ON A CONJECTURE OF R. P. STANLEY
PART II – QUOTIENTS MODULO MONOMIAL IDEALS

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Abstract. In 1982 Richard P. Stanley conjectured that any finitely generated $\mathbb{Z}^n$-graded module M over a finitely generated $\mathbb{N}^n$-graded $\mathbb{K}$-algebra R can be decomposed in a direct sum $M = \bigoplus_{i=1}^{t} \nu_i S_i$ of finitely many free modules $\nu_i S_i$ which have to satisfy some additional conditions. Besides homogeneity conditions the most important restriction is that the S_i have to be subalgebras of R of dimension at least *depth* M.

We will study this conjecture for modules $M = R/I$, where R is a polynomial ring and I a monomial ideal. We will present a necessary and sufficient condition for squarefree monomial Cohen-Macaulay rings to fulfill Stanley's Conjecture. Moreover, we prove that Stanley's Conjecture holds for the quotient modulo any generic monomial ideal, the quotient modulo any monomial ideal in at most three variables, and for any cogeneric Cohen-Macaulay ring. The presented theorems are constructive and allow to extract explicit algorithms for the computation of Stanley decompositions. Further, we will discuss some relationships between our algebraic results and combinatorial properties of related complexes. Finally, we will give an outlook to Stanley decompositions of arbitrary graded polynomial modules. In particular, we obtain a more general result in the 3-variate case.

1. Introduction

This is the second of two articles studying some aspects of a conjecture formulated by Richard P. Stanley in 1982. For the reader's convenience we repeat some parts of the introduction of Part I (see [A]).

Conjecture 1 ([St82], 5.1). *Let R be a finitely-generated $\mathbb{N}^n$-graded $\mathbb{K}$-algebra (where $R_0 = \mathbb{K}$ as usual), and let M be a finitely-generated $\mathbb{Z}^n$-graded R-module. Then there exist finitely many subalgebras $S_1, \ldots, S_t$ of R, each generated by algebraically independent $\mathbb{N}^n$-homogeneous elements of R, and there exist $\mathbb{Z}^n$-homogeneous elements $\nu_1, \ldots, \nu_t$ of M, such that*

$$M = \bigoplus_{i=1}^{t} \nu_i S_i,$$

where $\dim S_i \geq$ depth M for all i, and where $\nu_i S_i$ is a free S_i-module (of rank one). Moreover, if $\mathbb{K}$ is infinite and under a given specialization to an $\mathbb{N}$-grading of R is generated by R_1, then we can choose the ($\mathbb{N}^n$-homogeneous) generators of each S_i to lie in R_1.

Research at MSRI is supported in part by NSF grant DMS-9701755. Further, the author was supported by a research fellowship of the Deutsche Forschungsgemeinschaft (DFG).

In [A] this conjecture could be proved to hold for a series of cases where M is an ideal of a polynomial ring. In particular, the conjecture was shown for ideals in rings with at most three variables and for monomial ideals, which are generic with respect to one variable. In this paper we will study the case $M = \mathbb{K}[X]/I$ with an emphasize on Cohen-Macaulay rings. The verification of Conjecture 1 for Cohen-Macaulay modules would contribute to answers of further combinatorial problems, in particular, it would prove that Cohen-Macaulay complexes are partitionable (see [St79]).

Definition 1. *Let R be a finitely-generated $\mathbb{N}^n$-graded $\mathbb{K}$-algebra and let M be a finitely-generated $\mathbb{Z}^n$-graded R-module. For each nonnegative integer d let Δ_d denote the set of all direct sum decompositions $M = \bigoplus_{i=1}^{t} \nu_i S_i$ of M in finitely many free S_i-modules $\nu_i S_i \quad (i \in \{1, \ldots, t\})$, where the $S_1, \ldots, S_t$ are subalgebras of R each of them generated by at least d algebraically independent $\mathbb{N}^n$-homogeneous elements of R and the $\nu_1, \ldots, \nu_t$ are $\mathbb{Z}^n$-homogeneous elements of M.*

If not all sets Δ_d are empty then we call the largest d such that $\Delta_d \neq \emptyset$ the Stanley dimension *of M and denote it by $Sdim\, M$.*

For $\mathbb{N}^n$-graded polynomial rings $R = \mathbb{K}[x_1, \ldots, x_m]$ the existence of $Sdim\, M$ follows e.g. from results of Riquier related to the solution space of systems of partial differential equations ([Ri], see also [T]). In the context of the special case studied in this paper Stanley's Conjecture reads as follows.

Conjecture 2. *Let $R = \mathbb{K}[X]$ be a polynomial ring in the variables $X = \{x_1, \ldots, x_n\}$ over a field $\mathbb{K}$. Then for any monomial ideal $I \subset R$ holds*

$$Sdim\, R/I \geq depth\, R/I \; . \tag{1}$$

That is, there exists a finite decomposition (called Stanley decomposition *throughout this paper) of R/I of the following type*

$$R/I = \bigoplus_{i=1}^{t} u_i \mathbb{K}[Z_i], \tag{2}$$

where the u_i are residue classes of (monic) monomials modulo I, $Z_i \subseteq X$ and $|Z_i| \geq depth(XR, R/I) = depth\, R/I$ for all $i = 1, \ldots, t$.

