ON A CONJECTURE OF R. P. STANLEY
PART I – MONOMIAL IDEALS

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Abstract. In 1982 Richard P. Stanley conjectured that any finitely generated $\mathbb{Z}^n$-graded module M over a finitely generated $\mathbb{N}^n$-graded $\mathbb{K}$-algebra R can be decomposed in a direct sum $M = \bigoplus_{i=1}^{t} \nu_i S_i$ of finitely many free modules $\nu_i S_i$ which have to satisfy some additional conditions. Besides homogeneity conditions the most important restriction is that the S_i have to be subalgebras of R of dimension at least $depth\, M$.

We will study this conjecture for the special case that R is a polynomial ring and M an ideal of R, where we encounter a strong connection to generalized involutive bases. We will derive a criterion which allows us to extract an upper bound on $depth\, M$ from particular involutive bases. As a corollary we obtain that any monomial ideal M which possesses an involutive basis of this type satisfies Stanley's Conjecture and in this case the involutive decomposition defined by the basis is also a Stanley decomposition of M. Moreover, we will show that the criterion applies, for instance, to any monomial ideal of depth at most 2, to any monomial ideal in at most 3 variables, and to any monomial ideal which is generic with respect to one variable. The theory of involutive bases provides us with the algorithmic part for the computation of Stanley decompositions in these situations.

1. Introduction

This is the first of two articles studying some aspects of a conjecture formulated by Richard P. Stanley in 1982.

Conjecture 1 ([St82], 5.1). *Let R be a finitely-generated $\mathbb{N}^n$-graded $\mathbb{K}$-algebra (where $R_0 = \mathbb{K}$ as usual), and let M be a finitely-generated $\mathbb{Z}^n$-graded R-module. Then there exist finitely many subalgebras $S_1, \ldots, S_t$ of R, each generated by algebraically independent $\mathbb{N}^n$-homogeneous elements of R, and there exist $\mathbb{Z}^n$-homogeneous elements $\nu_1, \ldots, \nu_t$ of M, such that*

$$M = \bigoplus_{i=1}^{t} \nu_i S_i,$$

where $\dim S_i \geq depth\, M$ for all i, and where $\nu_i S_i$ is a free S_i-module (of rank one). Moreover, if $\mathbb{K}$ is infinite and under a given specialization to an $\mathbb{N}$-grading of R is generated by R_1, then we can choose the ($\mathbb{N}^n$-homogeneous) generators of each S_i to lie in R_1.

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Definition 1. *Let R be a finitely-generated $\mathbb{N}^n$-graded $\mathbb{K}$-algebra and let M be a finitely-generated $\mathbb{Z}^n$-graded R-module. For each non-negative integer d let Δ_d denote the set of all direct sum decompositions $M = \bigoplus_{i=1}^t \nu_i S_i$ of M in finitely many free S_i-modules $\nu_i S_i \quad (i \in \{1, \ldots, t\})$, where the $S_1, \ldots, S_t$ are subalgebras of R each of them generated by at least d algebraically independent $\mathbb{N}^n$-homogeneous elements of R and the $\nu_1, \ldots, \nu_t$ are $\mathbb{Z}^n$-homogeneous elements of M.*

If not all sets Δ_d are empty then we call the largest d such that $\Delta_d \neq \emptyset$ the Stanley dimension *of M and denote it by $Sdim\, M$.*

It is not obvious whether a R-module M satisfying the assumptions of the definition possesses a Stanley dimension or not since it might happen that M does not have any decomposition in finitely many direct summands $\nu_i S_i$ satisfying the above conditions. However, at least for the modules $M = I$ which are subject of this paper, i.e. monomial ideals in finitely many variables, the Stanley dimension always exist. Specialized to our studies Stanley's Conjecture reads as follows:

Conjecture 2. *Let $R = \mathbb{K}[X]$ be a polynomial ring in the variables $X = \{x_1, \ldots, x_n\}$ over a field $\mathbb{K}$. Then for each monomial ideal $I \subset R$ it holds*

$$Sdim\, I \geq depth(XR, I) = depth\, I\ .$$

That is, there exist a finite decomposition (called Stanley decomposition *throughout this paper) of I of the following type*

$$I = \bigoplus_{i=1}^{t} u_i \mathbb{K}[Y_i], \tag{1}$$

where the u_i are (monic) monomials, $Y_i \subseteq X$ and $|Y_i| \geq depth\, I$ for all $i = 1, \ldots, t$.

The notion $depth\, I$ could lead to a confusion since it is also a frequently used short cut for $depth(I, M)$, the depth of the ideal I on some R-module M. Therefore, we emphasize that in this paper $depth\, I$ will always stand for $depth(XR, I)$, the depth of the maximal homogeneous ideal of R on I considered as an R-module. Another remark on the denotation concerns the use of inclusion symbols, $A \subset B$ stands for A is a (not necessarily non-empty) subset of B but $A \neq B$. $A \subseteq B$ stands for $A = B$ or $A \subset B$.

There is a strong relationship between Stanley decompositions and the Riquier-Janet theory for solving systems of partial differential equations (see [Ri], [J]) which also produces a decomposition (1), however, without the additional assumption on the dimension of the subalgebras. In particular, this observation provides an existence proof for the Stanley dimension of an arbitrary monomial ideal. In general, the classical decompositions due to Janet, Thomas, or Pommaret will violate the crucial condition on the dimension of the subalgebras S_i. Here we will rely on a more general definition of involutive bases introduced in [A98]. We will prove a criterion (Corollary 1) showing that a particular type of involutive decompositions satisfies even the harder assumptions of Stanley's Conjecture and, hence, is a Stanley decomposition.

Our definition of involutive bases will turn out to be sufficiently general to allow the application of the above criterion to some large classes of monomial ideals. Among them there are all monomial ideals, in an arbitrary number of variables, of depth at most 2 (Corollary 3), all monomial ideals in at most three variables (Theorem 1), and all monomial ideals which are generic with respect to one variable, i.e. whenever two distinct minimal generators have the same degree in some fixed variable then there is another minimal generator of strictly smaller degree in this variable which divides the least common multiple of these minimal generators (Theorem 2). The proofs of the theorems give explicit hints how in these cases a Stanley decomposition can be constructed using the algorithmic methods provided by the theory of involutive bases.

We remark that Example 4 shows that also the notion of involutive bases introduced in [A98] is not powerful enough for the construction of Stanley decompositions for arbitrary monomial ideals.

Finally, we give an outlook to the studies contained in Part II of this project ([A01]). The most interesting situation Stanley had in mind are the Cohen-Macaulay modules. The verification of Conjecture 1 for Cohen-Macaulay modules would contribute to the solution of further combinatorial problems, in particular, it would prove that Cohen-Macaulay complexes are partitionable (see [St79]). We will study Stanley's Conjecture for factor rings of polynomial rings modulo monomial ideals with an emphasize on Cohen-Macaulay rings. Our main result will be to prove the inequality $Sdim\, M \geq depth\, M$ for all modules $M = R/I$ belonging to one of the following cases: (i) R has at most three variables, (ii) I is generic, (iii) I is cogeneric and Cohen-Macaulay.

