R-cyclic families of matrices in free probability

Alexandru Nica *
Department of Pure Mathematics
University of Waterloo
Waterloo, Ontario, N2L 3G1, Canada
(e-mail: anica@math.uwaterloo.ca)

Dimitri Shlyakhtenko †
Department of Mathematics
U.C.L.A.
Los Angeles, CA 90095-1555, USA
(email: shlyakht@math.ucla.edu)

Roland Speicher
Department of Mathematics and Statistics
Queen's University
Kingston, Ontario K7L 3N6, Canada
(email: speicher@mast.queensu.ca)

Abstract

We introduce the concept of "R-cyclic family" of matrices with entries in a non-commutative probability space; the definition consists in asking that only the "cyclic" non-crossing cumulants of the entries of the matrices are allowed to be non-zero.

Let $A_1, \ldots, A_s$ be an R-cyclic family of $d \times d$ matrices over a non-commutative probability space $(\mathcal{A}, \varphi)$. We prove a convolution-type formula for the explicit computation of the joint distribution of $A_1, \ldots, A_s$ (considered in $M_d(\mathcal{A})$ with the natural state), in terms of the joint distribution (considered in the original space $(\mathcal{A}, \varphi)$) of the entries of the s matrices. Several important situations of families of matrices with tractable joint distributions arise by application of this formula.

Moreover, let $A_1, \ldots, A_s$ be a family of $d \times d$ matrices over a non-commutative probability space $(\mathcal{A}, \varphi)$, let $\mathcal{D} \subset M_d(\mathcal{A})$ denote the algebra of scalar diagonal matrices, and let $\mathcal{C}$ be the subalgebra of $M_d(\mathcal{A})$ generated by $\{A_1, \ldots, A_s\} \cup \mathcal{D}$. We prove that the R-cyclicity of $A_1, \ldots, A_s$ is equivalent to a property of $\mathcal{C}$ – namely that $\mathcal{C}$ is free from $M_d(\mathbf{C})$, with amalgamation over $\mathcal{D}$.

*Research supported by a grant of NSERC, Canada.
†Partially supported by an NSF Postdoctoral Fellowship.

Introduction

In the influential paper [21], Voiculescu introduced the concepts of circular and semicircular systems, and used them to obtain results about the fundamental groups of the von Neumann algebras associated to free groups. There are three main properties of the circulars and semicirculars which are essential for the arguments in [21]:

(a) the compression of a semicircular system by a projection free from the system is again a semicircular system;

(b) in the polar decomposition of a circular element, the polar part is free from the positive part;

(c) one can obtain semicircular systems consisting of *matrices* over a non-commutative probability space, if the entries of these matrices are themselves chosen (in an appropriate way) to be circular/semicircular and free.

Each of (a), (b), (c) points to a direction of investigation in the combinatorics of free probability.

Concerning (a) and (b), the things are now pretty well understood. For (a), we know a general formula describing the distribution of the compression by a free projection (see [9]), or even more generally for what happens when we perform a compression by a free matrix unit (see [17], [8]). For (b), the relevant class of elements to be studied is the one of "R-diagonal elements", introduced in [10], and which turns out to have a lot of good properties (see e.g. [5], or [12]–[14]).

With (c) the situation is not that clear. If we look at the case of only one matrix, then the problem is to give effective methods for computing the distribution of the matrix, by starting from the joint distribution of its entries. Of course, the distribution of the matrix is always completely determined by the joint distribution of its entries; the issue is here about the word "effective". It is unlikely that one can give a nice formula which would work in full generality. The problem is more like this: to what kind of matrices can one generalize the nice facts known about matrices of free circular/semicircular elements? We look for a situation which is general enough to contain interesting examples, but also particular enough so that a nice formula does exist.

In this paper we propose the concept of R-cyclic matrix (or more generally, of R-cyclic family of matrices), which we believe is a good framework for studying the direction (c).

The definition is in terms of the joint R-transform of the entries of the matrix – where the R-transform is the free probabilistic counterpart for the characteristic function of the joint distribution. The coefficients of the R-transform are called non-crossing cumulants. The definition of an R-cyclic matrix goes by asking that only the cyclic non-crossing cumulants of the entries survive; see Definition 2.2 in Section 2 below, and see Sections 2.3-2.6 for examples.

If A is an R-cyclic matrix, then all the information about the distribution of A is stored in the family of cyclic cumulants of its entries. These cyclic cumulants can be in turn nicely stored in one formal power series f (in d non-commuting variables, where $d \times d$ is the size of A); the series f is called "the determining series" of A. Our problem is then to find an effective method for computing the distribution of the R-cyclic matrix A, in terms of its determining series f. In the Section 2 of the paper we show that this problem can be treated by using a convolution-type formula:

$$(I) \qquad R_A(z) \ = \ \frac{1}{d}(f \boxed{\star} H_d)(\underbrace{z, \ldots, z}_{d}),$$

where: R_A is the R-transform of A; H_d is a certain universal series in d indeterminates; and $\boxed{\star}$ is a convolution-type operation introduced in [9], which appears to play an important role in combinatorial free probability (see review in Section 1 below). The formula (I) can be extended to the case of an R-cyclic family of matrices (see Definition 2.9 and Theorem 2.10 in Section 2), and can be used to obtain various situations when one gets a family of matrices with computable joint distribution. Some applications are presented in the Section 3 of the paper.

Section 4 is about operations with matrices in an R-cyclic family. It is trivial from the definition that if $A_1, \ldots, A_s$ is an R-cyclic family (of $d \times d$ matrices over a non-commutative probability space $(\mathcal{A}, \varphi)$), then one can add to $A_1, \ldots, A_s$:

(a) a linear combination of $A_1, \ldots, A_s$, or

(b) any scalar diagonal matrix,

and the enlarged family is still R-cyclic. In Lemma 4.2 we show that a similar statement is true when one adds to $A_1, \ldots, A_s$ a product of some of the matrices in the family; this comes as a fairly easy application of a formula for non-crossing cumulants with products for entries, which was found in [6].

The considerations of Section 4 show that the property of a family of matrices $A_1, \ldots, A_s \in M_d(\mathcal{A})$ of being R-cyclic is really a property of the algebra $\mathcal{C}$ generated together by $A_1, \ldots, A_s$ and the set of scalar diagonal matrices. The rest of the paper is devoted to identifying what this property of $\mathcal{C}$ exactly is. The result turns out to be the following (Theorem 8.2):

$$(II) \qquad \left\{ \begin{array}{c} \text{the family} \\ A_1, \ldots, A_s \\ \text{is R-cyclic} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} \mathcal{C} \text{ is free from } M_d(\mathbf{C}), \\ \text{with amalgamation over} \\ \text{scalar diagonal matrices} \end{array} \right\},$$

where $A_1, \ldots, A_s$ and $\mathcal{C}$ are as above, and where the algebra $M_d(\mathbf{C})$ of scalar $d \times d$ matrices is identified as a subalgebra of $M_d(\mathcal{A})$ in the natural way.

In the paper [14] we had shown that an element $a \in \mathcal{A}$ is R-diagonal if and only if the matrix $\begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix} \in M_2(\mathcal{A})$ is free from $M_2(\mathbf{C})$, with amalgamation over scalar diagonal matrices. But it is easy to see, directly from the definitions, that a is R-diagonal if and only if the matrix $\begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix}$ is R-cyclic. Hence the above equivalence (II) can be viewed as an ample generalization of the named result from [14].

The equivalence in (II) is obtained by studying non-crossing *operator-valued* cumulants, in the sense of [19]; a few basic facts about operator-valued cumulants are reviewed in Section 5, and the proof of (II) is shown in Section 8. In between 5 and 8 we have two short sections where we derive some explicit formulas (used in Section 8) for operator-valued cumulants with respect to the algebra $M_d(\mathbf{C})$ (in Section 6), and with respect to the algebra of scalar diagonal matrices (in Section 7).

1. Basic concepts for the combinatorics of free probability

As a preparation for the theorems proved in Section 2, we review here a few basic concepts and facts used in combinatorial free probability. We use the framework of a *non-commutative probability space*, by which we will simply understand a pair $(\mathcal{A}, \varphi)$ where $\mathcal{A}$ is a complex unital algebra ("the algebra of random variables") and $\varphi : \mathcal{A} \to \mathbf{C}$ ("the expectation") is a linear functional, normalized by $\varphi(1) = 1$. We assume that the reader has some familiarity with the concept of freeness for families of elements in $(\mathcal{A}, \varphi)$ (see e.g. [22], Chapter 2).

In the combinatorial study of freeness, an important role is played by the concepts of *moment series* and *R-transform* of a family of non-commuting random variables. The definition of the first of these two concepts is straightforward: if $(\mathcal{A}, \varphi)$ is a non-commutative probability space, and if $a_1, \ldots, a_s$ are in $\mathcal{A}$, then the numbers of the form:

$$\varphi(a_{r_1} \cdots a_{r_n}), \quad n \geq 1, \ 1 \leq r_1, \ldots, r_n \leq s, \tag{1.1}$$

are called the *joint moments* of $a_1, \ldots, a_s$; the moment series of $a_1, \ldots, a_s$ is the power series in s non-commuting indeterminates $z_1, \ldots, z_s$ which has the joint moments as coefficients. That is:

$$M_{a_1,\ldots,a_s}(z_1, \ldots, z_s) \;:=\; \sum_{n=1}^{\infty} \sum_{r_1,\ldots,r_n=1}^{s} \varphi(a_{r_1} \cdots a_{r_n}) z_{r_1} \cdots z_{r_n}. \tag{1.2}$$

The (less straightforward) definition of the R-transform can be placed within the framework of a certain convolution operation on formal power series which will be used in Section 2, and is reviewed next (in Sections 1.1-1.2, followed by the definition of the R-transform in Section 1.3).

1.1 Non-crossing partitions. Let $\pi = \{B_1, \ldots, B_k\}$ be a partition of $\{1, \ldots, n\}$ – i.e. $B_1, \ldots, B_k$ are pairwisely disjoint non-void sets (called the *blocks* of π), and $B_1 \cup \cdots \cup B_k = \{1, \ldots, n\}$. We say that π is *non-crossing* if for every $1 \leq i < j < k < l \leq n$ such that i is in the same block with k and j is in the same block with l, it necessarily follows that all of i, j, k, l are in the same block of π. The set of non-crossing partitions of $\{1, \ldots, n\}$ will be denoted by $NC(n)$.

For $\pi, \rho \in NC(n)$, we write "$\pi \leq \rho$" if each block of ρ is a union of blocks of π. Then "$\leq$" is a partial order relation on $NC(n)$, called the refinement order. It turns out that $(NC(n), \leq)$ is in fact a lattice, i.e. every two partitions in $NC(n)$ have a lowest upper bound and a greatest lower bound with respect to $\leq$.

For $\pi \in NC(n)$ we will denote by perm_π the permutation of $\{1, \ldots, n\}$ which has the blocks of π as cycles, in such a way that if $B = \{k_1 < \cdots < k_{p-1} < k_p\}$ is a block of π then we have

$$\mathrm{perm}_\pi(k_1) = k_2, \ldots, \mathrm{perm}_\pi(k_{p-1}) = k_p, \mathrm{perm}_\pi(k_p) = k_1.$$

(For example, if $\pi = \{\ \{1, 2, 5\}, \{3, 4\}\ \} \in NC(5)$, then $\mathrm{perm}_\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix}$.) The set $\{\mathrm{perm}_\pi \mid \pi \in NC(n)\}$ has a nice interpretation in terms of the geometry of the Cayley graph of the symmetric group (see [1]), and can be a useful instrument in considerations about the lattice $NC(n)$.

Unlike the lattice of all partitions of $\{1, 2, \ldots, n\}$, $NC(n)$ is anti-isomorphic to itself. We will in fact make extensive use of a canonical anti-isomorphism $\mathrm{Kr} : NC(n) \to NC(n)$, introduced in [7] and called the Kreweras complementation map. The map Kr can be conveniently described by using the permutations associated to non-crossing partitions, via the following formula:

$$\mathrm{perm}_\pi \circ \mathrm{perm}_{\mathrm{Kr}(\pi)} \;=\; \gamma_n, \quad \forall\, \pi \in NC(n), \tag{1.3}$$

where γ_n is the forward cycle on $\{1, \ldots, n\}$ ($\gamma_n(1) = 2, \ldots, \gamma_n(n-1) = n, \gamma_n(n) = 1$).

1.2 The operation of boxed convolution. Let s be a positive integer. We denote by Θ_s the set of all series of the form appearing in Equation (1.2):

$$\Theta_s = \left\{ f \;\begin{array}{|l} f(z_1, \ldots, z_s) = \sum_{n=1}^{\infty} \sum_{r_1,\ldots r_n=1}^{s} \alpha_{r_1,\ldots,r_n} z_{r_1} \cdots z_{r_n} \\ \text{where } \alpha_{r_1,\ldots,r_n} \in \mathbf{C} \ (n \geq 1, \ 1 \leq r_1, \ldots, r_n \leq s) \end{array} \right\}. \tag{1.4}$$

For a series f as in (1.4), we will use the notation

$$[\text{ coef } (r_1, \ldots, r_n) \ ](f) \tag{1.5}$$

to denote the coefficient $\alpha_{r_1,\ldots,r_n}$ of $z_{r_1} \cdots z_{r_n}$ in f.

The operation of boxed convolution, $\boxed{\star}$, is an associative binary operation on the set Θ_s. Its definition is inspired from the combinatorial theory of convolution in a lattice, as developed by Rota and his collaborators (see e.g. [2]; the lattices of relevance for the definition of $\boxed{\star}$ are those of non-crossing partitions, $NC(n)$ for $n \geq 1$).

In order to state the definition of $\boxed{\star}$, it is convenient to first expand the notations for coefficients introduced in (1.5). If $n \geq 1$, $1 \leq r_1, \ldots, r_n \leq s$, and if $B = \{k_1 < k_2 < \cdots < k_p\}$ is a non-void subset of $\{1, \ldots, n\}$, then by "$(r_1, \ldots, r_n)|B$" we will understand the p-tuple $(r_{k_1}, r_{k_2}, \ldots, r_{k_p})$ (for example $(r_1, r_2, r_3, r_4, r_5)|\{2, 3, 5\} = (r_2, r_3, r_5)$). Then for a series $f \in \Theta_s$ we introduce the following "generalized coefficients":

$$[\text{coef } (r_1, \ldots, r_n); \pi](f) := \prod_{B \text{ block of } \pi} [\text{ coef } (r_1, \ldots, r_n)|B \ ](f), \tag{1.6}$$

for every $n \geq 1$, $1 \leq r_1, \ldots, r_n \leq s$, and for every $\pi \in NC(n)$. (For example if $n = 4$ and $\pi = \{\{1, 3\}, \{2\}, \{4\}\}$, then

$$[\text{coef } (r_1, r_2, r_3, r_4); \pi](f) = [\text{coef } (r_1, r_3)](f) \cdot [\text{coef } (r_2)](f) \cdot [\text{coef } (r_4)](f),$$

for any $1 \leq r_1, r_2, r_3, r_4 \leq s$.)

By using the notation introduced in (1.6), the boxed convolution $f \boxed{\star} g$ of two series $f, g \in \Theta_s$ is described by the formula:

$$[\text{coef } (r_1, \ldots, r_n)](f \boxed{\star} g) := \tag{1.7}$$

$$\sum_{\pi \in NC(n)} [\text{coef } (r_1, \ldots, r_n); \pi](f) \cdot [\text{coef } (r_1, \ldots, r_n); \text{Kr}(\pi)](g),$$

holding for every $n \geq 1$ and $1 \leq r_1, \ldots, r_n \leq s$, and where $\text{Kr}(\pi)$ is the Kreweras complement of the partition $\pi \in NC(n)$.

It can be shown that $\boxed{\star}$ is associative and unital, where the unit is the series $\Delta(z_1, \ldots, z_s) := z_1 + \cdots + z_s$. A series $f \in \Theta_s$ is invertible with respect to $\boxed{\star}$ if and only if its coefficients of degree 1, $[\text{coef } (r)](f)$, $1 \le r \le s$, are all different from 0 (see [9], Section 3).

1.3 R-transform and free cumulants. Let $a_1, \ldots, a_s$ be an s-tuple of elements in a non-commutative probability space $(\mathcal{A}, \varphi)$. The R-transform of the s-tuple, $R_{a_1,\ldots,a_s}$, is a series in the set Θ_s of Equation (1.4). A succinct way of introducing $R_{a_1,\ldots,a_s}$ goes by using the boxed convolution $\boxed{\star}$ and a special series $\text{Möb}_s \in \Theta_s$, called the Möbius series.

Möb_s is defined as the inverse under $\boxed{\star}$ of the "zeta series in s indeterminates",

$$\text{Zeta}_s(z_1, \ldots, z_s) \;:=\; \sum_{n=1}^{\infty} \sum_{r_1,\ldots,r_n=1}^{s} z_{r_1} \cdots z_{r_n}. \tag{1.8}$$

It is not hard to determine the coefficients of Möb_s explicitly:

$$\text{Möb}_s(z_1, \ldots, z_s) \;=\; \sum_{n=1}^{\infty} \sum_{r_1,\ldots,r_n=1}^{s} (-1)^{n+1} \frac{(2n-2)!}{(n-1)!n!}\, z_{r_1} \cdots z_{r_n} \tag{1.9}$$

(see e.g. [9], Remark 3.8).