In this paper we will ask for classes of modules $M = R/I$ for which there exist decompositions (2) satisfying the condition $|Z_i| \geq \min_{\mathfrak{p} \in \operatorname{Ass} M} \dim R/\mathfrak{p}$. In the most interesting case of Cohen-Macaulay rings M this is exactly Condition (1). However, in general, Condition (1) is weaker and it is not surprising that we will meet ideals, see e.g. Example 1, where our condition fails. After declaration of some notions and notations in Section 2 and deriving a trivial upper bound on the Stanley dimension in Section 3 we will start our considerations with quotients modulo squarefree ideals in Section 4. We will give a sufficient condition for squarefree monomial ideals I such that $M = R/I$ possesses a decompositions (2) satisfying $|Z_i| \geq \min_{\mathfrak{p} \in \operatorname{Ass} M} \dim R/\mathfrak{p}$. In the case of a pure dimensional I this condition is also necessary (Proposition 2). That reduces (or better reformulates) Stanley's conjecture for squarefree Cohen-Macaulay rings to our condition. Section 5 deals with nonsquarefree ideals. We will prove the inequality $Sdim\, R/I \leq Sdim\, R/\sqrt{I}$ for arbitrary monomial ideals (Theorem 1). Moreover, we will generalize Proposition 2 to the nonsquarefree case (Proposition 3 and Theorem 2). However, here we will obtain only sufficient conditions.

In particular, we will show that Stanley's Conjecture holds for the quotient R/I for any generic monomial ideal (Section 6). Moreover, we will prove Stanley's Conjecture for the case of at most three variables (Section 7) and for all cogeneric Cohen-Macaulay rings (Section 8). All presented propositions and theorems contain (at least indirect) also an algorithm for the construction of Stanley decompositions for ideals satisfying the assumptions. However often, the derived algorithms will be impractical by the dependence on a suitable enumeration of the primary components. But in the cases of generic, 3-variate, and cogeneric Cohen-Macaulay ideals the corresponding theorems provide explicite information about a suitable arrangement of the primary components. This leads to efficient algorithms for the construction of Stanley decompositions in these cases.

In Section 9 we will relate our results to combinatorial properties of the Stanley-Reisner complex and the Scarf complex corresonding to a monomial ideal. Finally, we will give an outlook to the general case of arbitrary graded polynomial modules in Section 10 and generalize our results in the 3-variate case.

2. Notation

First we will introduce some notions and notations used throughout this paper. By writing $A \subset B$ we mean that A is a (not necessarily nonempty) subset of B but $A \neq B$. $A \subseteq B$ stands for $A = B$ or $A \subset B$. $|A|$ denotes the number of elements of a finite set A.

A term $x_1^{i_1} \dots x_n^{i_n}$, i.e. a monomial with coefficient 1, is called a *monic* monomial. For a monic monomial $t = x_1^{i_1} \dots x_n^{i_n}$ we introduce the notation $\operatorname{supp} t = \{x_j \ : \ 1 \leq j \leq n \ , \ i_j > 0\}$ for the *Support* of t. Since we will need to work only with monic monomials throughout this paper, for the sake of simplicity, we will use the short cut *monomial* for *monic monomial*.

By a *minimal monomial* t of a set $A \subset \mathbb{K}[x_1, \dots, x_n]$ we will always mean that t is a monic monomial contained in A and that $\frac{t}{x} \notin A$ for all $x \in \operatorname{supp} t$. So a minimal generator t of a monomial ideal I is a monic monomial such that t but no proper divisor of t belongs to I.

Consider a nonzero monomial ideal $I \subset R = \mathbb{K}[x_1, \dots, x_n]$ with irredundant primary decomposition $I = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \dots \cap \mathfrak{q}_k$, that is, the intersection of any all but one of the primary ideals is strictly larger than I. Note, frequently the additional assumption that all primary ideals have to be maximal, i.e. the associated prime ideals $\mathfrak{p}_i$ of the primary components $\mathfrak{q}_i$ are pairwise distinct, is subsumed by the notion irredundant primary decomposition. We emphasize that we do not make this additional assumption, unless in places where it is explicitly stated, since for monomial ideals it is often much more convenient to work with the (upto the order of components uniquely determined) irredundant decomposition in irreducible ideals, i.e. primary ideals generated only by powers of variables.

Let Ass $M = \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$, where $\mathfrak{p}_i$ belongs to the primary ideal $\mathfrak{q}_i$, denote the set of associated primes of $M = R/I$. $\mathfrak{q}_i$ is called an *embedded component* if there exists $j \neq i$ such that $\mathfrak{p}_j \subset \mathfrak{p}_i$. By a *minimal associated prime* $\mathfrak{p}_i$ we mean that $\mathfrak{p}_j \not\subset \mathfrak{p}_i$ for all $\mathfrak{p}_j \in$ Ass M.

A set of variables $Y \subseteq X$ is called an *algebraically independent set* of I iff $I \cap \mathbb{K}[Y] = \{0\}$. If, in addition, no set of variables properly containing Y is an independent set of I then Y is called a *maximal algebraically independent set* of I. A monomial prime ideal[1] $\mathfrak{p}$ possesses exactly one maximal algebraically independent set, namely the set $Y = \{x \in X \mid x \notin \mathfrak{p}\}$. Any $\mathfrak{p}$-primary ideal $\mathfrak{q}$ has the same uniquely determined maximal algebraically independent set Y as $\mathfrak{p}$, $|Y|$ is called the (Krull-)dimension of $R/\mathfrak{p}$ and denoted by $\dim R/\mathfrak{p}$. Throughout this paper the notion Y_i will always refer to the maximal algebraically independent set of the associated prime $\mathfrak{p}_i$ corresponding to the primary component $\mathfrak{q}_i$ of I, $1 \leq i \leq k$. A set Y is algebraically independent for I iff it is algebraically independent for some associated prime ideal of M and it is maximal algebraically independent for I iff it is maximal algebraically independent for some minimal associated prime ideal of M. Accordingly, the dimension of M is defined by $\dim M = \max_{\mathfrak{p}_i \in \text{Ass } M} \dim R/\mathfrak{p}_i$. In this paper we will need to refer frequently to $\min_{\mathfrak{p}_i \in \text{Ass } M} \dim R/\mathfrak{p}_i$ which is an upper bound for the depth of M (c.f. [E, SV]). An ideal I satisfying $\dim M = \min_{\mathfrak{p}_i \in \text{Ass } M} \dim R/\mathfrak{p}_i$ is called *pure dimensional* or simply *pure*. Cohen-Macaulay rings M are defined by the condition $\dim M = \mathit{depth}\, M$ in which case we call I a Cohen-Macaulay ideal. From the previous observations it follows immediately that any Cohen-Macaulay ideal is pure.