2. Involutive Bases of Polynomial Ideals

Let $X = \{X_1, \dots, X_n\}$ be a finite set of indeterminates and $T = \langle X \rangle$ denote the free commutative monoid generated by X. By an *involutive division* on T we denote a certain type of subrelations of the conventional division which is defined in the following way.

Definition 2 ([A98], Def. 3.1). *Let $(Y_u)_{u\in T}$ be a family of subsets $Y_u \subseteq X$ of indeterminates. The family $\mathcal{M} = (u\,\langle Y_u\rangle)_{u\in T}$ is called the* involutive division *generated by $(Y_u)_{u\in T}$. The elements $u \in v\,\langle Y_v\rangle$ are called the $\mathcal{M}$-*multiples *of v and v is called a $\mathcal{M}$-*divisor *of $u \in v\,\langle Y_v\rangle$. Furthermore, we call the $x \in Y_u$ the $\mathcal{M}$-*multipliers *and the $y \in X \setminus Y_u$ the $\mathcal{M}$-*nonmultipliers *of $u \in T$. The number $|Y_u|$ of $\mathcal{M}$-multipliers is denoted by $invdim_{\mathcal{M}}\, u$ and called the* involutive dimension *of u with respect to $\mathcal{M}$.*

Let $V \subseteq T$ be a set of terms and $\sqsubset$ an order on V. The involutive division $\mathcal{M} = (M_u)_{u\in T}$ is called admissible for $(V, \sqsubset)$ *if for all $v, w \in V$ such that $w \sqsubset v$ it holds one of the conditions $M_w \subset M_v$ or $M_v \cap wT = \emptyset$. If the admissibility of $\mathcal{N} = (N_u)_{u\in T}$, where $M_u \subseteq N_u$ for all $u \in V$, implies that all inclusions have to be equalities then $\mathcal{M}$ is called a* maximal admissible involutive division for $(V, \sqsubset)$. *The short cut $\mathcal{M}$* is (maximal) admissible for V *will express the existence of $\sqsubset$ such that $\mathcal{M}$ is (maximal) admissible for $(V, \sqsubset)$.*

Now, let $R = \mathbb{K}[X]$ be the polynomial ring in the variables $X = \{X_1, \dots, X_n\}$ over a field $\mathbb{K}$. Then the above defined involutive divisions give raise to the distinction of a particular type of Gröbner bases of ideals of I.

Definition 3. *Let $\prec$ be an admissible term order, F a finite set of non-zero polynomials and $\mathcal{M}$ an involutive division admissible for the set* $\mathrm{lt}\, F$ *of leading terms of F. If* $\mathrm{lt}\,(I) = \bigcup_{f \in F} M_{\mathrm{lt}\, f}$*, where $I = FR$ denotes the ideal of R generated by F, then F is called a* $\mathcal{M}$-involutive basis of I with respect to $\prec$. *If, in addition, the union on the right hand side is disjoint we call F a* minimal $\mathcal{M}$-involutive basis *of I with respect to $\prec$.*

Obviously, any $\mathcal{M}$-involutive basis F of I is also a Gröbner basis of I. Using some well-known facts on Gröbner bases and taking into account the uniqueness of involutive reduction with respect to minimal involutive bases it follows

$$I = \bigoplus_{f \in F} f\mathbb{K}[Y_f] \tag{2}$$

for an arbitrary minimal $\mathcal{M}$-involutive basis of I, where $\mathcal{M}$ is generated by $(Z_u)_{u \in T}$ and $Y_f = Z_{\mathrm{lt}\, f}$.

For given I, $\prec$ and $\mathcal{M}$ there need not to exist any $\mathcal{M}$-involutive basis of I with respect to $\prec$. However, at least we have:

Proposition 1. *Let F be a finite set of non-zero polynomials generating the ideal $I \subset R$ and $\prec$ be an admissible term order. Then there exists a finite set $G \subset I \setminus \{0\}$, a linear order $\sqsubset$ of* $\mathrm{lt}\, G$ *and a (maximal) admissible involutive division $\mathcal{M}$ on* $(\mathrm{lt}\, G, \sqsubset)$ *such that G is a (minimal) $\mathcal{M}$-involutive basis of I with respect to $\prec$.*

Proof: Algorithm 4 from [A98] was proved to compute in a finite number of steps an involutive division $\mathcal{M}$ and a finite set G such that G is an $\mathcal{M}$-involutive basis of I w.r.t. $\prec$. $\sqsubset$ can be even prescribed as e.g. a reverse lexicographical order of the set T but can be altered during the completion process as well. It is an easy exercise to show that $\mathcal{M}$ can be maximized and G be minimized by possibly enlarging the cones $M_{\mathrm{lt}\, g}$ and by removing redundant elements from G. □

3. An Upper Bound Criterion for the Depth

In the previous section we recalled the existence of minimal $\mathcal{M}$-involutive bases with respect to maximal admissible involutive divisions. Now, we will study the properties of such bases in the case of monomial ideals in view of Conjecture 2. Note, the admissible term order $\prec$ is of no importance in the monomial case. If the reader likes he may assume an arbitrary $\prec$ to be fixed.

If some decomposition (2) arising from an involutive basis satisfies Stanley's Conjecture then, obviously, there is also a minimal $\mathcal{M}$-involutive basis with respect to a maximal admissible involutive division $\mathcal{M}$ having this property. In the rest of the paper by a *$\mathcal{M}$-involutive basis U* of a monomial ideal I we will always mean that U consists of monic monomials, i.e. monomials of coefficient 1 which we will freely identify with the elements of T, $\mathcal{M}$ is a maximal involutive division admissible on $(U, \sqsubset)$, and U is a minimal $\mathcal{M}$-involutive basis of I. Recall, each monomial ideal I possesses a uniquely determined minimal generating set formed by monic monomials. When the ideal I is clear from the context we will refer to this set by B. Unless stated differently we consider only nontrivial ideals I, i.e. $0 \subset I \subset R$. We start with the definition of some notions which will turn out to be useful during our studies of Stanley's Conjecture.