Now, if $(\mathcal{A}, \varphi)$ is a non-commutative probability space, and if $a_1, \ldots, a_s \in \mathcal{A}$, then we define:

$$R_{a_1,\ldots,a_s} \;:=\; M_{a_1,\ldots,a_s} \boxed{\star}\, \text{Möb}_s, \tag{1.10}$$

where $M_{a_1,\ldots,a_s}$ is the moment series from Equation (1.2). It is clear that $R_{a_1,\ldots,a_s}$ contains the same information about $a_1, \ldots, a_s$ as the moment series, since Equation (1.10) can be re-written equivalently as

$$M_{a_1,\ldots,a_s} \;=\; R_{a_1,\ldots,a_s} \boxed{\star}\, \text{Zeta}_s. \tag{1.11}$$

Following [18], it is customary to denote the coefficient of $z_{r_1} \cdots z_{r_n}$ in $R_{a_1,\ldots,a_s}$ by:

$$k_n(a_{r_1}, \ldots, a_{r_n}). \tag{1.12}$$

More generally, given $n \ge 1$, $1 \le r_1, \ldots, r_n \le s$, and a partition $\pi \in NC(n)$, we use the notation

$$k_\pi(a_{r_1}, \ldots, a_{r_n}) \tag{1.13}$$

for the "generalized coefficient" $[\text{coef } (r_1, \ldots, r_n); \pi](R_{a_1,\ldots,a_s})$ defined as in Equation (1.6). These generalized coefficients are called the *non-crossing cumulants* of the s-tuple $a_1, \ldots, a_s$. It is worth keeping in mind that for any $n \ge 1$ and $\pi \in NC(n)$, it makes sense to view k_π as a multilinear map from $\mathcal{A}^n$ to $\mathbf{C}$ (see [18]).

1.4 R-transform and freeness. The R-transform and the boxed convolution turn out to have very pleasant properties in connection to the addition and multiplication of free n-tuples – see [20], [9]. Even more importantly, R-transforms (or equivalently, non-crossing cumulants) can be used to provide a neat characterization of freeness. To be precise: let $a'_1, \ldots, a'_m, a''_1, \ldots, a''_n$ be elements of the non-commutative probability space $(\mathcal{A}, \varphi)$; then the freeness of the families $\{a'_1, \ldots, a'_m\}$ and $\{a''_1, \ldots, a''_n\}$ is equivalent to the equation

$$R_{a'_1,\ldots,a'_m,a''_1,\ldots,a''_n}(z'_1, \ldots, z'_m, z''_1, \ldots, z''_n) \;=\; \tag{1.14}$$

$$= \; R_{a'_1,\ldots,a'_m}(z'_1, \ldots, z'_m) + R_{a''_1,\ldots,a''_n}(z''_1, \ldots, z''_n).$$

It is obvious how Equation (1.14) extends by induction to the case of s (instead of just two) families of elements. Note that in the case of s families having one element each, we obtain the following: the elements $a_1, \ldots, a_s \in \mathcal{A}$ form a free family if and only if we have that

$$R_{a_1,\ldots,a_s}(z_1, \ldots, z_s) \;=\; R_{a_1}(z_1) + \cdots + R_{a_s}(z_s). \tag{1.15}$$

1.5 Extended boxed convolution. Let s and d be positive integers. Consider the set Θ_{sd} of power series in sd non-commuting indeterminates $z_{1,1}, \ldots, z_{r,i}, \ldots, z_{s,d}$. The same formula as in Equation (1.7) above can be used to define a "convolution operation", denoted in what follows by $\widetilde{\boxed{\star}}$, which gives a right action of Θ_d on Θ_{sd}. More precisely, if $f \in \Theta_{sd}$ and $g \in \Theta_d$ then we define $f \,\widetilde{\boxed{\star}}\, g \in \Theta_{sd}$ by the following formula:

$$[\text{coef } (\; (r_1, i_1), \ldots, (r_n, i_n) \;)](f \,\widetilde{\boxed{\star}}\, g) \;:=\; \tag{1.16}$$

$$\sum_{\pi \in NC(n)} [\text{coef } (\; (r_1, i_1), \ldots, (r_n, i_n) \;); \pi](f) \cdot [\text{coef } (i_1, \ldots, i_n); \text{Kr}(\pi)](g),$$

holding for every $n \geq 1$ and for every $1 \leq r_1, \ldots, r_n \leq s$, $1 \leq i_1, \ldots, i_n \leq d$. Some trivial adjustments of the considerations made in Section 4 of [9] for $\boxed{\star}$ show that $\widetilde{\boxed{\star}}$ is indeed a right action of Θ_d on Θ_{sd}, in the sense that the equation

$$(f \,\widetilde{\boxed{\star}}\, g) \,\widetilde{\boxed{\star}}\, h \;=\; f \,\widetilde{\boxed{\star}}\, (g \,\boxed{\star}\, h) \tag{1.17}$$

holds for every $f \in \Theta_{sd}$ and $g, h \in \Theta_d$.

Let us also record the fact that:

$$f \,\widetilde{\boxed{\star}}\, \text{Zeta}_d \;=\; f \,\boxed{\star}\, \text{Zeta}_{sd}, \quad \forall \, f \in \Theta_{sd} \tag{1.18}$$

(where on the right-hand side of (1.18), $\boxed{\star}$ denotes the boxed convolution operation on Θ_{sd}). This relation is obvious if one takes into account the fact that any Zeta series has all the coefficients equal to 1.

From (1.17) and (1.18) it is immediate that one also has:

$$f \,\tilde{\boxed{\star}}\, \text{Möb}_d \;=\; f \,\boxed{\star}\, \text{Möb}_{sd}, \quad \forall\, f \in \Theta_{sd}. \tag{1.19}$$

Note that, as a consequence, we can write the relation

$$M_{a_{1,1},\ldots,a_{r,i},\ldots,a_{s,d}} \,\tilde{\boxed{\star}}\, \text{Möb}_d \;=\; R_{a_{1,1},\ldots,a_{r,i},\ldots,a_{s,d}}, \tag{1.20}$$

holding for any family $\{a_{r,i} \mid 1 \leq r \leq s,\ 1 \leq i \leq d\}$ of elements in some non-commutative probability space $(\mathcal{A}, \varphi)$.

1.6 Dilations and scalar multiples of power series. Let s be a positive integer, let f be a series in Θ_s, and let α be a complex number. We denote by $f \circ D_\alpha$ the series in Θ_s which is defined by the equation: "$(f \circ D_\alpha)(z_1, \ldots, z_s) = f(\alpha z_1, \ldots, \alpha z_s)$", or more rigorously by the fact that:

$$[\text{coef}(r_1, \ldots, r_n)](f \circ D_\alpha) \;=\; \alpha^n \cdot [\text{coef}(r_1, \ldots, r_n)](f), \quad \forall\, n \geq 1,\ 1 \leq r_1, \ldots, r_n \leq s.$$

The formulas relating $\boxed{\star}$ with dilation and with scalar multiplication which are proved in [9] can be easily extended to the case of $\tilde{\boxed{\star}}$. Concerning dilation we have:

$$(f \circ D_\alpha) \,\tilde{\boxed{\star}}\, g \;=\; f \,\tilde{\boxed{\star}}\, (g \circ D_\alpha) \;=\; (f \,\tilde{\boxed{\star}}\, g) \circ D_\alpha, \tag{1.21}$$

for every $f \in \Theta_{ds}$, $g \in \Theta_d$, $\alpha \in \mathbf{C}$. Concerning scalar multiplication we have the formula:

$$(\alpha f) \,\tilde{\boxed{\star}}\, (\alpha g) \;=\; \alpha(\ (f \,\tilde{\boxed{\star}}\, g) \circ D_\alpha), \quad \forall\, f \in \Theta_{ds},\ g \in \Theta_d,\ \alpha \in \mathbf{C}. \tag{1.22}$$

It is sometimes convenient to use Equation (1.22) in the form:

$$(\alpha f) \,\tilde{\boxed{\star}}\, g \;=\; \alpha(\ f \,\tilde{\boxed{\star}}\, (\frac{1}{\alpha} g \circ D_\alpha)\ ), \tag{1.23}$$

holding for $f \in \Theta_{ds}$, $g \in \Theta_d$, and $\alpha \in \mathbf{C} \setminus \{0\}$.

1.7 The special series H_d. Let d be a positive integer. In this paper we also encounter the "geometric series in d separate indeterminates",

$$G_d(z_1, \ldots, z_d) \;=\; \sum_{n=1}^{\infty} \sum_{i=1}^{d} z_i^n \quad (= \frac{z_1}{1 - z_1} + \cdots + \frac{z_d}{1 - z_d}\ ), \tag{1.24}$$

and a series derived from G_d which can be described as follows:

$$H_d \ := \ G_d \boxed{\star} (d \cdot \text{Möb}_d \circ D_{1/d}). \tag{1.25}$$

To give an idea of how H_d looks like, here is its truncation to order three:

$$H_d(z_1, \ldots, z_d) \ = \ \sum_{i=1}^{d} z_i \ + \ \sum_{i_1,i_2=1}^{d} (\delta_{i_1,i_2} - \frac{1}{d}) z_{i_1} z_{i_2}$$

$$+ \ \sum_{i_1,i_2,i_3=1}^{d} \Big(\delta_{i_1,i_2,i_3} - \frac{1}{d}(\delta_{i_1,i_2} + \delta_{i_1,i_3} + \delta_{i_2,i_3}) + \frac{2}{d^2} \ \Big) z_{i_1} z_{i_2} z_{i_3} \ + \cdots$$

Note that a direct application of Equation (1.23) (in the particular case when $\widetilde{\boxed{\star}}$ is $\boxed{\star}$ on Θ_d, and $\alpha = 1/d$) gives the alternative formula:

$$H_d \ = \ d \cdot \Big(\ (\frac{1}{d} G_d) \boxed{\star} \text{Möb}_d \ \Big). \tag{1.26}$$

Furthermore, the latter equation has the following interpretation. Let tr_d denote the normalized trace on the algebra $M_d(\mathbf{C})$, and consider the matrices $P_1, \ldots, P_d \in M_d(\mathbf{C})$ where P_i has its (i,i)-entry equal to 1 and all the other entries equal to 0. Then, obviously:

$$M_{P_1,\ldots,P_d} \ = \ \frac{1}{d} G_d$$

(moment series considered in the non-commutative probability space $(M_d(\mathbf{C}), tr_d)$); hence:

$$(\frac{1}{d} G_d) \boxed{\star} \text{Möb}_d \ = \ M_{P_1,\ldots,P_d} \boxed{\star} \text{Möb}_d \ = \ R_{P_1,\ldots,P_d},$$

and the formula (1.26) for H_d takes the form

$$H_d \ = \ d \cdot R_{P_1,\ldots,P_d}. \tag{1.27}$$

An application of Equation (1.27) is that for every $n \geq 2$, every $k \in \{1, \ldots, n\}$, and every fixed indices $i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_n \in \{1, \ldots d\}$, we have:

$$\sum_{i=1}^{d} [\text{coef}(i_1, \ldots, i_{k-1}, i, i_{k+1}, \ldots, i_n)](H_d) = 0. \tag{1.28}$$

Indeed, the sum on the left-hand side of (1.28) is equal to:

$$d \cdot \sum_{i=1}^{d} k_n(P_{i_1}, \ldots, P_{i_{k-1}}, P_i, P_{i_{k+1}}, \ldots, P_{i_n}) \ \ \text{(by (1.27))}$$

$$= \ k_n(P_{i_1}, \ldots, P_{i_{k-1}}, I, P_{i_{k+1}}, \ldots, P_{i_n}) \ \ \text{(by the multilinearity of } k_n),$$

and the latter quantity equals 0 by (1.14) and the fact that the identity matrix I is free from $\{P_1, \ldots, P_d\}$ in $(M_d(\mathbf{C}), tr_d)$.

2. R-cyclic matrices and their R-transforms

2.1 Notation. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, and let d be a positive integer. Consider the algebra $M_d(\mathcal{A})$ of $d \times d$ matrices over $\mathcal{A}$. We denote by φ_d the linear functional on $M_d(\mathcal{A})$ defined by the formula:

$$\varphi_d(\ [a_{i,j}]_{i,j=1}^d \ ) \ = \ \frac{1}{d} \sum_{i=1}^d \varphi(a_{i,i}). \tag{2.1}$$

Then $(M_d(\mathcal{A}), \varphi)$ is a non-commutative probability space, too.

2.2 Definition. Let $(\mathcal{A}, \varphi)$ and d be as above. A matrix $A = [a_{i,j}]_{i,j=1}^d \in M_d(\mathcal{A})$ is said to be *R-cyclic* if the following condition holds:

$$k_n(a_{i_1,j_1}, \ldots, a_{i_n,j_n}) \ = \ 0$$

for every $n \geq 1$ and every $1 \leq i_1, j_1, \ldots, i_n, j_n \leq d$ for which it is not true that $j_1 = i_2, \ldots, j_{n-1} = i_n, j_n = i_1$.

If the matrix A is R-cyclic, then the series:

$$f(z_1, \ldots, z_d) \ := \ \sum_{n=1}^{\infty} \ \sum_{i_1,\ldots,i_n=1}^{d} k_n(a_{i_n,i_1}, a_{i_1,i_2}, \ldots, a_{i_{n-1},i_n}) z_{i_1} z_{i_2} \cdots z_{i_n} \tag{2.2}$$

is called the *determining series* of A.

2.3 Example. Consider a diagonal matrix,

$$A \ := \ \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_d \end{bmatrix} \in M_d(\mathcal{A}),$$

where $(\mathcal{A}, \varphi)$ and d are as above. An application of Equation (1.15) shows that A is R-cyclic if and only if the elements $a_1, \ldots, a_d$ form a free family; if this is the case, then the determining series of A coincides with the joint R-transform $R_{a_1,\ldots,a_d}$.

For more elaborate examples we will use the framework of a $*$-probability space, which is also the one most frequently encountered in applications. A $*$-probability space is a non-commutative probability space $(\mathcal{A}, \varphi)$ where $\mathcal{A}$ is a $*$-algebra, and φ has the property that $\varphi(a^*) = \overline{\varphi(a)}$, $\forall\, a \in \mathcal{A}$.

2.4 Example. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, and let $\{e_{i,j} \mid 1 \leq i, j \leq d\}$ be a family of elements of $\mathcal{A}$ which satisfy the following relations: $e_{i,j}^* = e_{j,i}$ for all $1 \leq i, j \leq d$, $e_{i,j}e_{k,l} = \delta_{j,k}e_{i,l}$ for all $1 \leq i, j, k, l \leq d$, and $\sum_{i=1}^{d} e_{i,i} = I$. We will assume in addition that $\varphi(e_{i,j}) = 0$ whenever $i \neq j$, and that $\varphi(e_{1,1}) = \cdots = \varphi(e_{d,d}) = 1/d$. We denote by $(\mathcal{C}, \psi)$ the compression of $(\mathcal{A}, \varphi)$ by $e_{1,1}$, i.e:

$$\mathcal{C} := e_{1,1}\mathcal{A}e_{1,1}, \quad \psi := d \cdot \varphi|\mathcal{C}.$$

Let now a be a selfadjoint element of $\mathcal{A}$, which is free from $\{e_{i,j} \mid 1 \leq i, j \leq d\}$. We compress a by the matrix unit formed by the $e_{i,j}$'s, and we move the compressions under the projection $e_{1,1}$; that is, we consider the family of elements:

$$c_{i,j} \ := \ e_{1,i}ae_{j,1} \in \mathcal{C}, \quad 1 \leq i, j \leq d.$$

One can compute explicitly the free cumulants of the family $\{c_{i,j} \mid 1 \leq i, j \leq d\}$, and obtain that for every $n \geq 1$ and $1 \leq i_1, j_1, \ldots, i_n, j_n \leq d$:

$$k_n(c_{i_1,j_1}, \ldots, c_{i_n,j_n}) \ = \ \begin{cases} d^{-(n-1)}k_n(a, \ldots, a) & \text{if } j_1 = i_2, \ldots, j_{n-1} = i_n, j_n = i_1 \\ \\ 0 & \text{otherwise} \end{cases}$$

(see Theorem 8.14 or Theorem 17.3 in the notes [11]). In other words, the matrix $C = [c_{i,j}]_{i,j=1}^{d} \in M_d(\mathcal{C})$ is R-cyclic, with determining series:

$$\begin{aligned} f(z_1, \ldots, z_d) \ &= \ \sum_{n=1}^{\infty} \ \sum_{i_1,\ldots,i_n=1}^{d} \ d^{-(n-1)}k_n(a, \ldots, a)z_{i_1} \cdots z_{i_n} \\ &= \ d \cdot \sum_{n=1}^{\infty} k_n(a, \ldots, a) \cdot \Big(\frac{z_1 + \cdots + z_d}{d}\Big)^n \\ &= \ d \cdot R_a\Big(\ \frac{z_1 + \cdots + z_d}{d} \ \Big), \end{aligned}$$

where R_a is the R-transform of a, in the original space $(\mathcal{A}, \varphi)$.

2.5 Example. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space. Let $a \in \mathcal{A}$ be an *R-diagonal element*, by which we mean that the joint R-transform of a and a^* is of the form

$$R_{a,a^*}(z_1, z_2) = \sum_{n=1}^{\infty} \alpha_n((z_1 z_2)^n + (z_2 z_1)^n)$$

for a sequence of real coefficients $(\alpha_n)_{n=1}^{\infty}$ (see [10]). The series $f(z) := \sum_{n=1}^{\infty} \alpha_n z^n$ is called the determining series of a.

Now consider the non-commutative probability space $(M_2(\mathcal{A}), \varphi_2)$ defined as in Section 2.1, and the selfadjoint matrix:

$$A = \begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix} \in M_2(\mathcal{A}).$$

One immediately checks that A is R-cyclic (and in fact that also conversely, the R-cyclicity of A implies the R-diagonality of a). Moreover, the determining series of A (as defined in Section 2.2) coincides with the determining series of the R-diagonal element a. A number of results known about R-diagonal elements can be incorporated in the theory of R-cyclic matrices by using this trick.

2.6 Example. The situation discussed in the Example 2.5 can be generalized to the one of a selfadjoint matrix with free R-diagonal entries. More precisely, let $(\mathcal{A}, \varphi)$ be a $*$-probability space, let d be a positive integer, and suppose that the elements $\{a_{i,j} \mid 1 \leq i, j \leq d\}$ of $\mathcal{A}$ have the following properties:

(i) $a_{i,j}^* = a_{j,i}$, $\forall\ 1 \leq i, j \leq d$;

(ii) $a_{i,j}$ is R-diagonal whenever $i \neq j$;

(iii) the $d(d+1)/2$ families: $\{a_{i,i}\}$ for $1 \leq i \leq d$, together with $\{a_{i,j}, a_{j,i}\}$ for $1 \leq i < j \leq d$, are free in $(\mathcal{A}, \varphi)$.