Finally, we will need the notions of (strongly) generic and cogeneric monomial ideals introduced by Bayer, Peeva and Sturmfels and generalized by Miller, Sturmfels and Yanagawa. A monomial ideal I is called *strongly generic* if no two distinct minimal generators m and m' of I have the same positive degree in any variable $x \in X$ (see [BPS]). If the monomial ideal I satisfies only the weaker assumption that for any two distinct minimal generators m and m' which have the same degree in some variable $x \in X$, there is a third minimal generator m'' which strictly divides $\text{lcm}(m, m')$, i.e. $\text{supp lcm}(m, m') = \text{supp } \frac{\text{lcm}(m,m')}{m''}$, then I is called *generic* (see [MSY]). Generic monomial ideals I allow a simple characterization for R/I being a Cohen-Macaulay ring, which is the case iff I is pure dimensional iff I has no embedded primary components (c.f. [MSY, Theorem 2.5]). If I is (strongly) generic and $\mathfrak{p}$ is an arbitrary monomial prime ideal then the ideal $I_{(\mathfrak{p})}$ obtained by localization of I at $\mathfrak{p}$ is (strongly) generic, too (c.f. [MSY, Remark 2.1]). If the Alexander dual I^a of the monomial ideal I is (strongly) generic then I is called *(strongly) cogeneric*. That is, I is cogeneric if whenever two distinct irreducible components $\mathfrak{q}_i$ and $\mathfrak{q}_j$ of I have a minimal generator in common then there exists an irreducible component $\mathfrak{q}_m$ of I which is contained in the ideal sum $\mathfrak{q}_i + \mathfrak{q}_j$ and has no minimal generator in common with this sum. Various characterizations of cogeneric Cohen-Macaulay monomial ideals were given by Miller, Sturmfels and Yanagawa (see [MSY, Theorem 4.9]). In this paper we will make use of one of them which in view of Alexander duality can be formulated as:

[1] Actually this implies that all minimal generators of $\mathfrak{p}$ are variables.

Lemma 1 ([MSY] Theorem 4.9 c). *A cogeneric monomial ideal I with irredundant decomposition $I = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_k$ in irreducible components is Cohen-Macaulay iff I is pure dimensional and for any irreducible components $\mathfrak{q}_i$ and $\mathfrak{q}_j$ such that* $\operatorname{codim}(\mathfrak{p}_i + \mathfrak{p}_j) > \operatorname{codim} I + 1$ *there exists an irreducible component $\mathfrak{q}_l$ of I, where $l \notin \{i, j\}$, satisfying $\mathfrak{q}_l \subset \mathfrak{q}_i + \mathfrak{q}_j$.*

It is easy to observe that we have the following analogue to the generic case. If I is (strongly) cogeneric and $\mathfrak{p}$ is an arbitrary monomial prime ideal then the ideal $I_{(\mathfrak{p})}$ obtained by localization of I at $\mathfrak{p}$ is (strongly) cogeneric, too.

3. An Upper Bound for *Sdim* M

Let $m_1, \ldots, m_l$ denote the minimal generators of I. We introduce the notion $S = \{u \in T \mid \forall i \in \{1, \ldots, l\} : m_i \nmid u\}$ for the set of all standard monomials modulo I. Recall that the set of minimal generators is the reduced Gröbner basis of I with respect to any admissible term order $\prec$. Furthermore, the residue classes of S modulo I form a $\mathbb{K}$-vector space basis of M. In the following we will freely identify M and $\operatorname{spann}_{\mathbb{K}} S$. Decomposition (2) has to satisfy $u_i v \in S$ for all monic monomials $v \in \mathbb{K}[Z_i]$, $i = 1, \ldots, t$. In particular, for arbitrary $i \in \{1, \ldots, t\}$ and $u \in u_i\mathbb{K}[Z_i]$ the set Z_i has to be algebraically independent for the ideal quotient $I : (u)$ and, consequently,

$$Sdim\, M \leq \min_{u \in S} \dim R/(I : (u)) = \min_{\mathfrak{p}_i \in \operatorname{Ass} M} \dim R/\mathfrak{p}_i \tag{3}$$

which shows that *depth* M and *Sdim* M share the upper bound at the right hand side. Let us start with a simple example illustrating that the above inequality can be strict.

Example 1. *Consider the ideal $I = (xz, yz, xu, yu) = (x, y) \cap (z, u) \subset \mathbb{K}[x, y, z, u]$. There are two possibilities for two-dimensional subalgebras of $\mathbb{K}[x, y, z, u]$ which can appear in a direct summand containing the monomial* 1*, namely either $\mathbb{K}[z, u]$ or $\mathbb{K}[x, y]$. Choosing $\mathbb{K}[z, u]$ there will be no way to find two-dimensional subalgebras for the direct summands containing the standard monomials x and y. Avoiding this problem by choosing $\mathbb{K}[x, y]$ will only shift the problem to the standard monomials z and u. A best decomposition (2), i.e. one with $|Z_i| = Sdim\, R/I$, is $R/I = \mathbb{K}[x, y] \oplus z\mathbb{K}[z, u] \oplus u\mathbb{K}[u]$. But, this is already a Stanley decomposition since* $\operatorname{depth} R/I = 1$.

