FREE FISHER INFORMATION FOR NON-TRACIAL STATES

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ABSTRACT. We extend Voiculescu's microstates-free definitions of free Fisher information and free entropy to the non-tracial framework. We explain the connection between these quantities and free entropy with respect to certain completely positive maps acting on the core of the non-tracial non-commutative probability space. We give a condition on free Fisher information of an infinite family of variables, which guarantees factoriality of the von Neumann algebra they generate.

1. INTRODUCTION.

Free entropy and free Fisher information were introduced by Voiculescu [13], [14], [16] in the context of his free probability theory [17] as analogs of the corresponding classical quantities. These quantities are usually considered in the framework of tracial non-commutative probability spaces; not surprisingly, the most striking applications of free entropy theory were to tracial von Neumann algebras (see e.g. [15], [2], [10]). Recently, however, it turned out that some type III factors associated with free probability theory [5] have certain properties in common with their type II_1 cousins [9]. This gives rise to a speculation that there is room for free entropy to exist outside of the context of tracial non-commutative probability spaces.

The goal of this paper is to initiate the development of free Fisher information, based on Voiculescu's microstates-free approach [16], in the non-tracial framework. The key idea is that all of the ingredients going into the definition of free Fisher information in this case must behave covariantly with respect to the modular group [11] of the non-tracial state. The principal example of a family of variables for which free Fisher information is non-trivial, and which belong to an algebra not having any traces, are semicircular generators of free Araki-Woods factors, taken with free quasi-free states [5].

We describe another route towards free Fisher information, which is based on first converting the non-tracial von Neumann algebra into a larger algebra, the core (having an infinite trace), and then considering free Fisher information relative to a certain completely positive map (in the spirit of [7]). We should point out that it is this approach that is most likely to connect with the microstates free entropy (as suggested by [3]; see also [8]), since it is at present unclear what a microstates approach to free entropy in the non-tracial framework should be.

We finish the paper with a look at free Fisher information on von Neumann algebras that have traces. Our first result is that once the algebra has a trace, the free Fisher information is automatically infinite when computed with respect to a non-tracial state. It is likely that on any von Neumann algebra, free Fisher information can be finite for only very special states (however, we do not have any results in this direction in the non-tracial category). Another result of the present paper is a statement guaranteeing factoriality of a tracial von Neumann algebra, once we know that it has an infinite generating family whose free Fisher information is bounded in a certain way.

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2. Free Fisher Information for Arbitrary KMS States.

2.1. **Free Brownian motion in the presence of a modular group.** Let M be a von Neumann algebra, $\phi : M \to \mathbb{C}$ be a normal faithful state on M. Denote by σ_t^ϕ the modular group of ϕ. Denote by $L^2(M^{\mathrm{sa}}, \phi) \subset L^2(M, \phi)$ the closure of the real subspace of self-adjoint elements $M^{\mathrm{sa}} \subset M$.

Let $X = X^* \in M$ and let $B \subset M$ be a subalgebra. Assume that $\sigma_t^\phi(B) \subset B$ for all $t \in \mathbb{R}$.

Consider the von Neumann algebra $\mathcal{M} = \Gamma(L^2(M^{\mathrm{sa}}, \phi) \subset L^2(M, \phi))$, taken with the free quasi-free state $\phi_{\mathcal{M}}$. Consider the element $Y = s(X) \in \mathcal{M}$ (see [5] for definitions and notation). Then $\phi_{\mathcal{M}}(Y \sigma_t^{\phi_{\mathcal{M}}}(Y)) = \phi(X \sigma_t^\phi(X))$, for all $t \in \mathbb{R}$.

Consider the algebra $\mathcal{N} = (M, \phi) * (\mathcal{M}, \phi_{\mathcal{M}})$, and denote by $\hat{\phi}$ the free product state on $\mathcal{N}$. Note that $\sigma_t^{\hat{\phi}} = \sigma_t^\phi * \sigma_t^{\phi_{\mathcal{M}}}$. The elements $X_\epsilon = X + \sqrt{\epsilon} Y$, $\epsilon \geq 0$ form a natural free Brownian motion, which behaves nicely under the action of the modular group. In particular, note that for all $\epsilon \geq 0$ and $t \in \mathbb{R}$,

$$\hat{\phi}(X_\epsilon \sigma_t^{\hat{\phi}}(X_\epsilon)) = \phi(X \sigma_t^\phi(X)) \cdot (1 + \epsilon).$$

Furthermore, for each $\epsilon > 0$, the distribution of X_ϵ is that of a free Brownian motion at time ϵ, starting at X; this is because Y is a semicircular variable, free from X.

2.2. **Conjugate variables.** Let $B[X]$ denote that algebra generated by B and all translates $\sigma_t^\phi(X)$, $t \in \mathbb{R}$. Assume that $\{\sigma_t^\phi(X)\}$ are algebraically free over B, i.e., satisfy no algebraic relations modulo B. Denote by $\partial_X : B[X] \to \mathcal{N}$ the derivation given by:

1. $\partial_X(\sigma_t^\phi(X)) = \sigma_t^{\hat{\phi}}(Y)$
2. $\partial_X(b) = 0$, $b \in B$.

Notice that the range of ∂_X actually lies in the subspace $B[X] Y B[X] \subset \mathcal{N}$. Note also that since $\partial_X(\sigma_t^\phi(X))$ is self-adjoint, we have that for $P \in B[X]$, $\partial_X(P^*) = \partial_X(P)^*$, i.e., ∂_X is a $*$-derivation. Observe finally that ∂_X is covariant with respect to the modular groups σ_t^ϕ and $\sigma_t^{\hat{\phi}}$:

$$\partial_X(\sigma_t^\phi(P)) = \sigma_t^{\hat{\phi}}(\partial_X(P)), \quad P \in B[X].$$

Define the conjugate variable $J_\phi(X : B) \in L^2(B[X], \phi)$ to be such a vector ξ that

$$\langle \xi, P \rangle_{L^2(B[X], \phi)} = \langle Y, \partial_X(P) \rangle_{L^2(\mathcal{N}, \hat{\phi})}, \quad \forall P \in B[X], \tag{2.1}$$

if a vector ξ satisfying such properties exists. Formally, this means that $\xi = \partial_X^*(Y)$, where $\partial_X : L^2(B[X], \phi) \to L^2(\mathcal{N}, \hat{\phi})$ is viewed as a densely defined operator.

