A CHARACTERIZATION OF FREENESS BY A FACTORIZATION PROPERTY OF R-TRANSFORM.

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ABSTRACT. Let M be a B-probability space. Assume that B itself is a D-probability space; then M can be viewed as a D-probability space as well. Let $X \in M$. We characterize freeness of X from B with amalgamation over D in terms of a certain factorization condition linking the B-valued and D-valued R-transforms of X. We give an application to random matrices.

1. OPERATOR-VALUED R-TRANSFORM.

1.1. **Multiplicative functions.** Let B be a unital algebra. Recall that a *B-probability space* $(M, E : M \to B)$ (see e.g. [8], [6]) is a pair consisting of an algebra M containing B as a unital subalgebra, and a *conditional expectation* $E : M \to B$. In other words, $E : M \to B$ is unital and B-bilinear:

$$E(1) = 1, \qquad E(bmb') = bE(m)b', \quad \forall b, b' \in B, \forall m \in M.$$

Elements of M are called B-valued random variables.

Recall that a *multiplicative map* $\langle \cdots \rangle : \bigcup M^n \to B$ is a $\mathbb{C}$-multilinear map, (i.e., a sequence of maps $M^n \ni m_1, \ldots, m_n \mapsto \langle m_1, \ldots, m_n \rangle \in B$) satisfying the B-linearity conditions

$$\begin{aligned} \langle bm_1, \ldots, m_n \rangle &= b\langle m_1, \ldots, m_n \rangle \\ \langle m_1, \ldots, m_n b \rangle &= \langle m_1, \ldots, m_n \rangle b \\ \langle m_1, \ldots, m_k b, m_{k+1}, \ldots, m_n \rangle &= \langle m_1, \ldots, m_k, bm_{k+1}, \ldots, m_n \rangle. \end{aligned}$$

(Here $m_i \in M$, $b \in B$).

Given a non-crossing partition $\pi \in NC(n)$ and an arbitrary multiplicative map $\langle \cdots \rangle$, we can construct a *bracketing* $\pi\langle \cdots \rangle : M^n \to B$, defined recursively by

$$\begin{aligned} 1_k\langle m_1, \ldots, m_k \rangle &= \langle m_1, \ldots, m_k \rangle \\ (\pi \sqcup \rho)\langle m_1, \ldots, m_k \rangle &= \pi\langle m_1, \ldots, m_p \rangle \cdot \rho\langle m_{p+1}, \ldots, m_k \rangle \\ \mathrm{ins}(p, \rho \to \pi)(m_1, \ldots, m_k) &= \pi\langle m_1, \ldots, m_p \rho\langle m_{p+1}, \ldots, m_{p+q} \rangle, \\ &\qquad m_{p+q+1}, \ldots, m_k \rangle. \end{aligned}$$

Here 1_k denotes the partition with the sole class $\{1, \ldots, k\}$, $\pi \sqcup \rho$ denotes disjoint union (with the equivalence classes of ρ placed after those of π), and $\mathrm{ins}(p, \rho \to \pi)$ denotes the

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partition obtained from π by inserting the partition ρ after the p-th element of the set on which π determines a partition.

In other words, each partition π is interpreted as a recipe for placing brackets $\langle \cdots \rangle$, and $\pi\langle \cdots \rangle$ is the value of the resulting expression.

1.2. **Moments and R-transform.** The B-probability space structure of the algebra M gives rise to one example of such a multiplicative map, namely, the *moments map*

$$[\cdots]_E : \bigcup M^n \to B,$$

given by

$$[m_1, \ldots, m_n]_E = E(m_1 \cdots m_n).$$

The reason for the name is that, having fixed B-random variables $X_1, \ldots, X_p \in M$, the following values of $[\cdots]_E$,

$$[b_0 X_{i_1} b_1, \cdots, X_{i_{n-1}} b_{n-1}, X_{i_n} b_n]_E$$

are called B-*valued moments* of the family $X_1, \ldots, X_n$.

