Non-commutative Poisson Boundaries and Compact Quantum Group Actions

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Abstract

We discuss some relationship between two different fields, a non-commutative version of the Poisson boundary theory of random walks and the infinite tensor product (abbreviated as ITP) actions of compact quantum groups on von Neumann algebras. In contrast to the ordinary compact group case, the ITP action of a compact quantum group on a factor may allow non-trivial relative commutant of the fixed point subalgebra. We give a probabilistic description of the relative commutant in terms of a non-commutative Markov operator. In particular, we show that the following three objects can be naturally identified in the case of the quantum group $SU_q(2)$: (1) the relative commutant of the fixed point algebra under the action, (2) the space of harmonic elements for some non-commutative Markov operator on the dual quantum group of $SU_q(2)$, and (3) the weak closure $L^\infty(\mathbf{T}\backslash SU_q(2))$ of one of the Podles quantum spheres. In view of the ordinary Poisson boundary theory of random walks on discrete groups, this shows that symbolically the quantum homogeneous space $\mathbf{T}\backslash SU_q(2)$ may be regarded as the "Poisson boundary" of a non-commutative random walk on the dual object of $SU_q(2)$. An analogy of the Poisson integral formula is also given.

1 Introduction

Group actions on von Neumann algebras and C*-algebras are of fundamental interest in the theory of operator algebras, and there are a number of fruitful results in history. In particular, several authors have studied compact group actions inspired by the gauge group actions in physics (see for example [1]). Among others, the best understood class so far is that of ITP (infinite tensor product) actions. They are prototypes of so-called minimal actions; an action of a compact group on a factor is said to be *minimal* if the relative commutant of the fixed point algebra is trivial. For compact group actions, minimality seems to be the right counterpart of freeness [36], which plays a fundamental role in classification of discrete group actions.

The notion of the ITP actions of compact quantum groups was first introduced by Y. Konishi, M. Nagisa, and Y. Watatani in the case of the quantum group $SU_q(2)$ for $-1 \leq q \leq 1$, $q \neq 0$ acting on the UHF algebra of type 2^∞ [24]. Konishi showed that the above action on the UHF algebra can be extended to the Powers factor [23], and Y. Nakagami generalized this result to $SU_q(N)$ case for general N [29]. For related results, see also [12] [42]. In contrast to the classical case, the above action on the Powers factor is *not* minimal for $q \neq \pm 1$. This can be observed from a well-known fact in subfactor theory. Namely, if it were minimal, the Jones subfactor of index larger than 4 would be irreducible because it could be obtained via A. Wassermann's type construction with the above action [43]. However, it is known that the Jones subfactor is reducible if its index is larger than 4 [20] [31] (see also [18]). The same reasoning using [37] implies that the above mentioned ITP action of $SU_q(N)$ is not minimal as well. Note that if one goes beyond the AFD case, there exist several minimal actions of $SU_q(N)$ constructed by Y. Ueda [40] [41]. Based on Ueda's actions, D. Shlyakhtenko and Ueda recently constructed irreducible subfactors of the free group factor $L(\mathbf{F}_\infty)$ for all index values larger than 4 [38]. Whether minimal actions can be realized in the AFD factors or not is still a fascinating open problem.

One of the main purposes of the present work is to investigate mathematical structure behind the above sharp contrast between the ordinary and quantum group cases about minimality. To the best knowledge of the author, this type of difference has never been observed as far as algebraic structure is concerned. Indeed, it turns out that the difference manifests itself only in non-commutative probability theoretical analysis. Symbolically, the relative commutant of the fixed point algebra under the ITP action of a compact quantum group G should be considered as the "L^∞-space" over the "Poisson boundary " of a "random walk" on the dual object of G. More precisely, we can show, under a mild assumption, that the relative commutant is completely positively isometric to the set of harmonic elements $H^\infty(\widehat{M}, P_\mu)$ of some non-commutative Markov operator P_μ on the dual Hopf von Neumann algebra $\widehat{M}$. Another purpose of the present work is to show that there exist rich examples of the Poisson boundary theory of non-commutative Markov operators on discrete quantum groups as well as ordinary random walks on discrete groups [22] [44]. Note that random walks on the dual of $SU(N)$ were discussed by Ph. Biane in [3] [4] [5], and it would be interesting to compare his results with our approach.

In the case of the ITP action of $G = SU_q(2)$, further detailed analysis is possible. Namely, we explicitly determine the structure of $H^\infty(\widehat{M}, P_\mu)$ using the full power of the representation theory of $SU_q(2)$. In consequence, we show that the relative commutant of the fixed point algebra is isomorphic to the weak closure $L^\infty(\mathbf{T}\backslash SU_q(2))$ of one of the Podles quantum spheres introduced in [32], both as a von Neumann algebra and as a $SU_q(2)$-comodule. One may be tempted to rephrase this fact as " the Poisson boundary of the random walk on the dual of $SU_q(2)$ can be identified with the quantum homogeneous space $\mathbf{T}\backslash SU_q(2)$." We also construct an explicit map from $L^\infty(\mathbf{T}\backslash SU_q(2))$ to $H^\infty(\widehat{M}, P_\mu)$, which should be interpreted as a $\widehat{SU_q(2)}$-version of the Poisson integral formula. The construction of this map makes sense for general compact quantum groups, and it implies that the ITP actions are never minimal for non-Kac compact quantum groups.

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2 Preliminaries

In this section, we collect basic notions for compact quantum groups and the Poisson boundaries of non-commutative Markov operators in order to fix notation. Since necessary materials for the subject of the present work are gathered from several different sources, we try to make exposition self-contained as much as possible. For general theory of compact quantum groups, our standard references are [2] [46]. For more details of the classical theory of the Poisson boundaries of random walks on discrete groups, readers are encouraged to consult [21] [22] [44] and references in them.

Throughout the paper, the symbol $\otimes$ stands for the minimal tensor product whenever we work on C*-algebras. For the von Neumann algebra tensor product and weak closure of spatial tensor product of operator systems, we use the symbol $\bar{\otimes}$. The algebraic tensor product is denoted by $\otimes_{alg}$ when it is necessary to be emphasized. For two densely defined closed operators T_1 and T_2 on a Hilbert space, we denote by $T_1 \bar{\otimes} T_2$ the closure of their tensor product $T_1 \otimes_{alg} T_2$ defined on the algebraic tensor product of their domains.

2-1. Compact Quantum Groups.
We start with S. L. Woronowicz' C*-algebra approach to compact quantum groups.

Let $G = (A, U, \delta)$ be a compact quantum group, a matrix pseudogroup in the sense of S. L. Woronowicz [46]. Namely, A is a unital C*-algebra, and $U = (U_{i,j})_{i,j \in I_U}$ is an A-valued unitary matrix in $B(\ell^2(I_U)) \otimes A$ with I_U a finite index set, such that the set $\{U_{i,j}\}_{i,j \in I_U}$ generate A as a C*-algebra. The coproduct δ is a $*$-homomorphism $\delta : A \longrightarrow A \otimes A$ satisfying

$$\delta(U_{i,j}) = \sum_{k \in I_U} U_{i,k} \otimes U_{k,j}.$$

We set $\delta^{(1)} = \delta$ and inductively define $\delta^{(n)} = (id \otimes \delta) \cdot \delta^{(n-1)}$ for every natural number n. Although one can define a compact quantum group without using the "fundamental corepresentation U", (or in other words, the finite generation condition is not necessary for the general context), we fix U as above in order to introduce the ITP action of the adjoint action of U. The dense $*$-algebra generated by $\{U_{i,j}\}_{i,j \in I_U}$ is denoted by $\mathcal{A}$.

An A-valued unitary matrix $w = (w_{i,j})_{i,j \in I_w} \in B(\ell^2(I_w)) \otimes A$ is called a *unitary corepresentation of* (A, δ) (called a unitary representation of G in [46]) if the following holds:

$$\delta(w_{i,j}) = \sum_{k \in I_w} w_{i,k} \otimes w_{k,j}.$$

Usual notions such as equivalence, direct sums, and irreducibility of unitary corepresentations were established in [46]. We denote by $\widehat{G}$ the set of equivalence classes of finite dimensional irreducible unitary corepresentations. For each $s \in \widehat{G}$, we fix a representative $w(s) = (w(s)_{i,j})_{i,j\in I_s} \in B(\ell^2(I_s)) \otimes A$. Let $W(s)$ be the linear span of $\{w(s)_{i,j}\}_{i,j\in I_s}$, which depends only on $s \in \widehat{G}$. Then as a linear space, $\mathcal{A}$ is the direct sum,

$$\mathcal{A} = \bigoplus_{s\in\widehat{G}} W(s). \tag{2.1.1}$$

The antipode κ, which is an invertible antimultiplicative linear map from $\mathcal{A}$ to itself, is defined by

$$\kappa(w(s)_{i,j}) = {w(s)_{j,i}}^*.$$

The counit ϵ is the character of $\mathcal{A}$ determined by

$$\epsilon(w(s)_{i,j}) = \delta_{i,j}, \quad s \in \widehat{G}.$$

Let h be the Haar measure, which is a unique state of A satisfying the following invariance condition:

$$(h \otimes id) \cdot \delta(x) = (id \otimes h) \cdot \delta(x) = h(x)1, \quad x \in A.$$

On $\mathcal{A}$, h is determined as follows: $h(w(s)_{i,j}) = 0$ unless s is the trivial representation $0 \in \widehat{G}$. We denote by (π_h, H_h, Ω_h) the GNS triple of h, and by Λ_h the natural map from A to H_h. We assume that π_h is faithful, and we will omit π_h in what follows. The multiplicative unitary V is defined by the bounded extension of the following operator [2]:

$$V(\Lambda_h(x) \otimes \xi) = \delta(x)(\Omega_h \otimes \xi), \quad x \in A,\ \xi \in H_h.$$

V satisfies the following pentagon equation:

$$V_{12}V_{13}V_{23} = V_{23}V_{12}.$$

For $s \in \widehat{G}$, we define a system of matrix units $\{e(s)_{ij}\}_{i,j\in I_s} \subset B(H_h)$ by

$$e(s)_{ij}\Lambda_h(w(t)_{k,l}) = \delta_{s,t}\delta_{j,l}\Lambda_h(w(t)_{k,i}). \tag{2.1.2}$$

(Note that the rank of $e(s)_{ii}$ is not 1.) Then, the dual Hopf algebra $\widehat{A}$ is the C*-direct sum of the finite dimensional simple C*-subalgebras $\{\widehat{A}(s)\}_{s\in\widehat{G}}$, where

$$\widehat{A}(s) = \mathrm{span}\{e(s)_{ij}\}_{i,j\in I_s}.$$

Note that we have

$$V = \sum_{s \in \widehat{G}} \sum_{i,j \in I_s} e(s)_{i,j} \otimes w(s)_{i,j} \in M(\widehat{A} \otimes A), \tag{2.1.3}$$

where $M(\widehat{A} \otimes A)$ stands for the multiplier algebra of $\widehat{A} \otimes A$.

We denote by M and $\widehat{M}$ the weak closure of A and $\widehat{A}$ respectively. In order to save notations, we use the same symbols δ and h for their weakly continuous extensions to M. Then, δ on M is given by

$$\delta(x) = V(x \otimes 1)V^*, \quad x \in M. \tag{2.1.4}$$

The dual coproduct $\widehat{\delta}$ on $\widehat{A}$ (and on $\widehat{M}$) is defined by

$$\widehat{\delta}(x) = V^*(1 \otimes x)V. \tag{2.1.5}$$

2-2. Unbounded Realization of the Dual Hopf Algebra.
Although it is enough to work on bounded operators in order to formulate compact quantum groups, the dual Hopf algebras realized as unbounded operator algebras are more useful for practical computation (it is easier to construct representations of Lie algebras than Lie groups, of course!). In this subsection, we discuss unbounded realization of the dual Hopf algebra of $\mathcal{A}$.

We denote by $\mathcal{A}^*$ the set of (not necessarily bounded) linear functionals on $\mathcal{A}$. Then, $\mathcal{A}^*$ has a natural Hopf $*$-algebra structure, and we use the symbol $\widehat{\mathcal{A}}$ when $\mathcal{A}^*$ is regarded as a Hopf algebra. The product of two elements $\phi, \psi \in \widehat{\mathcal{A}}$ is given by the following convolution product

$$(\phi * \psi)(x) = (\phi \otimes \psi) \cdot \delta(x), \quad x \in \mathcal{A}. \tag{2.2.1}$$

The antipode $\widehat{\kappa}$ and the $*$ operation $\#$ on $\phi \in \widehat{\mathcal{A}}$ are defined by

$$\widehat{\kappa}(\phi)(x) = \phi(\kappa(x)), \quad x \in \mathcal{A}, \tag{2.2.2}$$

$$\phi^{\#}(x) = \overline{\phi(\kappa(x)^*)}, \quad x \in \mathcal{A}. \tag{2.2.3}$$

We denote the coproduct of $\widehat{\mathcal{A}}$ by $\widehat{\delta}_0$, which is strictly speaking a map from $\widehat{\mathcal{A}}$ to $(\mathcal{A} \otimes_{alg} \mathcal{A})^*$, and is defined by

$$\widehat{\delta}_0(\phi)(x \otimes y) = \phi(xy), \quad x, y \in \mathcal{A}. \tag{2.2.4}$$

Note that thanks to (2.1.1) and (2.1.2), $\widehat{\mathcal{A}}$ is naturally identified with the direct product algebra

$$\prod_{s\in\widehat{G}} \widehat{A}(s),$$

which will be further identified with the algebra $\widetilde{\widehat{M}}$ of densely defined closed operators on H_h affiliated with $\widehat{M}$ as below.

Let $\mathcal{H}$ be the dense subspace $\Lambda_h(\mathcal{A}) \subset H_h$. For $x \in \mathcal{A}$ and $\phi \in \widehat{\mathcal{A}}$, we set

$$\phi * x = (id \otimes \phi) \cdot \delta(x),$$

$$x * \phi = (\phi \otimes id) \cdot \delta(x).$$

We introduce a linear operator $\mathcal{R}_0(\phi)$ of H_h, whose domain is $\mathcal{H}$, by

$$\mathcal{R}_0(\phi)\Lambda_h(x) = \Lambda_h(\phi * x), \quad x \in \mathcal{A}. \tag{2.2.5}$$

Then, it is easy to show that $\mathcal{R}_0(\phi)$ maps $\mathcal{H}$ into itself and the closure of $\mathcal{R}_0(\phi)$, denoted by $\mathcal{R}(\phi)$, is affiliated with the finite type I von Neumann algebra $\widehat{M}$. Therefore, there is no complicated problem about domains and every formal calculation involving $\mathcal{R}(\widehat{\mathcal{A}})$ can be justified. $\mathcal{R}_0$ is a representation of the algebra $\widehat{\mathcal{A}}$ satisfying $\mathcal{R}_0(\phi)^* = \mathcal{R}(\phi^{\#})$. In the same way, for $\psi \in (\mathcal{A} \otimes_{alg} \mathcal{A})^*$ we define an operator $\mathcal{R}_{00}(\psi)$ of $H_h \otimes H_h$, whose domain is $\mathcal{H} \otimes_{alg} \mathcal{H}$, and we denote its closure by $\mathcal{R}(\psi)$. Then, it is easy to show that the following holds:

$$(1 \otimes \mathcal{R}_0(\phi))V\xi = V\mathcal{R}_{00}(\widehat{\delta}_0(\phi))\xi, \quad \xi \in \mathcal{H} \otimes_{alg} \mathcal{H}.$$

As before, we abuse notation and use the same symbol $\widehat{\delta}$ for its natural extension to $\widetilde{\widehat{M}}$ defined by (2.1.5). Then, the above implies

$$\widehat{\delta}(\mathcal{R}(\phi)) = \mathcal{R}(\widehat{\delta}_0(\phi)).$$

In [46], Woronowicz introduced a unique family $\{f_z\}_{z\in\mathbf{C}} \subset \widehat{\mathcal{A}}$ of characters, called the *Woronowicz characters*, satisfying the following properties:

(F1) For every $a \in \mathcal{A}$, the mapping $\mathbf{C} \ni z \mapsto f_z(a)$ is an entire function of exponential growth on the right half-plane.

(F2) $f_z(1) = 1$ for all $z \in \mathbf{C}$.

(F3) $f_z * f_{z'} = f_{z+z'}$ for all $z, z' \in \mathbf{C}$.

(F4) $f_z(\kappa(x)) = f_{-z}(x)$, $f_z(x^*) = \overline{f_{-\overline{z}}(x)}$ for all $x \in \mathcal{A}$, $z \in \mathbf{C}$.

(F5) $\kappa^2(x) = f_{-1} * x * f_1$ for all $x \in \mathcal{A}$.

(F6) $h(xy) = h(y(f_1 * x * f_1))$, for all $x, y \in \mathcal{A}$.

Note that f_0 is nothing but the counit by construction in [46].

(F4) implies $f_t^{\#} = f_t$ for real t and that $\mathcal{R}(f_t) = \mathcal{R}(f_{\frac{t}{2}})^2$ is a positive self-adjoint operator for $t \in \mathbf{R}$. We set $\rho = \mathcal{R}(f_1)$, which satisfies $\mathcal{R}(f_z) = \rho^z$ for all $z \in \mathbf{C}$. Since f_1 is a character, we have

$$\widehat{\delta}(\rho) = \rho\bar{\otimes}\rho, \tag{2.2.6}$$

that is, ρ is a group-like element. (F5) implies

$$\widehat{\kappa}^2(\mathcal{R}_0(\phi)) = \rho\mathcal{R}_0(\phi)\rho^{-1}, \quad \phi \in \widehat{\mathcal{A}}.$$

The operator $\rho^{1/2}$ plays an important role in their analysis in [47]. ρ will be used to construct invariant states of the ITP actions later.

In the rest of this subsection, we introduce fundamental objects for the Woronowicz algebra structure of M. Since f_z is a character, the map

$$g_{z,w}(x) := f_z * x * f_w, \quad x \in \mathcal{A},$$

defines an h preserving algebra automorphism of $\mathcal{A}$ for each $z, w \in \mathbf{C}$. (F4) implies $g_{z,w}(x)^* = g_{-\overline{z},-\overline{w}}(x^*)$, which shows that if z and w are pure imaginary, $g_{z,w}$ extends to a $*$-automorphism of A and also M. Among others, there are two distinguished one parameter automorphism groups in this family of automorphisms: namely the modular automorphism group $\{\sigma_t^h\}_{t\in\mathbf{R}}$ of h and the deformation automorphism group $\{\tau_t\}_{t\in\mathbf{R}}$ introduced in [26]. They are given by

$$\sigma_t^h(x) = f_{it} * x * f_{it}, \quad x \in \mathcal{A}, \tag{2.2.7}$$

$$\tau_t(x) = f_{-it} * x * f_{it}, \quad x \in \mathcal{A}. \tag{2.2.8}$$

In a similar way using (F4) and (F5), we can introduce an involutive $*$-antiautomorphism j of A and also M [2, Section 5], called unitary antipode, by

$$j(x) = f_{\frac{1}{2}} * \kappa(x) * f_{-\frac{1}{2}}, \quad x \in \mathcal{A}. \tag{2.2.9}$$

Note that κ can be recovered from j and τ by

$$\kappa(x) = j \cdot \tau_{-\frac{i}{2}}(x) = \tau_{-\frac{i}{2}} \cdot j(x), \quad x \in \mathcal{A}. \tag{2.2.10}$$

2-3. Modules and Comodules.