It is clear, because of the density of $B[X]$ in $L^2(B[X], \phi)$, that ξ is unique, if it exists.

It is convenient to talk about $J_\phi(X : B)$ even in the case that $\{\sigma_t(X)\}_{t \in \mathbb{R}}$ are not algebraically free over B (such is the case, for example, when ϕ is a trace, and hence $\sigma_t^\phi(X) = X$ for all t). In this case, one can view ∂_X as a multi-valued map, the set of values given by the results of application of the definition of ∂_X in all possible ways; the definition of J_ϕ is then that (2.1) is valid for all values of ∂_X.

Note that $J_\phi(X : B)$ depends on more than just the joint distribution of X and B with respect to the state ϕ; it depends on the joint distribution of the family $B \cup \{\sigma_t^\phi(X) : t \in \mathbb{R}\}$.

We continue to denote by σ_t^ϕ the extension of σ_t^ϕ to the Hilbert space $L^2(M, \phi)$ (this is precisely the one-parameter group of unitaries Δ_ϕ^{it}, where Δ_ϕ is the modular operator). In particular, if ϕ is a trace, then the definition of J_ϕ is (up to a multiple) precisely that of the conjugate variable of Voiculescu [16].

Lemma 2.1. *Assume that $\xi = J_\phi(X : B)$ exists. Then $\xi \in L^2(M^{\mathrm{sa}}, \phi)$, and $\sigma_t^\phi(J_\phi(X : B)) = J_\phi(\sigma_t(X) : B)$.*

Proof. We note that, because ∂_X is a $*$-derivation,

$$\langle P^*, \xi\rangle = \overline{\langle Y, \partial_X(P^*)\rangle} = \langle \partial_X(P), Y\rangle = \overline{\langle \xi, P\rangle}.$$

From this it follows that ξ is in the domain of the S operator of Tomita theory, and moreover that $S\xi = \xi$. Hence $\xi \in L^2(M^{\mathrm{sa}})$.

One also has

$$\begin{aligned} \langle \sigma_t^\phi(\xi), P\rangle &= \langle Y, \partial_X(P)\rangle \\ &= \langle \sigma_t^\phi(Y), \partial_{\sigma_t^\phi(X)}(P)\rangle, \end{aligned}$$

since the joint distributions of $B[X]$ and $\{\sigma_t^{\hat\phi}(Y)\}_{t\in\mathbb{R}}$ is the same as $B[X]$ and $\{\sigma_{s+t}^{\hat\phi}(Y)\}_{t\in\mathbb{R}}$, for any s. It follows that $\sigma_t^\phi(J_\phi(X : B)) = J_\phi(\sigma_t^\phi(X) : B)$. □

Lemma 2.2. *Let $P, Q \in B[X]$, and assume that $\xi = J_\phi(X)$ exists and is in M. Then*

$$\phi(P\xi Q) = \hat\phi(PY\partial_X(Q)) + \hat\phi(\partial_X(P)YQ).$$

Proof. Recall that ϕ (and $\hat\phi$) satisfy the KMS condition: for all $a, b \in M$ (or $\in \mathcal{N}$), there exists a (unique) function $f(z)$, analytic on the strip $\{z : 0 < \Im z < 1\}$, and so that (writing σ_t for either σ_t^ϕ or $\sigma_t^{\hat\phi}$)

$$\begin{aligned} \phi(a\sigma_t(b)) &= f(t), \\ \phi(\sigma_t(b)a) &= f(t+i), \quad t \in \mathbb{R}. \end{aligned}$$

Fix $P, Q \in B[X]$ and let f be as above, so that

$$\begin{aligned} \phi(\sigma_t^\phi(P)\xi Q) &= f(t+i), \\ \phi(\xi Q\sigma_t^\phi(P)) &= f(t). \end{aligned}$$

Then

$$\begin{aligned} f(t) &= \langle \xi, Q\sigma_t^\phi(P)\rangle \\ &= \langle Y, \partial_X(Q\sigma_t^\phi(P))\rangle \\ &= \langle Y, \partial_X(Q)\sigma_t^\phi(P)\rangle + \langle Y, Q\sigma_t^{\hat\phi}(\partial_X(P))\rangle, \end{aligned}$$

where in the last step we used the fact that ∂_X intertwines σ_t^ϕ and $\sigma_t^{\hat\phi}$. Using the KMS-condition for $\hat\phi$, we then get

$$\begin{aligned} f(t+i) &= \hat\phi(\sigma_t^{\hat\phi}(P)Y\partial_X(Q)) + \hat\phi(\sigma_t^{\hat\phi}(\partial_X(P))YQ) \\ &= \hat\phi(\sigma_t^\phi(P)Y\partial_X(Q)) + \hat\phi(\partial_X(\sigma_t^\phi(P))YQ). \end{aligned}$$

Since $f(t+i) = \phi(\sigma_t^\phi(P)\xi Q)$, we get, setting $t = 0$, that

$$\phi(P\xi Q) = \hat\phi(PY\partial_X(Q)) + \hat\phi(\partial_X(P)YQ),$$

as claimed. □

2.3. **Conjugate variables as free Brownian gradients.** As pointed out above, $X+\sqrt{\varepsilon}Y$ is a natural free Brownian motion, which is covariant with respect to the appropriate modular groups. The following proposition shows that $J_\phi(X:B)$ plays the role of the free Brownian gradient of X.