In [6] and [7] the notion of B-valued R-transform was introduced (we follow the combinatorial approach of [6], see also [5]). Like the map $[\cdots]$, the R-transform map is a multiplicative map

$$\{\cdots\}_E : \bigcup M^n \to B.$$

The following combinatorial formula actually determines $\{\cdots\}_E$ uniquely:

$$\begin{aligned}[m_1, \ldots, m_n]_E &= \sum \text{all possible bracketings involving } \{\cdots\}_E \\ &= \sum_{\pi \in NC(n)} \pi\{m_1, \ldots, m_n\}_E.\end{aligned}$$

The uniqueness of the definition can be easily seen by observing that the right-hand side of the equation above involves $\{m_1, \ldots, m_n\}_E$ and that the rest of the terms are products of factors of smaller order (i.e., restrictions of $\{\cdots\}_E$ to M^k, $k < n$).

It is important to note that $[\cdots]$ determines $\{\cdots\}$ and vice-versa. Moreover, the value of $[\cdots]|_{M^n}$ depends only on $\{\cdots\}|_{M \cup \cdots \cup M^n}$, and vice-versa.

1.3. **Moment and cumulant series.** Given a multiplicative function $\langle \cdots \rangle : \bigcup M^k \to B$, and $X_1, \ldots, X_n \in B$, consider the family of multilinear maps

$$M^{\langle \cdots \rangle}_{i_1, \ldots, i_k} : B^{k-1} \to B, \quad i_1, \ldots, i_k \in \{1, \ldots, n\}$$

given by

$$M^{\langle \cdots \rangle}_{i_1, \ldots, i_k}(b_1, \ldots, b_{k-1}) = \langle X_{i_1} b_1, \cdots, b_{k-1} X_{i_k} \rangle.$$

In the particular examples above, we get the *moment series* of $X_1, \ldots, X_n$,

$$\mu^{X_1, \ldots, X_n}_{i_1, \ldots, i_k} = M^{[\cdots]_E}_{i_1, \ldots, i_k}$$

and the *cumulant series*,

$$k^{X_1, \ldots, X_n}_{i_1, \ldots, i_k} = M^{\{\cdots\}_E}_{i_1, \ldots, i_k}.$$

We will sometimes write

$$k^{X_1,\ldots,X_n}_{B;i_1,\ldots,i_k}$$

to emphasize that the series is valued in B.

1.4. **Freeness with amalgamation.** Let $M_1, M_2 \subset M$ be two subalgebras, each containing B. The importance of R-transform is apparent from the following theorem ([6], see also [1]):

Theorem. *Let S_1, S_2 be two subsets of M. Let M_i be the algebra generated by S_i and B, $i = 1, 2$. Then M_1 and M_2 are free with amalgamation over B iff whenever $X_1, \ldots, X_n \in S_1 \cup S_2$,*

$$\{X_1, \ldots, X_n\}_E = 0$$

unless either all $X_1, \ldots, X_n \in S_1$, or all $X_1, \ldots, X_n \in S_2$.

1.5. **Canonical random variables.** Let $X_1, \ldots, X_n \in M$ be fixed.

Then by [7] there exists a B-probability space $(\mathcal{F}, E_B : \mathcal{F} \to B)$, elements $\lambda_1^*, \ldots, \lambda_n^* \in \mathcal{F}$, and elements λ_p^k satisfying the following properties:

(i) $\lambda^*_{j_1} b_1 \lambda^*_{j_2} b_2 \ldots \lambda^*_{j_k} b_k \lambda^k_j = k^{X_1,\ldots,X_n}_{j_1,\ldots,j_k,j}(b_1, \ldots, b_k)$, $b_1, \ldots, b_k \in B$;

(ii) Let $w = b_0 a_1 b_1 a_2 b_2 \ldots a_n b_n$, where $b_i \in B$, $a_i = \lambda_p^k$ or $a_i = \lambda_p^*$. Then $E_B(w) = 0$ unless w can be reduced to an element of B using relation 1.5(i).