From $I : (u) = \bigcap_{i=1}^{k} (\mathfrak{q}_i : (u))$ and $\mathfrak{q}_i : (u) = \begin{cases} \mathfrak{q}_i : & \text{if } u \notin \mathfrak{p}_i \\ \mathfrak{q}_i' : & \text{if } u \in \mathfrak{p}_i \setminus \mathfrak{q}_i \\ R : & \text{if } u \in \mathfrak{q}_i \end{cases}$, where $\mathfrak{q}_i'$ is a $\mathfrak{p}_i$-primary ideal properly containing $\mathfrak{q}_i$, we deduce that the maximal algebraically independent sets of $I : (u)$ are exactly the maximal algebraically independent sets Y_i of the associated primes $\mathfrak{p}_i \in$ Ass R/I which satisfy $u \in \left(\bigcap_{\mathfrak{p}_j \subset \mathfrak{p}_i} \mathfrak{q}_j\right) \setminus \mathfrak{q}_i$.

Now, we can describe the problem we met in Example 1 as follows. Let I be a pure dimensional monomial ideal with irredundant decomposition $I = \mathfrak{q}_1 \cap \mathfrak{q}_2$ in maximal primary ideals. Assume, there are two elements $s, t \in \mathfrak{q}_1 \setminus \mathfrak{q}_2$ and two elements $s', t' \in \mathfrak{q}_2 \setminus \mathfrak{q}_1$ such that $\gcd(s, t) = \gcd(s', t') =: m$. Then it follows $m \notin \mathfrak{q}_1 \cup \mathfrak{q}_2$. We will show that under certain conditions we have $Sdim\, R/I < \dim R/I$. In the case $\frac{\mathrm{lcm}(s,t)}{\mathrm{gcd}(s,t)} \in \langle Y_2 \rangle$ it will follow $s\mathbb{K}[Y_2] \cap t\mathbb{K}[Y_2] \neq \{0\}$. Hence, in case $Sdim\, R/I = \dim R/I$ the Stanley decomposition must contain a direct summand $u\mathbb{K}[Y_2]$ such that $s, t \in u\mathbb{K}[Y_2]$. In particular, also $m \in u\mathbb{K}[Y_2]$. Now suppose, we have also $\frac{\mathrm{lcm}(s',t')}{\mathrm{gcd}(s',t')} \in \langle Y_1 \rangle$. Then application of the same arguments shows that the Stanley decomposition must contain a second direct summand $v\mathbb{K}[Y_1]$ such that $s', t', m \in v\mathbb{K}[Y_1]$. Obviously, this is impossible simultaneously.

In some sense this is the typical scenario for monomial ideals I with the property $Sdim\, R/I < \min_{\mathfrak{p}_i \in \mathrm{Ass}\, M} \dim R/\mathfrak{p}_i$. This behavior is always caused by contradicting requirements posed by a set of standard monomials modulo I to the direct summand which should contain their greatest common divisor.

4. The Squarefree Case

Let us first consider the case that the monomial ideal I is squarefree. Then all primary components $\mathfrak{q}_i$, $i = 1, \ldots, k$, of I are even prime, i.e. $\mathfrak{q}_i = \mathfrak{p}_i$, and I has no embedded components. In the pure dimensional case we have the following nice relationship.

Lemma 2. *Let $I \subset R = \mathbb{K}[X]$ be a pure squarefree monomial ideal such that $Sdim\, M = \dim M$, where $M = R/I$. Then the direct summands of an arbitrary Stanley decomposition are in a natural bijection with the associated primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ of M, more precisely, each Stanley decomposition of M has the form*

$$M = \bigoplus_{i=1}^{k} t_i\ \mathbb{K}[Y_i]\ . \tag{4}$$

Proof: Consider an arbitrary Stanley decomposition

$$M = \bigoplus_{i=1}^{m} t_i\ \mathbb{K}[Z_i].$$

From $Sdim\, M = \dim M$ it follows $|Z_i| \geq \dim M$ and in view of purity we obtain $Z_i \in \{Y_1, \ldots, Y_k\}$ for all $i = 1, \ldots, m$. Let $1 \leq j \leq k$ and consider a monomial $t \in M$ which belongs to all but the j-th associated prime of M. Furthermore, let i be such that $t \in t_i\ \mathbb{K}[Z_i]$. Then we must have $Z_i = Y_j$ since, otherwise, $t\mathbb{K}[Z_i] \cap I \neq \{0\}$. Hence, for each associated prime $\mathfrak{p}_j$ the Stanley decomposition contains at least one direct summand such that $Z_i = Y_j$.

Now, consider two direct summands $t_i\mathbb{K}[Z_i]$ and $t_l\mathbb{K}[Z_l]$ satisfying $Z_i = Z_l = Y_j$ for some $j \in \{1, \ldots, k\}$. It follows $\mathrm{supp}\ t_i \subseteq Y_j$ and $\mathrm{supp}\ t_l \subseteq Y_j$ since, otherwise, $(t_i\mathbb{K}[Z_i] + t_l\mathbb{K}[Z_l]) \cap I \neq \{0\}$. Hence, $\mathrm{lcm}(t_i, t_l) \in t_i\mathbb{K}[Z_i] \cap t_l\mathbb{K}[Z_l]$ which implies $i = l$ since the sum is direct. □

Proposition 1. *Let $I \subset R$ be a squarefree monomial ideal. If the associated primes of the module $M = R/I$ can be ordered in such a way that each set $\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_{i-1} \setminus \mathfrak{p}_i$ contains exactly one minimal monomial t_i, $i = 2, \ldots, k$, then it holds $Sdim\, M =$*

$\min_{p_i \in \operatorname{Ass} M} \dim R/\mathfrak{p}_i$. *Moreover, setting $t_1 = 1$ we obtain the Stanley decomposition*

$$M = \bigoplus_{i=1}^{k} t_i \, \mathbb{K}[Y_i] \, . \tag{5}$$

Proof: First, we want to show $t_i\mathbb{K}[Y_i] \cap t_j\mathbb{K}[Y_j] = \{0\}$ for all $1 \leq i < j \leq k$. From $t_i \notin \mathfrak{p}_i$ it follows $t_i\mathbb{K}[Y_i] \cap \mathfrak{p}_i = \{0\}$. Furthermore, $t_j\mathbb{K}[Y_j] \subseteq \mathfrak{p}_i$ since $t_j \in \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_i \cap \cdots \cap \mathfrak{p}_{j-1}$ and the claim follows. So far, we observed that the sum at the right hand side of (5) is direct. It remains to show the equality. Consider an arbitrary monomial $t \in M$. There is a uniquely determined index i such that $t \notin \mathfrak{p}_i$ but $t \in \mathfrak{p}_l$ for all $1 \leq l < i$. Then $t_i \mid t$. Moreover, $t \in \mathbb{K}[Y_i]$ since $\mathfrak{p}_i \cap \operatorname{supp} t = \emptyset$. This proves the inclusion "$\subseteq$" and the other inclusion is obvious.