Then the matrix $A := [a_{i,j}]_{i,j=1}^d \in M_d(\mathcal{A})$ is R-cyclic. Indeed, the freeness condition (iii) combined with the R-diagonality of $a_{i,j}$ for $i \neq j$ implies that the only free cumulants made with the entries of A which could possibly be non-zero are:

$$\begin{cases} k_n(a_{i,i}, \ldots, a_{i,i}) & \text{with } n \geq 1,\ 1 \leq i \leq d, \text{and} \\ k_n(a_{i,j}, a_{j,i}, \ldots, a_{i,j}, a_{j,i}) & \text{with } n \geq 1 \text{ even},\ 1 \leq i, j \leq d,\ i \neq j; \end{cases}$$

all these cumulants fall within the pattern allowed by the definition of R-cyclicity.

2.7 Remarks. 1) Variations of the Example 2.6 can be fabricated, such that non-selfadjoint matrices are obtained. For this purpose, it is more natural to use the concept of

R-cyclic family of matrices, given in Definition 2.9 below, and the $d \times d$ matrix which appears should be considered together with its adjoint. We mention that a particularly intriguing construction of this type – upper triangular $d \times d$ matrix with circular $*$-distribution – was studied recently in [3].

2) In the Example 2.6 one can take the $a_{i,j}$'s to be circular/semicircular, thus obtaining a matrix A as considered in [21]. Recall that $a \in \mathcal{A}$ is said to be *semicircular* of radius r if $a = a^*$ and if

$$\varphi(a^n) \ = \ \frac{2}{\pi r^2} \int_{-r}^{r} t^n \sqrt{r^2 - t^2} \ dt, \quad \forall \ n \geq 1;$$

and that $c \in \mathcal{A}$ is said to be *circular* of radius r if it is of the form $c = (a + ib)/\sqrt{2}$, where each of a, b is semicircular of radius r, and a is free from b. It can be shown (see e.g. [22], Chapter 3) that if $a \in \mathcal{A}$ is semicircular of radius r, then $k_2(a, a) = r^2/4$ and $k_n(a, a, \ldots, a) = 0$ for $n \neq 2$. As an easy consequence (see e.g. [10]), a circular element c of radius r is R-diagonal with $R_{c,c^*}(z_1, z_2) = (r^2/4) \cdot (z_1 z_2 + z_2 z_1)$. Thus an example of R-cyclic matrix $A = [a_{i,j}]_{i,j=1}^d$ is provided by the case when every $a_{i,i}$ is semicircular (of some radius $r_{i,i}$), every $a_{i,j}$ with $i \neq j$ is circular (of some radius $r_{i,j}$), and the conditions (i), (iii) of Example 2.6 are satisfied.

The following theorem indicates how the distribution of an R-cyclic matrix (considered in the non-commutative probability space $(M_d(\mathcal{A}), \varphi_d)$) can be obtained from the determining series of the matrix.

2.8 Theorem. Suppose that A is an R-cyclic matrix, and let f denote the determining series of A. Then we have the formulas:

$$M_A(z) \ = \ \frac{1}{d}(f \boxed{\star} G_d)(\underbrace{z, \ldots, z}_{d \ times}), \tag{2.3}$$

and

$$R_A(z) \ = \ \frac{1}{d}(f \boxed{\star} H_d)(\underbrace{z, \ldots, z}_{d \ times}), \tag{2.4}$$

where the series G_d and H_d are as defined in Section 1.7.

Before starting on the proof of Theorem 2.8, it is convenient to observe that the discussion about R-cyclicity can be generalized without much effort to the situation of a family of matrices, as follows.

2.9 Definition. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, and let d be a positive integer. Let $A_1 = [a_{i,j}^{(1)}]_{i,j=1}^d, \ldots, A_s = [a_{i,j}^{(s)}]_{i,j=1}^d$ be matrices in $M_d(\mathcal{A})$. We say that the family $A_1, \ldots, A_s$ is *R-cyclic* if the following condition holds:

$$k_n(a_{i_1,j_1}^{(r_1)}, \ldots, a_{i_n,j_n}^{(r_n)}) \ = \ 0,$$

for every $n \geq 1$, every $1 \leq r_1, \ldots, r_n \leq s$, and every $1 \leq i_1, j_1, \ldots, i_n, j_n \leq d$ for which it is not true that $j_1 = i_2, \ldots, j_{n-1} = i_n, j_n = i_1$.

If the family $A_1, \ldots, A_s$ is R-cyclic, then the power series in ds indeterminates:

$$f(z_{1,1}, \ldots, z_{s,d}) \ := \tag{2.5}$$

$$\sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n=1}^{d} \sum_{r_1,\ldots,r_n=1}^{s} k_n(\ a_{i_n,i_1}^{(r_1)}, a_{i_1,i_2}^{(r_2)}, \ldots, a_{i_{n-1},i_n}^{(r_n)} \ ) \cdot z_{r_1,i_1} z_{r_2,i_2} \cdots z_{r_n,i_n}$$

is called the *determining series* of the family.

2.10 Theorem. Suppose that $A_1, \ldots, A_s$ is an R-cyclic family of matrices, with determining series f. Then we have the formulas:

$$M_{A_1,\ldots,A_s}(z_1, \ldots, z_s) \ = \ \frac{1}{d}(f \ \widetilde{\boxed{\star}} \ G_d)(\ \underbrace{z_1, \ldots, z_1}_{d \ times}, \ldots, \underbrace{z_s, \ldots, z_s}_{d \ times} \ ), \tag{2.6}$$

and

$$R_{A_1,\ldots,A_s}(z_1, \ldots, z_s) \ = \ \frac{1}{d}(f \ \widetilde{\boxed{\star}} \ H_d)(\ \underbrace{z_1, \ldots, z_1}_{d \ times}, \ldots, \underbrace{z_s, \ldots, z_s}_{d \ times} \ ), \tag{2.7}$$

where the operation $\widetilde{\boxed{\star}}$ is as described in Section 1.5, and where the series G_d and H_d are as defined in Section 1.7.

In the proof of Theorem 2.10 we will use the following lemma:

2.11 Lemma. Consider the framework of Theorem 2.10. Let n be a positive integer, let π be in $NC(n)$, and consider some indices $1 \leq r_1, \ldots, r_n \leq s$, $1 \leq i_1, \ldots, i_n \leq d$. Then we have the equality:

$$k_\pi(\ a_{i_n,i_1}^{(r_1)}, a_{i_1,i_2}^{(r_2)}, \ldots, a_{i_{n-1},i_n}^{(r_n)} \ ) \ = \tag{2.8}$$

$$[\text{coef}(\ (r_1, i_1), \ldots, (r_n, i_n) \ ); \pi \ ](f) \cdot [\text{coef}(\ i_1, \ldots, i_n \ ); \text{Kr}(\pi) \ ](G_d).$$

Proof. We will work with the permutations associated to π and to $Kr(\pi)$ (as discussed in Section 1.1). We will use cyclic notations modulo n for indices – i.e, "i_{k+1}" will mean "i_1" if $k = n$ and "i_{k-1}" will mean "i_n" if $k = 1$.

Since every coefficient G_d is equal either to 0 or to 1, the generalized coefficient of G_d appearing on the right-hand side of (2.8) also is 0 or 1. So we have two cases.

Case 1: $[\text{coef}(\ i_1, \ldots, i_n\ ); \text{Kr}(\pi)\ ](G_d) = 1.$

By writing explicitly what the generalized coefficient of G_d is, we find that:

$$\left\{ \begin{array}{c} 1 \leq k, l \leq n, \\ k, l \text{ in the same block of } \text{Kr}(\pi) \end{array} \right\} \Longrightarrow i_k = i_l. \tag{2.9}$$

Under this assumption, we have to show that:

$$k_\pi(\ a_{i_n,i_1}^{(r_1)}, a_{i_1,i_2}^{(r_2)}, \ldots, a_{i_{n-1},i_n}^{(r_n)}\ ) \ = \ [\text{coef}(\ (r_1, i_1), \ldots, (r_n, i_n)\ ); \pi\ ](f). \tag{2.10}$$

Each of the two sides of (2.10) is a product of factors indexed by the blocks of π; we will prove (2.10) by showing that actually for any given block B of π, the factor corresponding to B on the left-hand side of (2.10) is equal to the factor corresponding to B on the right-hand side of (2.10).

So let us fix a block $B = \{k_1 < k_2 < \cdots < k_p\}$ of π. The factor corresponding to B on the left-hand side of (2.10) is:

$$k_p\Big(\ a_{i_{k_1-1},i_{k_1}}^{(r_{k_1})}, a_{i_{k_2-1},i_{k_2}}^{(r_{k_2})}, \ldots, a_{i_{k_p-1},i_{k_p}}^{(r_{k_p})}\ \Big) \tag{2.11}$$

(where recall that if $k_1 = 1$, then we use i_n for "i_{k_1-1}"); the factor corresponding to B on the right-hand side of (2.10) is $[\text{coef}(\ (r_{k_1}, i_{k_1}), \ldots, (r_{k_p}, i_{k_p})\ )\ ](f)$, i.e:

$$k_p\Big(\ a_{i_{k_p},i_{k_1}}^{(r_{k_1})}, a_{i_{k_1},i_{k_2}}^{(r_{k_2})}, \ldots, a_{i_{k_{p-1}},i_{k_p}}^{(r_{k_p})}\ \Big). \tag{2.12}$$

But now, let us notice that k_1 and $k_2 - 1$ belong to the same block of $\text{Kr}(\pi)$, and same for k_2 and $k_3 - 1, \ldots$, same for k_p and $k_1 - 1$. This is easily seen by looking at the permutations associated to π and $\text{Kr}(\pi)$: we have that

$$\text{perm}_\pi(k_1) = k_2, \ldots, \text{perm}_\pi(k_{p-1}) = k_p, \text{perm}_\pi(k_p) = k_1,$$

so from Eqn.(1.3) we get that:

$$\text{perm}_{\text{Kr}(\pi)}(k_2 - 1) = k_1, \ldots, \text{perm}_{\text{Kr}(\pi)}(k_p - 1) = k_{p-1}, \text{perm}_{\text{Kr}(\pi)}(k_1 - 1) = k_p.$$

As a consequence of this remark and of the implication stated in (2.9), we see that the expressions appearing in (2.11) and (2.12) are actually identical.

Case 2: $[\mathrm{coef}(\ i_1, \ldots, i_n\ ); \mathrm{Kr}(\pi)\ ](G_d) = 0$.

In this case we know that (2.9) does not hold, and we have to show that the left-hand side of (2.8) is equal to 0.

It is immediate that, under the current assumption, we can find $1 \leq k, l \leq n$ such that:

$$\mathrm{perm}_{\mathrm{Kr}(\pi)}(l) = k, \quad \text{and } i_k \neq i_l. \tag{2.13}$$

Indeed, if it were true that $i_k = i_l$ whenever $\mathrm{perm}_{\mathrm{Kr}(\pi)}(l) = k$, then by moving along the cycles of $\mathrm{perm}_{\mathrm{Kr}(\pi)}$ we would find that (2.9) holds.

By taking into account the relation between perm_π and $\mathrm{perm}_{\mathrm{Kr}(\pi)}$, we see that for k, l as in (2.13) we also have that $\mathrm{perm}_\pi(k) = l + 1$. Hence k and $l+1$ belong to the same block B of π; and moreover, if the block B is written as $B = \{k_1 < k_2 < \cdots < k_p\}$, then there exists an index j, $1 \leq j \leq p$ such that $k = k_j$ and $l + 1 = k_{j+1}$ (with the convention that if $k = k_p$, then $l + 1 = k_1$). But then the fact that $i_k \neq i_l$ reads: $i_{k_j} \neq i_{k_{j+1}-1}$, which in turn implies that

$$k_p\Big(\ a^{(r_{k_1})}_{i_{k_1-1}, i_{k_1}}, a^{(r_{k_2})}_{i_{k_2-1}, i_{k_2}}, \ldots, a^{(r_{k_p})}_{i_{k_p-1}, i_{k_p}}\ \Big) \ = \ 0$$

(by the definition of R-cyclicity). Since the latter expression is the factor corresponding to B in the product defining $k_\pi(\ a^{(r_1)}_{i_n, i_1}, a^{(r_2)}_{i_1, i_2}, \ldots, a^{(r_n)}_{i_{n-1}, i_n}\ )$, we conclude that the left-hand side of (2.8) is indeed equal to 0. **QED**

Proof of Theorem 2.10. Let n be a positive integer, and consider some indices $1 \leq r_1, \ldots, r_n \leq s$, $1 \leq i_1, \ldots, i_n \leq d$. By summing over $\pi \in NC(n)$ in the Equation (2.8) of Lemma 2.11, and by taking into account the properties of non-crossing cumulants and of boxed convolution, we get:

$$\varphi(a^{(r_1)}_{i_n, i_1} a^{(r_2)}_{i_1, i_2} \cdots a^{(r_n)}_{i_{n-1}, i_n}) \ = \ [\mathrm{coef}(\ (r_1, i_1), \ldots, (r_n, i_n)\ )\ ](f \ \widetilde{\boxed{\star}}\ G_d). \tag{2.14}$$

For every $1 \leq i \leq d$, let us denote by $P_i \in M_d(\mathcal{A})$ the matrix which has I (the unit of $\mathcal{A}$) on the (i,i)-entry, and has all the other entries equal to 0. It is immediately verified that

$$\varphi_d(A_{r_1} P_{i_1} \cdots A_{r_n} P_{i_n}) \ = \ \frac{1}{d} \varphi(a^{(r_1)}_{i_n, i_1} a^{(r_2)}_{i_1, i_2} \cdots a^{(r_n)}_{i_{n-1}, i_n}).$$

By combining this equation with (2.14), we get an equality of power series in ds variables, which is stated as follows:

$$M_{A_1 P_1, \ldots, A_r P_i, \ldots, A_s P_d} \ = \ \frac{1}{d}(f \ \widetilde{\boxed{\star}}\ G_d). \tag{2.15}$$

The Equation (2.6) is an immediate consequence of (2.15), since we have for every $n \geq 1$ and $1 \leq r_1, \ldots, r_n \leq s$:

$$
\begin{aligned}
\varphi_d(A_{r_1} \cdots A_{r_n}) &= \sum_{i_1,\ldots,i_n=1}^{d} \varphi_d(A_{r_1}P_{i_1} \cdots A_{r_n}P_{i_n}) \\
&= \sum_{i_1,\ldots,i_n=1}^{d} [\mathrm{coef}(\ (r_1,i_1),\ldots,(r_n,i_n)\ )](M_{A_1P_1,\ldots,A_sP_d}) \\
&= \frac{1}{d} \sum_{i_1,\ldots,i_n=1}^{d} [\mathrm{coef}(\ (r_1,i_1),\ldots,(r_n,i_n)\ )](f \ \tilde{\boxed{\star}}\ G_d);
\end{aligned}
$$

the latter quantity is easily seen to be the coefficient of $z_{r_1} \cdots z_{r_n}$ in the series:

$$\frac{1}{d}(f \ \tilde{\boxed{\star}}\ G_d)(\ \underbrace{z_1,\ldots,z_1}_{d\ times},\ldots,\underbrace{z_s,\ldots,z_s}_{d\ times}\ ),$$

hence (2.6) follows.

On the other hand let us $\tilde{\boxed{\star}}$-convolve with $\mathrm{Möb}_d$ on the right, on both sides of (2.15). On the left-hand side we get $M_{A_1P_1,\ldots,A_sP_d} \ \tilde{\boxed{\star}}\ \mathrm{Möb}_d$, which is equal to $R_{A_1P_1,\ldots,A_sP_d}$ (see Equation (1.20) in Section 1.5). On the right-hand side we get:

$$
\begin{aligned}
\Big(\frac{1}{d}(f \ \tilde{\boxed{\star}}\ G_d) \Big) \ \tilde{\boxed{\star}}\ \mathrm{Möb}_d &= \frac{1}{d}\Big(f \ \tilde{\boxed{\star}}\ G_d \ \tilde{\boxed{\star}}\ (d\mathrm{Möb}_d \circ D_{1/d}) \Big) \quad \text{(by Eqn.(1.23))} \\
&= \frac{1}{d}(f \ \tilde{\boxed{\star}}\ H_d) \quad \text{(by the definition of } H_d \text{ in Section 1.7).}
\end{aligned}
$$

So we obtain the equation:

$$R_{A_1P_1,\ldots,A_rP_i,\ldots,A_sP_d} = \frac{1}{d}(f \ \tilde{\boxed{\star}}\ H_d), \tag{2.16}$$

out of which (2.7) is obtained in the same way as (2.6) was obtained from (2.15). **QED**

2.12 Remark. The proof of Theorem 2.10 obtains the Equations (2.15) and (2.16), stronger than what was originally stated, and which show better the significance of the series $f \ \tilde{\boxed{\star}}\ G_d$ and $f \ \tilde{\boxed{\star}}\ H_d$.

3. Applications of Theorem 2.10.

We will concentrate on applications to a family $A_1, \ldots, A_s$ of selfadjoint $d \times d$ matrices over a $*$-probability space $(\mathcal{A}, \varphi)$. By keeping in mind the motivating example from [21], it is of particular interest to put into evidence situations where the family $A_1, \ldots, A_s$ is free in $(M_d(\mathcal{A}), \varphi_d)$, and where the individual R-transform of each of $A_1, \ldots, A_s$ is determined explicitly. It seems that some important situations of this kind appear as a consequence of a "partial summation condition", described in the next proposition.

3.1 Proposition. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, let d, s be positive integers, and let $A_1 = [a_{i,j}^{(1)}]_{i,j=1}^d, \ldots, A_s = [a_{i,j}^{(s)}]_{i,j=1}^d$ form an R-cyclic family of selfadjoint matrices in $M_d(\mathcal{A})$. We denote the determining series of $A_1, \ldots, A_s$ by f. Suppose that for every $n \geq 1$ and every $1 \leq r_1, \ldots, r_n \leq s$, $1 \leq i_1, \ldots, i_n \leq d$, the sum:

$$\sum_{i_1,\ldots,i_{n-1}=1}^{d} [\mathrm{coef}(\ (r_1, i_1), \ldots, (r_{n-1}, i_{n-1}), (r_n, i_n)\ )](f) \ =: \lambda_{r_1,\ldots,r_n} \tag{3.1}$$

does not depend on i_n (even though the sum is only over $i_1, \ldots, i_{n-1}$). Then:

$$R_{A_1,\ldots,A_s}(z_1, \ldots, z_s) \ = \ \sum_{n=1}^{\infty} \ \sum_{r_1,\ldots,r_n=1}^{s} \lambda_{r_1,\ldots,r_n} z_{r_1} \cdots z_{r_n}. \tag{3.2}$$

Proof. The Equation (3.2) is equivalent to the fact that for every $n \geq 1$ and every $1 \leq r_1, \ldots, r_n \leq s$ we have:

$$k_n(A_{r_1}, \ldots, A_{r_n}) \ = \ \lambda_{r_1,\ldots,r_n}. \tag{3.3}$$

We fix n and $r_1, \ldots, r_n$ about which we show that (3.3) is true. The case when $n = 1$ is trivial, so we will assume that $n \geq 2$.