(By convention, we take $\lambda_p^0 = E_B(X_p) \in B$). We should clarify that for each k, λ_p^k is just a formal variable, and we do not assume any relations between $\{\lambda_k^p\}_{k,p}$: for example, λ_p^k is not the k-th power of λ_p^1.

It is not hard to show that the properties listed above determine the restriction of E_B to the algebra generated by λ^* and $\{\lambda_p^k\}_{p,k}$.

Let

$$Y_j = \lambda_j^* + \sum_{k \geq 0} \lambda_j^k.$$

(This series is formal; however, $Y_1, \ldots, Y_n$ have moments, since each such moment involves only a finite number of terms from the series defining Y_j).

It turns out that

$$k^{X_1,\ldots,X_n}_{i_1,\ldots,i_k} = k^{Y_1,\ldots,Y_n}_{i_1,\ldots,i_k}, \quad \text{and } \mu^{X_1,\ldots,X_n}_{i_1,\ldots,i_k} = \mu^{Y_1,\ldots,Y_n}_{i_1,\ldots,i_k}.$$

In other words, given a cumulant series, $Y_1, \ldots, Y_n$ is an explicit family of B-valued random variables, whose cumulant series is equal to the one given.

2. Freeness from a subalgebra.

2.1. **D-cumulants vs. B-cumulants.** Let now $D \subset B$ be a unital subalgebra, and let $F : B \to D$ be a conditional expectation. If $(M, E : M \to B)$ is a B-probability space, then $(M, F \circ E : M \to D)$ is a D-probability space.

Theorem 2.1. *Let $X_1, \ldots, X_n \in M$. Assume that the B-valued cumulants of $X_1, \ldots, X_n$ satisfy*

$$k_{B;i_1,\ldots,i_k}^{X_1,\ldots,X_n}(d_1, \ldots, d_{k-1}) \in D, \quad \forall d_1, \ldots, d_{k-1} \in D.$$

Then the D-valued cumulants of $X_1, \ldots, X_n$ are given by the restrictions of the B-valued cumulants:

$$k_{D;i_1,\ldots,i_k}^{X_1,\ldots,X_n}(d_1, \ldots, d_{k-1}) = k_{B;i_1,\ldots,i_k}^{X_1,\ldots,X_n}(d_1, \ldots, d_{k-1}), \quad \forall d_1, \ldots, d_{k-1} \in D.$$

Proof. Let N be the algebra generated by D and $X_1, \ldots, X_n$. The condition on cumulants implies that

$$\{\cdots\}_E|_{\bigcup N^p}$$

is valued in D. It follows from the moment-cumulant formula that $[\cdots]_E|_{\bigcup N^p}$ is valued in D, and hence that

$$[\cdots]_{F\circ E}|_{\bigcup N^p} = [\cdots]_E|_{\bigcup N^p} = \sum_{\pi \in NC(p)} \pi\{\cdots\}_E|_{\bigcup N^p}.$$

Since the moment-cumulant formula determines $\{\cdots\}_{F\circ E}|_{\bigcup N^p}$, it follows that

$$\{\cdots\}_{F\circ E}|_{\bigcup N^p} = \{\cdots\}_E|_{\bigcup N^p}.$$

□

We record an equivalent formulation of the theorem above (which was implicit in the proof):

Theorem 2.2. *Let $N \subset M$ be a subalgebra, containing D. Assume that $\{\cdots\}_E|_{\bigcup N^p}$ is valued in D. Then*

$$\{\cdots\}_{F\circ E}|_{\bigcup N^p} = \{\cdots\}_E|_{\bigcup N^p}.$$

In general, in the absence of the condition that $k_B^{X_1,\ldots,X_n}$ restricted to $\cup D^p$ is valued in D, the expression of D-valued cumulants of $X_1, \ldots, X_n$ in terms of the B-valued cumulants is quite complicated. Note, for example, that if X is a B-valued random variable, and $b \in B$, then the B-valued cumulant series of bX are very easy to describe. On the other hand, the D-valued cumulant series of bX can have a very complicated expression in terms of the D-valued cumulant series of X and b.