Let K be a linear space. K is called a *right comodule* of A if there exists a linear map $\alpha : K \longrightarrow K \otimes A$ satisfying

$$(\alpha \otimes id) \cdot \alpha = (id \otimes \delta) \cdot \alpha,$$

where the tensor product should be understood in an appropriate way depending on which category we are working on. For $s \in \widehat{G}$, the *spectral subspace* of K for s is defined to be

$$\{\xi \in K; \alpha(\xi) \in K \otimes W(s)\}.$$

Left comodules are also define in a similar way, and comodules of $\mathcal{A}$, M, $\widehat{\mathcal{A}}$, $\widehat{A}$, and $\widehat{M}$ are defined in the same way.

Let $w = (w_{i,j})_{i,j \in I_w} \in B(\ell^2(I_w)) \otimes A$ be a unitary corepresentation, and $\{\xi_i\}_{i \in I_w}$ be the canonical orthonormal basis of $\ell^2(I_w)$. Then, $\ell^2(I_w)$ is a right $\mathcal{A}$ comodule with α given by

$$\alpha(\xi_i) = \sum_{j \in I_w} \xi_j \otimes w_{j,i}.$$

$\ell^2(I_w)$ naturally has left $\widehat{\mathcal{A}}$ module structure given by

$$\pi_w(\phi)\xi = (id \otimes \phi) \cdot \alpha(\xi), \quad \xi \in \ell^2(I_w). \tag{2.3.1}$$

We also regard π_w as a representation of $\widehat{M}$ in an obvious way. When $w = w(s)$ for some $s \in \widehat{G}$, we denote π_w by π_s for simplicity, which is identified with the irreducible representation of $\widehat{M}$ corresponding to the simple component $\widehat{A}(s)$. For two unitary corepresentations w_1 and w_2, the tensor product representation $(\pi_{w_1} \otimes \pi_{w_2}) \cdot \widehat{\delta}$ is simply denoted by $\pi_{w_1} \otimes \pi_{w_2}$, and the n-fold tensor product representation of π_w is denoted by $\pi_w^{\otimes n}$.

For $s, t, r \in \widehat{G}$, let $N_{s,t}^r$ be the multiplicity of π_r in $\pi_s \otimes \pi_t$. The quantum dimension $d(s)$ of s is defined by

$$d(s) = \mathrm{Tr}(\pi_s(\rho)),$$

where Tr stands for the usual trace. Then, $\widehat{G}$ has fusion algebra structure in the sense of [16] with the structure constant $N_{s,t}^r$ and the dimension function $d(s)$.

For a unitary corepresentation w, we introduce a positive linear functional $T_w \in \widehat{M}_*^+$ by

$$T_w(x) = \mathrm{Tr}(\pi_w(\rho x)), \quad x \in \widehat{M}.$$

T_w is sometimes called a *q-trace.* (2.2.6) implies

$$T_{w_1} * T_{w_2}(x) := (T_{w_1} \otimes T_{w_2}) \cdot \widehat{\delta}(x) = \mathrm{Tr}((\pi_{w_1} \otimes \pi_{w_2})(\rho x)),$$

and in particular,

$$T_s * T_t = \sum_{r \in \widehat{G}} N_{s,t}^r T_r, \quad s, t \in \widehat{G}. \tag{2.3.2}$$

2-4 Actions.
Let B be a C*-algebra. A right (respectively left) action α of (A, δ) on B is a $*$-homomorphism from B to $B \otimes A$ (respectively $A \otimes B$) satisfying the right (respectively left) comodule property above. The fixed point subalgebra of B under α is defined to be

$$B^\alpha = \{x \in B;\ \alpha(x) = x \otimes 1\}.$$

A linear functional $\varphi \in B^*$ is said to be invariant under α if the following condition is satisfied:

$$(\varphi \otimes id) \cdot \alpha(x) = \varphi(x)1, \quad x \in B.$$

We also define actions of $(\widehat{A}, \widehat{\delta})$, (M, δ), and $(\widehat{M}, \widehat{\delta})$ in the same way except that in the M and $\widehat{M}$ cases we assume that B is a von Neumann algebra and α is normal.

The simplest example of an action of (A, δ) is δ acting on A from either left or right. Using the same equations (2.1.4) and (2.1.5) and the pentagon equation, we can introduce a right action of (M, δ) on $\widehat{M}$ and a left action of $(\widehat{M}, \widehat{\delta})$ on M, which will be denoted by Φ and $\widehat{\Phi}$ respectively.

Now, we introduce the adjoint actions of unitary corepresentations.

Lemma 2.1 *Let w be a unitary corepresentation of (A, δ) and z be an element of the center $Z(\widehat{M})$ of $\widehat{M}$. Then,*

(1) *For* $x \in B(\ell^2(I_w))$ *we set*

$$\alpha_w(x) = w(x \otimes 1)w^*.$$

Then, α_w *is a right action of* (A, δ) *on* $B(\ell^2(I_w))$. *The linear span of* $\alpha_w(B(\ell^2(I_w)))(\mathbf{C} \otimes A)$ *coincides with* $B(\ell^2(I_w)) \otimes A$.

(2) *The fixed point algebra* $B(\ell^2(I_w))^{\alpha_w}$ *is given by*

$$B(\ell^2(I_w))^{\alpha_w} = \pi_w(\widehat{M})'.$$

(3) *For* $x \in \widehat{M}$, *we have*

$$\alpha_w \cdot \pi_w(x) = (\pi_w \otimes id) \cdot \Phi(x).$$

(4) *We define a linear functional* $\varphi \in B(\ell^2(I_w))^*$ *by*

$$\varphi_{w,z}(x) = \mathrm{Tr}(\pi_w(z\rho)x), \quad x \in B(\ell^2(I_w)).$$

Then, $\varphi_{w,z}$ *is invariant under* α_w.

Proof. (1) It is easy to show that α_w is an action. Let $\{e_{ij}\}_{i,j \in I_w}$ be the canonical system of matrix units of $B(\ell^2(I_w))$. Then,

$$\alpha_w(e_{ij}) = \sum_{k,l \in I_w} e_{kl} \otimes w_{k,i} w_{l,j}^*.$$

Since the matrix $(w_{p,q}^*)_{p,q}$ is invertible [46], we have

$$B(\ell^2(I_w)) \otimes \mathbf{C} \subset \alpha_w(B(\ell^2(I_w)))(\mathbf{C} \otimes A),$$

which proves the statement. (2) and (3) are easy consequences of the definitions. (4) Since π_w is a direct sum of irreducible representations, a simple argument shows that it suffices to prove the statements for irreducible w. Thus, we assume $z = 1$. Then

$$\begin{aligned}(\varphi_{w,1} \otimes id) \cdot \alpha_w(x) &= \sum_{i,j,k,l \in I_w} \mathrm{Tr}(x e_{lk} \pi_w(\rho) e_{ij}) w_{i,j} w_{k,l}^* \\ &= \sum_{i,j,k,l \in I_w} \mathrm{Tr}(x e_{lj}) f_1(w_{k,i}) w_{i,j} w_{k,l}^*.\end{aligned}$$

Thus, (F5) implies

$$\begin{aligned}
(\varphi_{w,1} \otimes id) \cdot \alpha_w(x) &= \sum_{m,j,k,l \in I_w} \mathrm{Tr}(xe_{lj}) f_1(w_{m,j}) \kappa^2(w_{k,m}) w^*_{k,l} \\
&= \sum_{m,j,k,l \in I_w} \mathrm{Tr}(xe_{lj}) f_1(w_{m,j}) \kappa(w_{l,k} w^*_{m,k}) \\
&= \sum_{j,l \in I_w} \mathrm{Tr}(xe_{lj}) f_1(w_{l,j}) 1 \\
&= \mathrm{Tr}(x\pi_w(\rho)) 1.
\end{aligned}$$

Q.E.D.

2-5. The Poisson Boundaries.
In view of the Poisson boundary theory of random walks [21] [22], we adopt the following algebraic definition of the Poisson boundaries, which makes sense also in the non-commutative case.

Let N be a von Neumann algebra. A unital complete positive map P from N to itself is called a *non-commutative Markov operator*. We say that $x \in N$ is *P-harmonic* if x is fixed by P. We denote by $H^\infty(N, P)$ the set of P-harmonic elements. Although $H^\infty(N, P)$ is *not* an algebra in general, it is a weakly closed operator system [8]: namely it is a unital self-adjoint subspace of N. We show that one can introduce a new product in $H^\infty(N, P)$ so that it becomes a von Neumann algebra.

We fix a free ultrafilter $\omega \in \beta\mathbf{N} \setminus \mathbf{N}$ and define a norm one projection E_ω from N to $H^\infty(N, P)$ by the weak limit

$$E_\omega(x) = w - \lim_{n \to \omega} \frac{1}{n} \sum_{k=0}^{n-1} P^k(x).$$

Then, the Choi-Effros product $E_\omega(xy)$ for $x, y \in H^\infty(N, P)$ gives $H^\infty(N, P)$ abstract von Neumann algebra structure [8]. We call the resulting von Neumann algebra *the non-commutative Poisson boundary of* (N, P) for the reason stated below. Note that the Choi-Effros product does not depend on ω because every completely positive isometry between two von Neumann algebras is actually an isomorphism. For a normal state φ of N, the restriction of φ to $H^\infty(N, P)$ induces a normal state of $H^\infty(N, P)$ considered as a von Neumann algebra, which is called the *harmonic state for the initial distribution* φ.

To justify the terminology, at this point we give a classical example, which in fact inspired the present work. Let Γ be a discrete group and μ be a probability measure on Γ. Then, the right convolution operator P_μ by μ is a Markov operator of the von Neumann algebra $N = \ell^\infty(\Gamma)$. In this case, $H^\infty(N, P)$ is nothing but the bounded harmonic function space $H^\infty(\Gamma, \mu)$ and the resulting von Neumann algebra is isomorphic to the L^∞-space over the Poisson boundary, (in fact this is even one of the equivalent definitions of the Poisson boundary adopted in [22]). Let ν be an initial distribution and (Ω, ϵ) be the Poisson boundary, where Ω is a Borel space and ϵ is the harmonic measure for the initial distribution ν. Though the Poisson boundary is usually constructed in a purely measure theoretic way, the functional analytic characterization is the following: under some mild assumption on μ and ν, there exists a surjective positive isometry

$$\theta : H^\infty(\Gamma, \mu) \longrightarrow L^\infty(\Omega, \epsilon),$$

which intertwines the left actions of Γ. The product of the right-hand side is recovered from the strong limit

$$\theta^{-1}(\theta(f_1)\theta(f_2)) = s - \lim_{n\to\infty} P_\mu^n(f_1 f_2), \quad f_1, f_2 \in H^\infty(\Gamma, \mu).$$

For $f \in H^\infty(\Gamma, \mu)$, the image $\theta(f)$ is interpreted as the restriction of f to the Poisson boundary. The harmonic measure ϵ is, by definition, determined by the following relation

$$\int_\Gamma f(g) d\nu(g) = \int_\Omega \theta(f)(\omega) d\epsilon(\omega).$$

The classical Poisson integral formula on the unit disc is a means to recover the values of a harmonic function f from its boundary values. In the discrete group situation, a concrete realization of the inverse of θ should be considered as its counterpart. When the initial distribution is concentrated on the neutral element e, it takes a very simple form, which is even carried to the case of the dual of $SU_q(2)$ (see Section 5). For $f \in H^\infty(\Gamma, \mu)$ and $g \in \Gamma$, we set $f_g(h) = f(gh)$, which is a harmonic function again. Then,

$$\begin{aligned} f(g) &= f_g(e) = \int_\Omega \theta(f_g)(\omega) d\epsilon(\omega) = \int_\Omega \theta(f)(g\omega) d\epsilon(\omega) \\ &= \int_\Omega \theta(f)(\omega) \frac{d\epsilon(g^{-1}\cdot)}{d\epsilon}(\omega) d\epsilon(\omega). \end{aligned}$$

This is regarded as the discrete group version of the Poisson integral formula and the Radon-Nikodym derivative should be considered as the Poisson kernel.

Now we introduce a family of non-commutative Markov operators on $\widehat{M}$, which generalize the convolution operators on discrete groups.

Lemma 2.2 *For $s \in \widehat{G}$, we define a normal completely positive map Q_s from $\widehat{M}$ to itself by*

$$Q_s(x) = (id \bar{\otimes} T_s) \cdot \widehat{\delta}(x), \quad x \in \widehat{M}.$$

Then,

(1) *The family $\{Q_s\}_{s \in \widehat{G}}$ satisfies*

$$Q_s \cdot Q_t = \sum_{r \in \widehat{G}} N_{s,t}^r Q_r, \quad s, t \in \widehat{G}.$$

(2) *We set*

$$P_s = \frac{1}{d(s)} Q_s.$$

For a probability measure μ on $\widehat{G}$, we define a non-commutative Markov operator P_μ by

$$P_\mu = \sum_{s \in \widehat{G}} \mu(s) P_s.$$

Then, we have

$$P_\mu \cdot P_\nu = P_{\mu * \nu},$$

where $\mu * \nu$ is the convolution product of two probability measures μ and ν on the fusion algebra $\widehat{G}$ introduced in [16].

(3) *For every $x \in \widehat{M}$, we have*

$$(P_\mu \bar{\otimes} id)(\Phi(x)) = \Phi(P_\mu(x)).$$

(4) *For every $x \in \widehat{M}$, we have*

$$(id \bar{\otimes} P_\mu)(\widehat{\delta}(x)) = \widehat{\delta}(P_\mu(x)).$$

Proof. (1) and (2) are direct consequences of (2.3.2), and (4) is immediate from the definitions. It suffices to show (3) for Q_s instead of P_μ. Let $x \in \widehat{M}$. Then,

$$\begin{aligned}
(Q_s \bar{\otimes} id)(\Phi(x)) &= (id \bar{\otimes} T_s \bar{\otimes} id) \cdot (\widehat{\delta} \bar{\otimes} id) \cdot \Phi(x) \\
&= (id \bar{\otimes} T_s \bar{\otimes} id) \cdot (Ad(V_{12}^* V_{23})(1 \otimes x \otimes 1)) \\
&= (id \bar{\otimes} T_s \bar{\otimes} id) \cdot (Ad(V_{13} V_{23} V_{12}^*)(1 \otimes x \otimes 1)) \\
&= V(id \bar{\otimes} T_s \bar{\otimes} id) \cdot (id \bar{\otimes} \Phi) \cdot (\widehat{\delta} \bar{\otimes} id)(x) V^*.
\end{aligned}$$

Thanks to Lemma 2.1, we have

$$(T_s \bar{\otimes} id) \cdot \Phi(y) = T_s(y) 1, \quad y \in \widehat{M}.$$

Thus,

$$(Q_s \bar{\otimes} id)(\Phi(x)) = V((id \bar{\otimes} T_s)(\widehat{\delta}(x)) \otimes 1) V^* = \Phi(Q_s(x)),$$

which proves the statement. Q.E.D.

3 ITP Actions

In this section, we show that as in the classical case, the von Neumann algebra structure of the space of harmonic elements $H^\infty(\widehat{M}, P_\mu)$ has a natural concrete realization for every probability measure μ on $\widehat{G}$ whose support is a finite generating set. The relative commutant of the fixed point subalgebra of the ITP action on the Powers factor (or the Araki-Woods factor) turns out to be the right realization.

First, we introduce the ITP actions in the C*-level following [24]. We concentrate on the ITP action of the adjoint action of the "fundamental corepresentation" $U = (U_{i,j})_{i,j \in I_U} \in B(\ell^2(I_U)) \otimes A$ (otherwise, we would consider the C*-subalgebra generated by the corepresentation we would start with.) Let B_n be the tensor product of n copies of $B(\ell^2(I_U))$, and B be the UHF algebra generated by the union $\cup_n B_n$, namely

$$B_n = \bigotimes_{k=1}^{n} B(\ell^2(I_U)),$$

$$B = \bigotimes_{k=1}^{\infty} B(\ell^2(I_U)).$$

Let $\{e^k_{ij}\}_{i,j\in I_U}$ be the canonical system of matrix units of the k-th tensor component, and we set

$$v_k = \sum_{i,j\in I_U} e^k_{ij} \otimes U_{i,j} \in B_k \otimes A \subset B \otimes A.$$

Let $V_n = v_1 v_2 \cdots v_n$, which is nothing but $(\pi_U^{\otimes n} \otimes id)(V)$. A simple argument using (1) of Lemma 2.1 implies the following:

Lemma 3.1 *Under the above notation, we have*

(1) *For every $x \in B$, the following limit exists in norm topology:*

$$\lim_{n\to\infty} V_n(x \otimes 1)V_n^*.$$

We denote by $\alpha(x)$ the above limit. Then, α is a right (A, δ) action of B.

(2) *The linear span of $\alpha(B)(\mathbf{C} \otimes A)$ is dense in $B \otimes A$.*

We slightly generalize the construction of the invariant states of the ITP actions in [23] [29] [42], which is probably well-known to (even non) specialists. We fix a positive central element $z \in Z(\widehat{M})$ such that $\pi_U(z)$ is invertible and $T_U(z) = 1$. As in Subsection 2-4, we define a faithful state φ_0 of $B(\ell^2(I_U))$ by

$$\varphi_0(x) = \mathrm{Tr}(\pi_U(z\rho)x), \quad x \in B(\ell^2(I_U)).$$

We set φ to be the infinite tensor product state of φ_0,

$$\varphi = \bigotimes_{k=1}^{\infty} \varphi_0.$$

Let $(\pi_\varphi, H_\varphi, \Omega_\varphi)$ be the GNS triple of φ, and Λ_φ be the natural map from A to H_φ. Since π_φ is faithful, we omit the symbol π_φ in what follows. Let N be the weak closure of A. We use the same symbol φ for their normal extensions to N.

The following lemma is standard. (See [11, Théorèm 2.9] for more general statements.)