Proposition 2.3. *Assume that $\xi = J_\phi(X:B)$ exists and belongs to $M\subset L^2(M,\phi)$. Let $P(Z_1,\dots,Z_n)$, be any non-commutative polynomial in n variables $Z_1,\dots,Z_n$, with coefficients from B. Write $X_t=\sigma_t^\phi(X)$, $Y_t=\sigma_t^{\hat\phi}(Y)$, $\xi_t=\sigma_t^\phi(\xi)$, $t\in\mathbb{R}$.*

Then for all $t_1,\dots,t_n\in\mathbb{R}$, we have

$$\begin{aligned}\hat\phi(P(X_{t_1}+\sqrt{\varepsilon}Y_{t_1},\dots,X_{t_n}+\sqrt{\varepsilon}Y_{t_n})) &= \frac{1}{2}\phi(P(X_{t_1}+\varepsilon\xi_{t_1},\dots,X_{t_n}+\varepsilon\xi_{t_n})\\ &\quad +O(\varepsilon^2).\end{aligned}$$

Proof. We may assume, by linearity, that P is a monomial, i.e., $P(Z_1,\dots,Z_n)=b_0Z_1b_1\cdots b_{n-1}Z_nb_n$, for $b_j\in B$. In this case, we have

$$\begin{aligned}\hat\phi(b_0(X_{t_1}+\sqrt{\varepsilon}Y_{t_1})b_1\cdots b_n) &= \phi(P(X_{t_1},\dots,X_{t_n}))+O(\varepsilon^2)\\ &\quad +\varepsilon\sum_{k<l}\hat\phi(b_0X_{t_1}\cdots b_lY_{t_{l+1}}b_{l+1}X_{t_{l+2}}\cdots b_kY_{t_{k+1}}\cdot\\ &\quad\ \cdot b_{k+1}X_{t_{k+2}}\cdots X_{t_n}b_n)\\ &= \phi(P(X_{t_1},\dots,X_{t_n}))+O(\varepsilon^2)\\ &\quad +\frac{1}{2}\varepsilon\sum_l\hat\phi(b_0X_{t_1}\cdots X_{t_l}b_lY_{t_{l+1}}\partial_{X_{t_{l+1}}}(b_{l+1}X_{t_{l+2}}\cdots X_{t_n}b_n))\\ &\quad +\frac{1}{2}\varepsilon\sum_k\hat\phi(\partial_{X_{t_{k+1}}}(b_0X_{t_1}\cdots X_{t_k}b_k)Y_{t_k}b_{k+1}X_{t_{k+2}}\cdots X_{t_n}b_n)\\ &= \phi(P(X_{t_1},\dots,X_{t_n}))+O(\varepsilon^2)\\ &\quad +\frac{1}{2}\sum_k\phi(b_0X_{t_1}\cdots X_{t_k}b_k\varepsilon\xi_{t_k}b_{k+1}X_{t_{k+2}}\cdots X_{t_n}b_n),\end{aligned}$$

the last equality by Lemma 2.2. This implies the statement of the Lemma. □

2.4. **Examples of conjugate variables.**

2.4.1. *Tracial case.* We have seen before that if ϕ is a trace, then the definition of $J_\phi(X:B)$ coincides with the definition of conjugate variables given by Voiculescu, up to a constant (which has to do with the fact that we choose Y so that $\|Y\|_{L^2(\hat\phi)}=\|X\|_{L^2(\phi)}$, and not 1). In particular,

$$J_\phi(X:B)=J(X:B)\cdot\frac{1}{\|X\|^2_{L^2(\phi)}},\quad \text{if } \phi \text{ is a trace.}$$

2.4.2. *Free quasi-free states.* Let μ be a positive finite Borel measure on $\mathbb{R}$, and let $\mathcal{H}_\mathbb{R}$ be the real Hilbert space $L^2(\mathbb{R},\mathbb{R},\mu)$ of μ -square-integrable real-valued functions. Denote by U_t the representation of $\mathbb{R}$ on $\mathcal{H}_\mathbb{R}$, given by

$$(U_tf)(x)=e^{2\pi itx}f(x),\quad x,t\in\mathbb{R}.$$

Let h denote the vector $1\in\mathcal{H}_\mathbb{R}$, and consider

$$M=\Gamma(\mathcal{H}_\mathbb{R},U_t)'',\quad \phi=\phi_U,\quad X=s(h)\in M$$

(see [5] for definitions and notation).

Then $X = J_\phi(X : \mathbb{C})$. Indeed, set $X_t = \sigma_t^\phi(X) = s(U_t h)$; then we have

$$\begin{aligned}\phi(X \cdot X_{t_1} \cdots X_{t_n}) &= \sum_k \phi(X X_{t_k})\phi(X_{t_1} \cdots X_{t_{k-1}}) \cdot \phi(X_{t_{k+1}} \cdots X_{t_n}) \\ &= \sum_k \hat{\phi}(Y \sigma_{t_k}^{\hat{\phi}}(Y))\phi(X_{t_1} \cdots X_{t_{k-1}}) \cdot \phi(X_{t_{k+1}} \cdots X_{t_n}) \\ &= \sum_k \hat{\phi}(Y X_{t_1} \cdots X_{t_{k-1}} Y X_{t_{k+1}} \cdots X_{t_n}) \\ &= \hat{\phi}(Y \partial_X(X_{t_1} \cdots X_{t_n})),\end{aligned}$$

so that X satisfies the defining property of $J_\phi(X : \mathbb{C})$, and hence $J_\phi(X : \mathbb{C})$ exists and equals X.

2.5. **Free Fisher information.** Following [16], we define the free Fisher information $\Phi_\phi^*(X : B)$ to be

$$\Phi_\phi^*(X : B) = \|J_\phi(X : B)\|_2^2 \cdot \|X\|_2^{-2}$$

(the extra factor $\|X\|_2^{-2}$ comes from the fact that $\partial_X(X)$ does not have unit norm in our definition). For several variables, we set

$$\Phi_\phi^*(X_1, \ldots, X_n) = \sum \Phi_\phi^*(X_i : W^*(\sigma_{t_1}^\phi(X_1), \ldots, \hat{X}_i, \ldots, \sigma_{t_n}^\phi(X_n) : t_1, \ldots, t_n \in \mathbb{R}))$$

(here $\hat{X}_i$ means that X_i is omitted).