In conclusion, $Sdim\, M \geq \min_{p_i \in \operatorname{Ass} M} \dim R/\mathfrak{p}_i$ which has to be equality in view of Inequality (3). $\square$

Example 2 ([Rei] Remark 3). *Consider Reisner's example* $I = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_1x_4x_6, x_1x_5x_6, x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, x_3x_4x_6) \subset \mathbb{K}[x_1, x_2, x_3, x_4, x_5, x_6]$ *of a monomial ideal which is Cohen-Macaulay iff the characteristic of* $\mathbb{K}$ *is not* 2. *Localization at the prime ideal* $\mathfrak{p} = (x_1, x_2, x_3, x_4, x_5)$ *yields* $I_{(\mathfrak{p})} = (x_1x_4, x_1x_5, x_2x_3, x_2x_5, x_3x_4)$. *The associated primes of the module* $M_{(\mathfrak{p})} = R_{(\mathfrak{p})}/I_{(\mathfrak{p})}$ *are* $\mathfrak{p}_1 = (x_1, x_2, x_3)$, $\mathfrak{p}_2 = (x_1, x_2, x_4)$, $\mathfrak{p}_3 = (x_1, x_3, x_5)$, $\mathfrak{p}_4 = (x_2, x_4, x_5)$, $\mathfrak{p}_5 = (x_3, x_4, x_5)$. *Application of Proposition 1 yields* $M_{(\mathfrak{p})} = \mathbb{K}[x_4, x_5] \oplus x_3\mathbb{K}[x_3, x_5] \oplus x_2\mathbb{K}[x_2, x_4] \oplus x_1\mathbb{K}[x_1, x_3] \oplus x_1x_2\mathbb{K}[x_1, x_2]$.

Now, we ask for a Stanley decomposition of the quotient $M = R/I$. *Since no confusion is possible we will denote the associated prime ideal* $\mathfrak{p}_i \cap R$, $i = 1, \ldots, 5$, *by the same symbol* $\mathfrak{p}_i$ *for the sake of better readability. The missing associated primes of* I *containing* x_6 *are* $\mathfrak{p}_6 = (x_1, x_4, x_6)$, $\mathfrak{p}_7 = (x_1, x_5, x_6)$, $\mathfrak{p}_8 = (x_2, x_3, x_6)$, $\mathfrak{p}_9 = (x_2, x_5, x_6)$, *and* $\mathfrak{p}_{10} = (x_3, x_4, x_6)$. *One easily observes that the sequence* $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_4, \mathfrak{p}_5$ *cannot be extended by appending one of the primes* $\mathfrak{p}_6, \ldots, \mathfrak{p}_{10}$ *towards a sequence allowing the application of the above proposition. Moreover, we checked using a computer program that our proposition is not applicable to the intersection of any more than* 5 *associated prime ideals of* M.

But it holds $Sdim\, M = \dim M$ *since* $M = \mathbb{K}[x_4, x_5, x_6] \oplus x_3\mathbb{K}[x_3, x_5, x_6] \oplus x_2\mathbb{K}[x_2, x_3, x_5] \oplus x_1\mathbb{K}[x_1, x_3, x_6] \oplus x_1x_2\mathbb{K}[x_1, x_2, x_5] \oplus x_1x_4\mathbb{K}[x_1, x_3, x_4] \oplus x_1x_5\mathbb{K}[x_1, x_4, x_5] \oplus x_2x_4\mathbb{K}[x_2, x_4, x_6] \oplus x_2x_6\mathbb{K}[x_1, x_2, x_6] \oplus x_3x_4\mathbb{K}[x_2, x_3, x_4]$ *is a Stanley decomposition of* M.

There is a strong relationship between Proposition 1 and shellable (nonpure) simplicial complexes. In view of this connection it is absolutely not surprising that we could not apply Proposition 1 in the previous example. We will discuss this relationship in more detail in Section 9.

The next remark concerns monomial ideals satisfying the assumption of Proposition 1 which are not pure dimensional. In this case the proposition proves the existence of a Stanley decomposition whose summands are in bijection with the associated primes but in contrast to Lemma 2 it does not imply that each Stanley decomposition has this nice property.

Example 3. *Examples for Stanley decompositions of $M = R/I$, where $I = (x, y) \cap (y, z, u) \subset \mathbb{K}[x, y, z, u]$, are $M = \mathbb{K}[z, u] \oplus x\mathbb{K}[x]$ and $M = \mathbb{K}[z] \oplus u\mathbb{K}[u] \oplus zu\mathbb{K}[z, u] \oplus x\mathbb{K}[x]$. While the first decomposition is of the type described in Proposition 1, the second is not.*

The generalization of Proposition 1 which could be applied successfully in Example 2 can be expressed as follows:

Proposition 2. *Let I be a squarefree monomial ideal with associated primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$. By setting*

$$\begin{aligned} J_1 &:= R, \\ J_{i+1} &:= J_i \cap \mathfrak{p}_i + (a \in J_i \ : \ t_i \nmid a)R, \quad (i = 1, \ldots, k), \end{aligned}$$

a sequence $t_1, t_2, \ldots, t_k$ of standard monomials modulo I defines a descending sequence $J_1 \supseteq J_2 \supseteq \cdots \supseteq J_{k+1}$ of overideals of I. If the associated primes of I can be ordered in such a way that for suitable monomials $t_1, t_2, \ldots, t_k$ we have