Lemma 3.2 *With the above notation, the following hold:*

(1) *φ is invariant under α.*

(2) *We define an isometry $V_\infty \in B(H_\varphi \otimes H_h)$ by the continuous extension of the following operator:*

$$V_\infty(\Lambda_\varphi(x) \otimes \xi) = \alpha(x)(\Omega_\varphi \otimes \xi), \quad x \in A, \quad \xi \in H_h.$$

Then, V_∞ is a unitary.

(3) *α extends to an action of (M, δ) on N.*

(4) *Let $\{\sigma_t^\varphi\}_{t\in\mathbf{R}}$ be the modular automorphism group of φ, and $\{\tau_t\}_{t\in\mathbf{R}}$ be the deformation automorphism group of M. Then,*

$$\alpha \cdot \sigma_t^\varphi = (\sigma_t^\varphi \bar{\otimes} \tau_{-t}) \cdot \alpha, \quad t \in \mathbf{R}.$$

(5) *Let J_φ be the modular conjugation of N with respect to Ω_φ, and $\widehat{J}$ be the antiunitary operator defined by*

$$\widehat{J} x \Omega_h = j(x^*)\Omega_h, \quad x \in M,$$

which is in fact the modular conjugation of $\widehat{M}$ [26]. Then,

$$(J_\varphi \otimes \widehat{J}) V_\infty (J_\varphi \otimes \widehat{J}) = V_\infty^*.$$

Proof. (1) follows from Lemma 2.1 and (2) follows from the previous lemma. Using V_∞, α can be expressed as

$$\alpha(x) = V_\infty(x \otimes 1)V_\infty^*, \quad x \in B,$$

which shows (3).

(4) Let $x \in B_n$. Since $(\otimes_{k=1}^n \pi_U(z)) \otimes 1$ commutes $(\pi_U^{\otimes n} \otimes id)(V)$, we have

$$(\sigma_{-t}^\varphi \otimes id) \cdot \alpha \cdot \sigma_t^\varphi(x) = Ad(\pi_U^{\otimes n} \otimes id)((\rho^{-it} \otimes 1)V(\rho^{it} \otimes 1))(x \otimes 1).$$

We claim that the following holds:

$$((\rho^{it} \otimes 1)V(\rho^{-it} \otimes 1)) = (id \otimes \tau_t)(V).$$

Indeed, for $s \in \widehat{G}$ we have

$$\begin{aligned}
\sum_{k,l\in I_s} \rho^{it} e(s)_{kl}\rho^{-it} \otimes w(s)_{k,l} &= \sum_{k,l,p,q,} e(s)_{pq} \otimes f_{it}(w(s)_{p,k}) f_{-it}(w(s)_{l,q}) w(s)_{k,l} \\
&= \sum_{p,q,} e(s)_{pq} \otimes \tau_t(w(s)_{p,q}).
\end{aligned}$$

Thus, (2.1.3) and (2.2.7) imply the claim, and so,

$$(\sigma^{\varphi}_{-t} \otimes id) \cdot \alpha \cdot \sigma^{\varphi}_t(x) = (id \otimes \tau_{-t}) \cdot \alpha(x).$$

Since $\cup_{n=1}^{\infty} B_n$ is dense in M, we get the statement.

(5) Let $x, y \in B_n$ and $a, b \in \mathcal{A}$. Note that x and y are analytic elements for σ^{φ}, and so are a and b for σ^h and τ^h. On one hand, we have

$$\begin{aligned}
&\langle (J_\varphi \otimes \widehat{J}) V_\infty (J_\varphi \otimes \widehat{J})(x\Omega_\varphi \otimes a\Omega_h), y\Omega_\varphi \otimes b\Omega_h \rangle \\
&= \langle \sigma^{\varphi}_{-\frac{i}{2}}(y^*)\Omega_\varphi \otimes j(b^*)\Omega_h, V_\infty(\sigma^{\varphi}_{-\frac{i}{2}}(x^*)\Omega_\varphi \otimes j(a^*)\Omega_h)\rangle \\
&= \langle \sigma^{\varphi}_{-\frac{i}{2}}(y^*)\Omega_\varphi \otimes j(b^*)\Omega_h, \alpha(\sigma^{\varphi}_{-\frac{i}{2}}(x^*))(\Omega_\varphi \otimes j(a^*)\Omega_h)\rangle \\
&= \langle \sigma^{\varphi}_{-\frac{i}{2}}(y^*)\Omega_\varphi \otimes j(b^*)\Omega_h, (\sigma^{\varphi}_{-\frac{i}{2}} \otimes \tau_{\frac{i}{2}})(\alpha(x^*))(\Omega_\varphi \otimes j(a^*)\Omega_h)\rangle \\
&= (\varphi \otimes h)((1 \otimes j(a))(\sigma^{\varphi}_{\frac{i}{2}} \otimes \tau_{-\frac{i}{2}})(\alpha(x))(\sigma^{\varphi}_{-\frac{i}{2}}(y^*) \otimes j(b^*))) \\
&= (\varphi \otimes h)((\sigma^{\varphi}_{\frac{i}{2}}(y^*) \otimes j(a\sigma^h_{-i}(b^*)))(\sigma^{\varphi}_{\frac{i}{2}} \otimes \tau_{-\frac{i}{2}})(\alpha(x))) \\
&= (\varphi \otimes h)((y^* \otimes \tau_{\frac{i}{2}} \cdot j(a\sigma^h_{-i}(b^*)))\alpha(x)) \\
&= (\varphi \otimes h)((y^* \otimes \kappa^{-1}(a\sigma^h_{-i}(b^*))\alpha(x)) \\
&= (\varphi \otimes h)((y^* \otimes 1)(id \otimes \kappa)(\alpha(x))(1 \otimes a\sigma^h_{-i}(b^*))).
\end{aligned}$$

In the above, we used the result of (4), the KMS condition of σ^{φ} and σ^h, $j \cdot \sigma^h_t = \sigma^h_{-t} \cdot j$, (2.2.10), and the fact that κ preserves h. On the other hand

$$\begin{aligned}
\langle V^*(x\Omega_\varphi \otimes a\Omega_h), y\Omega_\varphi \otimes b\Omega_h\rangle &= (\varphi \otimes h)((1 \otimes b^*)\alpha(y^*)(x \otimes a)) \\
&= (\varphi \otimes h)(\alpha(y^*)(x \otimes a\sigma^h_{-i}(b^*))).
\end{aligned}$$

Therefore, it suffices to show the following for $x, y \in B_n$:

$$(\varphi \otimes \kappa)((y \otimes 1)\alpha(x)) = (\varphi \otimes id)(\alpha(y)(x \otimes 1)).$$

Using the KMS condition of σ^{φ} and the property (F5) of the Woronowicz characters, we get

$$
\begin{aligned}
&(\varphi \otimes id)(\alpha(y)(x \otimes 1)) \\
&= \sum_{s\in\widehat{G}} \sum_{k,l,p,q\in I_s} \varphi(\pi_U^{\otimes n}(e(s)_{kl}) y \pi_U^{\otimes n}(e(s)_{qp})x) w(s)_{k,l} w(s)_{p,q}^* \\
&= \sum_{s\in\widehat{G}} \sum_{k,l,p,q\in I_s} \varphi(y \pi_U^{\otimes n}(e(s)_{qp}) x \sigma_{-i}^{\varphi}(\pi_U^{\otimes n}(e(s)_{kl}))) w(s)_{k,l} w(s)_{p,q}^* \\
&= \sum_{s\in\widehat{G}} \sum_{k,l,p,q\in I_s} \varphi(y \pi_U^{\otimes n}(e(s)_{qp}) x \pi_U^{\otimes n}(\rho e(s)_{kl}\rho^{-1})) w(s)_{k,l} w(s)_{p,q}^* \\
&= \sum_{s\in\widehat{G}} \sum_{i,j,p,q\in I_s} \varphi(y \pi_U^{\otimes n}(e(s)_{qp}) x \pi_U^{\otimes n}(e(s)_{ij}))(f_{-1} * w(s)_{i,j} * f_1) w(s)_{p,q}^* \\
&= \sum_{s\in\widehat{G}} \sum_{i,j,p,q\in I_s} \varphi(y \pi_U^{\otimes n}(e(s)_{qp}) x \pi_U^{\otimes n}(e(s)_{ij})) \kappa^2(w(s)_{i,j}) \kappa(w(s)_{q,p}) \\
&= \sum_{s\in\widehat{G}} \sum_{i,j,p,q\in I_s} \varphi(y \pi_U^{\otimes n}(e(s)_{qp}) x \pi_U^{\otimes n}(e(s)_{ij})) \kappa(w(s)_{q,p} w(s)_{j,i}^*) \\
&= (\varphi \otimes \kappa)((y \otimes 1)\alpha(x)),
\end{aligned}
$$

which finishes the proof. Q.E.D.

We use the same symbol α for their extensions to N. Let $R = N \cap N^{\alpha\prime}$. Then, we have

$$\alpha(R) \subset (N \bar{\otimes} M) \cap (N^{\alpha\prime} \bar{\otimes} \mathbf{C}) = R \bar{\otimes} M.$$

Thus, α induces an action of (M, δ) on R. We denote by α_R and φ_R the restriction of α and φ to R respectively. The following lemma shows that R has enough spectral subspaces for the (M, δ) action:

Lemma 3.3 *Let R, α_R, and φ_R be as above. Then,*

(1) *The linear span of $\alpha(R)(\mathbf{C} \otimes M)$ is weakly dense in $R \bar{\otimes} M$.*

(2) *For $s \in \widehat{G}$, we set*

$$R(s) = \{x \in R;\ \alpha(x) \in R \otimes W(s)\}.$$

Then, the linear span of $\cup_{s\in\widehat{G}} R(s)$ is weakly dense in R.

Proof. (1) Thanks to (4) of Lemma 3.2, the fixed point algebra is globally invariant under the modular automorphism group of φ, and so is the relative commutant. Therefore, Takesaki's theorem [39] assures that there exists a unique φ preserving normal conditional expectation E_R from M onto R. We claim that E_R commutes with α, that is, $\alpha \cdot E_R = (E_R \bar{\otimes} id) \cdot \alpha$. Indeed, let e_R be the projection onto the closure of $R\Omega_\varphi$. $\alpha(R) \subset R\bar{\otimes} M$ implies

$$V_\infty(e_R \otimes 1) = (e_R \otimes 1)V_\infty(e_R \otimes 1).$$

Since e_R commutes with J_φ, (4) of Lemma 3.2 implies that $e_R \otimes 1$ commutes with V_∞. Thus, for $x \in N$,

$$\begin{aligned}(E_R\bar{\otimes}id) \cdot \alpha(x)(\Omega_\varphi \otimes \Omega_h) &= (e_R \otimes 1)V_\infty(x\Omega_\varphi \otimes \Omega_h) = V_\infty(e_R x\Omega_\varphi \otimes \Omega_h) \\ &= \alpha(E_R(x))(\Omega_\varphi \otimes \Omega_h).\end{aligned}$$

Since $\Omega_\varphi \otimes \Omega_h$ is separating for $N\bar{\otimes}M$, we get the claim. Thanks to (2) of Lemma 3.1, the linear span of $\alpha(N)(\mathbf{C} \otimes M)$ is weakly dense in $N\bar{\otimes}M$. Applying $E_R\bar{\otimes}id$ to both sides, we get the statement.

(2) Since the Haar measure h is faithful on M and $\mathcal{A}$ is a dense $*$-subalgebra, the Hahn-Banach separating theorem implies that the following set is dense in the predual M_*:

$$\{h(\cdot a);\ a \in \mathcal{A}\}.$$

On the other hand, as in [33, Theorem 1.5], for $s \in \widehat{G}$ we have

$$(id\bar{\otimes}h)(\alpha(x)(1 \otimes w(s)^*_{i,j})) \in R(s), \quad x \in R.$$

Therefore, (1) implies the statement. Q.E.D.

Let n_s be the multiplicity of $w(s)$, $s \in \widehat{G}$ in U. Since z is a positive central element with $T_U(z) = 1$, there exist positive numbers $\{z_s\}_{s\in\widehat{G}}$ such that

$$z = \sum_{s\in\widehat{G}} z_s 1_s,$$

$$1 = \sum_{s\in\widehat{G}} z_s n_s d(s),$$

where 1_s is the unit of $\widehat{A}(s)$. We introduce a probability measure μ on $\widehat{G}$ by

$$\mu = \sum_{s\in\widehat{G}} z_s n_s d(s)\delta_s,$$

where δ_s is the Dirac measure at $s \in \widehat{G}$. We say that μ is *generating* if $\{U_{i,j}\}_{i,j \in I_U}$ generate $\mathcal{A}$ (not only as a $*$-algebra) as an algebra. The goal of this section is to give a precise description of the triple (R, α_R, φ_R) in terms of the space of P_μ-harmonic elements $H^\infty(\widehat{M}, P_\mu)$. As in the classical case, the key tool of our analysis is the non-commutative martingale convergence theorem (see [9, Theorem 2] for the proof).

Lemma 3.4 *Let E_n be the unique φ preserving conditional expectation from N to B_n. Then,*

(1) *For every element $x \in N$, the sequence $\{E_n(x)\}_{n=1}$ converges to x in strong $*$ topology.*

(2) *On the other hand, if $\{x_n\}_{n=1}^\infty \subset N$ is a bounded sequence satisfying $E_n(x_{n+1}) = x_n$ for every $n \in \mathbf{N}$, then there exists a unique element $x \in N$ such that $E_n(x) = x_n$.*

Note that E_n is given by an obvious slice map in the above. For a natural number n, we set

$$\widehat{A}_n = \bigoplus_{s \in \mathrm{supp}(\mu^n)} \widehat{A}(s),$$

where μ^n stands for the n-fold convolution of μ [16] and $\mathrm{supp}(\mu^n)$ stands for the support of μ^n. Let F_n be the natural projection from $\widehat{M}$ onto $\widehat{A}_n$. Then $\pi_U^{\otimes n}$ is injective on $\widehat{A}_n$ and the kernel of $\pi_U^{\otimes n}$ coincides with that of F_n. Let $x \in R$. Since x commutes with B_n^α, we have $E_n(x) \in B_n^{\alpha\prime} \cap B_n$. (2) of Lemma 2.1 shows that $E_n(x)$ belongs to the image of $\pi_U^{\otimes n}$, and so there exists a unique element $y_n \in \widehat{A}_n$ satisfying $\pi_U^{\otimes n}(y_n) = E_n(x)$. Therefore, all information about x is encoded in the sequence $\{y_n\}_{n=1}^\infty$. This is so called a time dependent harmonic sequence [21], and in a good situation, it actually comes from a single harmonic element.

Lemma 3.5 *Under the above notation, we have*

(1) *For every natural number n, the following holds:*

$$\pi_U^{\otimes n} \cdot P_\mu = E_n \cdot \pi_U^{\otimes (n+1)}.$$

(2) *We assume that for every $s \in \widehat{G}$, there exists some integer k such that π_s is contained in $\pi_U^{\otimes k}$, or in other words, μ is a generating measure. Then, for every bounded sequence $\{y_n\}_{n=1}^{\infty} \subset \widehat{M}$ satisfying $y_n \in \widehat{A}_n$ and $F_n \cdot P_\mu^m(y_{n+m}) = y_n$ for $m, n \in \mathbf{N}$, there exists a unique P_μ-harmonic element $y \in H^\infty(\widehat{M}, P_\mu)$ such that $y_n = F_n(y)$.*

Proof. (1) Using $P_\mu = (id\bar{\otimes}\mathrm{Tr}(\pi_U(z\rho\cdot))) \cdot \delta$, we get

$$E_n \cdot \pi_U^{\otimes(n+1)} = (\pi_U^{\otimes n}\bar{\otimes}\mathrm{Tr}(\pi_U(z\rho\cdot))) \cdot \delta = \pi_U^{\otimes n} \cdot P_\mu.$$

(2) We essentially follow the argument in [21]. Let p be the smallest natural number such that $\mathrm{supp}(\mu^{n+p}) \cap \mathrm{supp}(\mu^n)$ is not empty for some n. Such p always exists because there exists $m \in \mathbf{N}$ such that $\pi_U^{\otimes m}$ contains the trivial representation $0 \in \widehat{G}$ and $\mathrm{supp}(\mu) \cap \mathrm{supp}(\mu^{m+1})$ is not empty. For each $i = 0, 1, \cdots p-1$, we set

$$\widehat{G}_i = \bigcup_{n \in \mathbf{N}} \mathrm{supp}(\mu^{i+np}).$$

By assumption, we have

$$\widehat{G} = \bigcup_{i=1}^{m} \widehat{G}_i.$$

Now we apply Foguel's theorem [13] to μ^p, (which was originally proven for discrete groups but still works for fusion algebras), and get the following ℓ^1 norm estimate:

$$\lim_{n\to\infty} ||\mu^{p+n} - \mu^n||_1 = 0.$$

This implies

$$\lim_{n\to\infty} ||P_\mu^{p+n} - P_\mu^n|| = 0.$$

For every natural number k, we have

$$||F_k(y_{p+k}) - y_k|| = \lim_{n\to\infty} ||F_k \cdot P_\mu^n(y_{p+k+n}) - F_k \cdot P_\mu^{p+n}(y_{p+k+n})|| = 0.$$

Thus, there exist P_μ^m-harmonic elements y_i^0, $i = 0, 1, \cdots, p-1$ in

$$(\bigcup_{s \in \widehat{G}_i} \widehat{A}_s)'',$$

such that $y_{i+np} = F_{i+np}(y_i^0)$ for all $n \in \mathbf{N}$. $F_n \cdot P_\mu(y_{n+1}) = y_n$ implies

$$P_\mu(y_{i+1}^0) = y_i^0, \quad i = 0, 1, \cdots p-1,$$

where y_p^0 is understood as y_0^0.

We claim that $\widehat{G}_i$ and $\widehat{G}_j$ are disjoint for $0 \leq i < j \leq p-1$. Indeed, suppose that $\widehat{G}_i \cap \widehat{G}_j$ is not empty. Then there exist natural numbers a and b such that $\mathrm{supp}(\mu^{i+ap}) \cap \mathrm{supp}(\mu^{j+bp})$ is not empty. We take large enough n such that $\mathrm{supp}(\mu^{(n-a)p}) \cap \mathrm{supp}(\mu^{(n-b)p})$ is not empty. Note that such n exists thanks to the above norm estimate. Then, $\mathrm{supp}(\mu^{i+np}) \cap \mathrm{supp}(\mu^{j+np})$ is not empty. This contradicts the choice of p and we get the claim. Therefore, $y = y_0^0 + y_1^0 + \cdots + y_{p-1}^0$ is the desired P_μ-harmonic element. Q.E.D.

Remark. Wang's examples of ergodic ITP actions in [42] *do not* satisfy the assumption of (2). This is the reason our result does not apply to his case.

Now we state our main theorem in this section.