The sufficient condition in the theorem above is actually quite close to being necessary in the case that the conditional expectations are positive maps of $*$-algebras. As an illustration, consider the case that $D \subset B$ consists of scalar multiples of 1, and $F : B \to D$ is such that $\tau = F \circ E$ is a *trace* on M, satisfying $\tau(xy) = \tau(yx)$ for all $x, y \in M$.

Recall that X is called a B-semicircular variable if its cumulant series is given by

$$k^X_{\underbrace{1,\ldots 1}_{p}}(b_1, \ldots, b_{p-1}) = \delta_{p,2}\eta(b_1)$$

for some map $\eta : B \to B$. It is easily seen that if X is B-semicircular, then $\eta(b) = E(XbX)$.

Theorem 2.3. *Let $(M, E : M \to B)$ be a B-probability space, such that M and B are C^*-algebras. Let $F : B \to \mathbb{C} = D \subset B$ be a faithful state. Assume that $\tau = F \circ E$ is a faithful trace on M. Let X be a B-semicircular variable in M. Then the distribution of X with respect to τ is the semicircle law iff $E(X^2) \in \mathbb{C}$.*

Proof. If $E(X^2) = k_{11}^X(1) \in \mathbb{C}$, it follows that the B-valued cumulants of X, restricted to $D = \mathbb{C}$ are valued in D. Hence by Theorem 2.2, the D-valued cumulant series of X are the same as the restriction of the B-valued cumulant series; hence the only scalar-valued cumulant of X which is nonzero is the second cumulant k_{11}^X, so that the distribution of X is the semicircle law.

Conversely, assume that the distribution of X is the semicircle law. Let $\eta(b) = k_{11}^X(b)$, $b \in B$. Then we have

$$\begin{aligned} 2\tau(\eta(1))^2 &= 2\tau(X^2)^2 \\ &= \tau(X^4) = F \circ E(X^4) \\ &= F(E(\eta(\eta(1)))) + F(E(\eta(1)\eta(1))) \\ &= \tau(X\eta(1)X) + \tau(\eta(1)^2) \\ &= \tau(\eta(1)XX) + \tau(\eta(1)^2) \\ &= 2\tau(\eta(1)^2), \end{aligned}$$

so that

$$\tau(\eta(1)) = \tau(\eta(1)^2)^{1/2}.$$

By the Cauchy-Schwartz inequality, we have that if $\eta(1) \notin \mathbb{C}$,

$$\tau(\eta(1)) = \langle \eta(1), 1 \rangle < \|\eta(1)\|_2 \cdot \|1\|_2 = \tau(\eta(1)^2)^{1/2},$$

which is a contradiction. Hence $\eta(1) \in \mathbb{C}$. □

We mention a corollary, which is of interest to random matrix theory. Let $\sigma(x, y) = \sigma(y, x)$ be a non-negative function on $[0, 1]^2$, having at most a finite number of discontinuities in each vertical line. Let $G(n)$ be an $n \times n$ random matrix with entries g_{ij}, so that $\{g_{ij} : i \le j\}$ are independent complex Gaussian random variables, $g_{ij} = \overline{g_{ji}}$, the expectation $E(g_{ij}) = 0$ and the variance $E(|g_{ij}|^2) = \frac{1}{n}\sigma(\frac{i}{n}, \frac{j}{n})$. The matrices $G(n)$ are called Gaussian Random Band Matrices. Let μ_n be the expected eigenvalue distribution of $G(n)$, i.e.,

$$\mu_n([a, b]) = \frac{1}{n} \times \text{expected number of eigenvalues of } G(n) \text{ in } [a, b].$$