3. Free Fisher Information Relative to the Core.

Recall [11] that if (M, ϕ) is as above, its core is defined to be the von Neumann algebra crossed product $P = M \rtimes_{\sigma^\phi} \mathbb{R}$. There is a canonical inclusion $M \subset P$, and P is densely spanned by elements of the form

$$mU_t, \quad t \in \mathbb{R},$$

where $m \in M$, and U_t satisfy $U_t m U_t^* = \sigma_t^\phi(m)$. The elements $U_t : t \in \mathbb{R}$ generate a copy of the group von Neumann algebra $L(\mathbb{R}) \subset P$; the map

$$E^\phi : mU_t \mapsto \phi(m)U_t, \quad m \in M, t \in \mathbb{R}$$

extends to a normal conditional expectation from P onto $L(\mathbb{R})$.

For $X \in M$ self-adjoint, define the completely positive map $\eta_X : L(\mathbb{R}) \to L(\mathbb{R})$ by

$$\eta_X(g) = E^\phi(XgX), \quad g \in L(\mathbb{R}).$$

Identify $L(\mathbb{R})$ with $L^\infty(\mathbb{R})$ via Fourier transform. For each $t \in \mathbb{R}$, set

$$\eta(t) = \langle X, \sigma_t^\phi(X)\rangle = E^\phi(XU_tX).$$

Then $\eta_X(f) = \hat{\eta} * f$, if $f \in L^\infty(\mathbb{R}) \cong L(\mathbb{R})$; here $\hat{\eta}$ denotes Fourier transform.

Define on P an $L(\mathbb{R})$-valued inner product

$$\langle a, b\rangle_{L(\mathbb{R})} = E^\phi(a^*b), \quad a, b \in P.$$

Denote by $L^2(P, E^\phi)$ the $L(\mathbb{R})$-Hilbert bimodule arising from the completion of P with respect to the norm induced by this inner product. Note that the restriction of $\langle \cdot, \cdot \rangle_{L(\mathbb{R})}$ to $M \subset P$ is valued in the complex field, and coincides with the inner product $\langle a, b\rangle = \phi(a^*b)$ on $L^2(M)$.

Denote by $\langle \cdot, \cdot \rangle_\eta$ the $L(\mathbb{R})$-valued inner product on $P \otimes P$ (algebraic tensor product) given by

$$\langle a \otimes b, a' \otimes b'\rangle_\eta = E^\phi(b^* \eta(E^\phi(a^*a')b')), \quad a, a', b, b' \in P.$$

Denote by $1 \otimes_\eta 1$ the vector $1 \otimes 1 \in P \otimes P$.

Let $\delta_X : B[X]\cdot L(\mathbb{R}) \to P\otimes P$ be given by

$$\delta_X(X) = 1\otimes_\eta 1, \quad \delta_X(B\cdot L(\mathbb{R})) = 0$$

and the fact that δ_X is a derivation.

Theorem 3.1. *Let (M,ϕ) be as above, and let P be its core. Let $i : L^2(M,\phi) \to L^2(P,E^\phi)$ be the extension of the inclusion of $M\subset P$. Then $\zeta = i(J_\phi(X:B))$ satisfies*

$$\langle \zeta, Q\rangle_{L(\mathbb{R})} = \langle 1\otimes_\eta 1, \delta_X(Q)\rangle_{\eta X} \tag{3.1}$$

for all $Q\in B[X]\vee L(\mathbb{R})$. Conversely, if there exists a vector $\zeta\in L^2(P,E^\phi)$, so that (3.1) is satisfied, then $J_\phi(X:B)$ exists and $\zeta = i(J_\phi(X:B))$.

Proof. Assume first that $J_\phi(X:B)$ exists. Set $\zeta = i(J_\phi(X:B))$. We must verify that (3.1) holds. By linearity, and the fact that $L(\mathbb{R})BL(\mathbb{R})\subset BL(\mathbb{R})$, it is sufficient to consider the case when $Q = b_0U^{s_1}X_{t_1}b_1U^{s_2}\cdots X_{t_n}b_nU^{s_n}$, with $b_j\in B$ and $X_t = \sigma_t^\phi(X)$. Then $Q = P\cdot U^r$, where $r=\sum s_j$, and $P = b_0X_{t_1'}b_1'\cdots b_n'$, with $b_j' = \sigma^\phi_{s_{j-1}}\circ\cdots\sigma^\phi_{s_1}(b_j)$, $t_j' = s_{j-1}+\cdots+s_1+t_j$. Note that for $x,y,x',y'\in P$, $\langle x\otimes y, x'\otimes Uy'\rangle_\eta = \langle x\otimes y, x'U\otimes y\rangle_\eta$, and $\langle x\otimes y, U^r(x'\otimes y')U^s\rangle = \langle U^{-r}x\otimes yU^{-s}\rangle$. Using this , we get