Theorem 3.6 *Under the above notation, we assume that for every $s \in \widehat{G}$, there exists a natural number k such that π_s is contained in $\pi_U^{\otimes k}$, or equivalently $\{U_{i,j}\}$ generate $\mathcal{A}$ as an algebra (not just as a $*$-algebra). Then,*

(1) *For every $x \in H^\infty(\widehat{M}, P_\mu)$, the sequence $\{\pi_U^{\otimes n}(x)\}_{n=1}^\infty$ converges to an element in R in strong $*$ topology. We denote by $\theta(x)$ this limit. Then, θ is a surjective unital completely positive isometry from $H^\infty(\widehat{M}, P_\mu)$ to R.*

(2) *For any finite elements $x_1, x_2, \cdots, x_j \in H^\infty(\widehat{M}, P_\mu)$, the sequence $\{P_\mu^n(x_1 x_2 \cdots x_j)\}_{n=1}^\infty$ converges in strong $*$ topology, and*

$$\theta(\lim_{n\to\infty} P_\mu^n(x_1 x_2 \cdots x_j)) = \theta(x_1)\theta(x_2)\cdots\theta(x_j).$$

This means that θ gives a concrete realization of the non-commutative Poisson boundary of P_μ, and in consequence θ is a homeomorphism in σ-weak topology.

(3) *Let $\hat{\epsilon}$ be the counit of $\widehat{M}$. Then, we have*

$$\varphi_R \cdot \theta(x) = \hat{\epsilon}(x), \quad x \in H^\infty(\widehat{M}, P_\mu).$$

This means that φ_R is the harmonic state for the initial distribution $\hat{\epsilon}$.

(4) *For $s \in \widehat{G}$, we set*

$$H^\infty(\widehat{M}, P_\mu)(s) = \{x \in H^\infty(\widehat{M}, P_\mu);\ \Phi(x) \in H^\infty(\widehat{M}, P_\mu) \otimes W(s)\}.$$

Then, the linear span of the union $\cup_{s\in\widehat{G}} H^\infty(\widehat{M}, P_\mu)(s)$ is weakly dense in $H^\infty(\widehat{M}, P_\mu)$, and

$$\Phi(H^\infty(\widehat{M}, P_\mu)) \subset H^\infty(\widehat{M}, P_\mu)\bar{\otimes} M.$$

θ intertwines the (M, δ) actions, namely

$$(\theta\bar{\otimes} id) \cdot \Phi|_{H^\infty(\widehat{M}, P_\mu)} = \alpha_R \cdot \theta.$$

Proof. (1) Let $x \in H^\infty(\widehat{M}, P_\mu)$. Then $\{\pi_U^{\otimes n}(x)\}_{n=1}^\infty$ is a bounded sequence in N such that $\pi_U^{\otimes n}(x) \in B_n \cap B_n^{\alpha\prime}$ for all $n \in \mathbf{N}$. Moreover, (1) of Lemma 3.4 implies

$$E_n(\pi_U^{\otimes(n+1)}(x)) = \pi_U^{\otimes n}(x).$$

Therefore, the non-commutative martingale convergence theorem implies that the strong $*$ limit exists. Since $\theta(x)$ commutes with B_n^α for all n, $\theta(x) \in R$. Being the limit of unital completely positive maps, θ is unital completely positive. Since $\{\pi_U^{\otimes n}\}_{n=1}^\infty$ is a faithful family of representations of $\widehat{M}$, we get

$$||x|| = \sup_n ||\pi_U^{\otimes n}(x)|| = ||\theta(x)||.$$

The same argument for matrices with entries in $H^\infty(\widehat{M}, P_\mu)$ shows that θ is a complete isometry. Thanks to Lemma 3.5, θ is surjective and θ^{-1} is also completely positive.

(2) Let $x_1, x_2, \cdots, x_j \in H^\infty(\widehat{M}, P_\mu)$. Then, the non-commutative martingale convergence theorem implies

$$\begin{aligned}
\theta(x_1)\theta(x_2)\cdots\theta(x_k) &= \lim_{n\to\infty} E_n(\theta(x_1)\theta(x_2)\cdots\theta(x_k)) \\
&= \lim_{n\to\infty}\lim_{m\to\infty} E_n(\pi_U^{\otimes m}(x_1)\pi_U^{\otimes m}(x_2)\cdots\pi_U^{\otimes m}(x_k)) \\
&= \lim_{n\to\infty}\lim_{m\to\infty} E_n(\pi_U^{\otimes m}(x_1x_2\cdots x_k)) \\
&= \lim_{n\to\infty}\lim_{m\to\infty} \pi_U^{\otimes n}(P_\mu^{m-n}(x_1x_2\cdots x_k))
\end{aligned}$$

where every limit is taken in strong $*$ topology. In particular, for every fixed n the following limit exists

$$\lim_{m\to\infty} F_n(P_\mu^m(x_1x_2\cdots x_k)).$$

Since $\{P_\mu^m(x_1x_2\cdots x_k)\}_{m=1}^\infty$ is a bounded sequence, it converges in strong $*$ topology and we get the statement.

(3) For $x\in H^\infty(\widehat{M},P_\mu)$ we have

$$\varphi_R(\theta(x))=\varphi(E_1(\theta(x)))=\varphi(\pi_U(x))=\hat{\epsilon}(P_\mu(x))=\hat{\epsilon}(x),$$

which shows the statement.

(4) Let $x\in H^\infty(\widehat{M},P_\mu)$ and $\psi\in M_*$. Then, thanks to (3) of Lemma 2.2, $(id\bar{\otimes}\psi)(\Phi(x))$ is P_μ-harmonic. (3) of Lemma 2.1 implies

$$\begin{aligned}\theta((id\bar{\otimes}\psi)(\Phi(x))) &= \lim_{n\to\infty}(\pi_U^{\otimes n}\bar{\otimes}\psi)(\Phi(x))=\lim_{n\to\infty}(id\bar{\otimes}\psi)(\alpha(\pi_n^{\otimes n}(x)))\\ &= (id\bar{\otimes}\psi)(\alpha_R(\theta(x))).\end{aligned}$$

Thus, (2) of Lemma 3.5 and the above (2) imply that the preimage of $R(s)$ by θ is $H^\infty(\widehat{M},P_\mu)(s)$ and the linear span of the union $\cup_{s\in\widehat{G}}H^\infty(\widehat{M},P_\mu)(s)$ is dense in $H^\infty(\widehat{M},P_\mu)$ in σ-weak topology, which implies

$$\Phi(H^\infty(\widehat{M},P_\mu))\subset H^\infty(\widehat{M},P_\mu)\bar{\otimes}M.$$

Therefore, $(\theta\bar{\otimes}id)\cdot\Phi|_{H^\infty(\widehat{M},P_\mu)}$ makes sense and

$$(\theta\bar{\otimes}id)\cdot\Phi|_{H^\infty(\widehat{M},P_\mu)}=\alpha_R\cdot\theta.$$

Q.E.D.

Remark. Since the fixed point algebra $\widehat{M}^\Phi$ coincides with the center $Z(\widehat{M})$ of $\widehat{M}$, (3) of Lemma 2.2 implies that P_μ preserves $Z(\widehat{M})\cong\ell^\infty(\widehat{G})$. Restriction of P_μ to $Z(\widehat{G})$ is nothing but the Markov operator on the fusion algebra $\widehat{G}$ considered in [16]. Indeed, let 1_s, $s\in\widehat{G}$ be the unit of $\widehat{A}(s)$. Then,

$$T_r(P_s(1_t))=\frac{1}{d(s)}(T_r*T_s)(1_t)=N_{r,s}^t\frac{d(t)}{d(s)},$$

which shows

$$P_s(1_t)=\sum_{r\in\widehat{G}}N_{r,s}^t\frac{d(t)}{d(s)d(r)}1_r. \tag{3.1}$$

(4) of the above theorem implies that the Poisson boundary of $P_\mu|_{Z(\widehat{M})}$ can be realized by $R^{\alpha_R} = Z(N^\alpha)$. Therefore, using T. Hayashi's result [14, Corollary 3.5], we get the following corollary. Note that though Hayashi assumed μ to be symmetric in [14], his proof goes without any change in the general generating measure case.

Corollary 3.7 *Assume that the fusion algebra $\widehat{G}$ is commutative. Then with the same assumption as in Theorem 3.6, the Poisson boundary of the restriction of P_μ to $Z(\widehat{M})$ is trivial. In consequence, N^α is a factor.*

Let us show some examples.

Examples. (1) When M is commutative, there exists a compact Lie group G such that $M = L^\infty(G)$. In this case, the usual asymptotic abeliance argument [30] shows that R is trivial. Therefore, $H^\infty(\widehat{M}, P_\mu)$ is always trivial.
(2) When M is cocommutative, there exists a discrete group Γ such that $\widehat{M}$ is isomorphic to $\ell^\infty(\Gamma)$. In this case, $H^\infty(\widehat{M}, P_\mu) = H^\infty(\Gamma, \mu)$ as in Subsection 2.5. The above remark implies $R = Z(M^\alpha)$. The algebra M^α often appears as the core algebra of some subfactor coming from a Γ action [6] [7] [34] [35].
(3) The most interesting class of examples would be those non-trivial compact quantum groups with the commutative fusion algebras $\widehat{G}$, such as $SU_q(N)$. The above corollary shows that in this case N^α is a factor and α_R gives rise to an ergodic action: namely, the fixed point subalgebra of R under α_R is trivial. In next section, we work on $SU_q(2)$ and determine the structure of this ergodic action.

Before closing this section, we show a technical lemma, which will be used for the Poisson integral formula for $\widehat{SU_q(2)}$. We define a unital normal completely positive map $\Theta : M \longrightarrow \widehat{M}$ by

$$\Theta(x) = (id \otimes h) \cdot \widehat{\Phi}(x), \quad x \in M.$$

Note that Θ is trivial if M is either commutative or cocommutative, or more generally if M is a Kac algebra.

Lemma 3.8 *Let μ be a probability measure on $\widehat{G}$. Then,*

(1) *Every element in the image of Θ is P_μ-harmonic.*

(2) Θ *commutes with both of the right* (M, δ) *action and the left* $(\widehat{M}, \widehat{\delta})$ *actions, that is,*

$$(\Theta \otimes id) \cdot \delta = \Phi \cdot \Theta,$$
$$(id \otimes \Theta) \cdot \widehat{\Phi} = \widehat{\delta} \cdot \Theta.$$

Proof. (1) It suffices to show $Q_s \cdot \Theta = d(s)\Theta$ for all $s \in \widehat{G}$. For $x \in M$, we have

$$\begin{aligned} Q_s \cdot \Theta(x) &= (id \otimes T_s \otimes h) \cdot (\widehat{\delta} \otimes id) \cdot \widehat{\Phi}(x) \\ &= (id \otimes T_s \otimes h) \cdot (id \otimes \widehat{\Phi}) \cdot \widehat{\Phi}(x). \end{aligned}$$

We claim that for all $s \in \widehat{G}$ the following holds:

$$(T_s \otimes h) \cdot \widehat{\Phi}(x) = d(s)h(x), \quad x \in M,$$

which would prove the statement. Indeed, using $T_s(e(s)_{jk}) = f_1(w(s)_{k,j})$, we get

$$\begin{aligned} (T_s \otimes h) \cdot \widehat{\Phi}(x) &= \sum_{j,k,l \in I_s} T_s(e(s)_{jk}) h(w(s)^*_{l,j} x w(s)_{l,k}) \\ &= \sum_{j,k,l \in I_s} f_1(w(s)_{k,j}) h(\sigma^h_i(w(s)_{l,k}) w(s)^*_{l,j} x). \end{aligned}$$

Therefore, using the property (F6) of the Woronowicz characters, the right-hand side equals

$$\begin{aligned} &\sum_{j,k,l,p,q \in I_s} f_1(w(s)_{k,j}) f_{-1}(w(s)_{l,p}) f_{-1}(w(s)_{q,k}) h(w(s)_{p,q} w(s)^*_{l,j} x), \\ &= \sum_{j,l,p,q \in I_s} (f_{-1} * f_1)(w(s)_{q,j}) f_{-1}(w(s)_{l,p})) h(w(s)_{p,q} w(s)^*_{l,j} x), \\ &= \sum_{j,l,p \in I_s} f_{-1}(w(s)_{l,p})) h(w(s)_{p,j} w(s)^*_{l,j} x), \\ &= \sum_{l \in I_s} f_{-1}(w(s)_{ll})) h(x). \end{aligned}$$

This means that $(T_s \otimes h) \cdot \widehat{\Phi}$ is proportional to h, which proves the claim.

(2) Using the pentagon equation and invariance of h, we get the first equation:

$$\begin{aligned}(\Theta \otimes id) \cdot \delta(x) &= (id \otimes h \otimes id)(V_{12}^* V_{23}(1 \otimes x \otimes 1)V_{23}^* V_{12}) \\ &= (id \otimes h \otimes id)(V_{13} V_{23} V_{12}^*(1 \otimes x \otimes 1)V_{12} V_{23}^* V_{13}^*) \\ &= V(id \otimes h \otimes id) \cdot (id \otimes \delta) \cdot \widehat{\Phi}(x)V^* \\ &= V((id \otimes h) \cdot \widehat{\Phi}(x) \otimes 1)V^* \\ &= \Phi \cdot \Theta(x).\end{aligned}$$

Since $\widehat{\Phi}$ is a left action of $(\widehat{M}, \widehat{\delta})$, we have

$$\begin{aligned}(id \otimes \Theta) \cdot \widehat{\Phi}(x) &= (id \otimes id \otimes h) \cdot (id \otimes \widehat{\Phi}) \cdot \widehat{\Phi}(x) \\ &= (id \otimes id \otimes h) \cdot (\widehat{\delta} \otimes id) \cdot \widehat{\Phi}(x) \\ &= \widehat{\delta} \cdot \Theta(x).\end{aligned}$$

Q.E.D.

Corollary 3.9 *The image of Θ consists of only scalars if and only if M is a Kac algebra. In consequence, if M is not a Kac algebra, the relative commutant R is not trivial.*

Proof. It is well known that a compact Woronowicz algebra is a Kac algebra if and only if the Haar measure is a trace [2], or equivalently the Woronowicz character f_z does not depends on z, which is further equivalent to $\kappa^2 = id$. (2.1.3) implies that for $x \in M$ we have

$$\begin{aligned}\Theta(x) &= \sum_{s \in \widehat{G}} \sum_{j,k,l \in I_s} h(w(s)_{l,j}^* x w(s)_{l,k}) e(s)_{jk} \\ &= \sum_{s \in \widehat{G}} \sum_{j,k,l \in I_s} h(\sigma_i^h(w(s)_{l,k}) w(s)_{l,j}^* x) e(s)_{jk}.\end{aligned}$$

Therefore, if M is a Kac algebra we get

$$\begin{aligned}\Theta(x) &= \sum_{s \in \widehat{G}} \sum_{j,k,l \in I_s} h(w(s)_{l,k} w(s)_{l,j}^* x) e(s)_{jk} \\ &= \sum_{s \in \widehat{G}} \sum_{j,k,l \in I_s} h(\kappa(w(s)_{j,l} w(s)_{k,l}^*) x) e(s)_{jk} \\ &= h(x)1.\end{aligned}$$

Next, we assume that the image of Θ contains only scalars. Then, there exists $a \in \mathcal{A}$ such that for every $s \in \widehat{G}$ we have

$$\sum_{l \in \widehat{G}} \sigma_i^h(w(s)_{l,k}) w(s)_{l,j}^* = \delta_{j,k} a.$$

The $s = 0$ case shows $a = 1$. Thus, this implies

$$\begin{aligned}
\kappa^2(w(s)_{l,k}) &= \sum_{j,l \in I_s} \sigma_i^h(w(s)_{l,k}) w(s)_{l,j}^* \kappa^2(w(s)_{l,j}) \\
&= \sum_{j,l \in I_s} \sigma_i^h(w(s)_{l,k}) \kappa(w(s)_{j,l}^* w(s)_{j,l}) \\
&= \sigma_i^h(w(s)_{l,k}).
\end{aligned}$$

The properties (F5) and (F6) of the Woronowicz characters show that this is equivalent to $f_1 = f_{-1}$, which is further equivalent to $\rho = \rho^{-1}$. Since ρ is a positive self-adjoint operator, we get $\rho = 1$, which implies that M is a Kac algebra. Q.E.D.

4 Preliminaries for the $SU_q(2)$ Case

In the rest of this paper, we assume that G is the quantum group $SU_q(2)$ introduced by Woronowicz in [45]. For simplicity we assume $0 < q < 1$, though with a minor modification our argument is also valid for $-1 < q < 0$. We follow [25] for the notation, and in particular, the choice of the generators of $SU_q(2)$ for our exposition. Therefore some formulae we will present here appear different from those in [46] and [32] .

4-1. $C(SU_q(2))$. The "function algebra $A = C(SU_q(2))$ over $SU_q(2)$ " is the universal C*-algebra generated by four generators x, u, v, and y with the following relations:

$$ux = qxu, \quad vx = qxv, \quad yu = quy, \quad yv = qvy,$$

$$uv = vu, \quad xy - q^{-1}uv = yx - quv = 1,$$
$$x^* = y, \quad u^* = -q^{-1}v.$$

For a non-negative half integer s, we set

$$I_s = \{-s, -s+1, \cdots, s\}.$$

Then, the fundamental corepresentation $\{w(1/2)_{i,j}\}_{i,j\in I_{1/2}}$ is given by

$$w(1/2) = \begin{array}{c} \\ -1/2 \\ 1/2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} -1/2 & 1/2 \end{array} \\ \left(\begin{array}{cc} x & u \\ v & y \end{array}\right) \end{array}$$

The coproduct δ is determined by

$$\delta(w(1/2)_{i,j}) = \sum_{k\in I_{1/2}} w(1/2)_{i,k}\otimes w(1/2)_{k,j}, \quad i,j\in I_{1/2}.$$

As in the classical case, $\widehat{G}$ is identified with the set of non-negative half integers, and $s\in\widehat{G}$ is called *spin*. We denote by $w(s)_{i,j}$ the unitary corepresentation of spin s obtained in [25, Theorem 2.8].