The Equation (2.7) of Theorem 2.10 gives us the formula:

$$k_n(A_{r_1}, \ldots, A_{r_n}) \ =$$

$$\frac{1}{d} \sum_{i_1,\ldots,i_n=1}^{d} \ \sum_{\pi \in NC(n)} [\mathrm{coef}(\ (r_1, i_1), \ldots, (r_n, i_n)\ ); \pi](f) \cdot [\mathrm{coef}(i_1, \ldots, i_n); \mathrm{Kr}(\pi)](H_d).$$

We will write this in the form:

$$k_n(A_{r_1}, \ldots, A_{r_n}) \ = \ \sum_{\pi \in NC(n)} T_\pi, \tag{3.4}$$

where for every $\pi \in NC(n)$ we set:

$$T_\pi \ := \ \frac{1}{d} \sum_{i_1,\ldots,i_n=1}^{d} [\mathrm{coef}(\ (r_1, i_1), \ldots, (r_n, i_n)\ ); \pi](f) \cdot [\mathrm{coef}(i_1, \ldots, i_n); \mathrm{Kr}(\pi)](H_d). \tag{3.5}$$

We first consider the quantity T_π defined in (3.5) in the special case when $\pi = 1_n$, the partition of $\{1, \ldots, n\}$ which has only one block. In this case $\mathrm{Kr}(\pi)$ is the partition into n blocks of one element; since all the coefficients of degree 1 of H_d are equal to 1, it follows that

$$[\mathrm{coef}(i_1, \ldots, i_n); \mathrm{Kr}(1_n)](H_d) \;=\; 1, \;\; \forall\, i_1, \ldots, i_n \in \{1, \ldots, d\}.$$

We hence get:

$$T_{1_n} \;=\; \frac{1}{d} \sum_{i_1, \ldots, i_n = 1}^{d} [\mathrm{coef}(\ (r_1, i_1), \ldots, (r_n, i_n)\ )](f).$$

The partial summation property of the series f (given in Eqn.(3.1)) implies that the latter sum is equal to $\lambda_{r_1, \ldots, r_n}$. Thus, in view of (3.4), the proof will be over if we can show that $T_\pi = 0$ for every $\pi \neq 1_n$ in $NC(n)$.

So for the remaining of the proof we fix a partition $\pi \neq 1_n$ in $NC(n)$. Moreover, we will also fix a block B_o of π which is an interval, $B_o = [p, q] \cap \mathbf{Z}$ with $1 \leq p \leq q \leq n$ (every non-crossing partition has such a block). The considerations below, leading to the conclusion that $T_\pi = 0$, will be made by looking at the case when B_o has more than one element; the case when $|B_o| = 1$ (which is similar, and easier) is left as an exercise to the reader. We denote by "*Rest*" the set of blocks of π which are different from B_o.

Let us now look at at the Kreweras complement $\mathrm{Kr}(\pi)$. It is immediate that $\{p\}$, $\{p+1\}, \ldots, \{q-1\}$ are one-element blocks of $\mathrm{Kr}(\pi)$. We denote by B'_o the block of $\mathrm{Kr}(\pi)$ which contains q; observe that B'_o has more than one element – indeed, it is clear that $p-1$ also belongs to B'_o (where if $p = 1$, then "$p-1$" means "n"; even in this case we have that $p - 1 \neq q$, since it was assumed that $\pi \neq 1_n$). Moreover, let us denote by $Rest'$ the set of blocks of $\mathrm{Kr}(\pi)$ (if any) which remain after $\{p\}$, $\{p+1\}, \ldots, \{q-1\}$ and B'_o are deleted.

For any $i_1, \ldots, i_n \in \{1, \ldots, d\}$ we have:

$$\begin{aligned} &[\mathrm{coef}(\ (r_1, i_1), \ldots, (r_n, i_n)\ ); \pi](f) \cdot [\mathrm{coef}(i_1, \ldots, i_n); \mathrm{Kr}(\pi)](H_d) \;= \\ &\quad [\mathrm{coef}(\ (r_p, i_p), \ldots, (r_q, i_q)\ )](f) \cdot [\mathrm{coef}(i_1, \ldots, i_n) | B'_o](H_d) \cdot \qquad (3.6) \\ &\cdot \prod_{B \in Rest} [\mathrm{coef}(\ (r_1, i_1), \ldots, (r_n, i_n)\ ) | B](f) \cdot \prod_{B' \in Rest'} [\mathrm{coef}(i_1, \ldots, i_n) | B'](H_d) \end{aligned}$$

(we took into account that the factors $[\mathrm{coef}(i_p)](H_d), \ldots, [\mathrm{coef}(i_{q-1})](H_d)$, which should also appear on the right-hand side of (3.6), are all equal to 1). The indices $i_p, \ldots, i_{q-1}$ appear only in the factor "$[\mathrm{coef}(\ (r_p, i_p), \ldots, (r_q, i_q)\ )](f)$" of (3.6). Thus, if in (3.6) we sum over $i_p, \ldots, i_{q-1}$, and make use of the partial summation property from (3.1), then we get:

$$\lambda_{r_p, \ldots, r_q} \cdot [\mathrm{coef}(i_1, \ldots, i_n) | B'_o](H_d) \cdot \qquad (3.7)$$

$$\cdot \prod_{B \in Rest} [\text{coef}(\ (r_1, i_1), \ldots, (r_n, i_n)\ )|B](f) \cdot \prod_{B' \in Rest'} [\text{coef}(i_1, \ldots, i_n)|B'](H_d)$$

(expression depending on some arbitrary indices $i_1, \ldots, i_{p-1}, i_q, \ldots, i_n$, chosen from $\{1, \ldots, d\}$).

Next, in (3.7) we sum over the index i_q. The only factor in (3.7) which involves i_q is "$[\text{coef}(i_1, \ldots, i_n)|B'_o](H_d)$", so as a result of this new summation we get:

$$\lambda_{r_p, \ldots, r_q} \cdot \Big\{ \sum_{i_q=1}^{d} [\text{coef}(i_1, \ldots, i_n)|B'_o](H_d) \Big\} \cdot$$

$$\cdot \prod_{B \in Rest} [\text{coef}(\ (r_1, i_1), \ldots, (r_n, i_n)\ )|B](f) \cdot \prod_{B' \in Rest'} [\text{coef}(i_1, \ldots, i_n)|B'](H_d).$$

But, as an immediate consequence of the remark concluding Section 1.7, we have that $\sum_{i_q=1}^{d} [\text{coef}(i_1, \ldots, i_n)|B'_o](H_d) = 0$.

The conclusion that we draw from the preceding three paragraphs is the following: for any choice of the indices $i_1, \ldots, i_{p-1}, i_{q+1}, \ldots, i_n \in \{1, \ldots, d\}$, we have that

$$\sum_{i_p, \ldots, i_q=1}^{d} [\text{coef}(\ (r_1, i_1), \ldots, (r_n, i_n)\ ); \pi](f) \cdot [\text{coef}(i_1, \ldots, i_n); \text{Kr}(\pi)](H_d) \ = \ 0.$$

It only remains that we sum over $i_1, \ldots, i_{p-1}, i_{q+1}, \ldots, i_n$ in the latter equation, to obtain the desired fact that $T_\pi = 0$. **QED**

3.2 Corollary. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, let d, s be positive integers, and let $A_1 = [a_{i,j}^{(1)}]_{i,j=1}^{d}, \ldots, A_s = [a_{i,j}^{(s)}]_{i,j=1}^{d}$ form an R-cyclic family of selfadjoint matrices in $M_d(\mathcal{A})$. Suppose that the s families of entries $\{a_{i,j}^{(r)} \mid 1 \leq i, j \leq d\}$, with $1 \leq r \leq s$, are free in $(\mathcal{A}, \varphi)$. Moreover, for every $1 \leq r \leq s$ let $f_r \in \Theta_d$ be the determining series of A_r. We assume that for every $n \geq 1$ and for every $1 \leq r \leq s$, $1 \leq i \leq d$, the sum:

$$\sum_{i_1, \ldots, i_{n-1}=1}^{d} [\text{coef}(i_1, \ldots, i_{n-1}, i)](f_r) \ =: \ \lambda_n^{(r)} \tag{3.8}$$

does not depend on the choice of i (but only on n and r). Then the matrices $A_1, \ldots, A_s$ are free in $(M_d(\mathcal{A}), \varphi_d)$, and have R-transforms

$$R_{A_r}(z) \ = \ \sum_{n=1}^{\infty} \lambda_n^{(r)} z^n, \quad 1 \leq r \leq s. \tag{3.9}$$

Proof. Let f denote the determining series of the whole R-cyclic family $A_1, \ldots, A_s$. The condition of freeness between the families of entries of $A_1, \ldots, A_s$ implies the formula:

$$f(z_{1,1}, \ldots, z_{r,i}, \ldots, z_{s,d}) \ = \ \sum_{r=1}^{s} f_r(z_{r,1}, \ldots, z_{r,i}, \ldots, z_{r,d}),$$

where f_r is (as in the statement of the corollary) the determining series for just the R-cyclic matrix A_r. It is immediate that f satisfies the partial summation condition described in Equation (3.1) of Proposition 3.1, where we set:

$$\lambda_{r_1,\ldots,r_n} = \begin{cases} \lambda_n^{(r)} & \text{if } r_1 = \cdots = r_n = r \\ 0 & \text{otherwise.} \end{cases}$$

Thus the Proposition 3.1 can be applied, and gives us:

$$R_{A_1,\ldots,A_s}(z_1,\ldots,z_s) = \sum_{r=1}^{s} \sum_{n=1}^{\infty} \lambda_n^{(r)} z_r^n,$$

which (by virtue of Equation (1.15) in Section 1) is equivalent to saying that $A_1,\ldots,A_s$ are free and and have the indicated individual R-transforms. **QED**

The Corollary 3.2 can be in turn particularized to the situation of a family of matrices with free R-diagonal entries (on the line of Example 2.6). The precise spelling of this particular case goes as follows.

3.3 Corollary. Let $(\mathcal{A},\varphi)$ be a $*$-probability space, let d, s be positive integers, and suppose that the elements $\{a_{i,j}^{(r)} \mid 1 \leq i,j \leq d,\ 1 \leq r \leq s\}$ of $\mathcal{A}$ have the following properties:

(i) For every $1 \leq i \leq d$ and $1 \leq r \leq s$, the element $a_{i,i}^{(r)}$ is selfadjoint. We denote the R-transform of $a_{i,i}^{(r)}$ as $\sum_{n=1}^{\infty} \alpha_{i,i;n}^{(r)} z^n$.

(ii) For every $1 \leq i,j \leq d$ such that $i \neq j$, and for every $1 \leq r \leq s$, the element $a_{i,j}^{(r)}$ is R-diagonal and has $\left(a_{i,j}^{(r)}\right)^* = a_{j,i}^{(r)}$. We denote the determining series of $a_{i,j}^{(r)}$ as $\sum_{n=1}^{\infty} \alpha_{i,j;2n}^{(r)} z^n$; we also set $\alpha_{i,j;2n-1}^{(r)} := 0$, $\forall\ n \geq 1$.

(iii) The $sd(d+1)/2$ families: $\{a_{i,i}^{(r)}\}$ for $1 \leq i \leq d$, $1 \leq r \leq s$, together with $\{a_{i,j}^{(r)}, a_{j,i}^{(r)}\}$ for $1 \leq i < j \leq d$, $1 \leq r \leq s$ are free in $(\mathcal{A},\varphi)$.

Suppose moreover that for every $n \geq 1$ and every $1 \leq r \leq s$, $1 \leq i \leq d$, the sum:

$$\sum_{j=1}^{d} \alpha_{i,j;n}^{(r)} =: \lambda_n^{(r)} \tag{3.10}$$

does not actually depend on i. Then the matrices $A_1 = [a_{i,j}^{(1)}]_{i,j=1}^d, \ldots, A_s = [a_{i,j}^{(s)}]_{i,j=1}^d$ are free in $(M_d(\mathcal{A}),\varphi_d)$, and have R-transforms

$$R_{A_r}(z) = \sum_{n=1}^{\infty} \lambda_n^{(r)} z^n, \quad 1 \leq r \leq s.$$

3.4 Remark. The summation conditions (3.10) become extremely simple when the elements $a_{i,i}^{(r)}$ are semicircular, and the elements $a_{i,j}^{(r)}$ with $i \neq j$ are circular. Indeed, in this case we have that $\alpha_{i,j;n}^{(r)} = 0$ whenever $n \neq 2$, and that $\alpha_{i,j;2}^{(r)}$ is one quarter of the squared radius of the circular/semicircular element $a_{i,j}^{(r)}$ (compare to Remark 2.7.2). Thus in this case if we denote the radius of $a_{i,j}^{(r)}$ by $\gamma_{i,j}^{(r)}$, then (3.10) amounts to asking that for every $1 \leq r \leq s$ the matrix of squared radii $[\gamma_{i,j}^{(r)}]_{i,j=1}^{d}$ has constant sums along its columns:

$$\sum_{j=1}^{d} \left(\gamma_{1,j}^{(r)}\right)^2 = \cdots = \sum_{j=1}^{d} \left(\gamma_{d,j}^{(r)}\right)^2 =: \gamma_r^2.$$

The conclusion of Corollary 3.3 becomes that the matrices $A_1 = [a_{i,j}^{(1)}]_{i,j=1}^{d}, \ldots, A_s = [a_{i,j}^{(s)}]_{i,j=1}^{d}$ are free, and that A_r is semicircular of radius γ_r, for $1 \leq r \leq s$. This particular case of Corollary 3.3 is very close to Proposition 2.9 of [21], and can also be obtained by the methods used there (approximations in distribution by large Gaussian random matrices).

Another particularization of Proposition 3.1 covers a situation when the matrices $A_1, \ldots,$ A_s are not free, and which is motivated by results about free compressions (see Sections 8 and 17 of [11]; the case of only one matrix appeared in Example 2.4 above).

3.5 Corollary. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, let d, s be positive integers, and let $A_1 = [a_{i,j}^{(1)}]_{i,j=1}^{d}, \ldots, A_s = [a_{i,j}^{(s)}]_{i,j=1}^{d}$ form an R-cyclic family of selfadjoint matrices in $M_d(\mathcal{A})$. Suppose that the cyclic cumulants of the entries of these matrices depend only on the superscript indices:

$$k_n(a_{i_n,i_1}^{(r_1)}, a_{i_1,i_2}^{(r_2)}, \ldots, a_{i_{n-1},i_n}^{(r_n)}) =: \alpha_{r_1,\ldots,r_n}, \tag{3.11}$$

for every $n \geq 1$ and every $1 \leq r_1, \ldots, r_n \leq s$, $1 \leq i_1, \ldots, i_n \leq d$. Then:

$$R_{A_1,\ldots,A_s}(z_1, \ldots, z_s) = \sum_{n=1}^{\infty} \sum_{r_1,\ldots,r_n=1}^{s} d^{n-1} \alpha_{r_1,\ldots,r_n} z_{r_1} \cdots z_{r_n}. \tag{3.12}$$

Proof. If f denotes the determining series of $A_1, \ldots, A_s$, then the coefficients of f are:

$$[\text{coef}(\ (r_1, i_1), \ldots, (r_n, i_n)\ )](f) =: \alpha_{r_1,\ldots,r_n},$$

$\forall\ n \geq 1$, $\forall\ 1 \leq r_1, \ldots, r_n \leq s$, $1 \leq i_1, \ldots, i_n \leq d$. It is obvious that the partial summation condition of Equation (3.1) holds, where $\lambda_{r_1,\ldots,r_n} := d^{n-1} \alpha_{r_1,\ldots,r_n}$. **QED**

4. Algebras generated by R-cyclic families

4.1 Remark. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, let d be a positive integer, and let $A_1, \ldots, A_s$ be an R-cyclic family of matrices in $M_d(\mathcal{A})$. Directly from the definition of R-cyclicity, and by using some basic properties of the non-crossing cumulants, it is easy to observe several "operations" that can be performed on the family $A_1, \ldots, A_s$ without affecting its R-cyclicity. For instance, it is trivial that re-ordering the s matrices does not affect R-cyclicity, and same about the operation of deleting one of the matrices from the family. Another operation which clearly does not affect the R-cyclicity of $A_1, \ldots, A_s$ consists in arbitrarily re-scaling the entries of the matrices (multiply the (i,j)-entry of A_r by some constant $\lambda_{i,j}^{(r)}$, for every $1 \leq i, j \leq d$, $1 \leq r \leq s$). Let us also observe that:

(a) If we enlarge $A_1, \ldots, A_s$ with a matrix $A \in \text{span}\{A_1, \ldots, A_s\}$, then the enlarged family $A_1, \ldots, A_s, A$ is still R-cyclic. This is a direct consequence of the multilinearity of the cumulant functionals $k_n : \mathcal{A}^n \to \mathbf{C}$, $n \geq 1$.

(b) If we enlarge $A_1, \ldots, A_s$ with a scalar diagonal matrix D (which has the diagonal entries of the form $\lambda_i I$, $1 \leq i \leq d$, and the off-diagonal entries equal to 0), then the enlarged family $A_1, \ldots, A_s, D$ is still R-cyclic. This is a consequence of the fact that a non-crossing cumulant of $n \geq 2$ variables is 0 if at least one of its entries is in $\mathbf{C}I$ (same kind of argument as in the last phrase of Section 1).

In connection to (b) of Remark 4.1, note that we could not use there a scalar matrix which is not diagonal – indeed, the R-cyclicity condition asks in particular that every off-diagonal entry of every matrix in the family lies in the kernel of the state φ.

