D_{\mathfrak{p}}^{\nu_{\mathfrak{p}}}}{[D_{\mathfrak{p}}, D_{\mathfrak{p}}] \cap D_{\mathfrak{p}}^{\nu_{\mathfrak{p}}}(D_{\mathfrak{p}}^{\nu_{\mathfrak{p}}})^p}$ is the restriction of $D_{\mathfrak{p}}^{\nu_{\mathfrak{p}}}$ to the maximal abelian $p$-extension of $K_{\mathfrak{p}}$, and so has $p$-rank at most $[K_{\mathfrak{p}} : \mathbb{Q}_p] + 1$. $\square$

5.2. **Not all $G_{S,\nu}$ are finitely presentable.** The groups $G_{S,\nu}$ are finitely generated, so they are finitely presentable in the category of pro-$p$ groups if and only if they have finite Euler characteristic $\chi_2(G_{S,\nu}) = r_{S,\nu} - d_{S,\nu}$. Note that, for *finitely indexed* $(S, \nu)$, $\chi_2(G_{S,\nu})$ is non-negative since $G_{S,\nu}$ has finite abelianization in that case.

The main result of this part gives infinitely many examples (with $p = 2$, $\nu_{\mathfrak{p}}$ finite for some $\mathfrak{p}$ but infinite for others) for which $r_{S,\nu} = \chi_2(G_{S,\nu}) = \infty$.

**Theorem 26.** *Let $p = 2$, $q$ is a prime with $q \equiv 7 \ (mod\ 16)$. Put $K = \mathbb{Q}(\sqrt{-q})$, in which 2 splits: $2\mathcal{O}_K = \mathfrak{p}_1\mathfrak{p}_2$. Let $i$ be a positive integer, and define $\nu : S = \{\mathfrak{p}_1, \mathfrak{p}_2\} \to [0, \infty]$ by $\nu(\mathfrak{p}_1) = i, \nu(\mathfrak{p}_2) = \infty$. Then $r(G_{S,\nu}) = \infty$.*

*Remark* 5.1. $(i)$ If we take $\nu(\mathfrak{p}_1) = \nu(\mathfrak{p}_2) = 2$, then $G_{S,\nu}$ is finite for $q = 7$ by Theorem 18 and discriminant bounds. Assuming the Generalized Riemann Hypothesis, the same is true for $\nu(\mathfrak{p}_1) = \nu(\mathfrak{p}_2) = 3$.

$(ii)$ Note that $G_{S,\nu}$ (with $S, \nu$ as in the theorem) has $G_{\{\mathfrak{p}_2\}}$ has as a quotient, which in turn has $\mathbb{Z}_2$ as a quotient.

To show Theorem 26 we have to recall a result of Wingberg [W].

**Definition 27.** *Let $K$ be a number field, and $L/K$ be a pro-$p$-extension. We will say that $L/K$ is maximal local at $\mathfrak{p}$ if the natural composite map:*

$$D_{\mathfrak{p}} \overset{res}{\to} \mathrm{Gal}(L_{\mathfrak{P}}/K_{\mathfrak{p}}) \hookrightarrow \mathrm{Gal}(L/K),$$

*is an isomorphism.*

**Theorem 28** ([W])**.** *Let $S$ be a finite set of places of $K$ containing all the places above $p$ and the archimedean places. Let $K_S$ be the maximal $p$-extension of $K$ unramified outside $S$. Then, $K_S/K$ is maximal local at $\mathfrak{p} \in S_p$ if and only if the following conditions are satisfied:*

1. *for $p = 2$, $K$ is totally imaginary,*
2. $\sum_{\mathfrak{q} \in S \setminus \{\mathfrak{q}\}} \delta_{\mathfrak{q}} = \delta_K$,

3. $\Delta^{\{\mathfrak{p}\}}_{S-\{\mathfrak{p}\}} = \{1\}$,
4. $r_2 = [K_{\mathfrak{p}} : \mathbb{Q}_p]$, *where* $r_2$ *is the number of imaginary places of* $K$.