$$\begin{aligned}
\langle\zeta,Q\rangle_{L(\mathbb{R})} &= \langle\zeta,P\rangle_{L(\mathbb{R})}g\\
&= \langle\zeta,P\rangle_{L^2(M,\phi)}\cdot U^r\\
&= \hat\phi(Y\partial_X(P))\cdot U^r\\
&= \sum_j \phi(b_0'X_{t_1'}\cdots X_{t_j'}b_j')\phi(b_{j+1}'X_{t_{j+2}'}\cdots X_{t_n'}b_n')\cdot\hat\phi(YY_{t_j})U^r\\
&= \sum_j \phi(b_0'X_{t_1'}\cdots X_{t_j'}b_j')\phi(b_{j+1}'X_{t_{j+2}'}\cdots X_{t_n'}b_n')\cdot\phi(XU^{t_j'}XU^{-t_j'})U^r\\
&= \sum_j \phi(b_0'X_{t_1'}\cdots X_{t_j'}b_j')\phi(b_{j+1}'X_{t_{j+2}'}\cdots X_{t_n'}b_n')\cdot E^\phi(XU^{t_j'}XU^{-t_j'}U^r)\\
&= \sum_j \phi(b_0'X_{t_1'}\cdots X_{t_j'}b_j')\phi(b_{j+1}'X_{t_{j+2}'}\cdots X_{t_n'}b_n')\cdot\eta_X(U^{t_j'})U^{r-t_j'}\\
&= \sum_j \eta_X\circ E^\phi(b_0'X_{t_1'}\cdots X_{t_j'}b_j'U^{t_j'})\cdot E^\phi(U^{r-t_j'}b_{j+1}'X_{t_{j+2}'}\cdots X_{t_n'}b_n')\\
&= \sum_j\langle 1\otimes_\eta 1, b_0'X_{t_1'}\cdots X_{t_j'}b_j'U^{t_j'}\otimes U^{r-t_j}b_{j+1}'X_{t_{j+2}'}\cdots X_{t_n'}b_n'\rangle_\eta\\
&= \sum_j\langle 1\otimes_\eta 1, b_0U^{s_1}X_{t_1}\cdots X_{t_j}b_jU^{s_j}U^{t_j'}\otimes U^{r-t_j}U^{s_{j+1}}b_{j+1}X_{t_{j+2}}\cdots X_{t_n}b_nU^{s_n}U^{-r}\rangle_\eta\\
&= \sum_j\langle 1\otimes_\eta 1, b_0U^{s_1}X_{t_1}\cdots X_{t_j}b_jU^{s_j}\otimes U^{s_{j+1}}b_{j+1}X_{t_{j+2}}\cdots X_{t_n}b_nU^{s_n}\rangle_\eta\\
&= \langle 1\otimes_\eta 1,\delta_X(Q)\rangle_\eta.
\end{aligned}$$

Conversely, assume that ζ satisfying (3.1) exists. Since the argument above is reversible, it is sufficient to prove that ζ is in the image of $i: L^2(M)\to L^2(P,E^\phi)$. Let θ_t be the dual action of $\mathbb{R}$ on P, given by $\theta_t(U_s) = \exp(2\pi ist)$, $\theta_t(m) = m$, $m\in M$. It is sufficient to prove that $\theta_t(\zeta)=\zeta$, since $i(L^2(M))$ consists precisely of those vectors, which are left fixed by θ. It is sufficient to prove that $\theta_s(E^\phi(\zeta mU_t)) =$

$\exp(2\pi i s t)$ if $m \in M$. Since ζ is assumed to be in the closure of $B[X] \vee L(\mathbb{R})$, it is sufficient to check this for $m \in B[X]$. But then by (3.1),

$$\begin{aligned} E^{\phi}(\zeta m U_t) &= \langle \zeta, m U_t \rangle_{L(\mathbb{R})} \\ &= \langle 1 \otimes_{\eta} 1, \delta_X(mU) \rangle_{\eta} \\ &= \langle 1 \otimes_{\eta} 1, \delta_X(m) \rangle_{\eta} U \\ &\in M \cdot U_t, \end{aligned}$$

which gives the desired result, since θ_s acts trivially on M. □

Note that (3.1) means that ζ is equal to $J(X : B \vee \mathbb{R}, \eta)$ in the notation of [7]. (This is strictly speaking incorrect, since the setting of [7] presumes the existence of a finite trace on $B[X] \vee L(\mathbb{R})$; however, it is not hard to check that the arguments in [7] go through also in the case of a semifinite trace, which exists in our case).

This fact has many consequences for the conjugate variables $J_{\phi}(X : B)$, coming from the properties of $J(X : B \vee L(\mathbb{R}), \eta)$. Note in particular that if X is free from B with amalgamation over $D \subset B$ with respect to some conditional expectation $E : B \to D$, and E is ϕ-preserving, then X is free from $B \vee L(\mathbb{R})$ with amalgamation over $D \vee L(\mathbb{R})$ (see [12], [6]). We record this as

Theorem 3.2. *Assume that $E : B \to D$ is a ϕ-preserving conditional expectation. If X is free from B over D, then*

$$J_{\phi}(X : B) = J_{\phi}(X : D).$$

In a similar way, one can generalize to $J_{\phi}(X : B)$ all the properties of the conjugate variable $J(X : B, \eta)$ proved in [7].

Reformulating gives the following properties of Φ_{ϕ}, which we list for reader's convenience, since they are needed in the rest of the paper:

Theorem 3.3. *Let ϕ be a normal faithful state on M, $B \subset M$ be globally fixed by the modular group (i.e., $\sigma_t^{\phi}(B) = B$ for all t), and $X_i \in M$. Then:*
(a) $\Phi_{\phi}^*(\lambda X_1, \dots, \lambda X_n : B) = \lambda^{-2} \Phi_{\phi}^*(X_1, \dots, X_n : B)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$
(b) If $B \subset A \subset M$ and A is globally fixed by σ^{ϕ}, then $\Phi_{\phi}^*(X_1, \dots, X_n : A) \geq \Phi^*(X_1, \dots, X_n : B)$.
(c) If $C \subset M$ is globally fixed by σ^{ϕ}, and $W^*(X_1, \dots, X_n)$ and B are free with amalgamation over C (with respect to the unique ϕ-preserving conditional expectation from M onto C), then $\Phi_{\phi}^*(X_1, \dots, X_n : B \vee C) = \Phi_{\phi}^*(X_1, \dots, X_n : B)$.
(d) If $Y_i \in M$ are self-adjoint, $D \subset B$, $D \subset C$ subalgebras of M, which are globally fixed by σ^{ϕ}, and $B[X_1, \dots, X_n]$ is free from $C[X_1, \dots, X_n]$ over D (with respect to the unique ϕ-preserving conditional expectation from M onto D), then $\Phi_{\phi}^*(X_1, \dots, X_n, Y_1, \dots, Y_m : B \vee C) = \Phi_{\phi}^*(X_1, \dots, X_n : B) + \Phi_{\phi}^*(Y_1, \dots, Y_n : C)$.
(e) $\Phi_{\phi}^*(X_1, \dots, X_n, Y_1, \dots, Y_n : B) \geq \Phi_{\phi}^*(X_1, \dots, X_n : B) + \Phi_{\phi}^*(Y_1, \dots, Y_m : B)$.
(f) $\Phi_{\phi}^*(X_1, \dots, X_n : B) \cdot \phi(\sum X_i^* X_i)^2 \geq n^2$. Equality holds iff $\{\sigma_{t_1}^{\phi}(X_1), \dots, \sigma_{t_n}^{\phi}(X_n) : t_1, \dots, t_n \in \mathbb{R}\}$ have the same distribution as the semicircular family $\{\kappa s(\sigma_{t_1}^{\phi}(X_1)), \dots, \kappa s(\sigma_{t_n}^{\phi}(X_n)) : t_1, \dots, t_n \in \mathbb{R}\}$ with respect to the free quasi-free state, $\kappa > 0$.