Now, in the framework of the same R-cyclic family $A_1, \ldots, A_s$ as above, where we assume that $s \geq 2$, let us also observe that:

4.2 Lemma. If $A_{s+1} := A_1 A_2$, then the enlarged family $A_1, \ldots, A_s, A_{s+1}$ is still R-cyclic.

Proof. We will use a formula for free cumulants with products as entries, as developed in [6]. In fact we can set the proof by induction, in such a way that we only use a particular case of this formula, which had already appeared in [19]. The particular case in question says that for any $1 \le m < n$ and any $x_1, \dots, x_n$ in $\mathcal{A}$ we have:

$$
\begin{aligned}
& k_{n-1}(x_1, \dots, x_{m-1}, x_m x_{m+1}, x_{m+2}, \dots, x_n) \\
= \; & k_n(x_1, \dots, x_n) + k_m(x_1, \dots, x_m) \cdot k_{n-m}(x_{m+1}, \dots, x_n) \\
& + \sum_{j=2}^{m} k_{m-j+1}(x_j, \dots, x_m) \cdot k_{n-m+j-1}(x_1, \dots, x_{j-1}, x_{m+1}, \dots, x_n) \\
& + \sum_{j=m+1}^{n-1} k_{j-m}(x_{m+1}, \dots, x_j) \cdot k_{n-j+m}(x_1, \dots, x_m, x_{j+1}, \dots, x_n).
\end{aligned}
\tag{4.1}
$$

Now, let us return to the matrices $A_1, \dots, A_{s+1}$ appearing in the statement of the lemma. For $1 \le r \le s+1$ and $1 \le i, j \le d$ we denote by $a_{i,j}^{(r)}$ the (i,j)-entry of A_r. The hypothesis that $A_{s+1} = A_1 A_2$ thus says that

$$
a_{i,j}^{(s+1)} \;=\; \sum_{k=1}^{d} a_{i,k}^{(1)} a_{k,j}^{(2)}, \quad \forall\; 1 \le i, j \le d. \tag{4.2}
$$

We will prove by induction on $l \ge 0$ the following statement:

$$
\mathrm{St}(l) \quad \begin{cases} \text{For every } n \ge 1,\ r_1, \dots, r_n \in \{1, \dots, s+1\} \text{ and } i_1, j_1, \dots, i_n, j_n \in \{1, \dots, d\} \\ \text{such that } \{m \mid 1 \le m \le n,\ r_m = s+1\} \text{ has } l \text{ elements} \\ \text{and for which it is not true that } j_1 = i_2, \dots, j_{n-1} = i_n, j_n = i_1 \\ \text{we have that } k_n(a_{i_1,j_1}^{(r_1)}, \dots, a_{i_n,j_n}^{(r_n)}) = 0. \end{cases}
$$

If $l = 0$, the statement $\mathrm{St}(l)$ amounts precisely to the hypothesis that the family $A_1, \dots, A_s$ is R-cyclic. For the rest of the proof we fix an $l \ge 1$, for which we assume that the statements $\mathrm{St}(0), \dots, \mathrm{St}(l-1)$ are true, and for which we prove that the statement $\mathrm{St}(l)$ is also true.

Consider $n \ge 1$, $r_1, \dots, r_n \in \{1, \dots, s+1\}$ and $i_1, j_1, \dots, i_n, j_n \in \{1, \dots, d\}$ such that $\{m \mid 1 \le m \le n,\ r_m = s+1\}$ has l elements, and for which it is not true that $j_1 = i_2, \dots, j_{n-1} = i_n, j_n = i_1$. Moreover, let us fix an index m, $1 \le m \le n$, such that $r_m = s+1$. Our goal is to show that $k_n(a_{i_1,j_1}^{(r_1)}, \dots, a_{i_n,j_n}^{(r_n)}) = 0$, but in view of (4.2) and of the multilinearity of k_n it suffices to verify that:

$$
k_n(a_{i_1,j_1}^{(r_1)}, \dots, a_{i_{m-1},j_{m-1}}^{(r_{m-1})}, a_{i_m,k}^{(1)} a_{k,j_m}^{(2)}, a_{i_{m+1},j_{m+1}}^{(r_{m+1})}, \dots, a_{i_n,j_n}^{(r_n)}) = 0, \quad \forall\; 1 \le k \le d. \tag{4.3}
$$

Finally, let us also fix an index $k \in \{1, \dots, d\}$ about which we will show that (4.3) holds. This in fact will be an immediate application of the formula (4.1). Indeed, let us pick an index $p \in \{1, \dots, n\}$ such that $j_p \ne i_{p+1}$; for the sake of clarity of the presentation we will assume that we know the relative position of p and m – say for instance that $p < m-1$ (all the cases are treated similarly). We apply the formula (4.1) to the cumulant (4.3), and obtain a sum of $n+1$ terms $T_1, T_2, \ldots, T_{n+1}$ where each of these terms is either a cumulant or a product of two cumulants:

$$\begin{aligned} & k_n(a_{i_1,j_1}^{(r_1)}, \ldots, a_{i_p,j_p}^{(r_p)}, a_{i_{p+1},j_{p+1}}^{(r_{p+1})}, \ldots, a_{i_m,k}^{(1)} a_{k,j_m}^{(2)}, \ldots, a_{i_n,j_n}^{(r_n)}) \\ & \qquad = T_1 + T_2 + \cdots + T_{n+1}. \end{aligned} \tag{4.4}$$

The list of superscript indices on the left-hand side of (4.4) is $r_1, \ldots, r_{m-1}, 1, 2, r_{m+1}, \ldots, r_n$, containing $l-1$ occurrences of $s+1$. So the induction hypothesis will apply and will give us that $T_1 = \cdots = T_{n+1} = 0$ on the right-hand side of (4.4), provided that we make sure that each of $T_1, \ldots, T_{n+1}$ still violates the cyclicity condition of the subscript indices. The violation of cyclicity for subscript indices is trivial for all of $T_1, \ldots, T_{n+1}$ with one exception, because in general the neighboring indices $j_p \neq i_{p+1}$ will not be separated. The exception is for the term:

$$k_{m-p}(a_{i_{p+1},j_{p+1}}^{(r_{p+1})}, \ldots, a_{i_m,k}^{(1)}) \cdot k_{n+1-m+p}(a_{i_1,j_1}^{(r_1)}, \ldots, a_{i_p,j_p}^{(r_p)}, a_{k,j_m}^{(2)}, \ldots, a_{i_n,j_n}^{(r_n)});$$

but here the cyclicity condition of the subscript indices is still violated, since we must have that either $k \neq i_{p+1}$ or that $j_p \neq k$. **QED**

By combining the various "R-cyclicity preserving operations" which were observed in the Remark 4.1 and Lemma 4.2, we arrive to the following statement (which in some sense collects these observations together):

4.3 Theorem. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, let d be a positive integer, and let $A_1, \ldots, A_s$ be an R-cyclic family of matrices in $M_d(\mathcal{A})$. We denote by $\mathcal{D}$ the algebra of scalar diagonal matrices in $M_d(\mathcal{A})$, and by $\mathcal{C}$ the subalgebra of $M_d(\mathcal{A})$ which is generated by $\{A_1, \ldots, A_s\} \cup \mathcal{D}$. Then every finite family of matrices from $\mathcal{C}$ is R-cyclic.

This theorem will be put into a better perspective by the result in Section 8.

5. Review of operator-valued cumulants

Let $\mathcal{B}$ be a unital algebra over $\mathbf{C}$. By a *$\mathcal{B}$-probability space* we understand a pair $(\mathcal{M}, E)$, where:

– $\mathcal{M}$ is an algebra containing $\mathcal{B}$ as a unital subalgebra (by which we mean that $\mathcal{B}$ is identified as a unital subalgebra of $\mathcal{M}$, in some well-determined way);

– $E : \mathcal{M} \to \mathcal{B}$ is a linear map with the properties that $E(b) = b$ for every $b \in \mathcal{B}$, and $E(b_1 x b_2) = b_1 E(x) b_2$ for every $b_1, b_2 \in \mathcal{B}$, $x \in \mathcal{M}$.

If $(\mathcal{M}, E)$ is a $\mathcal{B}$-probability space and if $x_1, \ldots, x_s \in \mathcal{M}$, then the expressions of the form:

$$E(b_0 x_{r_1} b_1 \cdots x_{r_n} b_n), \text{ with } n \geq 1,\ 1 \leq r_1, \ldots, r_n \leq s,\ b_0, b_1, \ldots, b_n \in \mathcal{B}$$

are called *joint $\mathcal{B}$-moments* of the family $x_1, \ldots, x_s$. Moreover, if $(\widetilde{\mathcal{M}}, \widetilde{E})$ also is a $\mathcal{B}$-probability space and if $\widetilde{x}_1, \ldots, \widetilde{x}_s \in \widetilde{\mathcal{M}}$, we will say that the families $x_1, \ldots, x_s$ and $\widetilde{x_1}, \ldots, \widetilde{x_s}$ have *identical $\mathcal{B}$-distributions* if

$$E(b_0 x_{r_1} b_1 \cdots x_{r_n} b_n) \ = \ \widetilde{E}(b_0 \widetilde{x}_{r_1} b_1 \cdots \widetilde{x}_{r_n} b_n) \tag{5.1}$$

for every $n \geq 1$, $1 \leq r_1, \ldots, r_n \leq s$, and $b_0, b_1, \ldots, b_n \in \mathcal{B}$.

While the joint $\mathcal{B}$-moments generalize the joint moments appearing in Eqn.(1.1) of Section 1, it is in general inconvenient to introduce a concept of "$\mathcal{B}$-moment series" analogous to the one defined by Eqn.(1.2). Similarly, rather than introducing $\mathcal{B}$-valued R-transforms, it is more convenient to just consider the $\mathcal{B}$-valued generalization for the concept of non-crossing cumulant. Following the development of [19], this can be done as described in Proposition 5.2 below.

5.1 Notations. Let π, ρ be partitions in $NC(p)$ and $NC(q)$ respectively, where $p, q \geq 1$. Let k be in $\{0, 1, \ldots, q\}$. By $\mathrm{ins}(\pi \mapsto \rho; k)$ we will denote the non-crossing partition in $NC(p+q)$ which is obtained by "inserting π between the elements k and $k+1$ of ρ". Formally this means that the set $\{k+1, \ldots, k+p\}$ is a union of blocks of $\mathrm{ins}(\pi \mapsto \rho; k)$, and that:

(i) the restriction of $\mathrm{ins}(\pi \mapsto \rho; k)$ to $\{k+1, \ldots, k+p\}$ is naturally identified to π;

(ii) the restriction of $\mathrm{ins}(\pi \mapsto \rho; k)$ to $\{1, 2, \ldots, p+q\} \setminus \{k+1, \ldots, k+p\}$ is naturally identified to ρ.

For example, if $\pi = \{\ \{1\}, \{2,3\}\ \} \in NC(3)$ and $\rho = \{\ \{1,2\}\ \} \in NC(2)$, then: $\mathrm{ins}(\pi \mapsto \rho; 0) = \{\ \{1\}, \{2,3\}, \{4,5\}\ \}$; $\mathrm{ins}(\pi \mapsto \rho; 1) = \{\ \{1,5\}, \{2\}, \{3,4\}\ \}$; $\mathrm{ins}(\pi \mapsto \rho; 2)$ $= \{\ \{1,2\}, \{3\}, \{4,5\}\ \}$.

5.2 Proposition (see [19], Section 3.2). Let $(\mathcal{M}, E)$ be a $\mathcal{B}$-probability space. There exists a family of functionals $\{k_\pi^{(\mathcal{B})} \mid \pi \in \cup_{n=1}^\infty NC(n)\}$ uniquely determined by the following properties:

(1) For $\pi \in NC(n)$, $k_\pi^{(\mathcal{B})}$ is a multilinear functional from $\mathcal{M}^n$ to $\mathcal{B}$.

(2) If $\pi \in NC(p)$, $\rho \in NC(q)$, $k \in \{0, 1, \ldots, q\}$, and if $\sigma := \mathrm{ins}(\pi \mapsto \rho; k) \in NC(p+q)$, then for every $x_1, \ldots, x_{p+q} \in \mathcal{M}$ we have:

$$\left\{ \begin{array}{lcl} k_\sigma^{(\mathcal{B})}(x_1, \ldots, x_{p+q}) & = & k_\rho^{(\mathcal{B})}(x_1, \ldots, x_k b, x_{k+p+1}, \ldots, x_{p+q}) \\ & & \\ \text{where } b := k_\pi^{(\mathcal{B})}(x_{k+1}, \ldots, x_{k+p}). & & \end{array} \right. \tag{5.2}$$

(3) For every $n \geq 1$ and $x_1, \ldots, x_n \in \mathcal{M}$ we have:

$$\sum_{\pi \in NC(n)} k_\pi^{(\mathcal{B})}(x_1, x_2, \ldots, x_n) \;=\; E(x_1 x_2 \cdots x_n). \tag{5.3}$$

5.3 Remarks and Notations. 1) In the condition (1) of Proposition 5.2, by "multilinear" we understand **C**-multilinear. The functionals $k_\pi^{(\mathcal{B})}$ turn out to actually have $\mathcal{B}$-multilinearity properties, namely that:

$$\begin{aligned} k_\pi^{(\mathcal{B})}(bx_1, x_2, \ldots, x_n) &= b \cdot k_\pi^{(\mathcal{B})}(x_1, x_2, \ldots, x_n), \\ k_\pi^{(\mathcal{B})}(x_1, x_2, \ldots, x_n b) &= k_\pi^{(\mathcal{B})}(x_1, x_2, \ldots, x_n) \cdot b, \end{aligned}$$

also that

$$k_\pi^{(\mathcal{B})}(x_1, \ldots, x_i b, x_{i+1}, \ldots, x_n) \;=\; k_\pi^{(\mathcal{B})}(x_1, \ldots, x_i, b x_{i+1}, \ldots, x_n)$$

for every $\pi \in NC(n)$, $x_1, \ldots, x_n \in \mathcal{M}$, $b \in \mathcal{B}$ and $1 \leq i \leq n-1$. The **C**-multilinearity stated in (1) of Proposition 5.2 is however more convenient when using the uniqueness part of the proposition.

2) For every $n \geq 1$, we will denote by $k_n^{(\mathcal{B})} : \mathcal{M}^n \to \mathcal{B}$ the functional $k_{1_n}^{(\mathcal{B})}$, where 1_n is the partition of $\{1, \ldots, n\}$ into only one block.

The knowledge of the functionals $\{k_n^{(\mathcal{B})} \mid n \geq 1\}$ really determines the whole family $\{k_\pi^{(\mathcal{B})} \mid \pi \in \cup_{n=1}^\infty NC(n)\}$, via the Equation (5.2) and the observation that the only non-crossing partitions that are irreducible for the operation of insertion are those of the form 1_n. So in a certain sense the functionals $k_\pi^{(\mathcal{B})}$ with π not of the form 1_n are just some derived objects; but nevertheless, the $k_\pi^{(\mathcal{B})}$'s are important for stating the essential condition (3) of Proposition 5.2, which can be viewed as a $\mathcal{B}$-valued analogue for Eqn.(1.11) in Section 1.

3) Let $(\mathcal{M}, E)$ and $(\widetilde{\mathcal{M}}, \widetilde{E})$ be $\mathcal{B}$-probability spaces, and consider the families of elements $x_1, \ldots, x_s \in \mathcal{M}$, $\widetilde{x}_1, \ldots, \widetilde{x}_s \in \widetilde{\mathcal{M}}$. We say that the families $x_1, \ldots, x_s$ and $\widetilde{x}_1, \ldots, \widetilde{x}_s$ have *identical $\mathcal{B}$-cumulants* if:

$$k_n^{(\mathcal{B})}(x_{r_1}b_1, \ldots, x_{r_{n-1}}b_{n-1}, x_{r_n}) = k_n^{(\mathcal{B})}(\widetilde{x}_{r_1}b_1, \ldots, \widetilde{x}_{r_{n-1}}b_{n-1}, \widetilde{x}_{r_n}), \tag{5.4}$$

for every $n \geq 1$, $1 \leq r_1, \ldots, r_n \leq s$, and $b_1, \ldots, b_{n-1} \in \mathcal{B}$.

If $x_1, \ldots, x_s$ and $\widetilde{x}_1, \ldots, \widetilde{x}_s$ have identical $\mathcal{B}$-cumulants, then the Equations (5.4) actually hold with "$k_\pi^{(\mathcal{B})}$" instead of $k_n^{(\mathcal{B})}$; this is immediate from (5.2), by an induction argument. Another induction argument and the use of Equation (5.3) show that $x_1, \ldots, x_s$ and $\widetilde{x}_1, \ldots, \widetilde{x}_s$ have identical $\mathcal{B}$-cumulants if and only if the two families are identically $\mathcal{B}$-distributed in the sense of Equation (5.1). Hence, similarly to the scalar case reviewed in Section 1, the $\mathcal{B}$-cumulants offer an alternative to working with $\mathcal{B}$-moments.

It is useful to record the following generalization (in Proposition 5.5) of the uniqueness part of Proposition 5.2. In all the considerations of this paper, by "$\mathcal{B}$-bimodule" we will understand a left-and-right $\mathcal{B}$-module, where the left and the right action of $\mathcal{B}$ commute with each other.