**Corollary 29.** *With notation as above, and* $p = 2$, $K_S/K$ *is maximal local at* $\mathfrak{p} \in S_p$ *if and only if*

1. $K$ *is totally imaginary;*
2. $S = \{\mathfrak{p}, \mathfrak{q}\}$ *where* $2\mathcal{O}_K = \mathfrak{p}\mathfrak{q}$*;*
3. *There does not exist a non-trivial* 2*-extension of* $K$ *unramified outside* $\mathfrak{p}$ *and in which* $\mathfrak{q}$ *splits,*
4. $r_2 = [K_{\mathfrak{p}} : \mathbb{Q}_p]$.

*Proof.* There exists a non-trivial 2-extension of $K$, $\mathfrak{q}$-decomposed and unramified oustide $\mathfrak{p}$, if and only if $\Delta^{\{\mathfrak{q}\}}_{S-\{\mathfrak{q}\}} \neq \{1\}$, because $K$ contains $\mu_p$ (by Proposition 1). □

We need to verify the above properties for the fields appearing in Theorem 26. We do so in the following Proposition, which can be proved directly via a long calculation; we will instead derive it more easily by making use of a result in a book of Gras to appear [Gr2].

**Proposition 30.** *Let* $K = \mathbb{Q}(\sqrt{-q})$ *where* $q \equiv 7 \bmod 8$ *is a prime,* $2\mathcal{O}_K = \mathfrak{p}_1\mathfrak{p}_2$. *There exists a non-trivial* 2*-extension of* $K$ *unramified outside* $\mathfrak{p}_1$ *and for which* $\mathfrak{p}_2$ *splits completely if and only if* $p \equiv -1 \bmod 16$.

*Proof.* Let $n$ be the order of $\mathfrak{p}_1$ in $\mathrm{Cl}_K$: note that $n$ is odd. There exists integers $a$ and $b$, such that $\alpha = (a+b\sqrt{-q})/2$ ($a, b$ have the same parity, and are in fact odd) generates $\mathfrak{p}_1^n$: ${\mathfrak{p}_1}^n = (\alpha)$. Call the non-trivial element of $\mathrm{Gal}(K/\mathbb{Q})$ $\tau$, so that $\alpha^\tau = (a - b\sqrt{-q})/2$ and ${\mathfrak{p}_2}^n = (\alpha^\tau)$. We can handle the case $q = 7$ by hand. Now suppose $q > 7$. We find a sequence of equivalent conditions as follows.

**Claim.** For $q > 7$, the following are equivalent:

1. There exists non trivial 2-extension of $K$ unramified outside $\mathfrak{p}_1$ and for which $\mathfrak{p}_2$ splits completely,
2. $[K_{\mathfrak{p}_1}(\sqrt{-1}, \sqrt{\alpha^\tau}) : K_{\mathfrak{p}_1}] = 2$,
3. $(2, \alpha^\tau)_{\mathfrak{p}_1} = 1$,
4. $\alpha^\tau \equiv \pm 1 \pmod{\mathfrak{p}_1^3}$,
5. $a^2 \equiv 1 \bmod 16$,
6. $q \equiv 15 \bmod 16$.

*Proof of Claim.* The equivalence of 1 and 2 is from Gras' book (Chapter V). We then have $\alpha^\tau \in K_{\mathfrak{p}_1}(\sqrt{-1}) \iff \alpha^\tau \equiv \pm 1 \pmod{\mathfrak{p}_1^3} \iff (2, \alpha^\tau)_{\mathfrak{p}_1} = 1$, because $\alpha^\tau$ is a local unit in $\mathfrak{p}_1$ (see [Se1] for example). Now condition 4 is equivalent to $\alpha \equiv 0 \pmod{\mathfrak{p}_1^3}$ or $\alpha + \alpha^\tau \equiv \pm 1 \pmod{\mathfrak{p}_1^3}$, i.e. to $a \equiv \pm 1 \pmod 8$ or $a^2 \equiv 1 \bmod 16$. For the final step, we want to show that this is equivalent to $q \equiv 15 \bmod 16$. We take the norm of $2\alpha$: $4\mathbb{N}\alpha = a^2 + b^2 q = 2^{n+2}$, to see that $2^{n+2}$ is a square mod$b$, and as $n \geq 1$ is odd, 2 is a square mod $b$. Now by the quadratic reciprocity law, 2 is a square $\pmod b$ implies $b \equiv \pm 1 \mathrm{mod} 8$. As $q > 7$, 2 could not be a norm in $\mathcal{O}_K/\mathbb{Z}$, so $n \geq 3$. Now $a^2 + b^2 q = 2^{n+2}$ gives $a^2 + b^2 q \equiv 0 \pmod{16}$ or $a^2 + q \equiv 0 \pmod{16}$. This concludes the proof of the claim, of Proposition 30 and Theorem 26. □