We mention that all of the statements in sections 3 and 4 of [7] remain valid for Φ_{ϕ}^*; we leave details to the reader.

One can also define and study free entropy $\chi^*_\phi(X_1,\dots,X_n)$ by setting $X_i^\epsilon = X_i + Y_i$ to be the free Brownian motion described in the beginning of the paper, and letting

$$\chi^*_\phi(X_1,\dots,X_n) = \frac{1}{2}\int_0^\infty \left(\frac{n}{1+t} - \Phi^*_\phi(X_1^\epsilon,\dots,X_n^\epsilon)\right) dt.$$

The properties of $\chi^*(\cdots,\eta)$ once again generalize to χ^*_ϕ (compare section 8 of [7]).

4. States on a II_1 factor.

4.1. **Φ^*_ϕ vs. Φ^*_τ.** The following theorem is somewhat surprising, since it shows that Φ^*_ϕ is identically infinite for most states ϕ on a II_1 factor (the analogy with classical Fisher information would instead suggest that $\phi \mapsto \Phi^*_\phi$ would have some nice convexity properties). This, on the other hand, goes well with the "degenerate convexity" property of the microstates free entropy χ [15] (which is reflected in that it is identically $-\infty$ on generators of any von Neumann algebra which more than one unital trace).

Theorem 4.1. *Let M be a tracial von Neumann algebra, ϕ a faithful normal state on M, $B \subset M$ a subalgebra so that $\sigma_t^\phi(B) = B$ for all t, and $X = X^* \in M$. Then if $J_\phi(X : B)$ exists, the modular group of ϕ must fix X.*

Proof. Let $d \in M$ be a positive element, so that $\phi(x) = \tau(dx)$, where τ is a normal faithful trace on M, and d is an unbounded operator on $L^2(M,\tau)$, affiliated to M. The modular group of ϕ is then given by $\sigma_t^\phi(x) = d^{it}xd^{-it}$, $x \in M$. Denoting by X_t the element $\sigma_t^\phi(X)$, we then get

$$X = X_0 = d^{-it}X_t d^{it}, \quad t \in \mathbb{R}.$$

Consider

$$\phi(X_0^2) = \phi(J_\phi(X : B)\cdot X_0) = \phi(J_\phi(X : B)d^{-it}X_t d^{it}).$$

Let b_1 and b_2 be two elements in the domain of ∂_X, so that $b_1 = b_2^*$. Then we get, writing $Y_t = \sigma_t^{\hat\phi}(Y)$:

$$\begin{aligned}
\phi(J_\phi(X : B)b_1X_tb_2) &= \hat\phi(Y_0b_1Y_tb_2)\\
&\quad + \hat\phi(Y_0\partial_X(b_1)X_tb_2) + \hat\phi(Y_0b_1X_t\partial_X(b_2))\\
&= \phi(b_1)\phi(b_2)\hat\phi(Y_0Y_t)\\
&\quad + \hat\phi(Y_0\partial_X(b_1)X_tb_2) + \hat\phi([\partial_X(b_2^*)X_tb_1^*Y_0]^*)\\
&= \phi(b_1)\phi(b_1^*)\hat\phi(Y_0Y_t)\\
&\quad + \hat\phi(Y_0\partial_X(b_1)X_tb_1^*) + \hat\phi([\partial_X(b_1)X_tb_1^*Y_0]^*).
\end{aligned}$$

Now, for all $m, n \in M$, we have

$$\hat\phi(Y_0mY_0n) = \phi(m)\phi(n) = \hat\phi(mY_0nY_0),$$

so that

$$\begin{aligned}
\hat\phi(Y_0\partial_X(b_1)X_tb_1^*) + \hat\phi([\partial_X(b_1)X_tb_1^*Y_0]^*) &= \hat\phi(Y_0\partial_X(b_1)X_tb_1^*) + \hat\phi([Y_0\partial_X(b_1)X_tb_1^*]^*)\\
&= \hat\phi(Y_0\partial_X(b_1)X_tb_1^*) + \overline{\hat\phi(Y_0[\partial_X(b_1)X_tb_1^*])}\\
&\in \mathbb{R}.
\end{aligned}$$

It follows that

$$\Im\phi(J_\phi(X : B)b_1X_tb_1^*) = \Im\hat\phi(Y_0b_1Y_tb_1^*).$$

Now fix $t \in \mathbb{R}$ and choose a_n in the domain of ∂_X, $\|a_n\| \leq 1$, so that

$$a_n \to d^{it}, \quad a_n^* \to d^{-it} \qquad \text{strongly}.$$

One can choose a_n, for example, to be elements of the algebra $B[X]$. Then

$$\begin{aligned} 0 = \Im\phi(X_0^2) &= \Im\phi(J_\phi(X : B) \cdot X_0) \\ &= \Im\phi(J_\phi(X : B) d^{-it} X_t d^{it}) \\ &= \lim_{n\to\infty} \Im\phi(J_\phi(X : B) a_n X_t a_n^*) \\ &= \lim_{n\to\infty} \Im\hat{\phi}(Y_0 a_n Y_t a_n^*) \\ &= \lim_{n\to\infty} \Im(\phi(a_n)\phi(a_n^*)\hat{\phi}(Y_0 Y_t) \\ &= \lim_{n\to\infty} \phi(a_n)\phi(a_n^*)\Im\hat{\phi}(Y_0 Y_t) \\ &= \phi(d^{it})\phi(d^{-it})\Im\hat{\phi}(Y_0 Y_t). \end{aligned}$$