5.4 Definition. Let $\mathcal{X}$ be a $\mathcal{B}$-bimodule, and suppose that for every $n \geq 1$ and $\pi \in NC(n)$ we have a $\mathbf{C}$-multilinear functional $f_\pi : \mathcal{X}^n \to \mathcal{B}$. We say that the family of functionals $\{f_\pi \mid \pi \in \cup_{n=1}^{\infty} NC(n)\}$ has the *insertion property* if the following holds: if $\sigma = \mathrm{ins}(\pi \mapsto \rho; k)$ with $\pi \in NC(p)$, $\rho \in NC(q)$, $k \in \{0, 1, \ldots, q\}$, and if $x_1, \ldots, x_{p+q} \in \mathcal{X}$, then:

$$\begin{cases} f_\sigma(x_1, \ldots, x_{p+q}) = f_\rho(x_1, \ldots, x_k \cdot b, x_{k+p+1}, \ldots, x_{p+q}) \\ \text{where } b := f_\pi(x_{k+1}, \ldots, x_{k+p}) \in \mathcal{B}. \end{cases} \tag{5.5}$$

5.5 Proposition. Let $\mathcal{X}$ be a $\mathcal{B}$-bimodule, and suppose that for every $n \geq 1$ and $\pi \in NC(n)$ we have two $\mathbf{C}$-multilinear functionals $f_\pi, g_\pi : \mathcal{X}^n \to \mathcal{B}$. If both the families $\{f_\pi \mid \pi \in \cup_{n=1}^{\infty} NC(n)\}$ and $\{g_\pi \mid \pi \in \cup_{n=1}^{\infty} NC(n)\}$ have the insertion property, and if:

$$\sum_{\pi \in NC(n)} f_\pi(x_1, \ldots, x_n) = \sum_{\pi \in NC(n)} g_\pi(x_1, \ldots, x_n), \tag{5.6}$$

for every $n \geq 1$ and $x_1, \ldots, x_n \in \mathcal{X}$, then we must have that $f_\pi = g_\pi$ for all $\pi \in \cup_{n=1}^{\infty} NC(n)$.

The proof of Proposition 5.5 is done by induction on n (where $\pi \in NC(n)$), and is an immediate adaptation of arguments in [19], Section 3.2.

The main use of $\mathcal{B}$-cumulants is as tool for studying freeness with amalgamation over $\mathcal{B}$. Recall that this is defined as follows (cf. e.g. [22], Section 3.8).

5.6 Definition. Let $(\mathcal{M}, E)$ be a $\mathcal{B}$-probability space and let $\mathcal{M}_1, \ldots, \mathcal{M}_s$ be subalgebras of $\mathcal{M}$ such that $\mathcal{M}_1, \ldots, \mathcal{M}_s \supset \mathcal{B}$. We say that $\mathcal{M}_1, \ldots, \mathcal{M}_s$ are *free with amalgamation over* $\mathcal{B}$ if for every $n \geq 1$ and every $r_1, \ldots, r_n \in \{1, \ldots, s\}$ such that $r_1 \neq r_2, r_2 \neq r_3, \ldots, r_{n-1} \neq r_n$ we have:

$$\left.\begin{array}{l} x_1 \in \mathcal{M}_{r_1}, x_2 \in \mathcal{M}_{r_2}, \ldots, x_n \in \mathcal{M}_{r_n} \\ \\ E(x_1) = E(x_2) = \cdots = E(x_n) = 0 \end{array}\right\} \Rightarrow E(x_1 x_2 \cdots x_n) = 0. \tag{5.7}$$

5.7 Remark. The important characterization of freeness described in Remark 1.4 can be generalized to the $\mathcal{B}$-valued framework. More precisely: if $(\mathcal{M}, E)$ and $\mathcal{B} \subset \mathcal{M}_1, \ldots, \mathcal{M}_s \subset \mathcal{M}$ are as above, then the freeness of $\mathcal{M}_1, \ldots, \mathcal{M}_s$ with amalgamation over $\mathcal{B}$ is equivalent to the following condition:

$$\left\{\begin{array}{c} k_n^{(\mathcal{B})}(x_1, \ldots, x_n) = 0 \\ \text{whenever } x_1 \in \mathcal{M}_{r_1}, \ldots, x_n \in \mathcal{M}_{r_n} \\ \text{are such that } \exists\ 1 \leq k < l \leq n \text{ with } r_k \neq r_l. \end{array}\right.$$

See [19], Section 3.3.

5.8 Notations. For the remaining of this section we will suppose that besides the algebra $\mathcal{B}$ (which was fixed from the beginning of the section) we have also fixed:

– a unital subalgebra $\mathcal{D} \subset \mathcal{B}$;

– a linear map $\tau : \mathcal{B} \to \mathcal{D}$ with the properties that $\tau(d) = d$ for every $d \in \mathcal{D}$, and that $\tau(d_1 b d_2) = d_1 \tau(b) d_2$ for every $d_1, d_2 \in \mathcal{D}$, $b \in \mathcal{B}$.

We will assume moreover that τ is faithful (or non-degenerate) in the sense that if $b \in \mathcal{B}$ has the property that $\tau(bb') = 0$ for all $b' \in \mathcal{B}$, then $b = 0$.

In the Notations 5.8, observe that any $\mathcal{B}$-probability space $(\mathcal{M}, E)$ induces a $\mathcal{D}$-probability space $(\mathcal{M}, E_{\mathcal{D}})$, where we set $E_{\mathcal{D}} := \tau \circ E$.

5.9 Proposition. Let $(\mathcal{M}, E)$ and $(\widetilde{\mathcal{M}}, \widetilde{E})$ be $\mathcal{B}$-probability spaces, and consider the corresponding $\mathcal{D}$-probability spaces $(\mathcal{M}, E_{\mathcal{D}})$ and $(\widetilde{\mathcal{M}}, \widetilde{E}_{\mathcal{D}})$. Suppose that $\mathcal{C} \subset \mathcal{M}$ and

$\widetilde{\mathcal{C}} \subset \widetilde{\mathcal{M}}$ are subalgebras which contain $\mathcal{D}$, and suppose that each of $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ is free from $\mathcal{B}$ with amalgamation over $\mathcal{D}$ (in its corresponding space). Let $x_1, \ldots, x_s$ be in $\mathcal{C}$, and let $\widetilde{x}_1, \ldots, \widetilde{x}_s$ be in $\widetilde{\mathcal{C}}$. If the families $x_1, \ldots, x_s$ are identically $\mathcal{D}$-distributed, then the two families are also identically $\mathcal{B}$-distributed.

Proof. We have to show that:

$$E_{\mathcal{B}}(b_0 x_{r_1} b_1 \cdots x_{r_n} b_n) \ = \ \widetilde{E}_{\mathcal{B}}(b_0 \widetilde{x}_{r_1} b_1 \cdots \widetilde{x}_{r_n} b_n),$$

for every $n \geq 1$, $1 \leq r_1, \ldots, r_n \leq s$, and $b_0, b_1, \ldots, b_n \in \mathcal{B}$. In view of the faithfulness of $\tau : \mathcal{B} \to \mathcal{D}$, this will follow if we can show that:

$$\tau(\ E_{\mathcal{B}}(b_0 x_{r_1} b_1 \cdots x_{r_n} b_n) b' \ ) \ = \ \tau(\widetilde{E}_{\mathcal{B}}(b_0 \widetilde{x}_{r_1} b_1 \cdots \widetilde{x}_{r_n} b_n) b' \ ), \tag{5.8}$$

(for every $n, r_1, \ldots, r_n, b_0, b_1, \ldots, b_n$ as before, and for every $b' \in \mathcal{B}$). By absorbing b' into $E_{\mathcal{B}}$ and into $\widetilde{E}_{\mathcal{B}}$, and by taking into account that $\tau \circ E_{\mathcal{B}} = E_{\mathcal{D}}$, $\tau \circ \widetilde{E}_{\mathcal{B}} = \widetilde{E}_{\mathcal{D}}$, we reduce (5.8) to:

$$E_{\mathcal{D}}(b_0 x_{r_1} b_1 \cdots x_{r_n} b_n) \ = \ \widetilde{E}_{\mathcal{D}}(b_0 \widetilde{x}_{r_1} b_1 \cdots \widetilde{x}_{r_n} b_n), \tag{5.9}$$

for every $n \geq 1$, $1 \leq r_1, \ldots, r_n \leq s$, and $b_0, b_1, \ldots, b_n \in \mathcal{B}$. Finally, (5.9) follows from the definition of freeness with amalgamation plus an induction argument, by using the hypotheses that $x_1, \ldots, x_s$ and $\widetilde{x}_1, \ldots, \widetilde{x}_s$ have identical $\mathcal{D}$-distributions, and that $\mathcal{C}$, $\widetilde{\mathcal{C}}$ are free from $\mathcal{B}$ with amalgamation over $\mathcal{D}$. **QED**

6. Cumulants with respect to the algebra of d-by-d scalar matrices

6.1 Notations. In this section we fix a positive integer d, and we consider the algebra $\mathcal{B} := M_d(\mathbf{C})$. If $(\mathcal{A}, \varphi)$ is any non-commutative probability space, then the algebra $M_d(\mathcal{A})$ gets a structure of $\mathcal{B}$-probability space, where we view $\mathcal{B}$ as a subalgebra of $M_d(\mathcal{A})$ via the natural identification:

$$[\lambda_{i,j}]_{i,j=1}^d \ = \ [\lambda_{i,j} I]_{i,j=1}^d \tag{6.1}$$

(with $I =$ the unit of $\mathcal{A}$). The expectation $E_{\mathcal{B}} : M_d(\mathcal{A}) \to \mathcal{B}$ is defined by the formula:

$$E_{\mathcal{B}}(\ [a_{i,j}]_{i,j=1}^d \ ) \ := \ [\ \varphi(a_{i,j})\ ]_{i,j=1}^d. \tag{6.2}$$

Thus we are in the situation when we can consider $\mathcal{B}$-valued cumulants for families of matrices in $M_d(\mathcal{A})$.

The goal of the section is to give an explicit formula for the $\mathcal{B}$-valued cumulant of a family of matrices, in terms of the scalar cumulants of the entries of these matrices.

6.2 Theorem. In the framework considered above let $A_1, \ldots, A_n$ be matrices in $M_d(\mathcal{A})$, where $A_m = [a_{i,j}^{(m)}]_{i,j=1}^d$ for $1 \leq m \leq n$. Then for every $1 \leq i, j \leq d$, the (i,j)-entry $\lambda_{i,j}$ of the $\mathcal{B}$-valued cumulant $k_n^{(\mathcal{B})}(A_1, \ldots, A_n)$ is given by the formula:

$$\lambda_{i,j} = \sum_{i_1,\ldots,i_{n-1}=1}^{d} k_n(a_{i,i_1}^{(1)}, a_{i_1,i_2}^{(2)}, \ldots, a_{i_{n-2},i_{n-1}}^{(n-1)}, a_{i_{n-1},j}^{(n)}). \tag{6.3}$$

Proof. For every $n \geq 1$ and $\pi \in NC(n)$, we define a multilinear functional $f_\pi : (M_d(\mathcal{A}))^n \to \mathcal{B}$, by the following formula:

$$(i,j) - \text{entry of } f_\pi(A_1, \ldots, A_n) := \tag{6.4}$$

$$\sum_{i_1,\ldots,i_{n-1}=1}^{d} k_\pi(a_{i,i_1}^{(1)}, a_{i_1,i_2}^{(2)}, \ldots, a_{i_{n-2},i_{n-1}}^{(n-1)}, a_{i_{n-1},j}^{(n)}),$$

for every $A_1, \ldots, A_n \in M_d(\mathcal{A})$ and every $1 \leq i, j \leq d$ (and where $a_{k,l}^{(m)}$ stands for the (k,l)-entry of the matrix A_m). We will verify that the family of functionals $\{f_\pi \mid \pi \in \cup_{n=1}^\infty NC(n)\}$ satisfies the conditions (2) and (3) from Proposition 5.2, which determine uniquely the $\mathcal{B}$-valued cumulant functionals. Once this is done, the equality $f_\pi = k_\pi^{(\mathcal{B})}$ applied to the partition $\pi = 1_n$ (of $\{1, \ldots, n\}$ into only one block) will give the statement of the theorem.

We start with the verification of condition (3) (about summation). Given $n \geq 1$ and $A_1, \ldots, A_n \in M_d(\mathcal{A})$, we look at:

$$\sum_{\pi \in NC(n)} f_\pi(A_1, \ldots, A_n). \tag{6.5}$$

For every $1 \leq i, j \leq d$, the (i,j)-entry of the matrix appearing in (6.5) is equal to:

$$\sum_{\pi \in NC(n)} \sum_{i_1,\ldots,i_{n-1}=1}^{d} k_\pi(a_{i,i_1}^{(1)}, a_{i_1,i_2}^{(2)}, \ldots, a_{i_{n-2},i_{n-1}}^{(n-1)}, a_{i_{n-1},j}^{(n)})$$

$$= \sum_{i_1,\ldots,i_{n-1}=1}^{d} \Big(\sum_{\pi \in NC(n)} k_\pi(a_{i,i_1}^{(1)}, a_{i_1,i_2}^{(2)}, \ldots, a_{i_{n-2},i_{n-1}}^{(n-1)}, a_{i_{n-1},j}^{(n)}) \Big)$$

$$= \sum_{i_1,\ldots,i_{n-1}=1}^{d} \varphi(a_{i,i_1}^{(1)} a_{i_1,i_2}^{(2)} \cdots a_{i_{n-2},i_{n-1}}^{(n-1)} a_{i_{n-1},j}^{(n)})$$

(by the relation between scalar cumulants and moments). It is clear that the latter quantity is equal to φ of the (i,j)-entry of $A_1 A_2 \cdots A_n$. Hence the matrix in (6.5) is equal to $E_{\mathcal{B}}(A_1 A_2 \cdots A_n)$ (as desired).

We now move to the verification of condition (2) (about insertion). Suppose that $\sigma = \mathrm{ins}(\pi \mapsto \rho; k)$, where $\pi \in NC(p)$, $\rho \in NC(q)$, $0 \le k \le q$, and where the Notations 5.1 are used. Given matrices $A_1, \ldots, A_{p+q} \in M_d(\mathcal{A})$, we want to verify that:

$$f_\sigma(A_1, \ldots, A_{p+q}) \;=\; f_\rho(A_1, \ldots, A_{k-1}, A_k B, A_{k+p+1}, \ldots, A_{p+q}), \tag{6.6}$$

where

$$B \;:=\; f_\pi(A_{k+1}, \ldots, A_{k+p}). \tag{6.7}$$

We fix i and j in $\{1, \ldots, d\}$, and we work on the (i,j)-entry of the left-hand side of (6.6). By the definition of f_σ this equals:

$$\sum_{i_1, \ldots, i_{p+q-1}=1}^{d} k_\sigma(a^{(1)}_{i,i_1}, a^{(2)}_{i_1,i_2}, \ldots, a^{(p+q-1)}_{i_{p+q-2},i_{p+q-1}}, a^{(p+q)}_{i_{p+q-1},j}),$$

so by using the insertion property for scalar cumulants, we can re-write it as:

$$\sum_{i_1, \ldots, i_{p+q-1}=1}^{d} k_\pi(a^{(k+1)}_{i_k,i_{k+1}}, \ldots, a^{(k+p)}_{i_{k+p-1},i_{k+p}}) \cdot \tag{6.8}$$

$$\cdot k_\rho(a^{(1)}_{i,i_1}, \ldots, a^{(k)}_{i_{k-1},i_k}, a^{(k+p+1)}_{i_{k+p},i_{k+p+1}}, \ldots, a^{(p+q-1)}_{i_{p+q-2},i_{p+q-1}}, a^{(p+q)}_{i_{p+q-1},j}).$$

Now, let us denote:

$$B =: [\beta_{i,j}]_{i,j=1}^{d} \in \mathcal{B}, \quad BA_k =: [x_{i,j}]_{i,j=1}^{d} \in M_d(\mathcal{A}), \tag{6.9}$$

where B is the matrix defined by (6.7). If in the summation of (6.8) we first sum over the indices $i_{k+1}, \ldots, i_{k+p-1}$, we arrive to:

$$\sum_{i_1, \ldots, i_k, i_{k+p}, \ldots, i_{p+q-1}=1}^{d} \Big(\sum_{i_{k+1}, \ldots, i_{k+p-1}=1}^{d} k_\pi(a^{(k+1)}_{i_k,i_{k+1}}, \ldots, a^{(k+p)}_{i_{k+p-1},i_{k+p}}) \Big) \cdot$$

$$\cdot k_\rho(a^{(1)}_{i,i_1}, \ldots, a^{(k)}_{i_{k-1},i_k}, a^{(k+p+1)}_{i_{k+p},i_{k+p+1}}, \ldots, a^{(p+q-1)}_{i_{p+q-2},i_{p+q-1}}, a^{(p+q)}_{i_{p+q-1},j})$$

$$= \sum_{i_1, \ldots, i_k, i_{k+p}, \ldots, i_{p+q-1}=1}^{d} \beta_{i_k,i_{k+p}} \cdot k_\rho(a^{(1)}_{i,i_1}, \ldots, a^{(k)}_{i_{k-1},i_k}, a^{(k+p+1)}_{i_{k+p},i_{k+p+1}}, \ldots, a^{(p+q-1)}_{i_{p+q-2},i_{p+q-1}}, a^{(p+q)}_{i_{p+q-1},j})$$

(by taking into account the definition of f_π, the Equation (6.7), and the notation in (6.9))

$$= \sum_{i_1, \ldots, i_k, i_{k+p}, \ldots, i_{p+q-1}=1}^{d} k_\rho(a^{(1)}_{i,i_1}, \ldots, a^{(k)}_{i_{k-1},i_k}\beta_{i_k,i_{k+p}}, a^{(k+p+1)}_{i_{k+p},i_{k+p+1}}, \ldots, a^{(p+q-1)}_{i_{p+q-2},i_{p+q-1}}, a^{(p+q)}_{i_{p+q-1},j})$$

$$= \sum_{i_1,\ldots,i_{k-1},i_{k+p},\ldots,i_{p+q-1}=1}^{d} k_\rho(a^{(1)}_{i,i_1},\ldots,a^{(k-1)}_{i_{k-2},i_{k-1}},x_{i_{k-1},i_{k+p}},a^{(k+p+1)}_{i_{k+p},i_{k+p+1}},\ldots,a^{(p+q)}_{i_{p+q-1},j})$$

(by summing over i_k and by taking into account the definition of the $x_{i,j}$'s in (6.9)). The last expression is exactly the (i,j)-entry of the matrix on the right-hand side of (6.6), and this concludes the proof. **QED**

6.3 Remark. We actually arrived to prove a stronger formula than originally announced in Theorem 6.2, namely that

$$(i,j) - \text{entry of } k^{(\mathcal{B})}_\pi(A_1,\ldots,A_n) \; = \tag{6.10}$$

$$\sum_{i_1,\ldots,i_{n-1}=1}^{d} k_\pi(a^{(1)}_{i,i_1},a^{(2)}_{i_1,i_2},\ldots,a^{(n-1)}_{i_{n-2},i_{n-1}},a^{(n)}_{i_{n-1},j}),$$

for every non-crossing partition $\pi \in NC(n)$ and every $A_1,\ldots,A_n \in M_d(\mathcal{A})$ (and where $a^{(m)}_{k,l}$ denotes the (k,l)-entry of the matrix A_m).