*Proof.* [Theorem 26] With $S = \{\mathfrak{p}_1, \mathfrak{p}_2\}$, $K_S/K$ is maximal local at $\mathfrak{p}_2$ by Proposition 30 and Corollary 29. Thus, the global Galois group $G_{S,\nu}$ is isomorphic to the local group appearing in Gordeev's Theorem 24, which has infinite relation rank. $\square$

5.3. **Variation of $r_{S,\nu}$ for fixed $S$.** Let us consider what happens as we vary $\nu$ for $K = \mathbb{Q}(\sqrt{-q})$, $S = \{\mathfrak{p}_1, \mathfrak{p}_2\}$ as above. We introduce some notation. For integers $i, j > 1$, let $\nu_{i,j} : S = \{\mathfrak{p}_1, \mathfrak{p}_2\} \to [0, \infty)$ be defined by $\nu_{i,j}(\mathfrak{p}_1) = i$, $\nu_{i,j}(\mathfrak{p}_2) = j$. For simplification, we write

$$G_{i,j} = G_{S,\nu_{i,j}}.$$

**Definition 31.** *For a finitely generated pro-p-group $G$, the partial Euler-Poincare characteristic $\chi_2(G)$ is defined by :*

$$\chi_2 = p\text{-rk}H^2(G, \mathbb{F}_p) - p\text{-rk}H^1(G, \mathbb{F}_p) = r(G) - p\text{-rk}G.$$

**Corollary 32.** *For fixed $i$, we have $\sup_j r(G_{i,j}) = \sup_j \chi_2(G_{i,j}) = \infty$.*

*Proof.* We have $G_{i,\infty} = \varprojlim_j G_{i,j}$. Consequently,

$$H^2(G_{i,\infty}, \mathbb{F}_p) = \varinjlim_j H^2(G_{i,j}, \mathbb{F}_p).$$

If for some $j$, $r(G_{i,j}) = \infty$, then there is nothing to prove. Now suppose $r(G_{i,j}) \leq M < \infty$ for all $j$. Then $H^2(G_{i,\infty}, \mathbb{F}_p)$ is a direct limit of abelian groups of cardinality at most $M$, hence is finite. This contradicts Theorem 26 ($r(G_{i,\infty}) = \infty$). The claim for $\sup_j \chi_2(G_{i,j})$ follows since $d(G_{i,j})$ is bounded by $d(G_{i,\infty})$. $\square$

Note that $\sup_j(G_{i,j}) = \infty$ implies that $G$ is not analytic. This is a particular instance of the following theorem.

**Theorem 33.** *Let $\Gamma = \varprojlim_i \Gamma_i$ be a finitely generated projective limit of pro-p-groups $\Gamma_i$, such that the natural map of restriction from $\Gamma$ to $\Gamma_i$ is surjective for sufficiently large $i$. Assume that $\sup_i \chi_2(\Gamma_i) = \infty$. Then $\Gamma$ is not p-adic analytic.*

*Proof.* The homomorphism of restriction from $\Gamma$ to $\Gamma_i$ is continuous and the kernel $H_i$ is the inverse image of a closed subgroup of $\Gamma_i$ and so is closed in $\Gamma$. Thus, for large $i$, we have:

$$1 \longrightarrow H_i \longrightarrow \Gamma \longrightarrow \Gamma_i \longrightarrow 1.$$

By using the Hochschild-Serre sequence plus duality we have:

$$\left(\ker(H^2(\Gamma_i, \mathbb{F}_p) \to H^2(\Gamma, \mathbb{F}_p))\right)^* \hookrightarrow \frac{H_i}{H_i^p[\Gamma, H_i]} \longrightarrow \frac{\Gamma}{\Gamma^p[\Gamma, \Gamma]} \longrightarrow \frac{\Gamma_i}{\Gamma_i^p[\Gamma_i, \Gamma_i]} \longrightarrow 1.$$