Explicit expression of two kinds of inner products of $\mathcal{A}$ with respective to the Haar measure h are obtained in ([25, (3.30), (3.31)]):

$$h(w(s)^*_{i,j}w(t)_{k,l}) = \delta_{s,t}\delta_{i,k}\delta_{j,l}q^{2(s+i)}\frac{1-q^2}{1-q^{2(2s+1)}}, \tag{4.1.1}$$

$$h(w(t)_{k,l}w(s)^*_{i,j}) = \delta_{s,t}\delta_{i,k}\delta_{j,l}q^{2(s-j)}\frac{1-q^2}{1-q^{2(2s+1)}}, \tag{4.1.2}$$

The Woronowicz characters $\{f_z\}$ and the the modular automorphism group $\{\sigma^h_t\}_{t\in\mathbf{R}}$ are given by

$$f_z(w(s)_{k,l}) = \delta_{k,l}q^{-2kz}, \tag{4.1.3}$$

$$\sigma^h_t(w(s)_{k,l}) = q^{-2i(k+l)t}w(s)_{k,l}, \tag{4.1.4}$$

For later use, we present explicit formulae of $w(s)_{i,s}$ and $w(s)_{i,-s}$. As in [25], we use the following symbols: for two numbers a,q and a natural number l,

$$(a;q)_0 := 1, \quad (a;q)_l := \prod_{i=1}^{l-1}(1-aq^i).$$

For two non-negative integers $m\le n$,

$$\left[\begin{array}{c} n \\ m \end{array}\right]_q := \frac{(q;q)_n}{(q;q)_m(q;q)_{n-m}}.$$

Then, $w(s)_{i,s}$ and $w(s)_{i,-s}$ are given by ([25, Theorem 1.8])

$$w(s)_{i,s} = \left[\begin{array}{c} 2s \\ s+i \end{array} \right]_{q^2}^{1/2} u^{s-i} y^{s+i}, \quad w(s)_{i,-s} = \left[\begin{array}{c} 2s \\ s+i \end{array} \right]_{q^2}^{1/2} x^{s-i} v^{s+i}. \quad (4.1.5)$$

4-2. $U_q(su(2))$. The quantum universal enveloping algebra $U_q(su(2))$ is the $*$-algebra generated by k, k^{-1}, e, and f with the following relations [10][19]:

$$kk^{-1} = k^{-1}k = 1,$$

$$kek^{-1} = qe, \quad kfk^{-1} = q^{-1}f,$$

$$ef - fe = \frac{k^2 - k^{-2}}{q - q^{-1}},$$

$$k^* = k, \quad k^{-1*} = k^{-1}, \quad e^* = f.$$

$U_q(su(2))$ is regarded as a subalgebra of $\widehat{\mathcal{A}}$ with the following relations:

$$k^{\pm}(w(s)_{i,j}) = \delta_{i,j} q^{\mp j},$$

$$e(w(s)_{i,j}) = \delta_{j,i+1} \sqrt{(s+j)_q (s-j+1)_q},$$

$$f(w(s)_{i,j}) = \delta_{j,i-1} \sqrt{(s-j)_q (s+j+1)_q}. \quad (4.2.1)$$

where for a half integer n, n_q stands for

$$n_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Note that the Woronowicz character f_1 is given by k^2, and so,

$$\rho = \mathcal{R}(k^2).$$

Therefore, the quantum dimension $d(s)$ for $s \in \widehat{G}$ is given by

$$d(s) = q^{-2s} + q^{-2s+2} + \cdots + q^{2s} = (2s+1)_q.$$

The coproduct $\widehat{\delta}_0$ and the antipode $\widehat{\kappa}$ on the generators are as follows:

$$\widehat{\delta}_0(k^{\pm}) = k^{\pm} \otimes k^{\pm},$$

$$\widehat{\delta}_0(e) = e \otimes k + k^{-1} \otimes e, \quad \widehat{\delta}_0(f) = f \otimes k + k^{-1} \otimes f,$$
$$\widehat{\kappa}(k^{\pm}) = k^{\mp},$$
$$\widehat{\kappa}(e) = -qe, \quad \widehat{\kappa}(f) = -q^{-1}f.$$

For $s \in \widehat{G}$, the spin s irreducible representation π_s of $U_q(su(2))$ on $\ell^2(I_s)$ with respect to the canonical basis $\{\xi_j^s\}_{j \in I_s}$ is given by

$$\pi_s(k)\xi_j^s = q^{-j}\xi_j^s,$$
$$\pi_s(e)\xi_j^s = \sqrt{(s+j)_q(s-j+1)_q}\,\xi_{j-1}^s,$$
$$\pi_s(f)\xi_j^s = \sqrt{(s-j)_q(s+j+1)_q}\,\xi_{j+1}^s. \tag{4.2.2}$$

For a left $U_q(su(2))$-module L, $\xi \in L \setminus \{0\}$ is called a *highest weight vector of spin* $s \in \widehat{G}$, if the following holds:

$$e \cdot \xi = 0, \quad k \cdot \xi = q^s \xi.$$

If ξ is such a vector, $U_q(su(2)) \cdot \xi$ is equivalent to the spin s module.

The Casimir element C is a central element of $U_q(su(2))$ defined by

$$C = \left(\frac{q^{1/2}k - q^{-1/2}k^{-1}}{q - q^{-1}}\right)^2 + fe,$$

which satisfies

$$\pi_s(C) = (s + \frac{1}{2})_q^2 1_s, \quad s \in \widehat{G}.$$

4-3. The Adjoint Representation. For $X \in U_q(su(2))$, the coproduct $\widehat{\delta}_0(X)$, which is a priori defined as an element of $(\mathcal{A} \otimes \mathcal{A})^*$, actually belongs to the algebraic tensor product $U_q(su(2)) \otimes U_q(su(2))$, as it is the case for the generators. Therefore, $\widehat{\delta}_0(X)$ has the form $\sum_i X_i^{(1)} \otimes X_i^{(2)}$. Using this fact, we introduce the *adjoint representation* $ad(X)$ of $U_q(su(2))$ on $\widehat{\mathcal{A}}$ by

$$ad(X)Y := \sum_i X_i^{(1)} Y \widehat{\kappa}(X_i^{(2)}), \quad Y \in \widehat{\mathcal{A}}.$$

It is easy to show that $ad(X)$ is well defined and gives a representation of $U_q(su(2))$. For the generators, $ad(X)$ is given by

$$ad(k)Y = kYk^{-1},$$
$$ad(e)Y = eYk^{-1} - qk^{-1}Ye, \quad ad(f)Y = fYk^{-1} - q^{-1}k^{-1}Yf.$$

Lemma 4.1 *With the above notation, the following hold:*

(1) *Let n be a non-negative integer. Then, $e^n k^n$ is a highest weight vector of spin n with respect to the adjoint representation of $U_q(su(2))$. For $s \in \widehat{G}$ with $2s \geq n$, $\pi_s(e^n k^n)$ is not zero.*

(2) *There exists a $U_q(su(2))$-module map from $\ell^2(I_1)$ to $U_q(su(2))$ that sends ξ^1_{-1}, ξ^1_0, and ξ^1_1 to*

$$\chi_{-1} = ek, \quad \chi_0 = \frac{fe - q^{-2}ef}{\sqrt{2_q}}, \quad \chi_1 = -q^{-1}kf.$$

Proof. (1) follows from the definition of the highest weight vectors and the above formulae. (2) follows from (4.2.2) and

$$\frac{1}{\sqrt{(2)_q}} ad(f)ek = \frac{fe - q^{-2}ef}{\sqrt{2_q}},$$

$$\frac{1}{\sqrt{(2)_q}} ad(f)(\frac{fe - q^{-2}ef}{\sqrt{2_q}}) = -q^{-1}kf.$$

Q.E.D.

Let Φ be the right (M, δ) action on $\widehat{M}$ introduced in Section 2. Thanks to (2.1.3), we have

$$\Phi(\widehat{A}(s)) \subset \widehat{A}(s) \otimes W(s)W(s)^*, \quad s \in \widehat{G}.$$

We denote by Φ_s the restriction of Φ to $\widehat{A}(s)$, which is given by

$$\Phi_s(a) = \sum_{i,j,l,m \in I_s} e(s)_{ij} a e(s)_{ml} \otimes w(s)_{i,j} w(s)^*_{lm}, \quad a \in \widehat{A}(s).$$

Every finite dimensional corepresentation of A is self-contragredient, and the product $W(s)W(s)$ decomposed into

$$W(0) \oplus W(1) \oplus \cdots \oplus W(2s).$$

Therefore, as a right (A, δ)-comodule (or equivalently left $U_q(su(2))$-module), $\widehat{A}(s)$ is decomposed into the direct sum of the spin n comodules (or the spin n modules) $n = 0, 1, \cdots, 2s$ with multiplicity one.

As stated in Section 2, the algebra $\widetilde{\widehat{M}}$ of the elements affiliated with $\widehat{M}$ can be identified with $\prod_{s\in\widehat{G}}\widehat{A}(s)$. Therefore we can extend the definition of Φ to $\widetilde{\widehat{M}}$ by setting

$$\Phi(a)=(\Phi_s(a_s))\in\prod_{s\in\widehat{G}}\widehat{A}(s)\otimes W(s)W(s)^*,\quad a=(a_s)\in\prod_{s\in\widehat{G}}\widehat{A}(s).$$

For $X\in\widehat{\mathcal{A}}$, $(id\otimes X)(\Phi(a))$ makes sense as an element of $\widetilde{\widehat{M}}$.

Lemma 4.2 *Let $X\in U_q(su(2))$ and $Y\in\widehat{\mathcal{A}}$. Then, we have*

$$(id\otimes X)(\Phi(\mathcal{R}(Y)))=\mathcal{R}(ad(X)Y).$$

Proof. For $s\in\widehat{G}$, the $\widehat{A}(s)$-component of the left-hand side is

$$\begin{aligned}
&\sum_{i,j\in I_s}Y(w(s)_{i,j})(id\otimes X)(\Phi(e(s)_{ij}))\\
&=\sum_{i,j,l,m\in I_s}X(w(s)_{l,i}w(s)^*_{m,j})Y(w(s)_{i,j})e(s)_{lm}\\
&=\sum_{i,j,l,m\in I_s}(id\otimes\widehat{\kappa})(\widehat{\delta}_0(X))(w(s)_{l,i}\otimes w(s)_{j,m})Y(w(s)_{i,j})e(s)_{lm}\\
&=\sum_{i,j,l,m\in I_s}\sum_p(X^{(1)}_p\otimes Y\otimes\widehat{\kappa}(X^{(2)}_p))(w(s)_{l,i}\otimes w(s)_{i,j}\otimes w(s)_{j,m})e(s)_{lm}\\
&=\sum_{l,m,\in I_s}(ad(X)Y)(w(s)_{l,m})e(s)_{lm},
\end{aligned}$$

which is the $\widehat{A}(s)$-component of the right-hand side. Q.E.D.

Let ad_s be the representation of $U_q(su(2))$ on $\widehat{A}(s)$ corresponding to Φ_s for $s\in\widehat{G}$; namely $ad_s(X)a=(id\otimes X)(\Phi_s(a))$ for $a\in\widehat{A}(s)$. The above two lemmas imply

Corollary 4.3 *Let $\mathcal{E}_s$, $s\in\widehat{G}$ be a natural projection from $\widetilde{\widehat{M}}$ onto $\widehat{A}(s)$. Then, for an integer n with $0\le n\le 2s$, $\mathcal{E}_s(\mathcal{R}(e^nk^n))$ is a unique (up to scalar multiple) spin n highest weight vector in $\widehat{A}(s)$ with respect to ad_s.*

4-4. Quantum Spheres. Let D be a C*-algebra with a right action β of (A, δ). For each $s \in \widehat{G}$, we set

$$D(s) = \{x \in D; \beta(x) \in D \otimes W(s)\}.$$

In [32, Theorem 1], P. Podles completely classified those (D, β) satisfying the following three properties: (1) $D(0) = \mathbf{C}$, (2) the multiplicity of $D(1)$ is one, and (3) $D(1)$ generates D. Such (D, β) is called a *quantum sphere*.

Assume that (D, β) satisfies the above three conditions and take a basis $\{X_i\}_{i \in I_1}$ of $D(1)$ satisfying

$$\beta(X_i) = \sum_{j \in I_1} X_j \otimes w(1)_{j,i}.$$

Note that $\{X_i\}_{i \in I_1}$ are generators of D by assumption. Then, either of the following occurs up to scalar multiple of the generators [32] (Our expression of the matrix $w(1)$ is different from that in [32], and the following formulae are taken from [27]):
Case I.

$$\begin{aligned} q^{-1}X_0X_{-1} - qX_{-1}X_0 &= q^{-1}X_1X_0 - qX_0X_1 \\ &= (q^{-1} - q)X_0^2 + X_1X_{-1} - X_{-1}X_1 \\ &= 0. \end{aligned}$$

D is denoted by $C(S^2_{q,\infty})$ in this case.
Case II. There exists a real number c in the set

$$J_q = \{-\frac{1}{(q^n + q^{-n})^2};\ n = 2, 3, \cdots\} \cup [0, \infty),$$

such that

$$X_0^* = X_0, \quad X_{-1}^* = -qX_1, \quad X_1^* = -q^{-1}X_{-1},$$

$$q^{-1}X_0X_{-1} - qX_{-1}X_0 = (q^{-1} - q)X_{-1},$$

$$(q^{-1} - q)X_0^2 + X_1X_{-1} - X_{-1}X_1 = (q^{-1} - q)X_0,$$

$$q^{-1}X_1X_0 - qX_0X_1 = (q^{-1} - q)X_1,$$

$$X_0^2 - qX_{-1}X_1 - q^{-1}X_1X_{-1} = 1 + (q + q^{-1})^2c.$$

D is denoted by $C(S^2_{q,c})$ in this case.

The quantum spheres are completely classified by the parameter c, (in the first case, c should be regarded as ∞), and for every value $c \in J_q \cup \{\infty\}$, $C(S^2_{q,c})$ actually exists. Since $C(S^2_{q,c})(0) = \mathbf{C}$, there exists a unique β-invariant state on $C(S^2_{q,c})$, which will be also denoted by h and called the Haar measure. The weak closure of $C(S^2_{q,c})$ in the GNS representation of h will be denoted by $L^\infty(S^2_{q,c})$.

Among other quantum spheres, there is a distinguished one coming from a "quantum homogeneous space". Since A is a universal algebra with the generators and relations described at the beginning of this section, there exists a homomorphism γ from A to $C(T)$ defined by

$$\gamma(x) = z, \quad \gamma(y) = \overline{z}, \quad \gamma(u) = \gamma(v) = 0,$$

where z is the coordinate function of $\mathbf{T}$. γ is actually a Hopf algebra homomorphism and is considered to be the restriction map to the "subgroup $\mathbf{T}$ of $SU_q(2)$". It is easy to show that γ on $\mathcal{A}$ is given by

$$\gamma(w(s)_{i,j}) = \delta_{i,j} z^{-2i}.$$

We set

$$C(\mathbf{T}\backslash SU_q(2)) = \{x \in A;\ (\gamma \otimes id) \cdot \delta(x) = 1 \otimes x\}.$$

Then, $C(\mathbf{T}\backslash SU_q(2))$ is a C*-subalgebra of A, which is the closure of the linear span of

$$\bigcup_{s \in \widehat{G}, j \in I_s} \{w(s)_{0,j}\}.$$

$(C(\mathbf{T}\backslash SU_q(2)), \delta_{|C(\mathbf{T}\backslash SU_q(2))})$ is a quantum sphere with $c = 0$, which can be directly verified with the choice of generators

$$X_{-1} = w(1)_{0,-1} = \sqrt{1+q^2}xv, \quad X_1 = w(1)_{0,1} = \sqrt{1+q^2}uy,$$

$$X_0 = w(1)_{0,0} = 1 + (q + q^{-1})uv.$$

The weak closure of $C(\mathbf{T}\backslash SU_q(2))$ in the GNS representation of h will be denoted by $L^\infty(\mathbf{T}\backslash SU_q(2))$, which can be regarded as the von Neumann subalgebra of M.

Let $\widehat{\Phi}$ be the left $(\widehat{M}, \widehat{\delta})$ action on M defined in Subsection 2-4. The following lemma shows that $\widehat{\Phi}$ preserves $C(\mathbf{T}\backslash SU_q(2))$:

Lemma 4.4 *Let $M(\widehat{A} \otimes C(\mathbf{T}\backslash SU_q(2)))$ be the multiplier algebra of $\widehat{A} \otimes C(\mathbf{T}\backslash SU_q(2))$. Then,*

$$\widehat{\Phi}(C(\mathbf{T}\backslash SU_q(2))) \subset M(\widehat{A} \otimes C(\mathbf{T}\backslash SU_q(2))).$$

In consequence, the restriction of $\widehat{\Phi}$ to $L^\infty(\mathbf{T}\backslash SU_q(2))$ is a left action of $(\widehat{M}, \widehat{\delta})$ on $L^\infty(\mathbf{T}\backslash SU_q(2))$.

Proof. Let $a \in C(\mathbf{T}\backslash SU_q(2)))$. Then thanks to (2.1.3), $\widehat{\Phi}(a)$ has the form

$$\sum_{s\in\widehat{G}, i,j\in I_s} e(s)_{ij} \otimes a_{ij},$$

where $a_{ij} = \sum_{l\in I_s} w(s)^*_{li} a w(s)_{l,j}$. To prove the statement, it suffices to show $a_{ij} \in \widehat{A}(s) \otimes C(\mathbf{T}\backslash SU_q(2))$. Indeed,

$$\begin{aligned}
&(\gamma \otimes id) \cdot \delta(a_{ij}) \\
&= \sum_{l,\in I_s} (\gamma \otimes id) \cdot \delta(w(s)^*_{li} a w(s)_{l,j}) \\
&= \sum_{l,m,p,\in I_s} (\gamma \otimes id)((w(s)^*_{l,p} \otimes w(s)^*_{pi})\delta(a)(w(s)_{lm} \otimes w(s)_{m,j})) \\
&= \sum_{l,m,p,\in I_s} \gamma(w(s)^*_{l,p}\gamma(w(s)_{lm}) \otimes w(s)^*_{pi} a w(s)_{m,j} \\
&= \sum_{l,p,\in I_s} 1 \otimes w(s)^*_{pi} a w(s)_{p,j} \\
&= 1 \otimes a_{ij}.
\end{aligned}$$

Q.E.D.

Remark. As one can see in the proof, the above lemma actually holds for an arbitrary pair of a compact quantum group and its compact quantum subgroup.

4-5. Some equalities. We end this section with a few equalities involving the q-binomial coefficients, which will be used in next section. The following are elementary and easily shown by induction:

Lemma 4.5 *Let n be a non-negative integer. Then,*

(1) *For an indeterminate z,*

$$(z;q)_n = \sum_{j=0}^{n} q^{j(j-1)/2} \begin{bmatrix} n \\ j \end{bmatrix}_q (-z)^j.$$

(2) *For two elements a, b in an algebra satisfying $ba = qab$,*

$$(a+b)^n = \sum_{j=0}^{m} \begin{bmatrix} n \\ j \end{bmatrix}_q a^j b^{n-j}.$$

The next lemma is less elementary than the previous one though it is still easy to show.