1. *$t_1 = 1$ and t_i is a minimal monomial of the set $J_i \setminus \bigcup_{j=1}^{i-1} t_j\mathbb{K}[Y_j]$ for $i = 2, \ldots, k$,*
2. *$J_{k+1} = I + UR$, where $U = \bigcup_{j=1}^{k-1} (t_j\mathbb{K}[Y_j] \cap J_{j+1})$, and*
3. *for each monomial $t \in J_{k+1} \setminus I$ there exists a uniquely determined index $i \in \{1, \ldots, k\}$ such that $t \in t_i\ \mathbb{K}[Y_i]$.*

Then it holds $Sdim\, M = \min_{p_i \in \operatorname{Ass} M} \dim R/\mathfrak{p}_i$ and $M = \bigoplus_{i=1}^{k} t_i\ \mathbb{K}[Y_i]$ is a Stanley decomposition of M.

Moreover, if I is pure and the above conditions are not satisfiable for any permutation of associated primes then we have $Sdim\, M < \dim M$.

Before we come to the proof let us explain the idea behind the rather technical conditions formulated in the proposition. If I satisfies even the stronger assumptions of Proposition 1 then defining t_i, $i \in \{2, \ldots, k\}$, to be the unique minimal monomial contained in $\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_{i-1} \setminus \mathfrak{p}_i$ leads to the inclusion $(a \in J_i \ : \ t_i \nmid a)R \subseteq J_i \cap \mathfrak{p}_i$ and, hence, $J_{i+1} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_i$ for all $i \in \{1, \ldots, k\}$. If I does not satisfy the assumptions of Proposition 1 then possible additional minimal elements of $J_i \setminus \mathfrak{p}_i$ have to be kept in J_{i+1} for later use. Note that the difference between the ideal $(a \in J_i \ : \ t_i \nmid a)R$ and the quotient $J_i/(t_i)$ is larger than only the fact that both ideals live in different rings since the ideal $(a \in J_i \ : \ t_i \nmid a)R$ typically contains nonzero multiples of t_i. If there are such multiples which do not belong to $\mathfrak{p}_i$ they cause an unpleasant behavior, namely, $t_i\ \mathbb{K}[Y_i]$ will intersect with J_{i+1} nontrivially. The importance of Condition 3 becomes obvious already in the simple Example 1. Condition 2 ensures that no element of the intersection $t_i\ \mathbb{K}[Y_i] \cap J_{i+1}$ will belong to a second module $t_j\ \mathbb{K}[Y_j]$. Note, that this can only happen for "high" dimensional M, more precisely, only if $\dim M \geq |\text{supp}\ t_i| + 2$. Similarly, while in "small" dimensions all monomials t_i are minimal generators of J_i this need not to be the case in higher dimensions. The problems are demonstrated in the following example of a four-dimensional strongly generic monomial Cohen-Macaulay ideal.

Example 4. *Consider the ideal $I = (x_1x_3x_5, x_2x_4x_6)$ with irredundant prime decomposition $I = (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4) \cap (x_4, x_5) \cap (x_5, x_6) \cap (x_1, x_4) \cap (x_2, x_5) \cap (x_3, x_6) \cap (x_1, x_6)$.*

The following two ideal sequences $J_1, \dots, J_{10}$ are induced by different sequences of standard terms.

i	t_i	Y_i	J_i
1	1	$\{x_3, x_4, x_5, x_6\}$	(1)
2	x_1	$\{x_1, x_4, x_5, x_6\}$	(x_1, x_2)
3	x_2	$\{x_1, x_2, x_5, x_6\}$	(x_2, x_1x_3)
4	x_1x_3	$\{x_1, x_2, x_3, x_6\}$	(x_1x_3, x_2x_3, x_2x_4)
5	x_2x_4	$\{x_1, x_2, x_3, x_4\}$	$(x_2x_3, x_2x_4, x_1x_3x_4, x_1x_3x_5)$
6	x_2x_3	$\{x_2, x_3, x_5, x_6\}$	$(x_2x_3, x_1x_3x_4, x_1x_3x_5, x_2x_4x_5, x_2x_4x_6)$
7	$x_1x_3x_4$	$\{x_1, x_3, x_4, x_6\}$	$(x_2x_3x_4, x_1x_2x_3, x_1x_3x_4, x_1x_3x_5, x_2x_4x_5, x_2x_4x_6)$
8	$x_2x_4x_5$	$\{x_1, x_2, x_4, x_5\}$	$(x_1x_2x_3, x_1x_3x_5, x_2x_3x_4, x_2x_4x_5, x_2x_4x_6)$
9	$x_2x_3x_4x_5$	$\{x_2, x_3, x_4, x_5\}$	$(x_1x_2x_3, x_1x_3x_5, x_2x_3x_4, x_2x_4x_6)$
10			$(x_1x_2x_3, x_1x_3x_5, x_2x_3x_4, x_2x_4x_6)$
1	1	$\{x_3, x_4, x_5, x_6\}$	(1)
2	x_1	$\{x_1, x_4, x_5, x_6\}$	(x_1, x_2)
3	x_2	$\{x_1, x_2, x_5, x_6\}$	(x_2, x_1x_3)
4	x_1x_3	$\{x_1, x_2, x_3, x_6\}$	(x_1x_3, x_2x_3, x_2x_4)
5	x_2x_3	$\{x_1, x_2, x_3, x_4\}$	$(x_2x_3, x_2x_4, x_1x_3x_4, x_1x_3x_5)$
6	$x_2x_3x_6$	$\{x_2, x_3, x_5, x_6\}$	$(x_2x_4, x_1x_3x_4, x_1x_3x_5, x_2x_3x_5, x_2x_3x_6)$
7	$x_1x_3x_4$	$\{x_1, x_3, x_4, x_6\}$	$(x_2x_4, x_1x_3x_4, x_1x_3x_5, x_2x_3x_5, x_1x_2x_3x_6)$
8	x_2x_4	$\{x_1, x_2, x_4, x_5\}$	$(x_2x_4, x_1x_3x_5, x_2x_3x_5, x_1x_2x_3x_6)$
9	$x_2x_3x_5$	$\{x_2, x_3, x_4, x_5\}$	$(x_1x_3x_5, x_2x_3x_4, x_2x_4x_6, x_2x_3x_5, x_1x_2x_3x_6)$
10			$(x_1x_3x_5, x_2x_3x_4, x_2x_4x_6, x_2x_3x_5x_6, x_1x_2x_3x_6)$