Passing to the $p$-rank, we obtain:

$$\begin{aligned} p\text{-rk}H_i &\geq p\text{-rk}H_i/[H_i, \Gamma] \\ &\geq \chi_2(\Gamma_i) - \chi_2(\Gamma). \end{aligned} \tag{2}$$

If $\sup_i \chi_2(\Gamma_i) = \infty$, then $\sup_i p\text{-rk}H_i = \infty$, and as $H_i$ is closed in $\Gamma$, we find that

$$\sup\{p\text{-rk}H \mid H \subseteq \Gamma, H \text{ closed}\} = \infty.$$

By a result of Lubotzky and Mann [LM], this implies that $\Gamma$ is not analytic. $\square$

*Remark* 5.2. 1) Consider $G_S$, the Galois group over $K = \mathbb{Q}(\sqrt{-q})$ (with prime $q \equiv 7 \mod 16)$) of the maximal 2-extension unramified outside 2. Then $G_S$ is not analytic, since it has non-analytic quotients $G_{i,\infty}$. Note that we have $G_S = \varprojlim_j G_{i,\infty}$, with $\chi_2(G_{i,\infty}) = \infty$, but $\chi_2(G_S) = -(r_2+1) = -2$ is finite.

2) We give a simple example of the Theorem. Take $K = \mathbb{Q}, S = \{p\}$. Then $G_S$ is a pro-$p$ group with one generator and without relation, i.e. $G_S \simeq \mathbb{Z}_p$. For any finite indexing $\nu$ of $S$, we have $G_{S,\nu}$ is finite with vanishing Euler characteristic.

3) Theorem 33 suggests that the study of the growth of the Euler characteristic $\chi_2(\Gamma/H)$ for open subgroups $H$ of $\Gamma$, where $\Gamma = G_S$ or $\Gamma = G_{S,\nu}$ would yield interesting information about these groups. So far, one knows little about the growth of Euler characteristics for these groups. When $S = S_p$, Shafarevich's relation rank estimate together with the Golod-Shafarevich theorem show that $G_S$ is not analytic (unless it is isomorphic to $\mathbb{Z}_p$), so the Euler characteristic of finite quotients tends to infinity in that case.

# Part 3. The Fontaine-Mazur conjecture and $p$-adic representations of $G_{S,\nu}$

## 6. A Theorem of Sen

Suppose $F$ is a finite extension of $\mathbb{Q}_p$, and $E/F$ is a totally ramified normal extension such that $G = \mathrm{Gal}(E/F)$ is an infinite $p$-adic analytic group. The group $G$ has two natural filtrations given by its central series ($G_1 = G$, $G_{j+1} = [G, G_j]$), and the higher ramification groups $D^i(E/F)$. It was conjectured by Serre and proven by Sen [S] that these filtrations are closely related as follows.

**Theorem 34** (Sen)**.** *Let $F$ be a finite extension of $\mathbb{Q}_p$, with ramification index $e = e_{F/\mathbb{Q}_p}$. Suppose $E/F$ is a totally ramified $p$-extension with $p$-adic analytic Galois group $G$. Then there exists a constant $c$, such that for all $n \geq 0$,*

$$D^{ne+c}(E/F) \subseteq G_n \subseteq D^{ne-c}(E/F).$$

**Corollary 35.** *Suppose $F$ is a finite extension of $\mathbb{Q}_p$ and $E/F$ is a $p$-extension with $p$-adic analytic Galois group. Let $F^{un}$ be the maximal unramified $p$-extension of $F$, and assume that $E \cap F^{un}$ is of finite degree over $F$. Assume, moreover, that the ramification is bounded in $E/F$, i.e. there exists $i > 0$ such that $D^i(E/F) = \{1\}$. Then $E/F$ is finite.*