Corollary 2.4. *The eigenvalue distribution measures μ_n of the Gaussian Random Band Matrices $G(n)$ converge weakly to the semicircle law iff $\int_0^1 \sigma(x, y)dy$ is a.e. a constant, independent of x.*

The proof of this relies on a result from [2], showing that $G(n)$ has limit eigenvalue distribution μ, given as follows. Let X be the $L^\infty[0, 1]$-semicircular variable in an $L^\infty[0, 1]$-probability space $(M, E : M \to L^\infty[0, 1])$, so that $E(XfX)(x) = \int_0^1 f(y)\sigma(x, y)dy$. Let $F : L^\infty[0, 1] \to \mathbb{C}$ denote the linear functional $F(f) = \int_0^1 f(x)dx$, and denote by τ the trace $F \circ E$ on $W^*(X, L^\infty[0,1])$. Then μ is the scalar-valued distribution of X with respect to τ, i.e.,

$$\int t^k d\mu(t) = \tau(X^k).$$

It remains to apply Theorem 2.3, to conclude that μ is a semicircle law iff $E(X^2) \in \mathbb{C}$, i.e., $\int_0^1 \sigma(x,y)dy$ is a constant function of x.

2.2. **A characterization of freeness.** We are now ready to state the main theorem of this note. The following theorem was earlier proved for B-valued semicircular variables in [3], and found many uses in operator algebra theory.

Theorem 2.5. *Let $X_1, \dots, X_n \in M$. Assume that $F : B \to D$ satisfies the faithfullness condition that if $b_1 \in B$ and if $F(b_1 b_2) = 0$ for all $b_2 \in B$, then $b_1 = 0$. Then $X_1, \dots, X_n$ are free from B with amalgamation over D iff their B-valued cumulant series satisfies*

$$k^{X_1,\dots,X_n}_{B;i_1,\dots,i_k}(b_1, \dots, b_{k-1}) = F(k^{X_1,\dots,X_n}_{B;i_1,\dots,i_k}(F(b_1), \dots, F(b_{k-1})). \tag{2.1}$$

for all $b_1, \dots, b_{k-1} \in B$. In short, $k = F \circ k \circ F$. (Here the cumulant series are computed in the B-probability space $(M, E : M \to B)$). Equivalently,

$$k^{X_1,\dots,X_n}_{B;i_1,\dots,i_k}(b_1, \dots, b_{k-1}) = k^{X_1,\dots,X_n}_{D;i_1,\dots,i_k}(F(b_1), \dots, F(b_{k-1})).$$

We note that in the case that M is a C^*-probability space, the faithfulness assumption above is exactly the condition that the GNS representation of B with respect to the conditional expectation F is faithful.

Since $X_1, \dots, X_n$ are free from B with amalgamation over D iff the algebra N generated by $X_1, \dots, X_n$ and D is free from B over D, the theorem above can be equivalently stated as

Theorem 2.6. *Let $N \subset M$ be a subalgebra of M, containing D. Assume that $F : B \to D$ satisfies the faithfullness condition that if $b_1 \in B$ and if $F(b_1 b_2) = 0$ for all $b_2 \in B$, then $b_1 = 0$. Then N is free from B over D iff*

$$\{n_1 b_1, n_2 b_2, \dots, n_k\}_E = F\{n_1 F(b_1), n_2 F(b_2), \dots, n_k\}_E$$

for all k and $n_1, \dots, n_k \in N$, $b_1, \dots, b_{k-1} \in B$.

Proof. We prove the theorem in the first formulation.