Definition 4. *Let U be an $\mathcal{M}$-involutive basis of I, $u \in U$, and x an $\mathcal{M}$-nonmultiplier of u. An element $v \in U$ is called a* x-witness *for u iff $v \sqsubset u$, $u \nmid v$, $\deg_x v > \deg_x u$ and $\deg_y v \leq \deg_y u$ for all $\mathcal{M}$-nonmultipliers $y \neq x$ of u. If, in addition, $\deg_x v = \deg_x u + 1$ then v is called a* strong *x-witness.*

A set $W(u) \subset U$ consisting of a (strong) x-witness for u for each $\mathcal{M}$-nonmultiplier x will be called a (strong) witness set *for u.*

Definition 5. *Let U be an $\mathcal{M}$-involutive basis of I. The* involutive dimension of U with respect to $\mathcal{M}$ *is defined by $invdim_{\mathcal{M}} U := \min_{u \in U} invdim_{\mathcal{M}} u$. Moreover, the maximum*

$$invdim\, I := \max\{invdim_{\mathcal{M}} U \mid U \text{ is } \mathcal{M}\text{-involutive basis of } I\}$$

is called the involutive dimension of I.

Remark 1. *For an arbitrary monomial ideal I it holds $invdim\, I \leq Sdim\, I$.*

Recall some obvious facts on involutive bases. Every $\mathcal{M}$-involutive basis U of I contains all minimal generators of I. Any element u of an $\mathcal{M}$-involutive basis U of I possesses a witness set $W(u)$. For the last time let us emphasize our restriction to minimal involutive bases with respect to maximal admissible involutive divisions, both restriction are of course essential in the last statement.

Lemma 1. *Let U be an $\mathcal{M}$-involutive basis of I. Further, assume there exist $u \in U$ and $x \in Y_u$ such that $invdim_{\mathcal{M}} u = invdim_{\mathcal{M}} U$ and u possesses a strong witness set $W(u)$ satisfying $\deg_x u > \deg_x v$ for all $v \in W(u)$.*

Then the $\mathcal{M}$-nonmultipliers of u annihilate a common non-zero element $a \in I/xI$.

Proof: For each $\mathcal{M}$-nonmultiplier y and the corresponding strong y-witness $v_y \in W_u$ of u define $t_y := \frac{\mathrm{lcm}(u,v_y)}{y}$. By the property of a strong witness it follows that $\deg_y t_y = \deg_y v_y - 1 = \deg_y u$ and $\deg_z t_y = \deg_z u$ for all $\mathcal{M}$-nonmultipliers $z \neq y$ of u.

We will show that $a := \mathrm{lcm}_{y \in X \setminus Y_u} t_y$ satisfies the assertions of the lemma. $a \in I$ is obvious, it remains to show $a \notin xI$ and $ya \in xI$ for each $\mathcal{M}$-nonmultiplier y of u. By construction $yt_y = \mathrm{lcm}(u, v_y)$ divides ya. According to our assumptions on x we have $xv_y \mid yt_y$ and, hence, $xv_y \mid ya$. This implies $ya \in xI$, i.e. each $\mathcal{M}$-nonmultiplier y of u annihilates the element a.

Now, assume $a \in xI$, i.e. $\frac{a}{x} \in I$. Since U is a $\mathcal{M}$-involutive basis there exists $w \in U$ such that $\frac{a}{x} \in M_w$. From the inequalities $\deg_x w < \deg_x a = \deg_x u$ and $\deg_y w \leq \deg_y a = \deg_y u = \deg_y v_y - 1$ for all $y \in X \setminus Y_u$ we conclude (i) $\frac{a}{u}$ belongs to the monoid $\langle Y_u \rangle$, i.e. $a \in M_u$, (ii) $u \nmid w$ and $v_y \nmid w$ for all $y \in X \setminus Y_u$. By (i) x has to be a $\mathcal{M}$-nonmultiplier of w since, otherwise, $a \in M_w \cap M_u$. Moreover, from $a \in M_u \cap wT$ we deduce $u \sqsubset w$ by the properties of $\mathcal{M}$. Now, if some $\mathcal{M}$-nonmultiplier y of u would be an $\mathcal{M}$-multiplier of w then we had $y\frac{a}{x} \in M_w \cap v_y T$, in contradiction to the relation $v_y \sqsubset u \sqsubset w$. In conclusion, $a \in xI$ would imply $invdim_{\mathcal{M}} w \leq invdim_{\mathcal{M}} u - 1$ in contradiction to the construction of u. □

Corollary 1. *Let $I \subset R$ be a monomial ideal. If there exist a $\mathcal{M}$-involutive basis U of I, an element $u \in U$ of minimal involutive dimension with respect to $\mathcal{M}$, an $\mathcal{M}$-multiplier $x \in Y_u$ of u, and a strong witness set $W(u)$ for u such that $\deg_x u > \deg_x v_y$ for all $v_y \in W(u)$ then it holds $depth\, I \leq Sdim\, I$.*

Proof: x is a non-zero-divisor on the R-module I. Hence, $depth(XR, I) \leq depth((X \setminus Y_u)R + xR, I) + invdim_{\mathcal{M}} u - 1 = depth((X \setminus Y_u)R, I/xI) + invdim_{\mathcal{M}} u = invdim_{\mathcal{M}} u = invdim_{\mathcal{M}} U \leq invdim\, I \leq Sdim\, I$. □

In particular, Conjecture 2 holds for any monomial ideal I satisfying the assumptions of Corollary 1. Let us illustrate the statements of the lemma and its corollary by some examples.

Example 1. *Any Borel-fixed ideal (see* [E]*) $I \subset \mathbb{K}[x_1, \ldots, x_n]$ possesses an involutive basis satisfying the assumptions of Corollary 1 and, hence, satisfies $Sdim\, I \geq invdim\, I \geq depth\, I$.*

Let $t \in \{1, \ldots, n\}$ be the maximal index such that x_t is contained in some minimal generator of I. Since with $ux_l^d \in I$, where $u \in \mathbb{K}[x_1, \ldots, x_{l-1}]$ and $1 < l \leq n$, also $x_1^{\deg u} x_l^d \in I$ (see [E, Theorem 15.23b]*) it is easy to observe the existence of a minimal generator m of I which is a pure power of x_1 and $invdim_{\mathcal{M}}\, m = invdim_{\mathcal{M}}\, U = n - t + 1$ for any $\mathcal{M}$-involutive basis U of I, where the underlying order $\sqsubset$ is compatible with the pure lexicographical order $\prec$ extending $x_n \prec \cdots \prec x_1$ and U (for the compatibility notion see Definition 7). Now apply Corollary 1 with $u = m$ and $x = x_1$ and the first statement of this example follows.*

The above example refers to a well-known classical situation. Conjecture 1 is known to be true in the particular case $n = 1$ (c.f. [St82]), where decompositions due to Rees (see [Ree]) or Baclawski and Garsia (see [BG]) can be used. It is the nature of Borel-fixed ideals that the identity map can serve as the linear variable transformation occurring in Rees' method. So the decomposition obtained for the $\mathbb{N}$-grading satisfies even the stronger conditions posed by the $\mathbb{N}^n$-grading. We remark, that in general Rees' approach requires an infinite field $\mathbb{K}$, but in the particular situation of Borel-fixed ideals it will work also for an arbitrary field $\mathbb{K}$.