*Proof.* Let $F_0 = E \cap F^{un}$. Let $G = \mathrm{Gal}(E/F_0)$; $G$ is a $p$-adic analytic group. As $F_0/F$ is unramified, the groups $D^i(E/F_0)$ and $D^i(E/F)$ are the same (cf. Proposition 7), and by assumption, they vanish. By Theorem 34, there exists a constant $c$ such that

$$D^{ne+c}(E/F_0) \subseteq \mathrm{Gal}(E/F_0)_n \subseteq D^{ne-c}(E/F_0)).$$

Put $m = \lceil (i+c)/e \rceil$ and $j = me - c$. Then $j \geq i$, so $D^j(E/F_0)$ is trivial, hence $G_m$ (the $m$th term of the central series of $G = \mathrm{Gal}(E/F_0)$) is trival as well. A group with a finite central series filtration and whose open subgroups have finite abelianization must be finite, so we will know that $\mathrm{Gal}(E/F_0)$ (and hence $\mathrm{Gal}(E/F)$) is finite, once we prove the following lemma. □

**Lemma 36.** *Let $E$ be a Galois extension of local field $F$, totally ramified and with bounded ramification. Then for every open subgroup $H$ of $\mathrm{Gal}(E/F)$, $H^{ab}$ is finite.*

*Proof.* By Proposition 10, $E/E^H$ is totally ramified and has bounded ramification, so it suffices to prove that $G = \mathrm{Gal}(E/F)$ has finite abelianization. Let $F_1$ be the fixed field of $[G, G]$. By assumption, there exists $i$ such that $D^i(E/F) = \{1\}$. By the property of restriction, $D^i(F_1/F) = \{1\}$. As $E/F$ is totally ramified, $F_1/F$ is totally ramified (and abelian). By class field theory, $G^{ab}$ is generated by the image of the units $U$ of $F$ under the local reciprocity map $\rho$, and the image of $U^i$ is precisely $D^i(F_1/F)$, which is trivial. We have a map, therefore,

$$U/U^i \xrightarrow{\rho} G^{ab},$$

which implies the finiteness of $G^{ab}$. □

## 7. The Conjecture of Fontaine-Mazur

Let us call a finitely generated pro-$p$ group *$p$-adically finite* if it has no infinite $p$-adic analytic quotients. In other words, all $p$-adic representations of it into $GL_n(\mathbb{Q}_p)$ have open kernel. Recall the conjecture of Fontaine-Mazur.

*Conjecture* 1 ([FM], Conjecture 5a). Let $K$ be a number field, $S$ a finite set of places of $K$ disjoint from $S_p$. Then $G_S$ is $p$-adically finite.

For partial results on this topic, see Boston [B1], [B2], [B3] and Hajir [Ha]. The above conjecture of Fontaine and Mazur is only a special case of their vast program for characterizing the representations which "come from algebraic geometry," meaning those isomorphic to a subquotient of the action of $\mathrm{Gal}(\overline{K}/K)$ on an étale cohomology group $H^q_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r))$, where $X$ is a smooth projective variety over $K$. They call a $p$-adic representation of $\mathrm{Gal}(\overline{K}/K)$ "geometric" if is satisfies two conditions: 1) it is unramified outside a finite set of places of $K$, and 2) its restriction to every decomposition group is potentially semistable. Their main conjecture, then, is that an irreducible $p$-adic representation is geometric if and only if it is potentially semistable. One direction of this conjecture ("algebro-geometric representations are potentially semistable") has a much longer history, and is better understood (see, for example, Tsuji [T]).

By a theorem of Grothendieck, a tamely ramified $p$-adic representation is always potentially semistable; thus, when $S \cap S_p = \emptyset$, every $p$-adic representation $\rho$ of $G_S$ should come from algebraic geometry, and then the Hodge and Tate conjectures imply that $\rho$ has finite image. This is equivalent to the assertion of Conjecture 1.

Given the group-theoretic (Theorem 14) and arithmetic (Theorem 18) analogies between $G_S$ (when $S \cap S_p = \emptyset$) and $G_{S,\nu}$ (for finitely indexed $(S, \nu)$), as well as Sen's Theorem and its Corollary 35, it is tempting to guess, that the property of being $p$-adically finite which the tame groups $G_S$ are predicted to enjoy (by the Fontaine-Mazur conjecture), also applies to the groups $G_{S,\nu}$. We now list this and some closely related ideas as questions. Let $(S, \nu)$ be a finitely indexed set of places of $K$.

*Question* 2. Is $G_{S,\nu}$ $p$-adically finite?