Since $\hat{\phi}(Y_0 Y_t) = \phi(X_0 X_t)$, for t sufficiently close to zero (so that $\phi(d^{it}) \neq 0$), we get that

$$\phi(X X_t) \in \mathbb{R}.$$

Thus

$$\begin{aligned} 0 &= \tau(dXX_t) - \tau(d(XX_t)^*) \\ &= \tau(dXd^{it}Xd^{-it} - d^{it}Xd^{-it}Xd) \\ &= \tau((dX - Xd)d^{it}Xd^{-it}) \\ &= \tau([d, X]d^{it}Xd^{-it}). \end{aligned}$$

Differentiating this in t, and noting that $(d/dt)_{t=0}(d^{it}Xd^{-it}) = i[d, X]$ gives

$$i\tau([d, X]^2) = 0.$$

Since $[d, X]$ is anti-self-adjoint, this implies that $\tau(|[d, X]|^2) = 0$, so that $[d, X] = 0$, because τ is faithful. This means that $\sigma_t^\phi(X) = X$ for all t. □

Corollary 4.2. *Suppose that $X_1, \dots, X_n$ are self-adjoint generators of a II_1 factor M. Let ϕ be a normal faithful state on M, and denote by τ the unique faithful normal trace on M. Then $\Phi_\phi^*(X_1, \dots, X_n) < +\infty$ implies that:*
(1) $\Phi_\tau^*(X_1, \dots, X_n) < \infty$ and
(2) ϕ is a multiple of the trace τ on M.

Proof. Clearly, the second statement implies the first. To get the second statement, write $\phi(\cdot) = \tau(d\cdot)$ and apply the theorem to conclude that $[d, X_i] = 0$. Since $X_1, \dots, X_n$ generate M, d must be in the center of M, which must consist of multiples of identity, since M is a factor. But then d is a scalar multiple of identity, so that ϕ and τ are proportional. □

4.2. **Factoriality.** Voiculescu showed [15] that for his microstates entropy χ the following implication holds:

$$\chi(X_1, \dots, X_n) > -\infty \Rightarrow W^*(X_1, \dots, X_n) \text{ is a factor.}$$

In fact, the conclusion is stronger: not only is the center of $W^*(X_1, \dots, X_n)$ is trivial, but so is its asymptotic center. Unfortunately, we don't know if the same implication holds for the non-microstates free entropy χ^* introduced by Voiculescu in [16], or even under the stronger assumption that $\Phi^*(X_1, \dots, X_n)$ is finite. We prove a weaker version of the assertion above for $\Phi^* = \Phi_\tau^*$. We first need a technical lemma:

Lemma 4.3. *Let ϕ be a normal faithful state on M. Let $X \in M$ be self-adjoint and $B \subset M$ be a subalgebra, so that $\sigma_t^\phi(B) = B$ for all t. Assume that $p \in B$ is a self-adjoint projection, $\phi(p) = \alpha$, and so that $\sigma_t^\phi(p) = p$ for all p. Assume that $\|[X,p]\|_2 < \delta$. Then*

$$\Phi^*_\phi(X : B) > 4\frac{\alpha^2(1-\alpha)^2}{\delta^2}.$$

Proof. Let (A, τ) be a copy of $L(\mathbb{F}_2)$, free from $B[X]$. Since $\Phi^*_\phi(X : B) = \Phi^*_\phi(X : B \vee A)$, and since the centralizer of $\{\sigma_t^\phi(B)\}_t \vee A$ is a factor [1], we can find a projection $q \in B \vee A$, which is fixed by the modular group, and so that $\|[X,q]\|_2 < \delta$, and $\tau(q) = \beta = m/n$ is rational and close to α. We may moreover find a family of matrix units $e_{ij} \in B \vee A$, $1 \le i,j \le n$, fixed by the modular group, and so that

$$e_{ij}^* = e_{ji}, \qquad e_{ij}e_{kl} = \delta_{jk}e_{il}$$
$$\tau(e_{ii}) = \frac{1}{n}, \qquad q = \sum_{i=1}^m e_{ii}.$$

Denote by C the algebra generated in $B \vee A$ by $\{e_{ij}\}_{1\le i,j\le n}$. Note that $C \cong M_{n\times n}$, the algebra of $n \times n$ matrices. The restriction of $\phi * \tau$ to C is the usual matrix trace. Then

$$\Phi^*_\phi(X : B) = \Phi^*_{\phi*\tau}(X : B \vee A) \ge \Phi^*_{\phi*\tau}(X : C).$$

Write $X_{ij} = e_{1i}Xe_{j1}$. Then the inequality $\|[X,q]\|_2 < \delta$ implies that

$$\begin{aligned}
\delta &> \|xq - qx\|_2 \\
&= \|qxq + (1-q)xq - qxq - qx(1-q)\|_2 \\
&= \|(1-q)xq - qx(1-q)\|_2 \\
&= \sqrt{2} \cdot \|qx(1-q)\|_2,
\end{aligned}$$

since $(1-q)xq$ and $qx(1-q)$ are orthogonal. Hence

$$\|qX(1-q)\|_2 < \delta/\sqrt{2}.$$

It follows that

$$\sum_{1\le i\le m,\, m<j\le n} \phi(X_{ij}^*X_{ij}) + \sum_{m<i\le n,\, 1\le i\le n} \phi(X_{ij}^*X_{ij}) < \delta^2.$$