Example 2. *Consider the ideal $I = (xyz, x^2, y^2, z^2) \subset \mathbb{K}[x, y, z]$. An involutive basis of I is $U_1 = \{z^2, y^2z, x^2yz, x^2z, y^2, x^2y, x^2, xyz\}$, where the elements are in increasing order with respect to $\sqsubset$ and the involutive division is defined by the multiplier sets $Y_{z^2} = \{x, y, z\}$, $Y_{y^2z} = Y_{y^2} = \{x, y\}$, $Y_{x^2yz} = Y_{x^2z} = Y_{x^2y} = Y_{x^2} = \{x\}$, and $Y_{xyz} = \emptyset$. The only element of minimal involutive dimension is xyz, a possible set of witnesses for xyz is $W(xyz) = \{x^2, y^2, z^2\}$. This is even a strong set of witnesses, but Lemma 1 and its corollary are not applicable since the required $\mathcal{M}$-multiplier of xyz does not exist. Since $Y_{xyz} = \emptyset$ the same is of course true for any other set of witnesses for xyz.*

Another involutive basis of I is $U_2 = \{z^2, y^2z, xyz, x^2z, y^2, x^2y, x^2\}$. Here, $\sqsubset$ is the reverse lexicographical order extending $z \sqsubset y \sqsubset x$. The involutive division is defined by the multiplier sets $Y_{z^2} = \{x, y, z\}$, $Y_{y^2z} = Y_{y^2} = \{x, y\}$, and $Y_{xyz} = Y_{x^2z} = Y_{x^2y} = Y_{x^2} = \{x\}$. There are different elements of minimal involutive dimension but only x^2z

or x^2y allow to satisfy the assumption of Lemma 1. For instance, consider $u = x^2z$ and the strong witness set $W(x^2z) = \{xyz, z^2\}$. Then x^2z has larger degree in its $\mathcal{M}$-multiplier x than both witnesses.

Application of Corollary 1 proves that I has depth at most 1. In fact, it must be equality since depth 0 is impossible in this situation. A Stanley decomposition is $I = z^2\mathbb{K}[x, y, z] \oplus y^2z\mathbb{K}[x, y] \oplus y^2\mathbb{K}[x, y] \oplus xyz\mathbb{K}[x] \oplus x^2z\mathbb{K}[x] \oplus x^2y\mathbb{K}[x] \oplus x^2\mathbb{K}[x]$.

Example 3. *Consider the ideal $I = (yu, xu, yz, xz) \subset \mathbb{K}[x, y, z, u]$. Based on the reverse lexicographical order extending $u \sqsubset z \sqsubset y \sqsubset x$ the minimal basis is already an involutive basis, where $Y_{yu} = \{x, y, z, u\}$, $Y_{xu} = \{x, z, u\}$, $Y_{yz} = \{x, y, z\}$, and $Y_{xz} = \{x, z\}$ are the sets of multipliers. xz has minimal involutive dimension but neither its degree in x nor in z is strictly larger than those of both witnesses yz and xu. Note, yu can not be used as witness since it has higher degree in both nonmultipliers of xz. Hence, Lemma 1 is not applicable. It is easy to see that the minimal basis is the only (minimal) involutive basis of I. Considering all 24 possible linear orders of the 4 elements we obtain different sets of multipliers. Each time the minimal involutive dimension is 2 and none of the settings fulfills the assumptions of Lemma 1. Since $\operatorname{depth} I = 2$ the decomposition $I = yu\mathbb{K}[x, y, z, u] \oplus xu\mathbb{K}[x, z, u] \oplus yz\mathbb{K}[x, y, z] \oplus xz\mathbb{K}[x, z]$ satisfies Stanley's conditions. However, also $I = xyzu\mathbb{K}[x, y, z, u] \oplus yu\mathbb{K}[x, y, u] \oplus xu\mathbb{K}[x, z, u] \oplus yz\mathbb{K}[y, z, u] \oplus xz\mathbb{K}[x, y, z]$ is a Stanley decomposition of I and, hence, $\operatorname{Sdim} I = 3 > \operatorname{invdim} I = 2$.*

Note, the different natures of Examples 2 and 3. In both cases involutive bases proved to be suitable to construct a decomposition satisfying Stanley's Conjecture. But, while in the first example the decomposition comes along with a proof that it is of Stanley's type this is not so obvious in the second example. One could say, involutive bases are not optimal but still sufficient, i.e. $\operatorname{depth} I \leq \operatorname{invdim} I$, in Example 3. The next example will show the existence of monomial ideals where $\operatorname{depth} I > \operatorname{invdim} I$.

Example 4 ([Rei] Remark 3)**.** *Consider $I = (yvw, xvw, zuw, xuw, yzw, zuv, yuv, xzv, xyu, xyz) \subset \mathbb{K}[x, y, z, u, v, w]$. For any order $\sqsubset$ the largest minimal generator of I will have only three multipliers, namely, the three variables occurring in this generator. Hence no involutive basis can provide an involutive dimension higher than 3. An involutive basis yielding exactly 3 can be obtained e.g. by adding the elements $xuvw, yuvw$ and $zuvw$ to the minimal basis and ordering using the reverse lexicographical order extending $w \sqsubset \cdots \sqsubset x$. We obtain the decomposition $I = xyz\mathbb{K}[x, y, z] \oplus xyu\mathbb{K}[x, y, z, u] \oplus xzv\mathbb{K}[x, y, z, v] \oplus yuv\mathbb{K}[x, y, u, v] \oplus zuv\mathbb{K}[x, y, z, u, v] \oplus yzw\mathbb{K}[x, y, z, w] \oplus xuw\mathbb{K}[x, y, u, w] \oplus zuw\mathbb{K}[x, y, z, u, w] \oplus xvw\mathbb{K}[x, z, v, w] \oplus yvw\mathbb{K}[x, y, z, v, w] \oplus xuvw\mathbb{K}[x, u, v, w] \oplus yuvw\mathbb{K}[x, y, u, v, w] \oplus zuvw\mathbb{K}[x, y, z, u, v, w]$.*

However, we have $\operatorname{depth} I = 4 > \operatorname{invdim} I = 3$. But the decomposition $I = xyz\mathbb{K}[x, y, z, w] \oplus xyu\mathbb{K}[x, y, z, u] \oplus xzv\mathbb{K}[x, y, z, v] \oplus yuv\mathbb{K}[x, y, u, v] \oplus zuv\mathbb{K}[y, z, u, v] \oplus yzw\mathbb{K}[y, z, v, w] \oplus xuw\mathbb{K}[x, y, u, w] \oplus zuw\mathbb{K}[x, z, u, w] \oplus xvw\mathbb{K}[x, z, v, w] \oplus yvw\mathbb{K}[x, y, v, w] \oplus xuvw\mathbb{K}[x, u, v, w] \oplus yuvw\mathbb{K}[y, u, v, w] \oplus zuvw\mathbb{K}[z, u, v, w] \oplus yzuw\mathbb{K}[y, z, u, w] \oplus xzuv\mathbb{K}[x, z, u, v] \oplus xzuvw\mathbb{K}[x, z, u, v, w] \oplus xyuvw\mathbb{K}[x, y, u, v, w] \oplus yzuvw\mathbb{K}[x, y, z, u, v, w]$ illustrates $\operatorname{Sdim} I = 4 = \operatorname{depth} I$ and Stanley's Conjecture holds.