Only the first sequence satisfies the assumptions of our proposition and, hence, $M = t_1\mathbb{K}[Y_1] \oplus \cdots \oplus t_9\mathbb{K}[Y_9]$ is a Stanley decomposition of M. Note, the only minimal generator $x_2x_3x_4$ of J_9 which is not contained in (x_1, x_6) cannot be used as t_9 since it is already contained in the summand $x_2x_4\mathbb{K}[x_1, x_2, x_3, x_4]$. The second sequence violates Condition 2 since $x_1x_2x_3$ is contained in the intersection $x_1x_3\ \mathbb{K}[x_1, x_2, x_3, x_6] \cap x_2x_3\ \mathbb{K}[x_1, x_2, x_3, x_4]$. Nevertheless, Condition 3 holds.

It is clear that any violation of Condition 2 prevents the result from being a Stanley decomposition. However, if one could detect any "bad" sequence of prime ideals and/or choice of standard monomials by checking only the final Condition 3 one could use a simpler formulation of the proposition. Our example refutes such a possibility.

Nevertheless, it is an interesting question if the application of Condition 2 and the necessity of choosing monomials t_i which are not minimal generators of J_i can be avoided by application of "good" strategies for ordering the primes and choosing the monomials. At least in the above example, and in some more general situations discussed in Sections 6 - 8, the answer is positive.

Example 4 (continued). *After reordering the primes according to $I = (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4) \cap (x_1, x_4) \cap (x_4, x_5) \cap (x_2, x_5) \cap (x_5, x_6) \cap (x_3, x_6) \cap (x_1, x_6)$ even Proposition 1 becomes applicable and yields*

i	t_i	Y_i	J_i
1	1	$\{x_3, x_4, x_5, x_6\}$	(1)
2	x_1	$\{x_1, x_4, x_5, x_6\}$	(x_1, x_2)
3	x_2	$\{x_1, x_2, x_5, x_6\}$	(x_2, x_1x_3)
4	x_2x_3	$\{x_2, x_3, x_5, x_6\}$	(x_1x_3, x_2x_3, x_2x_4)
5	x_1x_3	$\{x_1, x_2, x_3, x_6\}$	(x_1x_3, x_2x_4)
6	$x_1x_3x_4$	$\{x_1, x_3, x_4, x_6\}$	$(x_2x_4, x_1x_3x_4, x_1x_3x_5)$
7	x_2x_4	$\{x_1, x_2, x_3, x_4\}$	$(x_2x_4, x_1x_3x_5)$
8	$x_2x_4x_5$	$\{x_1, x_2, x_4, x_5\}$	$(x_1x_3x_5, x_2x_4x_5, x_2x_4x_6)$
9	$x_2x_3x_4x_5$	$\{x_2, x_3, x_4, x_5\}$	$(x_1x_3x_5, x_2x_4x_6, x_2x_3x_4x_5)$
10			$(x_1x_3x_5, x_2x_4x_6)$

Proof of Proposition 2: First, we want to show that the sum $\bigoplus_{i=1}^{k} t_i\ \mathbb{K}[Y_i]$ is direct. So, in contrary, let us assume the existence of a monomial $t \in t_i\ \mathbb{K}[Y_i] \cap t_j\ \mathbb{K}[Y_j]$ for some integers i and j such that $1 \leq i < j \leq k$. At the one hand side $t \notin J_{k+1}$ according to condition 3. But at the other hand side from condition 2 and $t_j\ \mathbb{K}[Y_j] \subseteq J_j \subseteq J_{i+1}$ we can deduce $t \in J_{k+1}$. This is a contradiction and, hence, $t_i\ \mathbb{K}[Y_i] \cap t_j\ \mathbb{K}[Y_j] = \{0\}$ for all $i \neq j$.

The inclusion $M \supseteq \bigoplus_{i=1}^{k} t_i\ \mathbb{K}[Y_i]$ is an immediate consequence of $t_i \notin \mathfrak{p}_i$ for all $i = 1, \ldots, k$. It remains to show the inclusion $M \subseteq \bigoplus_{i=1}^{k} t_i\ \mathbb{K}[Y_i]$. Let $t \in M$ be an arbitrary standard monomial modulo I, i.e. $t \notin I$. If $t \in J_{k+1}$ then $t \in \bigoplus_{i=1}^{k} t_i\ \mathbb{K}[Y_i]$ by condition 3. Otherwise, there exists a uniquely determined index i, $1 \leq i \leq k$, such that $t \in J_i \setminus J_{i+1}$ and one easily observes $t \in t_i\ \mathbb{K}[Y_i]$. This completes the proof of the first part of the statement.