Denote by ϕ' the state $n(\phi * \tau)(e_{11} \cdot e_{11})$ on $e_{11}W^*(X,C)e_{11}$. Then

$$\begin{aligned}
\Phi^*_{\phi'}(\{X_{ij}\}) &\ge \sum_{i,j} \Phi^*_{\phi'}(\{X_{ij}\}) \\
&> 2m(n-m)\frac{1}{n(\delta^2/2m(n-m))} \\
&= \frac{(2m(n-m))^2}{n\delta^2} \\
&= n^3 4\frac{\beta^2(1-\beta)^2}{\delta^2}.
\end{aligned}$$

Arguing exactly as in [4, Proposition 4.1], we get that

$$\Phi^*_{\phi*\tau}(X : C) = \frac{1}{n^3}\Phi^*_{\phi'}(\{X_{ij}\}) > 4\frac{\beta^2(1-\beta)^2}{\delta^2}.$$

Since β was a rational number, arbitrarily close to α, we get the desired estimate for $\Phi^*_\phi(X : B)$. ∎

Theorem 4.4. *Assume that M is a von Neumann algebra with a faithful normal trace τ, and X_i are a family of self-adjoint elements in M, $\|X_i\| = 1$. Assume that B_i form an increasing sequence of subalgebras of M, so that $M = \overline{\cup B_i}^w$. Assume further that for some normal faithful state ϕ on M and for each j,*

$$\liminf_i \Phi^*_\phi(X_i : B_j) < +\infty.$$

Then M is a factor.

Proof. In view of Theorem 4.1, we may assume that ϕ is a trace, τ. Assume that M is not a factor. Then there exists a central projection $p \in M$ of some trace $\alpha = \tau(p)$, $\alpha(1-\alpha) \neq 0$. Moreover, $[p, X_i] = 0$ for all i. Since B_i increase to all of M, given $\delta > 0$, there is a large enough j and a projection $q \in B_j$, so that $\|q - p\|_2 < \delta/2$. Then for any k,

$$\begin{aligned} \|[q, X_k]\|_2 &= \|qX_k - X_k q\|_2 \\ &= \|(q-p)X_k - X_k(q-p) + pX_k - X_k p\|_2 \\ &\leq \|(q-p)X_k\|_2 + \|X_k(q-p)\|_2 + 0 \\ &\leq 2\|(q-p)\|_2 \|X_k\| \\ &< 2(\delta/2) = \delta. \end{aligned}$$

Now applying Lemma 4.3, we deduce that for any $i > j$,

$$\Phi^*_\tau(X_i : B_j) > 4\frac{\alpha^2(1-\alpha)^2}{\delta^2}.$$

Hence $\liminf_i \Phi^*_\tau(X_i : B_j) > 4\alpha^2(1-\alpha)^2/\delta^2$, which is a contradiction, since δ was arbitrary. □

The hypothesis of the theorem is satisfied for some von Neumann algebras. For example, let $M = L(\mathbb{F}_\infty)$ generated by an infinite semicircular family X_i, $i = 1, 2, 3, \ldots$. Then if $B_i = W^*(X_j : j < i)$, the assumptions of the theorem are satisfied. In fact if Y_i is *any* family of elements of a tracial von Neumann algebra, so that $\|Y_i\| = 1$, and X_i are a free semicircular family, then letting $Z_j(\epsilon) = Y_i + \epsilon X_i$, $M_\epsilon = W^*(Z_1(\epsilon), Z_2(\epsilon), \ldots)$ and $B_j = W^*(Z_1(\epsilon), \ldots, Z_j(\epsilon))$, we see that M_ϵ is a factor. In other words, generators of an arbitrary tracial von Neumann algebra can be perturbed (in a certain representation of this algebra) by an arbitrarily small amount ϵ in uniform norm, to produce a II_1 factor. Another way of putting it is to note that the free Brownian motion $\epsilon \mapsto Z_j(\epsilon)$ started at $\{Y_1, Y_2, \ldots\}$ generates a factor at any time $\epsilon > 0$.

4.3. **Factoriality in the non-tracial case.** In a similar way, we get the following:

Theorem 4.5. *Assume that M is a von Neumann algebra with a faithful normal state ϕ, and X_i are a family of self-adjoint elements in M, $\|X_i\| = 1$. Assume that B_i form an increasing sequence of subalgebras of M, $\sigma^\phi_t(B_i) = B_i$ for all t and i, and assume that $M^\phi = \overline{\cup(B_i \cap M^\phi)}^w$. Let R_i be the operator of right multiplication by X_i densely defined on $L^2(M, \phi)$. Assume that $\sup_i \|R_i\| = C < +\infty$. Assume further that for each j,*

$$\liminf_i \Phi^*_\phi(X_i : B_j) < +\infty.$$

Then M is a factor.

Proof. Assume that M is not a factor. Then there exists a central projection $p \in M$, $\alpha = \phi(p)$, $\alpha(1-\alpha) \neq 0$. Moreover, $[p, X_i] = 0$ for all i. Since automatically $p \in M^\phi$ and $B_i \cap M^\phi$ increase to all of M^ϕ, given

$\delta > 0$, there is a large enough j and a projection $q \in B_j \cap M^\phi$, so that $\|q - p\|_2 < \delta/(1+C)$. Then for any k,

$$\begin{aligned} \|[q, X_k]\|_2 &= \|qX_k - X_k q\|_2 \\ &= \|(q-p)X_k - X_k(q-p) + pX_k - X_k p\|_2 \\ &\leq \|(q-p)X_k\|_2 + \|X_k(q-p)\|_2 + 0 \\ &\leq \|R_k\| \|(q-p)\|_2 + \|(q-p)\|_2 \|X_k\| \\ &< (1+C)(\delta/(1+C)) = \delta. \end{aligned}$$

□

Note that the assumption on the norms of R_i is satisfied if each X_i is analytic for σ_t^ϕ and satisfies $\sup_k \|\sigma_i^\phi(X_k)\| = C < +\infty$.

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