Lemma 4.6 *The following equalities hold:*

(1) *For non-negative integers m, n, and p with $0 \le p \le m+n$,*

$$\begin{bmatrix} m+n \\ p \end{bmatrix}_q = \sum_j q^{j(j+n-p)} \begin{bmatrix} m \\ j \end{bmatrix}_q \begin{bmatrix} n \\ p-j \end{bmatrix}_q,$$

where j runs over all integers satisfying

$$\max\{0, p-n\} \le j \le \min\{m, p\}.$$

(2) *For an indeterminate z and an non-negative integer n,*

$$\frac{(q^{n+1};q)_n}{(z;q)_{n+1}} = \sum_{j=0}^{n} q^{j(2n+1)} \frac{(q^{-n};q)_j (q^{-n}z;q)_j}{(q;q)_j (z;q)_{j+1}}.$$

Proof. (1) Take two elements a, b in an algebra satisfying $ba = qab$. Lemma 4.6 implies

$$\begin{aligned}
(a+b)^{m+n} &= (a+b)^m (a+b)^n \\
&= \sum_{j=0}^{m} \sum_{i=0}^{n} \begin{bmatrix} m \\ j \end{bmatrix}_q \begin{bmatrix} n \\ i \end{bmatrix}_q a^{m-j} b^j a^i b^{n-i} \\
&= \sum_{j=0}^{m} \sum_{i=0}^{n} \begin{bmatrix} m \\ j \end{bmatrix}_q \begin{bmatrix} n \\ i \end{bmatrix}_q q^{ij} a^{m-j+i} b^{n-i+j},
\end{aligned}$$

which shows the statement.

(2) We introduce a polynomial $g(z)$ of degree at most n by

$$g(z)=\sum_{j=0}^{n}(-1)^j q^{j(j+2n+1)/2}\left[\begin{array}{c} n \\ j \end{array}\right]_q (q^{-n}z;q)_j(q^n z;q^{-1})_{n-j}.$$

Then, to prove the statement, it suffices to show $g(z)=(q^{n+1};q)_n$ for $n+1$ distinct z. Setting $z=q^{n-m}$ with $0\le m\le n$, we get

$$\begin{aligned} g(q^{n-m}) &= \sum_{j=0}^{m} q^{j(j+n-m)}\left[\begin{array}{c} n \\ j \end{array}\right]_q \frac{(q;q)_{2n-m}(q;q)_m}{(q;q)_{n+j-m}(q;q)_{m-j}} \\ &= (q;q)_n \sum_{j=0}^{m} q^{j(j+n-m)}\left[\begin{array}{c} m \\ j \end{array}\right]_q \left[\begin{array}{c} 2n-m \\ n-j \end{array}\right]_q . \end{aligned}$$

(1) implies that this is equal to

$$(q;q)_n\left[\begin{array}{c} 2n \\ n \end{array}\right]_q = \frac{(q;q)_{2n}}{(q;q)_n} = (q^{n+1};q)_n.$$

Q.E.D.

5 $SU_q(2)$, Spin 1/2 Case

The goal of the rest of this paper is to apply the general theory established in Section 3 to a concrete example; we determine the structure of the relative commutant R, or equivalently, the structure of $H^\infty(\widehat{M},P_\mu)$ for $G=SU_q(2)$. In this section, we work on the spin 1/2 Markov operator $P_{1/2}$. Based on the analysis of eigenvalues of $P_{1/2}$ in this section, we treat the general case in next section. For simplicity, $P_{1/2}$ and $Q_{1/2}$ will be denoted by P and Q respectively.

As we have already seen in (4) of Theorem 3.6, $H^\infty(\widehat{M},P)$ is the weak closure of the linear span of the spectral subspaces $H^\infty(\widehat{M},P)(s)$, $s\in\widehat{G}$. So, we first determine the structure of $H^\infty(\widehat{M},P)(s)$ for each s. For this purpose, it is enough to obtain the highest weight vectors because the fact

$$\Phi\cdot P=(P\otimes id)\cdot\Phi$$

implies that P commutes with the adjoint representation of $U_q(su(2))$. Thanks to Corollary 4.3, we already know all the integer spin highest weight vectors in each $\widehat{A}(t)$, $t \in \widehat{G}$ for the adjoint representation. (Note that Φ has only integer spins and it is actually an action of $SO_q(3)$ rather than $SU_q(2)$.) Therefore, our main concern is how to assemble them to make bounded P-harmonic elements.

From now on, we omit the symbol $\mathcal{R}$ when there is no possibility of confusion, and identify $\widehat{\mathcal{A}}$, $\widetilde{\widehat{M}}$, and $\prod_{s\in\widehat{G}} \widehat{A}(s)$. For every $X \in \widehat{\mathcal{A}}$ and q-trace T_s, $(id\otimes T_s)(\widehat{\delta}(X))$ makes sense as an element of $\widehat{\mathcal{A}}$. Therefore, we can extend P and Q to $\widehat{\mathcal{A}}$ in a natural way. The extensions will be denoted by $\widetilde{P}$ and $\widetilde{Q}$.

Lemma 5.1 *Let n be a non-negative integer and $e, k, C \in U_q(su(2))$ be as in the previous section. Then,*

$$\widetilde{Q}(e^n k^n) = (q^n + q^{-n})e^n k^n, \tag{1}$$

$$\widetilde{Q}(Ce^n k^n) = (q^{n+2} + q^{-(n+2)})Ce^n k^n + 2(n + \frac{3}{2})_q(\frac{1}{2})_q e^n k^n. \tag{2}$$

Proof. (1) Thanks to Lemma 4.5, we have

$$\begin{aligned}
\widehat{\delta}(e^n k^n) &= (e \otimes k + k^{-1} \otimes e)^n (k^n \otimes k^n) \\
&= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{q^2} (k^{-j} e^{n-j} k^n \otimes e^j k^{2n-j}) \\
&= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{q^2} q^{j(j-n)} (e^{n-j} k^{n-j} \otimes e^j k^{2n-j}).
\end{aligned}$$

Since $T_{1/2}(e^j k^{2n-j}) \neq 0$ only if $j = 0$, we get the formula.

(2) Direct computation shows

$$\begin{aligned}
\widehat{\delta}(C) &= C \otimes k^2 + k^{-2} \otimes C \\
&- \frac{(q + q^{-1})(k^{-2} \otimes k^2) + 2 - 2(1 \otimes k^2 + k^{-2} \otimes 1)}{(q - q^{-1})^2} \\
&+ k^{-1}e \otimes fk + fk^{-1} \otimes ke.
\end{aligned}$$

Thus, the only terms of $\widehat{\delta}(Ce^nk^n)$ that may survive in $(id \otimes T_{1/2})$ are the following:

$$
\begin{aligned}
&Ce^nk^n \otimes k^{2(n+1)} - \frac{2}{(q-q^{-1})^2} e^nk^n \otimes (k^{2n} - k^{2n+2}) \\
+ \quad & q^{-2n} e^n k^{n-2} \otimes \Big(q^{n+1} n_q fe + C + \frac{2-(q+q^{-1})k^2}{(q-q^{-1})^2} \Big) k^{2n}.
\end{aligned}
$$

In fact, the last term does not survive because $\widetilde{Q}(Ce^nk^n)$ should be a highest weight vector of spin n for the adjoint representation (it can be shown by direct computation as well). Thus, we obtain the statement. Q.E.D.

Now we determine the action of P on the highest weight vectors.

Lemma 5.2 *Let $\mathcal{E}_s$ be the natural projection from $\widehat{\mathcal{A}}$ onto $\widehat{A}(s)$, $s \in \widehat{G}$. For a natural number l, we set $x_{n,l} = \mathcal{E}_{\frac{l-1}{2}}(e^nk^n)$, (Note that $x_{n,l} \neq 0$, if and only if $l \geq n+1$). Then,*

$$
P(x_{n,n+1}) = \frac{1}{(q+q^{-1})(n+2)_q} x_{n,n+2},
$$

$$
P(x_{n,l}) = \frac{(l-n)_q}{(q+q^{-1})(l+1)_q} x_{n,l+1} + \frac{(l+n)_q}{(q+q^{-1})(l-1)_q} x_{n,l-1}, \quad l \geq n+2.
$$

Proof. As mentioned in the remark after Theorem 3.6, we have

$$
Q(\widehat{A}(s)) \subset \widehat{A}(s - \frac{1}{2}) \oplus \widehat{A}(s + \frac{1}{2}).
$$

Since $Q(x_{n,l})$ is again a highest weight vector of spin n for the adjoint representation, there exist two numbers $a_n(l+1,l)$, $l \geq n$ and $a_n(l-1,l)$, $l \geq n+1$ with $a_n(n,n+1) = 0$, $a_n(n+1,n) = 0$ satisfying

$$
Q(x_{n,l}) = a_n(l+1,l)x_{n,l+1} + a_n(l-1,l)x_{n,l-1}, \quad l \geq n+1.
$$

We claim that the following hold:

$$
\mathcal{E}_{\frac{l-1}{2}}(\widetilde{Q}(e^nk^n)) = (a_n(l,l-1) + a_n(l,l+1))x_{n,l}, \quad l \geq n+1,
$$

$$
\mathcal{E}_{\frac{l-1}{2}}(\widetilde{Q}(Ce^nk^n)) = [(\frac{l-1}{2})_q^2 a_n(l,l-1) + (\frac{l+1}{2})_q^2 a_n(l,l+1)]x_{n,l}, \quad l \geq n+1.
$$

Indeed, we set

$$\xi = (q + q^{-1})\Lambda_h(qx + q^{-1}y) \in H_h.$$

Then, (4.1.1) implies that for every $a \in \widehat{\mathcal{A}}$,

$$T_{1/2}(a) = \langle a\xi, \xi\rangle.$$

Thus, we get

$$\begin{aligned}
&\langle \widetilde{Q}(e^n k^n)\Lambda_h(w(s)_{0,i}), \Lambda_h(w(s)_{0,j})\rangle \\
&\quad = \langle V^*(1 \otimes \mathcal{R}(e^n k^n))V(\Lambda_h(w(s)_{0,i}) \otimes \xi), \Lambda_h(w(s)_{0,j}) \otimes \xi\rangle \\
&\quad = \sum_{p,r \in I_s} \langle \Lambda_h(w(s)_{0,p}) \otimes \mathcal{R}(e^n k^n) w(s)_{p,i}\xi, (\Lambda_h(w(s)_{0,r}) \otimes w(s)_{r,j}\xi)\rangle.
\end{aligned}$$

Since $w(s)_{p,i}\xi \in \Lambda_h(W(s+\frac{1}{2})) + \Lambda_h(W(s-\frac{1}{2}))$, we have

$$\mathcal{R}(e^n k^n) w(s)_{p,i}\xi = (\mathcal{E}_{s+\frac{1}{2}}(e^n k^n) + \mathcal{E}_{s-\frac{1}{2}}(e^n k^n)) w(s)_{p,i}\xi.$$

Thus, we can replace $e^n k^n$ with $x_{n,2s} + x_{2s+2}$ in the above equation, and so

$$\mathcal{E}_s(\widetilde{Q}(e^n k^n)) = \mathcal{E}_s(Q(x_{n,2s}) + Q(x_{n,2s+2})).$$

This proves the first equality of the claim, and the second one can be shown in the same way.

Now using Lemma 5.1, we get

$$a_n(l, l-1) + a_n(l, l+1) = q^{n+2} + q^{-(n+2)}, \quad l \geq n+1,$$

$$\begin{aligned}
&(\frac{l-1}{2})_q^2 a_n(l, l-1) + (\frac{l+1}{2})_q^2 a_n(l, l+1) \\
&\quad = (q^{n+2} + q^{-(n+2)})(\frac{l}{2})_q^2 + 2(n + \frac{3}{2})_q(\frac{1}{2})_q, \quad l \geq n+1.
\end{aligned}$$

Solving these, we get

$$a_n(l, l-1) = \frac{(l-n-1)_q}{l_q}, \quad a_n(l, l+1) = \frac{(l+n+1)_q}{l_q}.$$

Q.E.D.

Thanks to Corollary 4.3, $a \in \widehat{\mathcal{A}}$ is a highest weight vector of spin n if and only if there exists a sequence $\{a_l\}_{l=1}^{\infty}$ such that $a_l = 0$ for $l \leq n$ and

$$a = (a_{2s+1} x_{n,2s+1}) \in \prod_{s \in \widehat{G}} \widehat{A}(s).$$

Moreover, the above lemma implies that a is an eigenvector of $\widetilde{P}$ for an eigenvalue λ if and only if $\{a_l\}$ satisfies the following recurrence relation:

$$\lambda(q+q^{-1})(n+1)_q a_{n+1} = (2(n+1))_q a_{n+2},$$

$$\lambda(q+q^{-1}) l_q a_l = (l+n+1)_q a_{l+1} + (l-n-1)_q a_{l-1}, \quad l \geq n+2. \tag{5.1}$$

Therefore, for each eigenvalue λ, the eigenspace of $\widetilde{P}$ in $\widehat{\mathcal{A}}$ contains each integer spin spectral subspace for the adjoint representation with multiplicity exactly one. In this sense, the eigenvalue problem of $\widetilde{P}$ is completely solved. To settle the problem for P, namely to single out bounded eigenvectors from the above ones, we need some estimates.

It is easy to show the following lemma from (4.2.2):

Lemma 5.3 *For each non-negative integer n, there exist two positive constant C_1 and C_2, depending only on q and n, such that*

$$C_1 q^{-nl} \leq ||x_{n,l}|| \leq C_2 q^{-nl}, \quad l \geq n+1.$$

For a non-negative integer n and a non-zero complex number α with $|\alpha| \leq 1$, we introduce a sequence $\{a_l^{n,\alpha}\}_{l=n+1}^{\infty}$ determined by the following recurrence relation:

$$a_{n+1}^{n,\alpha} = 1, \quad a_{n+2}^{n,\alpha} = \frac{(1+\alpha^2)(1-q^{2(n+1)})}{1-q^{4(n+1)}} = \frac{1+\alpha^2}{1+q^{2(n+1)}},$$

$$\begin{aligned}&(\alpha+\alpha^{-1})(1-q^{2l}) a_l^{n,\alpha} \\ &= \alpha^{-1}(1-q^{2(l+n+1)}) a_{l+1}^{n,\alpha} + \alpha(1-q^{2(l-n-1)}) a_{l-1}^{n,\alpha}, \quad l \geq n+2.\end{aligned}$$

Lemma 5.4 *Let $\{a_l^{n,\alpha}\}_{l=n+1}^{\infty}$ be as above. Then,*

(1) *For $\alpha^2 \neq 1$, $\{a_l^{n,\alpha}\}_{l=n+1}^{\infty}$ is bounded. Moreover, if $|\alpha| = 1$, we have*

$$\lim_{l\to\infty} \Big(a_l^{n,\alpha} - \frac{\alpha^{2l}(q^{2(n+1)};q^2)_{n+1}}{\alpha^{2(n+1)}(\alpha^{-2};q^2)_{n+1}} \Big) = \frac{(q^{2(n+1)};q^2)_{n+1}}{(\alpha^2;q^2)_{n+1}}.$$

If $|\alpha| < 1$, we have

$$\lim_{l\to\infty} a_l^{n,\alpha} = \frac{(q^{2(n+1)}, q^2)_{n+1}}{(\alpha^2, q^2)_{n+1}}.$$

In particular, for $\alpha = q$,

$$\lim_{l\to\infty} a_l^{n,q} = \left[\begin{array}{c} 2n+1 \\ n \end{array}\right]_{q^2}.$$

(2) *For $\alpha^2 = 1$,*

$$\lim_{l\to\infty} \frac{a_l^{n,\alpha}}{l} = \frac{(q^2;q^2)_{2n+1}}{(q^2;q^2)_n^2}.$$

(3) *If α is a positive number, then $a_l^{n,\alpha} > 0$ for every $l \geq n+1$.*

Proof. (1) It is easy to show that for fixed n and α, there exists $c > 0$ such that $|a_l^{n,\alpha}| \leq c^l$ for all $l \geq n+1$. Thus, we can introduce an analytic function $g_{n,\alpha}(z)$ defined on a neighborhood of 0 by

$$g_{n,\alpha}(z) = \sum_{l=n+1}^{\infty} a_l^{n,\alpha} z^l.$$

The recurrence relation of $\{a_l^{n,\alpha}\}_{l=n+1}^{\infty}$ implies the following function equation:

$$(1-z)(1-\alpha^2 z)g_{n,\alpha}(z) = q^{2n}(1-q^{-2n}z)(1-q^{-2n}\alpha^2 z)g_{n,\alpha}(q^2 z) + (1-q^{2(2n+1)})z^{n+1}.$$

Therefore, $g_{n,\alpha}$ has analytic continuation to

$$D = \mathbf{C} \setminus (\{q^{-2j};\ j \in \mathbf{Z}_{\geq 0}\} \cup \{\alpha^{-2}q^{-2j};\ j \in \mathbf{Z}_{\geq 0}\}).$$

The above function equation implies

$$\begin{aligned}
&(1-q^{2(2n+1)})z^{n+1}q^{2l(2n+1)}\frac{(q^{-2n}z;q^2)_l(\alpha^2 q^{-2n}z;q^2)_l}{(z;q^2)_{l+1}(\alpha^2 z;q^2)_{l+1}} \\
= \ & \frac{(q^{-2n}z;q^2)_l(\alpha^2 q^{-2n}z;q^2)_l}{(z;q^2)_l(\alpha^2 z;q^2)_l}q^{2nl}g_{n,\alpha}(q^{2l}z) \\
- \ & \frac{(q^{-2n}z;q^2)_{l+1}(\alpha^2 q^{-2n}z;q^2)_{l+1}}{(z;q^2)_{l+1}(\alpha^2 z;q^2)_{l+1}}q^{2n(l+1)}g_{n,\alpha}(q^{2(l+1)}z).
\end{aligned}$$

It is easy to show that

$$\frac{(q^{-2n}z;q^2)_l(\alpha^2q^{-2n}z;q^2)_l}{(z;q^2)_l(\alpha^2z;q^2)_l}q^{2nl}g_{n,\alpha}(q^{2l}z)$$

uniformly converges to 0 on every compact subset of D when l tends to infinity. Therefore, we get

$$g_{n,\alpha}(z)=(1-q^{2(2n+1)})z^{n+1}\sum_{l=0}^{\infty}q^{2l(2n+1)}\frac{(q^{-2n}z;q^2)_l(\alpha^2q^{-2n}z;q^2)_l}{(z;q^2)_{l+1}(\alpha^2z;q^2)_{l+1}}.$$

If $\alpha^2\neq 1$ and $|\alpha|=1$, $z=1$ and $z=\alpha^{-2}$ are poles of order at most one. Indeed using Lemma 4.6, we get

$$\begin{aligned}\lim_{z\to 1}(1-z)g_{n,\alpha}(z) &= (1-q^{2(2n+1)})\sum_{l=0}^{n}q^{2l(2n+1)}\frac{(q^{-2n};q^2)_l(\alpha^2q^{-2n};q^2)_l}{(q^2;q^2)_l(\alpha^2;q^2)_{l+1}}\\ &= \frac{(q^{2(n+1)};q^2)_{n+1}}{(\alpha^2;q^2)_{n+1}},\end{aligned}$$

$$\begin{aligned}\lim_{z\to\alpha^{-2}}(1-\alpha^2z)g_{n,\alpha}(z) &= \frac{(1-q^{2(2n+1)})}{\alpha^{2(n+1)}}\sum_{l=0}^{n}q^{2l(2n+1)}\frac{(q^{-2n};q^2)_l(\alpha^{-2}q^{-2n};q^2)_l}{(q^2;q^2)_l(\alpha^{-2};q^2)_{l+1}}\\ &= \frac{(q^{2(n+1)};q^2)_{n+1}}{\alpha^{2(n+1)}(\alpha^{-2};q^2)_{n+1}}.\end{aligned}$$

Therefore, there exists an analytic function $g^1_{n,\alpha}(z)$ on $|z|<q^{-2}$ such that

$$\begin{aligned}g^1_{n,\alpha}(z) &= g_{n,\alpha}(z)-\frac{(q^{2(n+1)};q^2)_{n+1}}{(\alpha^2;q^2)_{n+1}}\frac{1}{1-z}-\frac{(q^{2(n+1)};q^2)_{n+1}}{\alpha^{2(n+1)}(\alpha^{-2};q^2)_{n+1}}\frac{1}{1-\alpha^2z}\\ &= \sum_{l=0}^{\infty}\big(a^{n,\alpha}_l-\frac{(q^{2(n+1)};q^2)_{n+1}}{(\alpha^2;q^2)_{n+1}}-\frac{\alpha^{2l}(q^{2(n+1)};q^2)_{n+1}}{\alpha^{2(n+1)}(\alpha^{-2};q^2)_{n+1}}\big)z^l,\end{aligned}$$

where we set $a^{n,\alpha}_l=0$ for $l\leq n$. This shows the statement for $|\alpha|=1$. The $|\alpha|<1$ case can be treated in a similar way.