Assume that the condition (2.1) is satisfied by the cumulant series of $X_1, \dots, X_n$. Let $Y_1, \dots, Y_n$ be as in Section 1.5. Since the freeness of $X_1, \dots, X_n$ from B with amalgamation over D is a condition on the B-moment series of $X_1, \dots, X_n$, and $Y_1, \dots, Y_n$ have the same B-moment series as $X_1, \dots, X_n$, it is sufficient to prove that $Y_1, \dots, Y_n \in \mathcal{F}$ are free with amalgamation over D from B.

Since (2.1) is satisfied, $\lambda_j^0 = E_B(Y_j) \in D$ and hence $Y_1, \dots, Y_n$ belong to the algebra $\mathcal{L}$ generated in $\mathcal{F}$ by λ_j^* and λ_p^q, $1 \le j, p \le n$, $q \ge 1$ and D. Therefore, it is sufficient to prove that $\mathcal{L}$ is free from B with amalgamation over D.

Let $w_1, \dots, w_s \in \mathcal{L}$, so that $F \circ E_B(w_j) = 0$, and let $b_0, \dots, b_s \in B$, so that $F(b_j) = 0$ (allowing also b_0 and/or b_s to be equal to 1). We must prove that

$$F \circ E_B(b_0 w_1 b_1 \cdots w_s b_s) = 0.$$

Note that the factorization condition (2.1) as well as the definition of the generators of $\mathcal{L}$ (see 1.5(i) and 1.5(ii)) imply that $E_B|_{\mathcal{L}}$ has values in D. It follows that we may assume that $E_B(w_j) = 0$ (since $F \circ E_B(w_j) = E_B(w_j) \in D$). By the definition of E_B, its kernel is spanned by irreducible non-trivial words in the generators λ_j^* and λ_p^q. Then

$$W = b_0 w_1 \cdots w_s b_s$$

is again a linear combination of words in the generators λ_j^* and λ_p^q. By linearity, we may reduce to the case that W is a single word. If W is irreducible, it must be non-trivial (since each w_i is non-trivial), hence $E_B(W) = 0$, so that $F \circ E_B(W) = 0$. So assume that W is not irreducible. Since each w_i is irreducible, this means that W contains a sub-word of the form

$$\begin{aligned} W &= W_1 \cdot d_0 \lambda_{i_1}^* d_1 \lambda_{i_2}^* d_2 \cdots \\ &\qquad d_s b_1 d_{s+1} \lambda_{i_{s+2}}^* \cdots b_j \cdots \lambda_{i_k}^* d_k b_r d_{k+1} \lambda_{i_{k+1}}^* \\ &\qquad \cdots d_p \lambda_{i_p}^* d_{p+1} \lambda_j^p \cdot W_2. \end{aligned}$$

Using the relation (1.5(i)) and the factorization condition (2.1), we get that

$$W = W_1 d_0 k_{i_1,\dots,i_p,j}(d_1, \dots, F(d_k b_r d_{k+1}), \dots, d_{p+1}) W_2 = 0,$$

since $F(d_k b_r d_{k+1}) = d_k F(b_r) d_{k+1} = 0$. Thus in any case, $F \circ E_B(W) = 0$.

We have therefore seen that the factorization condition implies freeness with amalgamation.

To prove the other implication, assume that $X_1, \dots, X_n$ are free with amalgamation over D from B. Let $Z_1, \dots, Z_n$ be B-valued random variables, so that

$$k_{B;i_1,\dots,i_k}^{Z_1,\dots,Z_n}(b_1, \dots, b_{k-1}) = F(k_{D,i_1,\dots,i_k}^{X_1,\dots,X_n}(F(b_1), \dots, F(b_{k-1}))).$$

where k_D denote D-valued cumulants. (Note that the first occurrence of F is actually redundant, as $k_{D,i_1,\dots,i_k}^{X_1,\dots,X_n}(F(b_1), \dots, F(b_{k-1}) \in D)$. Then by the first part of the proof, $Z_1, \dots, Z_n$ are free from B with amalgamation over D. Moreover, by Theorem 2.2, the D-valued distributions of $Z_1, \dots, Z_n$ and $X_1, \dots, X_n$ are the same. By assumption, $X_1, \dots, X_n$ are free with amalgamation over D from B. This freeness, together with the D-valued distribution of $X_1, \dots, X_n$, determines their B-valued distribution. Indeed, the freeness assumptions determine