6.4 Remark. For $A_1,\ldots,A_n \in M_d(\mathcal{A})$ as above, one sometimes denotes by $A_1 \odot A_2 \odot \cdots \odot A_n$ the matrix in $M_d(\mathcal{A}\otimes\mathcal{A}\otimes\cdots\otimes\mathcal{A})$ which has the (i,j)-entry equal to:

$$\sum_{i_1,\ldots,i_{n-1}=1}^{d} a^{(1)}_{i,i_1}\otimes a^{(2)}_{i_1,i_2}\otimes\cdots\otimes a^{(n)}_{i_{n-1},j}, \quad 1 \le i,j \le d.$$

The operation $\odot$ is for instance used in considerations on tensor products of operator spaces (see e.g. Section 8.1 of [4], or Section 3 of [16]).

The statement of Theorem 6.2 can be given a nice form if we use $\odot$, as follows: instead of viewing the scalar-valued cumulant k_n as a multilinear map from $\mathcal{A}^n$ to $\mathbf{C}$, let us view it as a linear map from the n-fold tensor product $\mathcal{A}\otimes\cdots\otimes\mathcal{A}$ into $\mathbf{C}$. When we go to $d\times d$ matrices, k_n then induces a linear application $\widetilde{k_n}$ from $M_d(\mathcal{A}\otimes\cdots\otimes\mathcal{A})$ to $M_d(\mathbf{C})$, hence to $\mathcal{B}$; this is given by the formula

$$\widetilde{k_n}(\;[x_{i,j}]^d_{i,j=1}\;) \;:=\; [k_n(x_{i,j})]^d_{i,j=1}, \quad \forall\;[x_{i,j}]^d_{i,j=1} \in M_d(\mathcal{A}\otimes\cdots\otimes\mathcal{A}).$$

It is immediate that with these notations, the statement of Theorem 6.2 takes the form:

$$k^{(\mathcal{B})}_n(A_1,\ldots,A_n) \;=\; \widetilde{k_n}(A_1\odot\cdots\odot A_n), \quad \forall\; A_1,\ldots,A_n \in M_d(\mathcal{A}). \tag{6.11}$$

7. Cumulants with respect to the algebra of scalar diagonal matrices

7.1 Notations. The framework for this section is similar to the one of Section 6, but where instead of the algebra $\mathcal{B} = M_d(\mathbf{C})$ we consider the algebra $\mathcal{D}$ of scalar diagonal $d \times d$ matrices. In other words $\mathcal{D} = \text{span}\{P_1, \ldots, P_d\}$, where P_i denotes the matrix which has its (i,i)-entry equal to 1 and all the other entries equal to 0.

If $(\mathcal{A}, \varphi)$ is any non-commutative probability space, then the algebra $M_d(\mathcal{A})$ gets a natural structure of $\mathcal{D}$-probability space, where we view $\mathcal{D}$ as a subalgebra of $M_d(\mathcal{A})$ via the natural identification:

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{bmatrix} = \begin{bmatrix} \lambda_1 I & & 0 \\ & \ddots & \\ 0 & & \lambda_d I \end{bmatrix} \tag{7.1}$$

(with $I =$ the unit of $\mathcal{A}$). The expectation $E_{\mathcal{D}} : M_d(\mathcal{A}) \to \mathcal{D}$ is defined by the formula:

$$E_{\mathcal{D}}(\ [a_{i,j}]_{i,j=1}^d\ ) \ := \ \begin{bmatrix} \varphi(a_{1,1}) & & 0 \\ & \ddots & \\ 0 & & \varphi(a_{d,d}) \end{bmatrix}. \tag{7.2}$$

Thus we are in the situation when we can consider $\mathcal{D}$-valued cumulants for families of matrices in $M_d(\mathcal{A})$.

Following the same line as in the preceding section, we consider the problem of expressing the $\mathcal{D}$-cumulants of a family of matrices from $M_d(\mathcal{A})$ in terms of the scalar cumulants of the entries of these matrices. It does not seem that there exists a nice formula holding in general, but it is still possible to get one in the case of R-cyclic families. In fact we will consider a class larger than the one of R-cyclic families, as described in the next theorem.

7.2 Theorem. In the framework considered above, let $A_1, \ldots, A_s$ be a family of matrices in $M_d(\mathcal{A})$, where $A_r = [a_{i,j}^{(r)}]_{i,j=1}^d$ for $1 \leq r \leq s$. Suppose that for every $n \geq 1$, $1 \leq r_1, \ldots, r_n \leq s$, $1 \leq i_1, \ldots, i_n, j \leq d$ we have:

$$j \neq i_n \ \Rightarrow \ k_n(a_{j,i_1}^{(r_1)}, a_{i_1,i_2}^{(r_2)}, \ldots, a_{i_{n-1},i_n}^{(r_n)}) = 0. \tag{7.3}$$

Then the $\mathcal{D}$-valued cumulants of the family $A_1, \ldots, A_s$ are described by the following formula:

$$k_n^{(\mathcal{D})}(A_{r_1}\Lambda_1, \ldots, A_{r_{n-1}}\Lambda_{n-1}, A_{r_n}) \ = \tag{7.4}$$

$$\sum_{i_1,\ldots,i_n=1}^{d} \lambda_{i_1}^{(1)} \cdots \lambda_{i_{n-1}}^{(n-1)} \cdot k_n(a_{i_n,i_1}^{(r_1)}, a_{i_1,i_2}^{(r_2)}, \ldots, a_{i_{n-1},i_n}^{(r_n)}) P_{i_n},$$

holding for $n \geq 2$, $r_1, \ldots, r_n \in \{1, \ldots, s\}$, and where

$$\Lambda_k := \begin{bmatrix} \lambda_1^{(k)} & & 0 \\ & \ddots & \\ 0 & & \lambda_d^{(k)} \end{bmatrix} \in \mathcal{D}, \quad 1 \leq k \leq n-1.$$

Proof. Let $\mathcal{X}$ be the free $\mathcal{D}$-bimodule with s generators $X_1, \ldots, X_s$. As a vector space over $\mathbf{C}$, $\mathcal{X}$ has dimension $d^2 s$, and has a natural basis given by the elements $P_i X_r P_j$, with $1 \leq i, j \leq d$ and $1 \leq r \leq s$.

For every $n \geq 1$ and $\pi \in NC(n)$ we consider the $\mathbf{C}$-multilinear functionals f_π and g_π from $\mathcal{X}^n$ to $\mathcal{D}$, determined as follows (by their action on the natural basis of $\mathcal{X}^n$):

$$f_\pi(P_{i_1} X_{r_1} P_{j_1}, \ldots, P_{i_n} X_{r_n} P_{j_n}) = k_\pi^{(\mathcal{D})}(P_{i_1} A_{r_1} P_{j_1}, \ldots, P_{i_n} A_{r_n} P_{j_n}) \tag{7.5}$$

and

$$\begin{aligned} g_\pi(P_{i_1} X_{r_1} P_{j_1}, \ldots, P_{i_n} X_{r_n} P_{j_n}) &= \\ \delta_{i_1, j_n} \delta_{i_2, j_1} \cdots \delta_{i_n, j_{n-1}} \cdot k_\pi(a_{j_n, j_1}^{(r_1)}, a_{j_1, j_2}^{(r_2)}, \ldots, a_{j_{n-1}, j_n}^{(r_n)}) P_{j_n}&, \end{aligned} \tag{7.6}$$

for $n \geq 1$ and $1 \leq r_1, \ldots, r_n \leq s$, $1 \leq i_1, \ldots, i_n, j_1, \ldots, j_n \leq d$. An immediate linearity argument shows that:

$$f_\pi(\Gamma_1 X_{r_1} \Lambda_1, \ldots, \Gamma_n X_{r_n} \Lambda_n) = k_\pi^{(\mathcal{D})}(\Gamma_1 A_{r_1} \Lambda_1, \ldots, \Gamma_n A_{r_n} \Lambda_n), \tag{7.7}$$

and

$$\begin{aligned} g_\pi(\Gamma_1 X_{r_1} \Lambda_1, \ldots, \Gamma_n X_{r_n} \Lambda_n) &= \\ \sum_{j_1, \ldots, j_n = 1}^{d} \gamma_{j_n}^{(1)} \cdot \lambda_{j_1}^{(1)} \gamma_{j_1}^{(2)} \cdots \lambda_{j_{n-1}}^{(n-1)} \gamma_{j_{n-1}}^{(n)} \cdot \lambda_{j_n}^{(n)} \cdot k_\pi(a_{j_n, j_1}^{(r_1)}, a_{j_1, j_2}^{(r_2)}, \ldots, a_{j_{n-1}, j_n}^{(r_n)}) P_{j_n}&, \end{aligned} \tag{7.8}$$

for every $n \geq 1$, $1 \leq r_1, \ldots, r_n \leq s$ and $\Gamma_1, \Lambda_1, \ldots, \Gamma_n, \Lambda_n \in \mathcal{D}$, where:

$$\Gamma_k := \begin{bmatrix} \gamma_1^{(k)} & & 0 \\ & \ddots & \\ 0 & & \gamma_d^{(k)} \end{bmatrix}, \quad \Lambda_k := \begin{bmatrix} \lambda_1^{(k)} & & 0 \\ & \ddots & \\ 0 & & \lambda_d^{(k)} \end{bmatrix}, \quad 1 \leq k \leq n.$$

Now, both the families of functionals $\{f_\pi \mid \pi \in \cup_{n=1}^\infty NC(n)\}$ and $\{g_\pi \mid \pi \in \cup_{n=1}^\infty NC(n)\}$ satisfy the insertion property considered in Definition 5.4. For the f_π's this is an immediate consequence of the corresponding property for the $\mathcal{D}$-valued cumulant functionals $\{k_\pi^{(\mathcal{D})} \mid \pi \in \cup_{n=1}^\infty NC(n)\}$. For the g_π's the insertion property follows from a calculation very similar in nature to the one shown in the proof of Theorem 6.2, and which, due to its routine character, will be left to the reader. (The reader who will have the patience to go through this calculation will notice that it effectively makes use of the implication (7.3) stated in the hypothesis of the current theorem.)

We next show that $f_\pi = g_\pi$ for every $\pi \in \cup_{n=1}^{\infty} NC(n)$. Proposition 5.5 combined with a linearity argument shows that all we need to check is the equality:

$$\sum_{\pi \in NC(n)} g_\pi(P_{i_1}X_{r_1}P_{j_1}, \ldots, P_{i_n}X_{r_n}P_{j_n}) = \sum_{\pi \in NC(n)} f_\pi(P_{i_1}X_{r_1}P_{j_1}, \ldots, P_{i_n}X_{r_n}P_{j_n})$$

(for some fixed $n \geq 1$, $1 \leq r_1, \ldots, r_n \leq s$, $1 \leq i_1, j_1, \ldots, i_n, j_n \leq d$). And indeed, we compute:

$$\sum_{\pi \in NC(n)} g_\pi(P_{i_1}X_{r_1}P_{j_1}, \ldots, P_{i_n}X_{r_n}P_{j_n}) =$$

$$\delta_{i_1,j_n}\delta_{i_2,j_1}\cdots\delta_{i_n,j_{n-1}} \cdot \Big(\sum_{\pi \in NC(n)} k_\pi(a^{(r_1)}_{j_n,j_1}, a^{(r_2)}_{j_1,j_2}, \ldots, a^{(r_n)}_{j_{n-1},j_n}) \Big) P_{j_n}$$

$$= \delta_{i_1,j_n}\delta_{i_2,j_1}\cdots\delta_{i_n,j_{n-1}} \cdot \varphi(a^{(r_1)}_{j_n,j_1} a^{(r_2)}_{j_1,j_2} \cdots, a^{(r_n)}_{j_{n-1},j_n}) P_{j_n}$$

$$= E_{\mathcal{D}}(P_{i_1}A_{r_1}P_{j_1}P_{i_2}A_{r_2}P_{j_2}\cdots P_{i_n}A_{r_n}P_{j_n})$$

$$= \sum_{\pi \in NC(n)} k_\pi^{(\mathcal{D})}(P_{i_1}A_{r_1}P_{j_1}, \ldots, P_{i_n}A_{r_n}P_{j_n})$$

$$= \sum_{\pi \in NC(n)} f_\pi(P_{i_1}X_{r_1}P_{j_1}, \ldots, P_{i_n}X_{r_n}P_{j_n}).$$

But if $f_\pi = g_\pi$, then one can equate the right-hand sides of the Equations (7.7) and (7.8). By doing this for $\pi = 1_n$ (the partition of $\{1, \ldots, n\}$ into only one block), and by appropriately choosing $\Gamma_1, \Lambda_1, \ldots, \Gamma_n, \Lambda_n \in \mathcal{D}$, one obtains the Equation (7.4) from the conclusion of the theorem. **QED**

7.3 Remark. In the framework of the Notations 7.1, let $A_1, \ldots, A_s$ be an R-cyclic family of matrices in $M_d(\mathcal{A})$, where $A_r = [a^{(r)}_{i,j}]_{i,j=1}^d$ for $1 \leq r \leq s$. Then Theorem 7.2 gives us an interpretation for the cyclic cumulants of the entries of $A_1, \ldots, A_s$ (i.e., for the coefficients of the determining series of the family $A_1, \ldots, A_s$). More precisely, for every $n \geq 1$, $1 \leq r_1, \ldots, r_n \leq s$ and $1 \leq i_1, \ldots, i_n \leq d$, we have that:

$$k_n(\, a^{(r_1)}_{i_n,i_1}, a^{(r_2)}_{i_1,i_2}, \ldots, a^{(r_{n-1})}_{i_{n-2},i_{n-1}}, a^{(r_n)}_{i_{n-1},i_n} \,) = \tag{7.9}$$

$$(i_n, i_n) - \text{entry of } k_n^{(\mathcal{D})}(A_{r_1}P_{i_1}, \ldots, A_{r_{n-1}}P_{i_{n-1}}, A_{r_n}).$$

7.4 Remark. In analogy to Remark 6.4, one can also reformulate the result of Theorem 7.2 by using the $\odot$-product. Let us denote by $\widetilde{k_n}^{\mathcal{D}}$ the counterpart of $\widetilde{k_n}$ (from Remark 6.4) which is suitable for working with $\mathcal{D}$. That is, $\widetilde{k_n}^{\mathcal{D}}$ is the linear application from $M_d(\mathcal{A} \otimes \cdots \otimes \mathcal{A})$ to $\mathcal{D}$ given by the formula:

$$\widetilde{k_n}^{\mathcal{D}}(\ [x_{i,j}]_{i,j=1}^d \ ) \ := \ [\delta_{i,j} k_n(x_{i,j})]_{i,j=1}^d, \ \ \forall \ [x_{i,j}]_{i,j=1}^d \in M_d(\mathcal{A} \otimes \cdots \otimes \mathcal{A}).$$

It is immediate that with these notations, the statement of Theorem 7.2 takes the following form (where we have set the matrices $\Lambda_1, \ldots, \Lambda_{n-1}$ from Equation (7.4) to be equal to the unit of $\mathcal{D}$):

$$k_n^{(\mathcal{D})}(A_{r_1}, \ldots, A_{r_n}) \ = \ \widetilde{k_n}^{\mathcal{D}}(A_{r_1} \odot \cdots \odot A_{r_n}), \ \ \forall \ 1 \leq r_1, \ldots, r_n \leq s. \tag{7.10}$$

It is a natural question if the same kind of formula is true when we consider other algebras of scalar $d \times d$ matrices (instead of $\mathcal{B}$ and $\mathcal{D}$, as we have in the Equations (6.11) and (7.10)). Let us consider the case of the smallest possible such algebra, namely $\mathbf{C}$ (corresponding to scalar multiples of the identity $d \times d$ matrix). Again we have a linear application $\widetilde{k_n}^{\mathbf{C}} : M_d(\mathcal{A} \otimes \cdots \otimes \mathcal{A}) \to \mathbf{C}$, given by the formula:

$$\widetilde{k_n}^{\mathbf{C}}(\ [x_{i,j}]_{i,j=1}^d \ ) \ := \ \frac{1}{d} \sum_{i=1}^{d} k_n(x_{i,i}), \ \ \forall \ [x_{i,j}]_{i,j=1}^d \in M_d(\mathcal{A} \otimes \cdots \otimes \mathcal{A}).$$

The question becomes: under what conditions on the matrices $A_1, \ldots, A_s \in M_d(\mathcal{A})$ can we infer that:

$$k_n(A_{r_1}, \ldots, A_{r_n}) \ = \ \widetilde{k_n}^{\mathbf{C}}(A_{r_1} \odot \cdots \odot A_{r_n}), \tag{7.11}$$

for every $1 \leq r_1, \ldots, r_n \leq s$? It turns out (see [15]) that (7.11) can be guaranteed if we know that $k_n^{(\mathcal{D})}(A_{r_1}, \ldots, A_{r_n})$, which is a priori an element in $\mathcal{D}$, is actually an element in $\mathbf{C}$. In the context of R-cyclic matrices, this amounts precisely to the situation discussed in Proposition 3.1; indeed, the "partial summation property" stated in Equation (3.1) asks that $k_n^{(\mathcal{D})}(A_{r_1}, \ldots, A_{r_n})$ is a scalar multiple of the identity $d \times d$ matrix, while on the other hand the conclusion of Proposition 3.1 (as appearing e.g. in Equation (3.3)) is tantamount to (7.11).

8. Characterization of R-cyclicity as freeness with amalgamation

In this section we combine the frameworks used in the Sections 6 and 7. That is, for a fixed integer $d \geq 1$ we will consider both the algebra $\mathcal{B} = M_d(\mathbf{C})$ and its subalgebra

$\mathcal{D}$ consisting of diagonal matrices. For every $1 \leq i, j \leq d$ we will denote by $V_{i,j} \in \mathcal{B}$ the matrix which has 1 on the (i, j)-entry and 0 on all the other entries. (Note that the matrices denoted up to now by "P_i" have become "$V_{i,i}$", for $1 \leq i \leq d$.)