Example 4 shows that there is no hope to prove Stanley's Conjecture only by means of involutive bases (in the sense of [A98]). However, there are at least some interesting subclasses of monomial ideals for which the validity of Stanley's Conjecture follows from Corollary 1. Such situations will be studied now.

4. Ideals of Small Depth ($depth\, I \leq 2$)

We will prove Stanley's Conjecture for monomial ideals of small depth. First of all, we want to proof that any monomial ideal has got an involutive basis such that each basis element has at least one multiplier. In fact, Example 2 shows that not all involutive bases have this property but at least for any reverse lexicographical order $\sqsubset$ this will turn out to be true.

Lemma 2. *Let $I \subset \mathbb{K}[X]$ be a monomial ideal, $\sqsubset$ a reverse lexicographical order, and U an $\mathcal{M}$-involutive basis of I, where $\mathcal{M}$ is admissible for $(U, \sqsubset)$. Then all sets Y_u, $u \in U$, are non-empty.*

Proof: Assume, there exists $u \in U$ such that $Y_u = \emptyset$. By maximality of $\mathcal{M}$ there exists a x-witness v_x for u, where x is the maximal element of X (w.r.t. $\sqsubset$). The witness property yields at the one hand side $\deg_x u < \deg_x v_x$ and $\deg_y u \geq \deg_y v_x$ for all $y \neq x$. But at the other hand side it also requires $v_x \sqsubset u$, i.e. $\deg_z u < \deg_z v_x$ for the smallest variable $z \in X$ (w.r.t. $\sqsubset$) for which the degrees of u and v_x are different. This implies $x = z$ and $u \mid v_x$, a contradiction to the witness property. $\square$.

Corollary 2. *Any monomial ideal I of depth 1 possesses a Stanley decomposition.*

Definition 6. *A monomial ideal I is called* involutively irreducible *if for any involutive division $\mathcal{M}$ and any $\mathcal{M}$-involutive basis U of I the maximal element u of U (w.r.t. the underlying order $\sqsubset$ of $\mathcal{M}$) satisfies $invdim_{\mathcal{M}}\, u \leq invdim\, I$.*

Let $b \in B$ be a minimal generator of I. For any involutive division $\mathcal{M}$ and any $\mathcal{M}$-involutive basis U of I such that b is the maximal element of the minimal generating set B (and, hence, also of the full involutive basis U) with respect to $\sqsubset$ we have $invdim_{\mathcal{M}}\, b \leq \dim A_b$, where $\dim A_b$ denotes the Krull-dimension of the quotient modulo the ideal $A_b := (B \setminus \{b\}) : (b)$ (see [A98, Theorem 3.1]). An immediate consequence is

$$invdim\, I \leq \max_{b \in B} \dim A_b \tag{3}$$

and equality holds if and only if I is involutively irreducible. Moreover, we have

$$invdim\, I \leq \max_{b \in B'} \dim A'_b \tag{4}$$

for each subset $B' \subseteq B$ containing at least two elements, where $A'_b := (B' \setminus \{b\}) : (b)$. This fact follows easily since the elements of B' have to be placed in some relative order by $\sqsubset$.

Lemma 3. *Any monomial ideal I can be decomposed in a direct sum*

$$I = J \oplus \left(\bigoplus_{i=1}^{t} u_i \mathbb{K}[Y_i] \right) , \tag{5}$$

where J is an involutively irreducible ideal, t is a non-negative integer, and $|Y_i| > invdim\, I$ for all $i = 1, \dots, t$.

Proof: For involutively irreducible I the setting $J := I$ and $t := 0$ fulfills the assertion.

So assume, that there exists an $\mathcal{M}$-involutive basis U of I such that $invdim_{\mathcal{M}}\, v \geq invdim\, I$ for all $v \in U$ and the inequality is strict for the maximal element $u = \max_{\sqsubset} U$. Recall that the maximal element of any $\mathcal{M}$-involutive basis U of I is always a minimal generator of I. I can be written as a direct sum $(U \setminus \{u\})R \oplus u\mathbb{K}[Y_u]$. If the ideal $(U \setminus \{u\})R$ is involutively irreducible we are done. Otherwise, we proceed to decompose $I' := (U \setminus \{u\})R$ in the above way.

There remain two questions. At first we have to show $invdim\, I' \geq invdim\, I$. But this is obvious, because $U \setminus \{u\}$ is an $\mathcal{M}$-involutive basis of J by $M_u \cap vT = \emptyset$ for all $v \in U \setminus \{u\}$. At second we have to show that the process terminates. However, all monomials contained in an involutive basis of I divide the least common multiple of the minimal generators of I. Therefore, the number of iterations is bounded by the number of monomials dividing this least common multiple. □

The ideals considered in Examples 2-4 are involutively irreducible.

Example 5. *Consider the ideal $I \subset \mathbb{K}[x, y, z, u]$ generated by the set $B = \{yx^2, y^2, xyz^2, u^2, xyzu, yz^2u\}$. While inequality (3) yields $invdim\, I \leq \dim\,(yx^2, y^2, xyz, yz^2) = 3$ we get the better bound $invdim\, I \leq \dim\,(x^2, xz^2, xzu, uz^2) = 2$ by application of inequality (4) to the subset $B' = \{yx^2, y^2, xyz^2, xyzu, yz^2u\}$. Hence, I is not involutively irreducible and by the algorithm in the proof of Lemma 3 we get after one step $I = (yx^2, y^2, xyz^2, yu^2, xyzu, yz^2u) \oplus u^2\mathbb{K}[x, z, u]$. Note, that the first summand is not simply the ideal generated by B' but that yu^2 had to be added to the generators. The best bound we can get using (3) and (4) for $J = (yx^2, y^2, xyz^2, yu^2, xyzu, yz^2u)$ is $invdim\, J \leq 2$*

Suppose, $invdim\, J = 2$. y^2 is the only candidate for the largest (w.r.t. $\sqsubset$) element of an involutive basis of J of involutive dimension 2. In addition, $J' = (yx^2, xy^2, uy^2, xyz^2, yu^2, xyzu, yz^2u)$ has to satisfy $invdim\, J' = 2$. But, $invdim\, J' \leq 1$ by (3), a contradiction.

Therefore, we have $invdim\, I = invdim\, J = 1$ and J is still not involutively irreducible. $I = (yx^2, xy^2, uy^2, xyz^2, yu^2, xyzu, yz^2u) \oplus y^2\mathbb{K}[y, z] \oplus u^2\mathbb{K}[x, z, u]$ is a final decomposition of type (5).