$$F \circ E(b' b_0 X_{i_1} b_1 X_{i_2} \cdots b_{n-1} X_{i_n} b_n), \quad b', b_i \in B$$

which in view of the assumptions on F determines

$$E(b_0 X_{i_1} b_1 X_{i_2} \cdots b_{n-1} X_{i_n} b_n), \quad b_i \in B.$$

It follows that the B-valued distributions of $X_1, \dots, X_n$ and $Z_1, \dots, Z_n$ coincide. Hence the B-valued cumulants of $X_1, \dots, X_n$ satisfy (2.1). □

As an application, we have the following proposition (see e.g. [4, Lemma 2.7]):

Proposition 2.7. *Let $N \subset M$ be a subalgebra. Let $B \subset C \subset D \subset M$ be subalgebras, and $E_C : M \to C$, $E_B : M \to B$, $E_D : M \to D$ be conditional expectations, so that $E_B = E_B \circ E_C$, $E_C = E_C \circ E_D$. Assume that E_C and E_B and E_D satisfy the faithfullness assumptions of Theorem 2.5. Assume that N is free from C with amalgamation over B, and* *also free from D with amalgamation over C. Then N is free from D with amalgamation over B.*

Proof. Since N is free from D with amalgamation over C, we have that for all $n_j \in N$ and $d_j \in D$,

$$\{n_1 d_1, n_2 d_2, \dots, n_k\}_{E_D} = E_C\{n_1 E_C(d_1), \dots, n_k\}_{E_C}.$$

Since N is free from C with amalgamation over D, we get similarly that for all $c_j \in C$,

$$\{n_1 c_1, n_2 c_2, \dots, n_k\}_{E_C} = E_B\{n_1 E_B(c_1), \dots, n_k\}_{E_B}.$$

Applying this with $c_j = E_C(d_j)$ and combining with the previous equation gives

$$\{n_1 d_1, n_2 d_2, \dots, n_k\}_{E_D} = E_B\{n_1 E_B(d_1), \dots, n_k\}_{E_B},$$

since $E_B = E_B \circ E_C$. Hence N is free from D with amalgamation over B. $\square$

We mention as a corollary the following identity. Let $B \subset C \subset D$ be C^*-algebras, $E_B^N : N \to B$, $E_B^C : C \to B$ and $E_C^D : D \to C$ be conditional expectations, having faithful GNS representations. Consider the reduced free product

$$(((N, E_B^N) *_B (C, E_B^C)), E_B^N * \mathrm{id}) *_C (D, E_C^D),$$

where $E_B^N * \mathrm{id}$ denotes the canonical conditional expectation from the free product $(N, E_B^N) *_B (C, E_B^C)$ onto C. Then

$$\begin{aligned}(((N, E_B^N) *_B (C, E_B^C)), E_B^N * \mathrm{id}) *_C (D, E_C^D)& \\ \cong (N, E_B^N) *_B (D, E_B^C \circ E_C^D).&\end{aligned}$$

To see this, it is sufficient to prove that $N \subset (((N, E_B^N) *_B (C, E_B^C)), E_B^N * \mathrm{id}) *_C (D, E_C^D)$ is free from D with amalgamation over B, since both $(((N, E_B^N) *_B (C, E_B^C)), E_B^N * \mathrm{id}) *_C (D, E_C^D)$ and $(N, E_B^N) *_B (D, E_B^C \circ E_C^D)$ are generated by N and D as C^*-algebras. But N is free from D over C, and from C over D, by construction. Hence by the proposition above, N is free from D over B.

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