(2) In a similar way as above, we get the following for $\alpha^2=1$:

$$\begin{aligned}\lim_{l\to\infty}(1-z)^2g_{n,\alpha}(z) &= (1-q^{2(2n+1)})\sum_{l=0}^{n}q^{2l(2n+1)}\frac{(q^{-2n};q^2)_l^2}{(q^2;q^2)_l^2}\\ &= \frac{(q^2;q^2)_{2n+1}}{(q^2;q^2)_n^2}\end{aligned}$$

which shows the statement.

(3) Assume that α is positive number less than or equal to 1. Let r be a positive number. Then, the recurrence relation is equivalent to

$$\alpha^{-1}(1-q^{2(l+n+1)})(a_{l+1}^{n,\alpha}-ra_l^{n,\alpha})=\alpha r^{-1}(1-q^{2(l-n-1)})(a_l^{n,\alpha}-ra_{l-1}^{n,\alpha})+b_l a_l^{n,\alpha},$$

$$b_l:=\alpha(1-r^{-1})+\alpha^{-1}(1-r)-q^{2l}(\alpha+\alpha^{-1}-\alpha^{-1}rq^{2(n+1)}-\alpha r^{-1}q^{-2(n+1)}),$$

where $l\geq n+1$ and $a_n^{n,\alpha}$ is understood as 0. Therefore, to prove the statement by induction, it suffices to show that there exists r such that $b_l>0$ for all $l\geq n+1$, which is equivalent to the following two:

$$\alpha(1-r^{-1})+\alpha^{-1}(1-r)=\frac{(1-r)(r-\alpha^2)}{r\alpha}\geq 0.$$

$$b_{n+1}=\frac{(1-q^{4(n+1)})}{\alpha}\left(\frac{1+\alpha^2}{1+q^{2(n+1)}}-r\right)>0.$$

Such a choice of r is always possible. Q.E.D.

For a non-negative integer n and a complex number λ, we introduce a sequence $\{b_l^{n,\lambda}\}_{l=1}^{\infty}$ with $b_l^{n,\lambda}$=0 for $l\leq n$ satisfying the following recurrence relation (See [26] for a similar recurrence relation appearing in the spherical functions on the quantum spheres):

$$b_{n+1}^{n,\lambda}=1,\quad b_{n+2}^{n,\lambda}=\frac{\lambda(q+q^{-1})}{1+q^{2(n+1)}},$$

$$\lambda(q+q^{-1})(1-q^{2l})b_l^{n,\lambda}=(1-q^{2(l+n+1)})b_{l+1}^{n,\lambda}+(1-q^{2(l-n-1)})b_{l-1}^{n,\lambda},\quad l\geq n+1.$$

Note that if α is a complex number with $|\alpha|\leq 1$ satisfying

$$\alpha+\alpha^{-1}=\lambda(q+q^{-1}),$$

then, $b_l^{n,\lambda}=\alpha^{n+1-l}a_l^{n,\alpha}$. Moreover, $a_l=q^{(n+1)l}b_l^{n,\lambda}$ satisfies (5.1). Thus, we can construct eigenvectors $\{h^\lambda(n)_j\}_{j\in I_n}$ of $\widetilde{P}$ for λ by setting

$$h^\lambda(n)_{-n}=(q^{2s(n+1)}b_{2s+1}^{n,\lambda}x_{n,2s+1})\in\prod_{s\in\widehat{G}}\widehat{A}(s),$$

$$h^\lambda(n)_j=\frac{1}{\sqrt{(n+1-j)_q(n+j)_q}}ad(f)h^\lambda(n)_{j-1}.$$

Note that Lemma 5.3 and Lemma 5.4 imply that $h^\lambda(n)_j$ is bounded if and only if $q\leq|\alpha|\leq 1$. For $\lambda=1$, $h^\lambda(n)_j$ will be simply denoted by $h(n)_j$. Summing up the results we have obtained so far, we get the following theorem:

Theorem 5.5 *For a non-negative integer n and a complex number λ, we denote by $\widehat{M}(P,\lambda)(n)$ and $\widehat{\mathcal{A}}(\widetilde{P},\lambda)(n)$ the eigenspaces of P and $\widetilde{P}$ for the eigenvalue λ with spin n for the adjoint representation. Then,*

(1) *$\{h^\lambda(n)_j\}_{j\in I_n}$ is a basis of $\widehat{\mathcal{A}}(\widetilde{P},\lambda)(n)$.*

(2) *$\widehat{M}(P,\lambda)(n) = \widehat{\mathcal{A}}(\widetilde{P},\lambda)(n)$, namely $h^\lambda(n)_j$ is bounded for $j \in I_n$ if and only if $\lambda = \xi + i\eta$, $\xi,\eta \in \mathbf{R}$ satisfies*

$$\xi^2 + (\frac{1+q^2}{1-q^2})^2\eta^2 \leq 1.$$

As a corollary, we get

Corollary 5.6 *Let n be a non-negative integer. Then,*

(1) *The multiplicity of spectral subspace $H^\infty(\widehat{M},P)(n)$ for the adjoint representation is exactly one, and $\{h(n)_j\}_{j\in I_n}$ is a basis of $H^\infty(\widehat{M},P)(n)$.*

(2) *A complex number λ is an eigenvalue of P restricted to the spin n spectral subspace of $\widehat{M}$ if and only if it is an eigenvalue of P restricted to the center $Z(\widehat{M})$.*

(3) *$\widetilde{P}$ restricted to the center $Z(\widehat{\mathcal{A}})$ has a positive eigenvector for eigenvalue λ if and only if*

$$\lambda \geq \frac{2}{q+q^{-1}}.$$

Now we decide the algebra structure of R using the Choi-Effros product on $H^\infty(\widehat{M},P)$. Let χ_i, $i=-1,0,1$ be as in Lemma 4.1. Then, we have

$$h(1)_j = (a^{1,q}_{2s+1}q^{2s+1}\mathcal{E}_s(\chi_j)) \in \prod_{s\in\widehat{G}} \widehat{A}(s),$$

$$h(n)_{-n} = (a^{n,q}_{2s+1}q^{(2s+1)n}x_{n,2s+1}) \in \prod_{s\in\widehat{G}} \widehat{A}(s).$$

It is straightforward to show the following:

Lemma 5.7 *Let C be the Casimir element. Then,*

$$\chi_0^* = \chi_0, \quad \chi_{-1}^* = -q\chi_1, \quad \chi_1^* = -q^{-1}\chi_1,$$

$$q^{-1}\chi_0\chi_{-1} - q\chi_{-1}\chi_0 = \frac{1-q^{-2}}{\sqrt{2_q}}(C + \frac{2}{(q-q^{-1})^2})(q^{-1}-q)\chi_{-1},$$

$$\chi_0^2 - q\chi_{-1}\chi_1 - q^{-1}\chi_1\chi_{-1} = \frac{(1-q^{-2})^2}{2_q}[(C + \frac{2}{(q-q^{-1})^2})^2 - \frac{(q+q^{-1})^2}{(q-q^{-1})^4}].$$

Thanks to (3) of Lemma 5.4, for each non-negative integer n we introduce a family of nearest neighbor transition probability on the set $\mathbf{N}_{\geq n+1}$ of natural numbers larger than n by

$$p_n(l,l+1) = \frac{(1-q^{2(l+n+1)})a_{l+1}^{1,q}}{(1+q^2)(1-q^{2l})a_l^{1,q}}, \quad l \geq n+1,$$

$$p_n(l,l-1) = \frac{q^2(1-q^{2(l-n-1)})a_{l-1}^{1,q}}{(1+q^2)(1-q^{2l})a_l^{1,q}}, \quad l \geq n+2.$$

Since

$$p(l,l+1) \sim \frac{1}{1+q^2} > \frac{1}{2} \quad (l \to \infty),$$

these random walks are transient. Lemma 5.2 implies

$$\begin{aligned} & P(q^{nl}a_l^{n,q}x_{n,l}) \\ &= p_n(l+1,l)q^{n(l+1)}a_{l+1}^{n,q}x_{n,l+1} + p_n(l-1,l)q^{n(l-1)}a_{l-1}^{n,q}x_{n,l-1}, \quad l \geq n+1, \end{aligned}$$

where $p_n(n,n+1)$ should be understood as 0. We denote by $p_n^{(m)}(j,l)$ the m-step transition probability from j to l.

Lemma 5.8 *With the above notation, we have*

(1) $$h(1)_0^* = h(1)_0, \quad h(1)_{-1}^* = -qh(1)_1, \quad h(1)_1^* = -q^{-1}h(1)_1,$$

(2)
$$s-\lim_{m\to\infty} P^m(q^{-1}h(1)_0h(1)_{-1} - qh(1)_{-1}h(1)_0) = \frac{-(1-q^6)}{(1-q^2)^2\sqrt{2_q}}(q^{-1}-q)h(1)_{-1}.$$

(3) $$s-\lim_{m\to\infty} P^m(h(1)_0^2 - qh(1)_{-1}h(1)_1 - q^{-1}h(1)_1h(1)_{-1}) = [\frac{(1-q^6)}{(1-q^2)^2\sqrt{2_q}}]^2.$$

(4) $$s-\lim_{m\to\infty} P^m(h(1)_{-1}{}^n) = \frac{q^{n(n-1)/2}(1-q^6)^n}{(1-q^2)^n \left[\begin{array}{c} 2n+1 \\ n \end{array}\right]_{q^2}} h(n)_{-n}.$$

Proof. (1) is an immediate consequence of the previous Lemma 5.7. (2) For $l \geq n+1$, we set

$$\xi(l) = -\frac{q^{-1}(1+q^{2l})a_l^{1,q}}{\sqrt{2}_q}.$$

Then, Lemma 5.7 implies

$$q^{-1}h(1)_0h(1)_{-1} - qh(1)_{-1}h(1)_0 = (\xi(2s+1)a_{2s+1}^{1,q}q^{2s+1}\mathcal{E}_s(\chi_{-1})) \in \prod_{s\in\widehat{G}} \widehat{A}(s).$$

Therefore we get

$$\begin{aligned} &P^m(q^{-1}h(1)_0h(-1) - qh(1)_{-1}h(1)_0) \\ &= \ (\xi^{(m)}(2s+1)a_{2s+1}^{1,q}q^{2s+1}\mathcal{E}_s(\chi_{-1})) \in \prod_{s\in\widehat{G}} \widehat{A}(s), \end{aligned}$$

where

$$\xi^{(m)}(l) = \sum_{j=n+1}^{\infty} p_1^{(m)}(l,j)\xi(j).$$

As the random walk determined by $p_1(l,j)$ is transient, (which is in fact more than enough), for fixed l and a finite set F we have

$$\lim_{m\to\infty} \sum_{j\in F} p_1^{(m)}(l,j) = 0.$$

Since $\xi(l)$ has limit as l goes to infinity, we get

$$\lim_{m\to\infty} \xi^{(m)}(l) = \lim_{j\to\infty} \xi(j) = -\frac{q^{-1}}{\sqrt{2}_q}\left[\begin{array}{c} 3 \\ 1 \end{array}\right]_{q^2} = -\frac{1-q^6}{q(1-q^2)\sqrt{2}_q},$$

where we use Lemma 5.4. Therefore, we get the statement.

(3) In a similar way, we set

$$\eta(l) = \frac{[(1+q^{2l})^2 - (q+q^{-1})q^{2l}](a_l^{1,q})^2}{2_q(1-q^2)^2}.$$

Then,

$$h(1)_0^2 - qh(1)_{-1}h(1)_1 - q^{-1}h(1)_1h(1)_{-1} = (\eta(2s+1)1_s).$$

(Note that $a_l^{0,q} = 1$). The same argument using $p_0(l,j)$ implies the statement from

$$\lim_{j\to\infty} \eta(j) = \frac{(1-q^6)^2}{(1-q^2)^4 2_q}.$$

(4) We set

$$\zeta(l) = q^{n(n-1)/2}\frac{(a_l^{1,q})^n}{a_l^{n,q}}.$$

Then,

$$h(1)_{-1}{}^n = (\zeta(2s+1)a_{2s+1}^{n,q}q^{n(2s+1)}x_{n,2s+1}).$$

The same argument using $p_n(l,j)$ implies the statement from the limit

$$\lim_{j\to\infty} \zeta(j) = \frac{q^{n(n-1)/2}(1-q^6)^n}{(1-q^2)^n \begin{bmatrix} 2n+1 \\ n \end{bmatrix}_{q^2}}.$$

Q.E.D.

Using Podles's classification of quantum spheres [32], we get the main result of this section.

Theorem 5.9 *With the above notation, the following hold:*

(1) *There exists a $*$-isomorphism θ_1 from the relative commutant R onto $L^\infty(\mathbf{T}\backslash SU_q(2))$ such that*

$$\varphi_R = h \cdot \theta_1,$$

$$(\theta_1 \otimes id) \cdot \alpha_R = \delta \cdot \theta_1.$$

(2) *There exists a completely positive isometry θ_0 from $H^\infty(\widehat{M}, P)$ onto $L^\infty(\mathbf{T}\backslash SU_q(2))$ commuting with the (M, δ) actions such that for every non-negative integer n and $j \in I_n$, we have*

$$\theta_0(h(n)_j) = \frac{(-1)^n q^{n/2}(1-q^{2(2n+1)})}{(1-q^2)^n(1-q^{2(n+1)})} \begin{bmatrix} 2n \\ n \end{bmatrix}_{q^2}^{1/2} w(n)_{0,j}.$$

Proof. (1) Let θ be the map constructed in Theorem 3.6. We set

$$X_i = -\frac{q^{-1/2}(1+q^2)^{1/2}(1-q^2)^2}{(1-q^6)}\theta(h(1)_i), \quad i=-1,0,1.$$

Since θ commutes with the (M,δ) actions, $\{X_i\}_{i\in I_1}$ is a basis of the spectral subspace $R(1)$ satisfying

$$\alpha_R(X_i) = \sum_{j\in I_1} X_j \otimes w(1)_{j,i}.$$

Let D be the C*-algebra generated by $\{X_i\}_{i\in I_1}$. Then, $(D, \alpha_{|D})$ is isomorphic to one of the quantum spheres. Lemma 5.8 implies

$$X_0^* = X_0, \quad X_{-1}^* = -qX_1, \quad X_1^* = -q^{-1}X_{-1},$$

$$q^{-1}X_0X_{-1} - qX_{-1}X_0 = (q^{-1}-q)X_{-1},$$
$$X_0^2 - qX_{-1}X_1 - q^{-1}X_1X_{-1} = 1,$$

which shows that $(D, \alpha_{|D})$ is isomorphic to $C(\mathbf{T}\backslash SU_q(2))$. In particular, D contains each integer spin spectral subspace with multiplicity one. Since the multiplicity of $R(s)$ is the same as that of $H^\infty(\widehat{M},P)(s)$ for every $s\in\widehat{G}$, Corollary 5.7 implies that $R(s)$ has multiplicity one for every integer s and 0 for every half-integer s. Thus, $R(n)\subset D$ for every non-negative integer n. Lemma 3.3 implies that D is weakly dense in R. Since φ_R is a faithful normal state on R that is invariant under α_R, the isomorphism θ_1 from D to $C(\mathbf{T}\backslash SU_q(2))$ defined by

$$\theta_1(X_i) = w(1)_{0,i}, \quad i=-1,0,1,$$

extends to a $*$-isomorphism from R onto $L^\infty(\mathbf{T}\backslash SU_q(2))$ satisfying the desired properties.

(2) We set $\theta_0 = \theta_1\cdot\theta$, which is a completely positive isometry from $H^\infty(\widehat{M},P)$ to $L^\infty(\mathbf{T}\backslash SU_q(2))$. Using (4.1.5), we get

$$w(n)_{0,-n} = \begin{bmatrix} 2n \\ n \end{bmatrix}_{q^2}^{1/2} x^n v^n = \frac{q^{-n(n-1)/2}}{(1+q^2)^{n/2}} \begin{bmatrix} 2n \\ n \end{bmatrix}_{q^2}^{1/2} (w(1)_{0,1})^n.$$

Thus,

$$\begin{aligned}\theta_0(h(1)_{-1})^n &= \frac{(-1)^n q^{n/2}(1-q^6)^n}{(1-q^2)^{2n}(1+q^2)^{n/2}}(w(1)_{0,-1})^n \\ &= \frac{(-1)^n q^{n^2/2}(1-q^6)^n}{(1-q^2)^{2n}\begin{bmatrix} 2n \\ n \end{bmatrix}_{q^2}^{1/2}} w(n)_{0,-n}.\end{aligned}$$

On the other hand, (2) of Theorem 3.6 and (4) of Lemma 5.8 imply

$$\theta_0(h(1)_{-1})^n = \frac{q^{n(n-1)/2}(1-q^6)^n}{(1-q^2)^n \begin{bmatrix} 2n+1 \\ n \end{bmatrix}_{q^2}} \theta_0(h(n)_{-n}).$$

Thus, we get

$$\begin{aligned}\theta_0(h(n)_{-n}) &= \frac{(-1)^n q^{n/2} \begin{bmatrix} 2n+1 \\ n \end{bmatrix}_{q^2}}{(1-q^2)^n \begin{bmatrix} 2n \\ n \end{bmatrix}_{q^2}^{1/2}} w(n)_{0,-n} \\ &= \frac{(-1)^n q^{n/2}(1-q^{2(2n+1)})}{(1-q^2)^n(1-q^{2(n+1)})} \begin{bmatrix} 2n \\ n \end{bmatrix}_{q^2}^{1/2} w(n)_{0,-n}.\end{aligned}$$

The general statement for $\{h(n)_j\}_{j\in I_n}$ follows from the fact that θ_0 commutes with the (M,δ) actions. Q.E.D.