The importance of reverse lexicographical orders for the theory of involutive bases has been demonstrated in [A98]. But, in view of Lemma 1 reverse lexicographical orders have a serious drawback. They do not provide any control to put elements of high degree in a certain variable at top of an involutive basis. For this purpose we would prefer to use a pure lexicographical order $\sqsubset$. However, if a monoid well-order $\sqsubset$ of T is applied then admissible involutive divisions, and hence involutive bases, will not exist in most cases. We will introduce a special class of linear orders on finite monomial sets which carry the advantages and avoid the drawbacks of the above two order types.

Definition 7. *Let $U \subset T$ be a finite set of monomials and $\prec$ a monoid well-order of T. A linear order $\sqsubset$ on U is called* compatible with $\prec$ and U *iff it satisfies the following three conditions*

1. *$u \prec v \iff u \sqsubset v$ for all elements $u, v \in U$ such that U contains no proper divisor of either of them,*
2. *$u \sqsubset v$ for all proper divisors $v \in U$ of $u \in U$,*
3. *whenever $u \sqsubset v$ and $v \prec u$ holds for two elements $u, v \in U$ such that $v \nmid u$ then there exists $w \in U$ such that $w \mid u$, $w \prec v$, and $w \sqsubset v$.*

In what follows by using the symbol $\sqsubset_x$ for a linear order on a finite set $U \subset T$ we will indicate that this order is compatible with some pure lexicographical order (always denoted by $\prec_x$ in this context) with largest variable $x \in X$ and the set U. Note, according to this convention the ordered set $(V, \sqsubset_x)$ is not obtained as the restriction of the ordered set $(U, \sqsubset_x)$ for a proper subset V of U. But since the corresponding set will be always clear from the context we use this short cut symbol for the sake of simplicity. Such orders have the following easy to prove properties:

Remark 2. *Let U be a finite set of monomials and $\mathcal{M}$ an involutive division admissible on $(U, \sqsubset_x)$. If U is an $\mathcal{M}$-involutive basis of the monomial ideal $I = UR$ then the maximal element $u = \max_{\sqsubset_x} U$ belongs to the minimal basis of I (in fact, this holds for any involutive division) and has maximal degree in x among all elements of U. Moreover, $\mathrm{invdim}_{\mathcal{M}}\, v \geq 1$ for all $v \in U$.*

In particular, there exist involutive divisions $\mathcal{M}$ admissible on $(U, \sqsubset_x)$ where $x \in Y_v$ for all $v \in U$. These involutive divisions $\mathcal{M}$ have the nice property that for an arbitrary $\mathcal{M}$-involutive basis U the membership relation $xt \in U$ implies $t \notin I$ for any monomial $t \in T$.

For the proof of the last statement let us assume the contrary, i.e. $xt \in U$ and $t \in I$ for some $t \in T$. Then there exists $u \in U$ such that $t \in M_u$ and since $x \in Y_u$ it follows $xt \in M_u$ in contradiction to the assumption $xt \in U$.

Lemma 4. *Let I be an involutively irreducible monomial ideal $I \subset R$ of $\mathrm{invdim}\, I = 1$. If U is an $\mathcal{M}$-involutive basis of I, where $\mathcal{M}$ is admissible on $(U, \sqsubset_x)$, then the maximal variable $x = \max_{\prec_x} X$ is the only $\mathcal{M}$-multiplier of the maximal element $u = \max_{\sqsubset_x} U$. Moreover, for each variable $y \neq x$ there exists a y-witness v_y for u which is a minimal generator of I and has a strictly smaller degree in x than u.*

Proof: Suppose, $\deg_x v_y = \deg_x u$ for all minimal generators $v_y \in B$ which are a y-witness for u and consider an arbitrary y-witness $v_y \in B$ for u which has maximal degree in y.[1] Then no minimal generator $v \in B \setminus \{v_y\}$ can satisfy $\frac{\mathrm{lcm}(v_y, v)}{v_y} \in \langle x, y\rangle$. Consequently, $\{x, y\}$ is an algebraically independent set for the ideal quotient $A_{v_y} := (B \setminus \{v_y\}) : (v_y)$. Hence, $\dim A_{v_y} \geq 2$ in contradiction to I being involutively irreducible. $\square$

[1] Note, the assumption $u = \max_{\sqsubset_x} U$ is essential at this point since, otherwise, all y-witnesses may belong to $U \setminus B$.

Lemma 4 brings us close to Lemma 1. But there is still a gap, namely, we have not proved yet that there are enough strong witnesses for u.

Lemma 5. *Let $I \subset \mathbb{K}[x_1, \ldots, x_n]$ be an involutively irreducible monomial ideal. Further, let invdim $I = 1$ and U an $\mathcal{M}$-involutive basis of I, where $\mathcal{M}$ is admissible on $(U, \sqsubset_{x_1})$ and x_1 is $\mathcal{M}$-multiplier for all $v \in U$. Then U contains an element u whose only $\mathcal{M}$-multiplier is x_1 and which for each $i \in \{2, \ldots, n\}$ possesses a strong x_i-witness v_i satisfying $\deg_{x_1} u > \deg_{x_1} v_i$.*

Proof: Let $u \in U$ be the minimal (w.r.t. $\sqsubset_{x_1}$) element such that for all $i \in \{2, \ldots, n\}$ there exists a x_i-witness v_i for u which has the property $\deg_{x_1} u > \deg_{x_1} v_i$. In addition, for each $i \in \{2, \ldots, n\}$ let the x_i-degree of the witness v_i be minimal among all possible choices. According to Lemma 4 such elements u and v_i exist.

Suppose, $k := \deg_{x_i} v_i - \deg_{x_i} u > 1$ for some $i \in \{2, \ldots, n\}$ and consider the element $w \in U$ such that $ux_i^{k-1} \in M_w$. From $M_w \cap uT \neq \emptyset$ we obtain $w \sqsubset u$. Further, we deduce $w \nmid u$ and $\deg_{x_i} v_i > \deg_{x_i} w > \deg_{x_i} u$. We must have $\deg_{x_1} w = \deg_{x_1} u$ since in case $\deg_{x_1} w < \deg_{x_1} u$ one observes easily that w would be a x_i-witness of u which contradicts the choice of v_i.

Next, we will show $v_j \sqsubset_{x_1} w$ for all $j \in \{2, \ldots, n\}$. So, let us assume the contrary, i.e. $w \sqsubset_{x_1} v_j$ for some $j \in \{2, \ldots, n\}$. Since $v_j \prec_{x_1} w$ with respect to the pure lexicographical order $\prec_{x_1}$ there must exist an element w' such that $w' \mid w$ and $w' \prec_{x_1} v_j$. In particular, $\deg_{x_1} w' \leq \deg_{x_1} v_j < \deg_{x_1} w$. Now, $\deg_{x_i} w' > \deg_{x_i} u$ is impossible since it would contradict to the choice of v_i. But $\deg_{x_i} w' \leq \deg_{x_i} u$ is impossible as well since it will follow $w' \mid u$ in contradiction to $w' \sqsubset_{x_1} u$. In conclusion $v_j \sqsubset_{x_1} w$ for all $j \in \{2, \ldots, n\}$.