If $(\mathcal{A}, \varphi)$ is any non-commutative probability space, then $M_d(\mathcal{A})$ is at the same time a $\mathcal{B}$-probability space and a $\mathcal{D}$-probability space, where the identifications $\mathcal{D} \subset \mathcal{B} \subset M_d(\mathcal{A})$ and the expectations $E_{\mathcal{B}} : M_d(\mathcal{A}) \to \mathcal{B}$, $E_{\mathcal{D}} : M_d(\mathcal{A}) \to \mathcal{D}$ are as described in the Sections 6 and 7. Note that the restriction of $E_{\mathcal{D}}$ to $\mathcal{B}$ is faithful; this implies that the discussion concluding the Section 5 (and in particular the Proposition 5.9) can be applied in this framework.

8.1 Lemma. In the framework considered above, let $C_1, C_2, \ldots, C_n \in M_d(\mathcal{A})$ form an R-cyclic family, where $n \geq 2$. Suppose that:

(i) for $m \in \{1, n\}$ we have that either $E_{\mathcal{D}}(C_m) = 0$ or that C_m is the unit of $M_d(\mathcal{A})$; and

(ii) for $m \in \{2, 3, \ldots, n-1\}$ we have that $E_{\mathcal{D}}(C_m) = 0$.

Consider also some indices $i_1, j_1, \ldots, i_{n-1}, j_{n-1} \in \{1, \ldots, d\}$ such that $i_1 \neq j_1, \ldots, i_{n-1} \neq j_{n-1}$. Then:

$$E_{\mathcal{D}}(\ C_1 V_{i_1,j_1} \cdots C_{n-1} V_{i_{n-1},j_{n-1}} C_n\ ) \ = \ 0. \tag{8.1}$$

Proof. We will denote by $c_{i,j}^{(m)}$ the (i, j)-entry of C_m $(1 \leq i, j \leq d,\ 1 \leq m \leq n)$. The hypotheses (i) and (ii) given above show that:

$$\begin{cases} \varphi(c_{i,i}^{(m)}) = 0 \text{ or } c_{i,i}^{(m)} = I & \text{if} \quad m \in \{1, n\},\ 1 \leq i \leq d \\ \varphi(c_{i,i}^{(m)}) = 0 & \text{if} \quad m \in \{2, 3, \ldots, n-1\},\ 1 \leq i \leq d. \end{cases} \tag{8.2}$$

In connection to this, let us also record the fact that:

$$\varphi(c_{i,j}^{(m)}) = 0, \quad \forall\, 1 \leq m \leq n, \forall\, 1 \leq i, j \leq d \text{ such that } i \neq j, \tag{8.3}$$

which follows from R-cyclicity ($\varphi(c_{i,j}^{(m)}) = k_1(c_{i,j}^{(m)}) = 0$ for $i \neq j$).

We will present the proof under the assumption that $n \geq 3$. The (similar, and simpler) case $n = 2$ is left as an exercise to the reader.

If we write explicitly the (i, i)-entry of the scalar diagonal matrix on the left-hand side of (8.1), it becomes clear that what we have to do in this proof is to fix an $i \in \{1, \ldots, d\}$, and show that

$$\varphi(\ c_{i,i_1}^{(1)} c_{j_1,i_2}^{(2)} \cdots c_{j_{n-2},i_{n-1}}^{(n-1)} c_{j_{n-1},i}^{(n)}\ ) \ = \ 0. \tag{8.4}$$

By using the relation between moments and non-crossing cumulants, the quantity on the left-hand side of (8.4) can be written as:

$$\sum_{\pi \in NC(n)} k_\pi(\ c^{(1)}_{i,i_1}, c^{(2)}_{j_1,i_2}, \ldots, c^{(n-1)}_{j_{n-2},i_{n-1}}, c^{(n)}_{j_{n-1},i} \ ). \tag{8.5}$$

We will actually prove that every term of the sum in (8.5) is equal to 0.

So, besides $i \in \{1, \ldots, d\}$, let us also fix a partition $\pi \in NC(n)$, and let us examine the non-crossing cumulant $k_\pi(\ c^{(1)}_{i,i_1}, c^{(2)}_{j_1,i_2}, \ldots, c^{(n-1)}_{j_{n-2},i_{n-1}}, c^{(n)}_{j_{n-1},i} \ )$. Recall from Section 1.3 that this cumulant is defined as a product having as many factors as there are blocks in π. For the sake of brevity, we will denote it in the rest of the proof by just "k_π".

Denoting by B the block of π which contains the number 2, we distinguish four cases:

Case 1. $B = \{2\}$. In this case, k_π has a factor "$k_1(c^{(2)}_{j_1,i_2})$", which is equal to 0 by (8.2), (8.3). So k_π itself is equal to 0.

Case 2. $B = \{1, 2\}$. In this case, k_π has a factor "$k_2(c^{(1)}_{i,i_1}, c^{(2)}_{j_1,i_2})$", which is equal to 0 by R-cyclicity and the hypothesis that $i_1 \neq j_1$. So again $k_\pi = 0$.

Case 3. $B \ni 3$. In this case, k_π has a factor "$k_{|B|}(\ldots, c^{(2)}_{j_1,i_2}, c^{(3)}_{j_2,i_3}, \ldots)$", which is equal to 0 by R-cyclicity and the hypothesis that $i_2 \neq j_2$. So again $k_\pi = 0$.

Case 4. B does not fall in any of the Cases 1-3. In this case B intersects $\{4, \ldots, n\}$; let us denote $m := \min(B \cap \{4, \ldots, n\})$. The set $\{3, 4, \ldots, m-1\}$ is a union of blocks of π; because π is non-crossing, there is one of these blocks, B_1, which has to be an interval-block – say that $B_1 = [p, q] \cap \mathbf{Z}$, with $3 \leq p \leq q \leq m-1$ ($\leq n-1$). The cumulant k_π has a factor $k_{|B_1|}(\cdots)$ corresponding to the block B_1. If B_1 has only one element (i.e. $p = q$), then the factor $k_{|B_1|}(\cdots)$ is equal to 0 by the same argument as in Case 1; while if $|B_1| > 1$ (i.e. $p < q$), then the factor $k_{|B_1|}(\cdots)$ is equal to 0 by the same argument as in Cases 2, 3. Either way, k_π is equal to 0. **QED**

8.2 Theorem. Let $A_1, \ldots, A_s$ be a family of matrices in $M_d(\mathcal{A})$, and let $\mathcal{C}$ denote the subalgebra of $M_d(\mathcal{A})$ generated by $\{A_1, \ldots, A_s\} \cup \mathcal{D}$. The family $A_1, \ldots, A_s$ is R-cyclic if and only if $\mathcal{C}$ is free from $\mathcal{B}$, with amalgamation over $\mathcal{D}$.

Proof. "$\Rightarrow$". We will verify that $\mathcal{C}$ is free from $\mathcal{B}$, with amalgamation over $\mathcal{D}$, by using the definition of freeness with amalgamation. That is: we consider an alternating sequence

$X_1, X_2, \ldots, X_k$ of matrices from $\mathcal{B}$ and from $\mathcal{C}$, such that $E_{\mathcal{D}}(X_1) = \cdots = E_{\mathcal{D}}(X_k) = 0$, and we want to show that $E_{\mathcal{D}}(X_1 X_2 \cdots X_k) = 0$.

If the alternating sequence of matrices considered above does not begin with a matrix from $\mathcal{C}$, let us add on the left end of the sequence one more matrix, equal to the identity of $M_d(\mathcal{A})$, and viewed as belonging to $\mathcal{C}$. Let us also use this procedure at the right end of the alternating sequence. With these adjustments we can assume that k (the number of matrices in the sequence) is odd, $k = 2n - 1$, and that $X_1, X_{2n-1} \in \mathcal{C}$. On the other hand the hypothesis which we have on X_1 and X_{2n-1} has to be weakened to the fact that they either have zero $\mathcal{D}$-expectation, or they are equal to the identity of $M_d(\mathcal{A})$.

We re-denote the matrices $X_1, X_3, \ldots, X_{2n-1}$ by $C_1, \ldots, C_n (\in \mathcal{C})$. The family $C_1, \ldots, C_n$ is R-cyclic, by Theorem 4.3.

On the other hand, let us look at the matrices $X_2, X_4, \ldots, X_{2n-2}$, which belong to $\mathcal{B}$ and have $\mathcal{D}$-expectation equal to 0. It is clear that each of these matrices belongs to $\operatorname{span}\{V_{i,j} \mid 1 \leq i, j \leq d,\ i \neq j\}$. An immediate argument with linear combinations allows us to assume without loss of generality that in fact we have $X_2 = V_{i_1,j_1}, \ldots, X_{2n-2} = V_{i_{n-1},j_{n-1}}$ for some $i_1, j_1, \ldots, i_{n-1}, j_{n-1} \in \{1, \ldots, d\}$ such that $i_1 \neq j_1, \ldots, i_{n-1} \neq j_{n-1}$.

With the above adjustments, the product $X_1 X_2 \cdots X_k$ now reads: $C_1 V_{i_1,j_1} \cdots C_{n-1} V_{i_{n-1},j_{n-1}} C_n$. The fact that this product has zero $\mathcal{D}$-expectation is exactly what was proved in Lemma 8.1.

"$\Leftarrow$" In a different non-commutative probability space $(\mathcal{N}, \psi)$ we construct a family of elements $\{x_{i,j}^{(r)} \mid 1 \leq i, j \leq d,\ 1 \leq r \leq s\}$ such that:

$$k_n(x_{i_1,j_1}^{(r_1)}, \ldots, x_{i_n,j_n}^{(r_n)}) \ = \ 0 \tag{8.6}$$

for every $n \geq 1$ and $1 \leq r_1, \ldots, r_n \leq s$, $1 \leq i_1, j_1, \ldots, i_n, j_n \leq d$ for which it is not true that $j_1 = i_2, \ldots, j_{n-1} = i_n, j_n = i_1$; and such that

$$k_n(x_{i_n,i_1}^{(r_1)}, x_{i_1,i_2}^{(r_2)}, \ldots, x_{i_{n-1},i_n}^{(r_n)}) \ = \tag{8.7}$$

$$(i_n, i_n) - \text{entry of } k_n^{(\mathcal{D})}(A_{r_1} V_{i_1,i_1}, \ldots, A_{r_{n-1}} V_{i_{n-1},i_{n-1}}, A_{r_n}),$$

for every $n \geq 1$ and $1 \leq r_1, \ldots, r_n \leq s$, $1 \leq i_1, \ldots, i_n \leq d$. Such a construction is possible because one can in general construct families of elements with any prescribed family of scalar cumulants, via an abstract free product construction (see e.g. [22], Chapter 1).

For $1 \leq r \leq s$, we consider the matrix $X_r = [x_{i,j}^{(r)}]_{i,j=1}^d \in M_d(\mathcal{N})$. The Equations (8.6) and (8.7) tell us that the family $X_1, \ldots, X_s$ is R-cyclic.

Observe that for every $n \geq 1$ and every $1 \leq r_1, \ldots, r_n \leq s$, $1 \leq i_1, \ldots, i_n \leq d$ we have that:

$$k_n^{(\mathcal{D})}(X_{r_1} V_{i_1,i_1}, \ldots, X_{r_{n-1}} V_{i_{n-1},i_{n-1}}, X_{r_n}) = \tag{8.8}$$
$$k_n^{(\mathcal{D})}(A_{r_1} V_{i_1,i_1}, \ldots, A_{r_{n-1}} V_{i_{n-1},i_{n-1}}, A_{r_n}).$$

Indeed, the scalar diagonal matrices on both sides of the Equation (8.8) have their (j,j)-entry equal to the cumulant $k_n(x_{j,i_1}^{(r_1)}, x_{i_1,i_2}^{(r_2)}, \ldots, x_{i_{n-1},j}^{(r_n)})$ (for the right-hand side this is just (8.7), while for the left-hand side we invoke Equation (7.9) from Remark 7.3). By taking linear combinations with respect to $V_{i_1,i_1}, \ldots, V_{i_{n-1},i_{n-1}}$ in (8.8) we find that the family $X_1, \ldots, X_s$ has identical $\mathcal{D}$-cumulants with the family $A_1, \ldots, A_s$. Or in other words, the families $A_1, \ldots, A_s$ and $X_1, \ldots, X_s$ have identical $\mathcal{D}$-distributions.

Now, our current hypothesis is that the algebra $\mathcal{C}$ generated by $\{A_1, \ldots, A_s\} \cup \mathcal{D}$ is free from $\mathcal{B}$, with amalgamation over $\mathcal{D}$. On the other hand, the same is true about the algebra $\widetilde{\mathcal{C}} \subset \mathcal{N}$ generated by $\{X_1, \ldots, X_s\} \cup \mathcal{D}$; this follows from the fact that the family $X_1, \ldots, X_s$ is R-cyclic, and the implication "$\Rightarrow$" (proved above!) of the current theorem. But then we are in the position to apply the Proposition 5.9, which gives us that the families $A_1, \ldots, A_s$ and $X_1, \ldots, X_s$ actually have identical $\mathcal{B}$-distributions. The latter fact implies in turn that the families $A_1, \ldots, A_s$ and $X_1, \ldots, X_s$ have identical $\mathcal{B}$-cumulants.

Finally, let us fix $n \geq 1$, $r_1, \ldots, r_n \in \{1, \ldots, s\}$ and $i_1, j_1, \ldots, i_n, j_n \in \{1, \ldots, d\}$, and suppose it is not true that $j_1 = i_2, \ldots, j_{n-1} = i_n, j_n = i_1$. Then:

$$k_n(a_{i_1,j_1}^{(r_1)}, \ldots, a_{i_n,j_n}^{(r_n)})$$

$$= (i_1, j_n) - \text{entry of } k_n^{(\mathcal{B})}(A_{r_1} V_{j_1,i_2}, \ldots, A_{r_{n-1}} V_{j_{n-1},i_n}, A_{r_n}) \text{ (by Theorem 6.2)}$$
$$= (i_1, j_n) - \text{entry of } k_n^{(\mathcal{B})}(X_{r_1} V_{j_1,i_2}, \ldots, X_{r_{n-1}} V_{j_{n-1},i_n}, X_{r_n})$$

(since the families $A_1, \ldots, A_s$ and $X_1, \ldots, X_s$ have identical $\mathcal{B}$-cumulants)

$$= k_n(x_{i_1,j_1}^{(r_1)}, \ldots, x_{i_n,j_n}^{(r_n)}) \text{ (again by Theorem 6.2)}$$

$$= 0 \text{ (by Equation (8.6)).} \textbf{ QED}$$

References

[1] P. Biane. Some properties of crossings and partitions, Discrete Mathematics 175 (1997), 41-53.

[2] P. Doubilet, G.-C. Rota, R. Stanley. On the foundations of combinatorial theory (VI): The idea of generating function, Proceedings of the sixth Berkeley symposium on mathematical statistics and probability, Lucien M. Le Cam et al. editors, University of California Press, 1972, 267-318.

[3] K. Dykema, U. Haagerup. Invariant subspaces for Voiculescu's circular operator, preprint of the Odense University (http://www.imada.sdu.dk), March 2000.

[4] E. Effros, Z.-J. Ruan. Operator spaces, Oxford University Press, 2000.

[5] U. Haagerup, F. Larsen. Brown's spectral distribution for R-diagonal elements in finite von Neumann algebras, Journal of Functional Analysis 176 (2000), 331-367.

[6] B. Krawczyk, R. Speicher. Combinatorics of free cumulants, Journal of Combinatorial Theory Series A 90 (2000), 267-292.

[7] G. Kreweras. Sur les partitions non-croisées d'un cycle, Discrete Mathematics 1 (1972), 333-350.

[8] A. Nica. R-diagonal pairs arising as free off-diagonal compressions, Indiana University Mathematics Journal 45 (1996), 529-544.

[9] A. Nica, R. Speicher. On the multiplication of free n-tuples of non-commutative random variables. With an Appendix by D. Voiculescu: Alternative proofs for the type II free Poisson variables and for the compression results. American Journal of Mathematics 118 (1996), 799-837.

[10] A. Nica, R. Speicher. R-diagonal elements – a common approach to Haar unitaries and to circular elements, in Free Probability Theory (D.-V. Voiculescu editor), Fields Institute Communications 12 (1997), 149-188.

[11] A. Nica, R. Speicher. Notes of lectures at the Poincare Institute during the special term in Fall 1999.

[12] A. Nica, D. Shlyakhtenko, R. Speicher. Some minimization problems for the free analogue of the Fisher information, Advances in Mathematics 141 (1999), 282-321.

[13] A. Nica, D. Shlyakhtenko, R. Speicher. Maximality of the microstates free entropy for R-diagonal elements, Pacific Journal of Mathematics 187 (1999), 333-347.

[14] A. Nica, D. Shlyakhtenko, R. Speicher. R-diagonal elements and freeness with amalgamation, preprint of the Erwin Schrödinger Institute (http://www.esi.ac.at), February 1999. To appear in the Canadian Journal of Mathematics.

[15] A. Nica, D. Shlyakhtenko, R. Speicher. A characterization of freeness by a factorization property of the R-transform, preprint, December 2000.

[16] G. Pisier. An introduction to the theory of operator spaces, Preliminary version, September 1997.

[17] D. Shlyakhtenko. R-transforms of certain joint distributions, in Free Probability Theory (D.-V. Voiculescu editor), Fields Institute Communications 12 (1997), 253-256.

[18] R. Speicher. Multiplicative functions on the lattice of non-crossing partitions and free convolution, Math. Annalen 298(1994), 611-628.

[19] R. Speicher. Combinatorial theory of the free product with amalgamation and operator-valued free probability theory, Memoirs of the American Mathematical Society 132 (1998), x+88.

[20] D. Voiculescu. Addition of certain non-commuting random variables, Journal of Functional Analysis 66 (1986), 323-346.

[21] D. Voiculescu. Circular and semicircular systems and free product factors, in Operator algebras, unitary representations, enveloping algebras, and invariant theory, A. Connes et al. editors, Birkhäuser, 1990.

[22] D. Voiculescu, K. Dykema, A. Nica. Free random variables, CRM Monograph Series, volume 1, AMS, 1992.