*Question* 3. Is every irreducible $p$-adic representation of $G_{S,\nu}$ everywhere potentially semistable?

*Question* 4. Does every irreducible $p$-adic representation of $\mathrm{Gal}(\overline{K}/K)$ coming from algebraic geometry and factoring through $G_{S,\nu}$ have finite image?

*Question* 5. Suppose $L/K$ is an infinite $p$-extension in which the root discriminant remains bounded. Does every irreducible $p$-adic representation of $\mathrm{Gal}(L/K)$ have finite image?

*Remark* 7.1.
1. Positive answers to Questions 3 and 4 would show that the groups $G_{S,\nu}$ with bounded ramification, and not just the tame ones, are predicted by the main conjecture of Fontaine-Mazur ([FM], Conjecture 1) to be $p$-adically finite.
2. By theorem 18, Question 5 is more general than Question 4. When $L/K$ is abelian, Question 5 has a positive answer (cf. Frey, Perret, Stichtenoth [FPS]).
3. The formulation of these questions in the introduction suggests that what is required is a better understanding the ramification in Galois extensions of global fields with $p$-adic analytic Galois group.

In the remainder of this section, we provide some partial evidence for a positive answer to Question 2.

7.1. **Global extensions of local type.** In section 6, we applied a fruitful notion introduced and studied by Wingberg [W]; here we will use further results from that paper.

**Definition 37.** *A $p$-extension $L$ of a number field $K$ is said to be of local type if there exists a place $\mathfrak{p}$ of $K$ such that the natural morphism:*

$$D_{\mathfrak{p}} \overset{res}{\to} \mathrm{Gal}(L_{\mathfrak{P}}/K_{\mathfrak{p}}) \hookrightarrow \mathrm{Gal}(L/K),$$

*is surjective (for some $\mathfrak{P}$ of $L$ dividing $\mathfrak{p}$).*

Let $S$ be a finite set of primes of $K$ and let $\tilde{S}$ be $S$ if $p \neq 2$ and the union of $S$ with the real infinite places of $K$ if $p = 2$. Let $K_{\tilde{S}}$ the maximal $p$-extension of $K$, unramified for primes outside $S$ (if $p = 2$, real places are allowed to become complex in $K_{\tilde{S}}$). Wingberg [W] gives a necessary and sufficent condition for $K_{\tilde{S}}/K$ to be of local type.

**Theorem 38.** *The extension $K_{\tilde{S}}/K$ is of local type at $\mathfrak{p}_0$ if and only if*

$$\sum_{\mathfrak{p}\in\tilde{S}\setminus\{\mathfrak{p}_0\}} \delta_{\mathfrak{p}} - \delta_K + d(\Delta^{\{\mathfrak{p}_0\}}_{\tilde{S}\setminus\{\mathfrak{p}_0\}}) + r_2 = \begin{cases} [K_{\mathfrak{p}_0} : \mathbb{Q}_p], & \mathfrak{p}_0 | p, \\ 0, & \textit{otherwise.} \end{cases}$$

One corollary of this theorem is that if $L/K$ is of local type, then for all intermediate fields $K \subseteq F \subseteq L$, $F/K$ is of local type.

*Remark* 7.2. There is a slight misprint in the statement of this theorem in [W].

7.2. **Structure of** $\mathrm{Gal}(K_{S,\nu}/K)$**.** We now give some evidence for Conjecture 2, namely examples of infinite wild extensions with bounded ramification admitting no infinite $p$-adic analytic quotients.

**Theorem 39.** *Suppose $(S,\nu)$ is a finitely indexed set and let $S \cap S_p = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_s\}$ be the places in $S$ above $p$. Define recursively: $S_0 = S_{S-S_p}$, $S_i = S_{i-1} \cup \{\mathfrak{p}_i\}$, $S_s = S$. We equip each $S_i$ with natural indexing maps $\nu_i$, for $i = 1, \cdots, s$: $\nu_i$ agrees with $\nu$ on $S_i$ and equals 0 outside $S_i$. Assume that for $i = 0, \cdots, s-1$, the extensions $K_{S_{i+1},\nu_{i+1}}/K_{S_i,\nu_i}$ are of local type at $\mathfrak{P}_i$, some prime of $K_{S_i,\nu_i}$ dividing $\mathfrak{p}_i$. If $K_{S-S_p,\nu_0}/K$ is finite (or if Conjecture 1 holds), then $G_{S,\nu}$ is $p$-adically finite.*