Therefore, by the admissibility of $\mathcal{M}$ we must have $M_w \cap v_jT = \emptyset$ for all $j = 2, \ldots, n$. In particular, no variable x_j, $j = 2, \ldots, n$, can be an $\mathcal{M}$-multiplier of w since otherwise, $ux_i^{k-1}x_j^l \in M_w \cap v_jT$, where l is the positive integer originating from the equation $\mathrm{lcm}(u, v_j) = ux_j^l$. Since $ux_i^{k-1} \in M_w$ requires $\deg_{x_j} u = \deg_{x_j} w$ for all $\mathcal{M}$-nonmultipliers $x_j \neq x_i$ of w, we can deduce that all v_j, $j = 2, \ldots, n$, are also x_j-witnesses for w in contradiction to the choice of u.

In conclusion, we proved $\deg_{x_i} v_i = \deg_{x_i} u + 1$ for all $i \in \{2, \ldots, n\}$ for the chosen u and v_i, $i \in \{2, \ldots, n\}$. □

Corollary 3. *Any monomial ideal $I \subset \mathbb{K}[X]$ of depth $I \geq 2$ has at least involutive dimension 2.*

Proof: Assume the contrary, i.e. *invdim* $I = 1$ for some ideal of depth at least 2. Decompose I in a direct sum $I = J \oplus \left(\bigoplus_{i=1}^t u_i\mathbb{K}[Y_i]\right)$ according to Lemma 3 and apply Lemma 5 to the involutively irreducible part J. This yields an $\mathcal{N}$-involutive basis V of J satisfying the assumptions of Lemma 1. There is an obvious involutive division $\mathcal{M}$ such that the union $U = V \cup \{u_1, \ldots, u_t\}$ becomes an $\mathcal{M}$-involutive basis of the ideal I which satisfies the assumptions of Lemma 1, too. Hence, by Corollary 1 we deduce *depth* $I \leq 1$ in contradiction to our assumptions on I. □

5. The 3-variate Case

Applying our results from the previous section it is now an easy exercise to prove Conjecture 2 for the first non-trivial case $n = 3$.

Theorem 1. *Let $R = \mathbb{K}[x_1, x_2, x_3]$ be a three-variate polynomial ring over a field $\mathbb{K}$. Then for any non-zero monomial ideal $I \subset R$ we have*

$$Sdim\, I \geq invdim\, I \geq depth\, I =: d\ ,$$

i.e. there exist an involutive division $\mathcal{M}$ generated by a family $Y = (Y_t)_{t \in T}$ and an $\mathcal{M}$-involutive basis U of I such that $|Y_u| \geq d$ for all $u \in U$. In particular, $I = \bigoplus_{u \in U} u\mathbb{K}[Y_u]$ is a Stanley decomposition of I.

Proof: The only possible values for d are 1, 2 or 3. For $d = 1$ and $d = 2$ the assertion follows from Corollary 2 and Corollary 3, respectively. The case $d = 3$ is trivial, since I has to be a principal ideal for which each $\mathcal{M}$-involutive basis is a singleton. □

For the most interesting case of monomial ideals I in three variables, i.e. $depth\, I = 2$, we only proved the existence of a "good" involutive decomposition. But since the given proof is indirect and not at all constructive, unless we have further a-priori information we are left with the brute force algorithm which consists in applying Algorithm [A98, Fig. 4] using different strategies and orders until an involutive basis of involutive dimension greater or equal 2 or allowing the application of Corollary 1 is found. At least it is clear that such a method will terminate since the number of minimal involutive bases of I is finite.

6. Generic Monomial Ideals and More

Definition 8. *Let $I \subset \mathbb{K}[X]$ be a monomial ideal and $\{m_1, \ldots, m_l\} \subset T$ its set of minimal generators.*

I is called generic *if the following condition holds: if two distinct minimal generators m_i and m_j have the same positive degree in some variable $x \in X$, there is a third minimal generator m_k which strictly divides* $\mathrm{lcm}(m_i, m_j)$*, i.e.* $\deg_y m_k \leq \deg_y \mathrm{lcm}(m_i, m_j)$ *for all $y \in X$ and the inequality is strict for all variables y such that* $\deg_y \mathrm{lcm}(m_i, m_j) > 0$.

If no two distinct minimal generators m_i and m_j, $1 \leq i < j \leq k$, have the same positive degree in any variable $x \in X$, then I is called strongly generic.

Here, we follow the notion of generic monomial ideals due to Miller, Sturmfels and Yanagawa ([MSY]) and recall that it generalizes that of Bayer, Peeva and Sturmfels ([BPS]), which corresponds to the strongly generic case. Typically, the so-defined generic monomial ideals are far from being (their own) generic initial ideals (see [E]). The latter are covered by our Example 1.

$\mathcal{M}$-involutive bases U of strongly generic monomial ideals with respect to the involutive divisions introduced in Remark 2, i.e. $\mathcal{M}$ is admissible on $(U, \sqsubset_x)$ and $x \in Y_u$ for all $u \in U$, have the particular nice property that by looking at the x-degree of an element $u \in U$ one can determine its "origin", i.e. the minimal generator m_i

of I from which u was obtained by a series of nonmultiplicative prolongations (see [A98]). This origin m_i is the (uniquely determined) minimal generator of I which has the same degree in x as u.

Lemma 6. *Let I be a monomial ideal whose minimal generators have pairwise distinct degrees in x. Further, let U be an $\mathcal{M}$-involutive basis of I, where $\mathcal{M}$ is admissible on $(U, \sqsubset_x)$ and $x \in Y_v$ for all $v \in U$. Moreover, let $u \in U$ be minimal with respect to $\sqsubset_x$ such that $invdim_{\mathcal{M}}\, u = invdim_{\mathcal{M}}\, U$. Then u possesses a strong set of witness $W(u)$, where $\deg_x u > \deg_x v_y$ for all $v_y \in W(u)$.*

Proof: It is impossible that all y-witnesses v for u satisfy $\deg_x u \leq \deg_x v$ for some $\mathcal{M}$-nonmultiplier y of u since in this case they had to originate all from minimal generators of I which are larger or equal (w.r.t. $\sqsubset_x$) than m_i, where m_i is the minimal generator of the same x-degree as u. We leave to the reader the simple but tedious proof that this would contradict the maximality of $\mathcal{M}$ or the minimality of U.