In view of the classical theory of the Poisson boundaries, for a harmonic element $a \in H^\infty(\widehat{M}, P)$, $\theta_0(a)$ is considered to be the "boundary value" of a on $\mathbf{T}\backslash SU_q(2)$. The following theorem provides a way to recover a from its "boundary value" using integration by the Haar measure.

Theorem 5.10 (Poisson Integral) *Let Θ be the map from M to $H^\infty(\widehat{M}, P)$ introduced in Section 3, that is,*

$$\Theta(a) = (id \otimes h) \cdot \widehat{\Phi}(a), \quad a \in M,$$

and let Θ_0 be restriction of Θ to $L^\infty(\mathbf{T}\backslash SU_q(2))$. Then, Θ_0 is the inverse of θ_0. In particular, Θ_0 is a complete positive isometry from $L^\infty(\mathbf{T}\backslash SU_q(2))$ onto $H^\infty(\widehat{M}, P)$ commuting both of the right (M,δ) actions and the left $(\widehat{M},\widehat{\delta})$ actions. (Note that the $(\widehat{M},\widehat{\delta})$ action on $L^\infty(\mathbf{T}\backslash SU_q(2))$ is well defined thanks to Lemma 4.4.)

Proof. Since $\theta_0 = \theta_1 \cdot \theta$ is a normal map, so is $\Theta_0 \cdot \theta_0$. Thus, to prove the statement it suffices to show $\Theta_0 \cdot \theta_0(h(n)_{-n}) = h(n)_{-n}$ for each non-negative integer n because $\Theta_0 \cdot \theta_0$ commutes with the right (M, δ) actions and the linear span of $\{h(n)_j\}_{n,j}$ is dense in weak topology. Since $\Theta_0 \cdot \theta(h(n)_{-n})$ is a highest weight vector of spin n, Corollary 5.6 implies that $\Theta_0 \cdot \theta(h(n)_{-n})$ is proportional to $h(n)_{-n}$. Therefore, we compare the $e(n/2)_{-n/2,n/2}$ components of $h(n)_{-n}$ and $\Theta_0 \cdot \theta(h(n)_{-n})$ to show that they are equal. The former is obtained from (4.2.1), and it is

$$\frac{q^{3n/2}(q^2;q^2)_n}{(1-q^2)^n}.$$

To obtain the latter, first we compute the $e(n/2)_{-n/2,n/2}$ component of $\Theta_0(w(n)_{0,-n})$, which is, by definition, given by

$$\begin{aligned}
&\sum_{j \in I_{n/2}} h(w(n/2)^*_{j,-n/2} w(n)_{0,-n} w(n/2)_{j,n/2}) \\
= &\sum_{j \in I_{n/2}} h(\sigma^h_i(w(n/2)_{j,n/2}) w(n/2)^*_{j,-n/2} w(n)_{0,-n}) \\
= \ & h([\sum_{j \in I_{n/2}} w(n/2)_{j,-n/2} \sigma^h_i(w(n/2)_{j,n/2})^*]^* w(n)_{0,-n}).
\end{aligned}$$

Using (4.1.4), (4.1.5), and Lemma 4.5, we get

$$\begin{aligned}
&\sum_{j \in I_{n/2}} w(n/2)_{j,-n/2} \sigma^h_i(w(n/2)_{j,n/2})^* \\
= &\sum_{l=0}^{n} q^{2l} \begin{bmatrix} n \\ l \end{bmatrix}_{q^2} x^{n-l} v^l (u^{n-l} y^l)^* \\
= \ &(-q^{-1})^n \sum_{l=0}^{n} q^{l(l-1)} (-q^4)^l \begin{bmatrix} n \\ l \end{bmatrix}_{q^2} x^n v^n \\
= \ &(-q^{-1})^n (q^4;q^2)_n x^n v^n \\
= \ &\frac{(-1)^n (q^2;q^2)_{n+1}}{q^n(1-q^2) \begin{bmatrix} 2n \\ n \end{bmatrix}_{q^2}^{1/2}} w(n)_{0,-n}.
\end{aligned}$$

This shows that the $e(n/2)_{-n/2,n/2}$ component of $\Theta_0(w(n)_{0,-n})$ is

$$\frac{(-1)^n (q^2;q^2)_{n+1}}{q^n(1-q^2)\begin{bmatrix} 2n \\ n \end{bmatrix}_{q^2}^{1/2}} h(w(n)^*_{0,-n} w(n)_{0,-n}).$$

Thus, (4.1.1) and Theorem 5.9 imply that the $e(n/2)_{-n/2,n/2}$ component of $\Theta_0 \cdot \theta(h(n)_{-n})$ is

$$\frac{(q^2;q^2)_n(1-q^{2(2n+1)})}{q^{n/2}(1-q^2)^{n+1}} h(w(n)^*_{0,-n} w(n)_{0,-n}) = \frac{q^{3n/2}(q^2;q^2)_n}{(1-q^2)^n}.$$

Q.E.D.

6 $SU_q(2)$, General Case

Now, we proceed to the general case; namely let $\mu \neq \delta_0$ be a finitely supported probability measure on $\widehat{G}$. The goal of this section is to show that the "Poisson boundary" for P_μ does not depends on the choice of μ and it is actually $L^\infty(\mathbf{T}\backslash SU_q(2))$.

Let $\{U_n(z)\}_{n=0}^\infty$ be the Tchebichef polynomials of second kind, which are defined by the recurrence relation

$$U_0(z) = 1, \quad U_1(z) = 2z,$$

$$U_{n+1}(z) = 2zU_n(z) - U_{n-1}(z), \quad n \geq 1,$$

or equivalently defined by

$$U_n(\cosh t) = \frac{\sinh(n+1)t}{\sinh t}.$$

Comparing the above recurrence relation with the fusion rules of $\widehat{G}$, we get

$$Q_s = U_{2s}(\frac{Q}{2}),$$

$$d(s) = (2s+1)_q = U_{2s}(\frac{q+q^{-1}}{2}).$$

Lemma 6.1 *Let $U_n'(z)$ be the derivative of $U_n(z)$. Then,*

$$U_n'(\frac{q+q^{-1}}{2}) > 0, \quad n \geq 1.$$

Proof. This follows from

$$U_{n+2}(\cosh t)' - U_n(\cosh t)' = 2(n+2)\sinh(n+2)t, \quad n \geq 0.$$

Q.E.D.

In the $SU_q(2)$ case, μ is a generating measure if and only if the support of μ contains a half-integer spin. Therefore, if μ is supported only by integer spins, the action α should be treated as an $SO_q(3)$ action rather than $SU_q(2)$. We first treat the generating case.

Theorem 6.2 *Let μ be a finitely supported generating measure on $\widehat{G}$. Then,*

$$H^\infty(\widehat{M}, P_\mu) = H^\infty(\widehat{M}, P).$$

In consequence, there exits an isomorphism from R of the corresponding ITP action onto $L^\infty(\mathbf{T}\backslash SU_q(2))$ commuting with the (M, δ) actions.

Proof. We define a polynomial $f(z)$ by

$$f(z) = \sum_{s \in \widehat{G}} \frac{\mu(s) U_{2s}(\frac{(q+q^{-1})z}{2})}{d(s)}.$$

Then, we have $P_\mu = f(P)$ and $f(1) = 1$. Thus, we get

$$H^\infty(\widehat{M}, P_\mu) \supset H^\infty(\widehat{M}, P).$$

Thanks to Lemma 6.1, there exists a polynomial $g(z)$ such that

$$f(z) - 1 = (z-1)g(z), \quad g(1) \neq 0.$$

This implies that for every non-negative integer n and a P_μ-harmonic element a of spin n,

$$g(P)(a) \in H^\infty(\widehat{M}, P)(n).$$

Since $g(1) \neq 0$, there exist $b \in H^\infty(\widehat{M}, P)(n)$ such that $g(P)(a) = g(P)(b)$. We set $a_1 = a - b$, which satisfies $g(P)(a_1) = 0$. Suppose $a_1 \neq 0$. Then, there would exist $1 \neq \lambda \in \mathbf{C}$ such that $g(\lambda) = 0$ and λ would be an eigenvalue of P restricted to the spectral subspace of $\widehat{M}$ of spin n. However, this implies, thanks to (2) of Corollary 5.6, that $h^\lambda(0)$ is bounded and

$$P_\mu(h^\lambda(0)) - h^\lambda(0) = (P - I)(g(P)(h^\lambda(0))) = 0,$$

that is, $h^\lambda(0)$ is a non-trivial harmonic element of P_μ restricted to $Z(\widehat{M})$. This contradicts Corollary 3.7, and so we get $a = b \in H^\infty(\widehat{M}, P)$. Since the Choi-Effros product is uniquely determine by the operator system structure of $H^\infty(\widehat{M}, P)$, we get the result of R from that of the spin $1/2$ case. Q.E.D.

Let $\widehat{M}_I$ be the von Neumann algebra direct sum of $\widehat{A}(n)$ over all non-negative integers n. Then, $\widehat{M}_I$ is regarded as the dual Hopf von Neumann algebra of $SO_q(3)$. We denote by E_I the projection from $\widehat{M}$ onto $\widehat{M}_I$. When μ is supported only by integer spins, $P_{\mu,I}$ stands for the restriction of P_μ to $\widehat{M}_I$. We set

$$H^\infty(\widehat{M}, P)_I = \{E_I(a) \in \widehat{M}_I;\ a \in H^\infty(\widehat{M}, P)\},$$

which actually coincides with $H^\infty(\widehat{M}_I, P_{1,I})$.

Theorem 6.3 *Let $\mu \neq \delta_0$ be a finitely supported measure on $\widehat{G}$ whose support contains integer spins only. Then, we have*

$$H^\infty(\widehat{M}_I, P_{\mu,I}) = H^\infty(\widehat{M}, P)_I.$$

Moreover, there exits an isomorphism from R of the corresponding ITP action onto $L^\infty(\mathbf{T}\backslash SU_q(2))$ commuting with the (M, δ) actions.

Proof. In the same way as above, we introduce a polynomial $f(z)$ by

$$f(z) = \sum_{s \in \widehat{G}} \frac{\mu(s) U_{2s}(\frac{(q+q^{-1})z}{2})}{d(s)}.$$

Note that $f(z)$ is a polynomial of z^2 because it is the case for $U_n(z)$ for even n. As before, we have $P_\mu = f(P)$ and $f(1) = 1$, and so

$$H^\infty(\widehat{M}_I, P_{\mu,I}) \supset H^\infty(\widehat{M}, P)_I.$$

$f(1) = 1$ implies that there exists a polynomial $g(z)$ such that

$$f(z) - 1 = (z^2 - 1)g(z^2), \quad g(1) \neq 0.$$

This implies that for every non-negative integer n and a $P_{\mu,I}$-harmonic element a of spin n,

$$P^2(g(P^2)(a)) = g(P^2)(a).$$

Thus, $g(P^2)(a) + P(g(P^2)(a))$ is P-harmonic. As before, there exists $b \in H^\infty(\widehat{M}, P)(n)$ such that

$$g(P^2)(a) + P(g(P^2)(a)) = g(P^2)(b).$$

We set $a_1 = a - E_I(b)$, which satisfies $g(P^2)(a_1) = 0$. Suppose $a_1 \neq 0$. Then, there would exist $1 \neq \lambda \in \mathbf{C}$ such that $g(\lambda) = 0$ and λ would be an eigenvalue of P^2 restricted to the spectral subspace of $\widehat{M_I}$ of spin n for the adjoint representation. Let a_2 be the corresponding eigenvector. We claim $\lambda \neq 0$. Indeed, if $g(0) = 0$, we would have $f(0) = 1$. However, for $n \neq 0$

$$|\frac{U_{2n}(0)}{d(n)}| < 1.$$

Since μ is a probability measure that is not equal to δ_0, we get

$$|f(0)| < 1,$$

which shows the claim. We set

$$a_3 = a_2 + \frac{P(a_2)}{\sqrt{\lambda}}.$$

Then, a_3 is an eigenvector of P for the eigenvalue $\sqrt{\lambda}$ such that a_3 has spin n for the adjoint representation. Thus, (2) of Corollary 5.6 implies that $h^{\sqrt{\lambda}}(0)$ is bounded and

$$P_\mu(E_I(h^{\sqrt{\lambda}}(0))) - E_I(h^{\sqrt{\lambda}}(0)) = (P^2 - I)(g(P^2)(E_I(h^{\sqrt{\lambda}}(0))) = 0,$$

that is, $E_I(h^{\sqrt{\lambda}}(0))$ is a harmonic element of $P_{\mu,I}$ restricted to $Z(\widehat{M_I})$. Therefore, Corollary 3.7 implies that $E_I(h^{\sqrt{\lambda}}(0))$ is a scalar. However, the definition of $h^{\sqrt{\lambda}}(0)$ shows that $E_I(h^{\sqrt{\lambda}}(0))$ is a scalar only if $\sqrt{\lambda} = \pm 1$, which is contradiction. Thus, we get $a = E_I(b) \in H^\infty(\widehat{M}, P)_I$. Since the Choi-Effros product is uniquely determine by the operator system structure of $H^\infty(\widehat{M}, P)_I$, we can compute it using P^2 instead of P_μ, and so the statement for R follows from that of the spin 1/2 case. Q.E.D.

7 Questions and Comments

We end the paper with a few questions and comments.

(1) Let λ be a positive number larger than or equal to $2/(q+q^{-1})$. Then (3) of Lemma 5.4 implies that $h^\lambda(0)$ is a positive invertible element affiliated with $Z(\widehat{M})$. Therefore, we can introduce the "Doob transformation P^λ" by setting

$$P^\lambda(a) = \frac{\widetilde{P}(h^\lambda(0)a)}{\lambda h^\lambda(0)}, \quad a \in \widehat{M}.$$

P^λ is again a Markov operator of $\widehat{M}$ commuting with the (M, δ) action. $a \in \widehat{M}$ is a P^λ-harmonic element if and only if $h^\lambda(0)a$ is an eigenvector of $\widetilde{P}$ for λ. (2) of Lemma 5.4 shows that $\{a_l^{n,\alpha}\}$ and $\{a_l^{0,\alpha}\}$ have the same asymptotics, up to a non-zero scalar multiple, as l tends to infinity. Therefore for every $s \in \widehat{G}$, $H^\infty(\widehat{M}, P^\lambda)(s)$ has the same multiplicity as $H^\infty(\widehat{M}, P)(s)$ does. This means that the weak closure of the linear span of

$$\bigcup_{s \in \widehat{G}} H^\infty(\widehat{M}, P^\lambda)(s),$$

with the Choi-Effros product gives rise to the quantum sphere $L^\infty(S^2_{q,c})$ for some c. Note that we do not know if this operator system coincides with $H^\infty(\widehat{M}, P^\lambda)$ in the $\lambda \neq 1$ case because we do not have a statement corresponding to (2) of Lemma 3.3 in general, or in other words, we do not have a natural realization of the Poisson boundaries for $\lambda \neq 1$. The same argument as above actually implies $c = 0$ for all λ. It would be interesting to seek for a natural realization of the von Neumann algebra structure of $H^\infty(\widehat{M}, P^\lambda)$.

(2) As we have seen in (4) of Lemma 2.2 (or Lemma 4.4), there is a natural left $(\widehat{M}, \widehat{\delta})$ action on $H^\infty(\widehat{M}, P)$ (or on $L^\infty(\mathbf{T}\backslash SU_q(2))$). Therefore, there exists a left $(\widehat{M}, \widehat{\delta})$ action on the relative commutant R. What does this action mean? Probably, the origin of the action will be the dual action $\widehat{\alpha}$ acting on the relative commutant

$$(N \rtimes_\alpha SU_q(2)) \cap N'.$$

(3) We go back to the general situation in Section 3. Let L be the linear span of

$$\bigcup_{s \in \widehat{G}, j,k \in I_s} \sum_{l \in I_s} \sigma_i^h(w(s)_{l,k}) w(s)^*_{l,j}.$$

Then, it is easy to show that L is a left coideal which is closed under $*$. In fact, L is a dense $*$-subalgebra of $C(\mathbf{T}\backslash SU_q(2))$ in the $SU_q(2)$ case. Is L an algebra in general? In view of Corollary 3.9 and the $SU_q(2)$ case, the weak closure of L is a candidate of the realization of $H^\infty(\widehat{M}, P_\mu)$ when the fusion algebra $\widehat{G}$ is commutative. We introduce an element $\mathcal{K} \in \prod_{s\in\widehat{G}}(\widehat{A}(s)\otimes L)$ by

$$\mathcal{K} = (\sum_{j,k,l\in I_s} e(s)_{jk} \otimes \sigma_i^h(w(s)_{l,k})w(s)_{l,j}^*) \in \prod_{s\in\widehat{G}}(\widehat{A}(s)\otimes L).$$

The proof of Corollary 3.9 implies that the map Θ is given by

$$\Theta(a) = (id\otimes h)(\mathcal{K}(1\otimes a)) = (id \otimes h)((1\otimes a)\mathcal{K}^*).$$

Therefore, at least in the $SU_q(2)$ case, $\mathcal{K}$ deserves to be called the "Poisson kernel". It is easy to show that $\mathcal{K}$ is also given by

$$\mathcal{K} = (id\otimes \kappa)(\Phi(\rho^{-2})).$$

(4) It is known that if a discrete group Γ allows a generating probability measure whose Poisson boundary is trivial, Γ is necessarily amenable [22]. (The converse is not true.) Is it also true in our situation? More precisely, if $H^\infty(\widehat{M}, P_\mu)$ is trivial for some generating measure, is the fusion algebra $\widehat{G}$ amenable in the sense of [16]? Note that if $\widehat{G}$ is amenable, M is necessarily a Kac algebra [15] (cf. Corollary 3.9.)

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