**Corollary 40.** *We consider the situation of theorem 26: $p = 2$, $K = \mathbb{Q}(\sqrt{-q})$, $q \equiv 7 \bmod 16$ is prime, $2\mathcal{O}_K = \mathfrak{p}_1\mathfrak{p}_2$. Take $0 < i, j < \infty$, $S = \{\mathfrak{p}_1, \mathfrak{p}_2\}$, and put $\nu_{i,j}(\mathfrak{p}_1) = i$, $\nu_{i,j}(\mathfrak{p}_2) = j$. Then $G_{S,\nu_{i,j}}$ is 2-adically finite.*

*Remark* 7.3. For large $i, j$ , $G_{S,\nu_{i,j}}$ is a group with three generators.

**Corollary 41.** *Suppose $p$ be an odd regular prime, and put $K = \mathbb{Q}(\zeta_p)$. We have the factorization: $p\mathcal{O}_K = \mathfrak{p}^{p-1}$. Let $1 < i < \infty$, $S = \{\mathfrak{p}\}$ and $\nu_i$ defined by $\nu_i(\mathfrak{p}) = i$. The group $G_{S,\nu_i}$ has no infinite $p$-adic analytic quotients. (For large $i$, $G_{S,\nu_i}$ is a $p$-group with $(p+1)/2$ generators).*

These corollaries follow from Theorems 38 and 39. Note that we do not need to assume Conjecture 1 here, since the unramified part is trivial.

*Proof.* of Theorem 39. Let $K \subseteq L \subseteq K_{S,\nu}$ is an intermediate field such that $\mathrm{Gal}(L/K)$ is $p$-adic analytic. Put $L_i = L \cap K_{S_i,\nu_i}$; in particular, $L_0 = L \cap K_{S-S_p}$, and $L_s = L$. The groups $\mathrm{Gal}(L_i/K)$ are all analytic. We proceed by induction on $s$.

• $s = 0$: By Conjecture 1, $L_0/K$ is finite.

• Suppose that $L_i/K$ is finite; we will show that $L_{i+1}/K$ is finite. By hypothesis, the extension $K_{S_{i+1},\nu_{i+1}}/K_{S_i}$ is of local type at $\mathfrak{P}_i$, and this implies that $L_{i+1}/L_i$ is of local type at $\mathfrak{P}_i \cap L_i$.

Let $F_i$ be the subfield of $L_{i+1}/L_i$ fixed by an inertia group of $\mathfrak{P}_i \cap L_i$ in this extension. Then $\mathrm{Gal}(F_i/L_i)$ is cyclic or pro-$p$-cyclic and is unramified at $\mathfrak{p}_i$. We will show that $\mathrm{Gal}(F_i/L_i)$ is finite and then cyclic. In the extension $F_i/K$ the ramification is bounded for all places $\mathfrak{p}$ of $K$. By Proposition 10 the ramification is bounded in $F_i/L_i$ also, because $[L_i : K] < \infty$. And so $\mathrm{Gal}(F_i/L_i)$ is isomorphic to the ray group for a certain finite modulus and so is finite. We apply Corollary 35 to conlcude that $[L_{i+1} : L_i] < \infty$. □

*Remark* 7.4. 1. Further evidence for Conjecture 2 can be obtained by adapting to our case the group-theoretic method of Boston [B1] and [B1].

2. Let $L$ be an intermediate field of $K_{S,\nu}/K$ such that $\mathrm{Gal}(L/K)$ is $p$-adic analytic. Suppose that for all primes $\mathfrak{p} \in S_p$, the inertia groups $D^0(\mathfrak{P}/\mathfrak{p})$ in $L/K$ are all torsion. Since $GL_n(\mathbb{Q}_p)$ has no infinite torsion subgroups, these inertia groups are finite and so $L/K$ is potentially tamely ramified: there exists a finite extension $F/K$, such that $L/F$ is tamely ramified. It then would follow from the Fontaine-Mazur Conjecture that $L/F$ is finite.

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