Hence, for each $\mathcal{M}$-nonmultiplier y of u there exists a y-witness v_y for u such that $\deg_x u > \deg_x v_y$. What remains to show is that there is even a strong one. Let $W(u)$ be a set of witness for u which for each $\mathcal{M}$-nonmultiplier y of u contains a y-witness v_y for u satisfying $\deg_x u > \deg_x v_y$ and having minimal y-degree among all these y-witnesses for u.

Suppose, $k := \deg_y v_y - \deg_y u > 1$ for some $v_y \in W(u)$. Let $t \in \langle Y_u \rangle$ be an arbitrary monomial such that $\deg_\xi ut \geq \deg_\xi v_z$ for all $\mathcal{M}$-multipliers $\xi \in Y_u$ of u and all witnesses $v_z \in W(u)$ and consider $w \in U$ such that $uty^{k-1} \in M_w$. From $M_w \cap uT \neq \emptyset$ we deduce $w \sqsubset_x u$. Moreover, $\deg_y w > \deg_y u$ since, otherwise, $M_u \cap wT \neq \emptyset$. From the choice of v_y it will follow $\deg_x w = \deg_x u$. The above properties refute the possibility $w \mid u$ but, next, we will observe that $w \nmid u$ is impossible, too.

Suppose $w \sqsubset_x v_z$ for some $\mathcal{M}$-nonmultiplier z of u. From $\deg_x w > \deg_x v_z$ it follows $v_z \prec_x w$. Hence, there exists $w' \in U$ such that $w' \mid w$, $w' \sqsubset_x v_z$ and $w' \prec_x v_z$. In particular, we have $\deg_x w' \leq \deg_x v_z < \deg_x u$. $w' \mid u$ is impossible because of $w' \sqsubset_x v_z \sqsubset_x u$, $u \mid w'$ is impossible by comparison of x-degrees. But also $u \nmid w' \wedge w' \nmid u$ is impossible since by the choice of v_y the monomial w' cannot be a y-witness for u. Now, suppose $v_z \sqsubset_x w$ for all $\mathcal{M}$-nonmultipliers z of u. No $\mathcal{M}$-nonmultiplier z of u can be an $\mathcal{M}$-multiplier of w since, otherwise, we had the contradiction $M_w \cap v_z T \neq \emptyset$. Therefore, $invdim_{\mathcal{M}}\, w \leq invdim_{\mathcal{M}}\, u$ and $w \sqsubset_x u$ in contradiction to the construction of u.

In conclusion, we proved $\deg_y v_y - \deg_y u = 1$ for all $v_y \in W(u)$. This completes the proof. □

Corollary 4. *Any monomial ideal $I \subset \mathbb{K}[X]$ whose minimal generators have pairwise distinct degrees in some variable $x \in X$ satisfies $depth\, I \leq invdim\, I \leq Sdim\, I$.*

The assertion follows immediately from Corollary 1 and Lemma 6. □

Obviously, Corollary 4 applies to any strongly generic monomial ideal I. Now, we will generalize the results in such a way that all generic monomial ideals are covered.

Lemma 7. *Let U be an $\mathcal{M}$-involutive basis of the monomial ideal I, where $\mathcal{M}$ is admissible on $(U, \sqsubset_x)$ and $x \in Y_v$ for all $v \in U$. Furthermore, assume that I has the following property: for any two distinct minimal generators m_i and m_j such that $\deg_x m_i = \deg_x m_j =: d$ there is a third minimal generator m_k which divides $\operatorname{lcm}(m_i, m_j)$ and satisfies $\deg_x m_k < d$.*

Then the minimal element $u \in U$ with respect to $\sqsubset_x$ which satisfies $invdim_{\mathcal{M}}\, u = invdim_{\mathcal{M}}\, U$ possesses a strong set of witness $W(u)$ such that $\deg_x u > \deg_x v_y$ for all $v_y \in W(u)$.

Proof: It suffices to disprove the possibility that for some $\mathcal{M}$-nonmultiplier y of u all y-witnesses v_y for u have x-degree larger or equal than u. The rest of the proof will be identical to that of Lemma 6.

If all y-witnesses would originate from minimal generators larger or equal (w.r.t. $\sqsubset_x$) than the origin m_i of u then we had a contradiction to the maximality of $\mathcal{M}$ or the minimality of U. So, suppose all y-witnesses of u have the same x-degree as u and consider an arbitrary y-witness v_y for u such that $m_j \mid v_y$ and $\deg_x m_i = \deg_x m_j = \deg_x u$ for some minimal generator $m_j \sqsubset_x m_i$ of I. By the assumptions on I it follows the existence of a minimal generator m_k such that $m_k \mid \operatorname{lcm}(m_i, m_j)$ and $\deg_x m_k < \deg_x u$. Suppose, $u \sqsubset_x m_k$. Since, $m_k \prec_x u$ there must exist $v \in U$ such that $v \prec_x m_k$ and $v \mid u$. Therefore, $\frac{u}{x} \in I$ and since x is an $\mathcal{M}$-multiplier for any element of U it follows $u \in M_w$, where $w \in U$ is such that $\frac{u}{x} \in M_w$. This contradicts the minimality of U, hence, we can deduce $m_k \sqsubset_x u$. Since $\deg_z \operatorname{lcm}(m_i, m_j) \leq \deg_z \operatorname{lcm}(u, v_y)$ for all $z \in X$ we have also $\deg_z m_k \leq \deg_z u$ for all $\mathcal{M}$-nonmultipliers $z \neq y$ of u. From $m_k \sqsubset_x u$ we obtain $M_u \cap m_k T = \emptyset$ and, therefore, $\deg_y m_k > \deg_y u$. Hence, m_k is a y-witness of u which has smaller x-degree than u and we are through. □

Theorem 2. *Let $I \subset \mathbb{K}[X]$ be a monomial ideal. Furthermore, assume that for some variable $x \in X$ the ideal I has the following property: for any two distinct minimal generators m_i and m_j such that $\deg_x m_i = \deg_x m_j =: d$ there is a third minimal generator m_k which divides $\operatorname{lcm}(m_i, m_j)$ and satisfies $\deg_x m_k < d$. Then it holds $depth\, I \leq invdim\, I \leq Sdim\, I$.*

The proof is obvious. This theorem generalizes Corollary 4 and its statement applies, in particular, to all generic monomial ideals. Let us draw the attention to an amazing fact. We never had to assume that U has maximal involutive dimension. Hence, any $\mathcal{M}$-involutive basis U of the type introduced in Remark 2 provides a Stanley decomposition of I. So, in spite our purely existential proofs, we are in a very comfortable position when the algorithmic construction of Stanley decompositions is concerned. We just need to run Algorithm [A98, Fig. 4] by using only involutive divisions of the type introduced in Remark 2. Note, Example 2 shows that we avoid at least the "worst" involutive bases by this restriction of the class of involutive divisions. But we should be aware also, sometimes to loose the "best" involutive bases by the same restriction.

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