The second part of the statement is a corollary to Lemma 2. Assume $Sdim\, M = \dim M$. Then by the lemma each Stanley decomposition of M has the form $M = \bigoplus_{i=1}^{k} t_i\ \mathbb{K}[Y_i]$. Now, let $j_1, \ldots, j_k$ be a permutation of the integers $\{1, \ldots, k\}$ such that $t_{j_l} \nmid t_{j_m}$ for all $1 \leq m < l \leq k$. Then the permutation $\mathfrak{p}_{j_1}, \ldots, \mathfrak{p}_{j_k}$ of associated primes and the monomial sequence $t_{j_1}, \ldots, t_{j_k}$ fulfill the assumptions of the proposition. □

Example 2 (continued). *Based on the permutation $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_6, \mathfrak{p}_4, \mathfrak{p}_{10}, \mathfrak{p}_9, \mathfrak{p}_8, \mathfrak{p}_3, \mathfrak{p}_5, \mathfrak{p}_7$ of* Ass M *the sequence $1, x_3, x_2, x_1, x_1x_2, x_1x_4, x_1x_5, x_2x_4, x_2x_6, x_3x_4$ of monomials fulfills the assumptions of Proposition 2 and confirms the Stanley decomposition presented earlier.*

Proposition 2 reduces the question if at least all Cohen-Macaulay rings R/I defined by a squarefree monomial ideal I satisfy Stanley's conjecture 2 to the question if each squarefree monomial Cohen-Macaulay ideal possesses a permutation of associated primes such that conditions 1-3 are satisfied for a suitable choice of standard monomials $t_1, \ldots, t_k$.

Besides purity another important property of Cohen-Macaulay rings consists in the connectedness in codimension 1, i.e. for any two associated primes $\mathfrak{p}$ and $\mathfrak{p}'$ of R/I, there is a sequence $\mathfrak{p} = \mathfrak{p}_{j_1}, \mathfrak{p}_{j_2}, \ldots, \mathfrak{p}_{j_r} = \mathfrak{p}'$ of associated primes of R/I such that $\operatorname{codim}\left(\mathfrak{p}_{j_i} + \mathfrak{p}_{j_{i+1}}\right) = 1 + \operatorname{codim} I$ for all $i \in \{1, \ldots, r-1\}$. Unfortunately, even the condition that I is a pure dimensional squarefree monomial ideal which is connected in codimension 1 is not strong enough to ensure $\mathit{Sdim}\, R/I = \dim R/I$. For a counter example consider $I' = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{p}_3 \cap \mathfrak{p}_4 \cap \mathfrak{p}_5 \cap \mathfrak{p}_6$, where the prime ideals are defined as in Example 2. Here we have $\mathit{Sdim}\, R/I' = 2 < \dim R/I' = 3$. At the other hand side Example 11 will show that connectedness in codimension 1 is also not necessary for the validity of $\mathit{Sdim}\, R/I = \dim R/I$.

5. The Nonsquarefree Case

In this section we study quotients modulo monomial ideals which are not necessarily squarefree. Typically, we loose the bijection between associated primes of $M = R/I$ and the direct summands of a Stanley decomposition of M even in the case that I is pure and $\mathit{Sdim}\, M = \dim M$. But since an element contained in exactly one primary component of I can only belong to a direct summand which makes use of the maximal algebraically independent set of this primary ideal, at least we can conclude that the correspondence remains surjective.

Theorem 1. *Let $I \subset R$ be a monomial ideal and $M = R/I$. Furthermore, let $\sqrt{I} = \bigcap_{\mathfrak{p} \in \operatorname{Ass} M} \mathfrak{p}$ be the radical of I and $\tilde{M} = R/\sqrt{I}$. Then it holds $\mathit{Sdim}\, M \leq \mathit{Sdim}\, \tilde{M}$.*

Proof: Let S denote the set of standard monomials modulo I and $\tilde{S}$ the set of standard monomials modulo $\sqrt{I}$. Recall the identifications $M = \operatorname{spann}_{\mathbb{K}} S$ and $\tilde{M} = \operatorname{spann}_{\mathbb{K}} \tilde{S}$. Consider an arbitrary Stanley decomposition $M = \bigoplus_{i=1}^{m} t_i\, \mathbb{K}[W_i]$ of M. For each monomial $u \in \tilde{M}$ we define $Z_u := W_i$, where $i \in \{1, \ldots, m\}$ is the uniquely determined index such that $u^\nu \in t_i\, \mathbb{K}[W_i]$ for all sufficiently high powers of u. Note, $\operatorname{supp} u \subseteq Z_u$ for the so-defined sets Z_u and, therefore, $\tilde{M} \supseteq \sum_{u \in \tilde{M}} u\, \mathbb{K}[Z_u]$. The other inclusion holds by construction.

Finally, we have to show that any two summands $u\, \mathbb{K}[Z_u]$ and $v\, \mathbb{K}[Z_v]$ either intersect trivially or are contained in the summand $\gcd(u,v)\mathbb{K}[Z_{\gcd(u,v)}]$. By construction there exist uniquely determined $i, j \in \{1, \ldots, m\}$ such that $u^r \in t_i\, \mathbb{K}[W_i]$ and $v^s \in t_j\, \mathbb{K}[W_j]$ for large enough exponents r and s. Moreover, $Z_u = W_i$ and $Z_v = W_j$. Since the summands originate from a Stanley decomposition either we have $i = j$ or $t_j\, \mathbb{K}[Z_v] \cap t_i\, \mathbb{K}[Z_u] = \{0\}$. In the latter case it follows $v^s\, \mathbb{K}[Z_v] \cap u^r\, \mathbb{K}[Z_u] = \{0\}$ and, hence, $v\, \mathbb{K}[Z_v] \cap u\, \mathbb{K}[Z_u] = \{0\}$. In the remaining case $i = j$ we have $u^r, v^s \in t_i\, \mathbb{K}[W_i]$ for all sufficiently large integers r and s. Hence, also $\gcd(u,v)^r = \gcd(u^r, v^r) \in t_i\, \mathbb{K}[W_i]$ for all sufficiently large exponents r. By construction $Z_{\gcd(u,v)} = W_i$ and, consequently, $u\, \mathbb{K}[Z_u] + v\, \mathbb{K}[Z_v] \subseteq \gcd(u,v)\, \mathbb{K}[Z_{\gcd(u,v)}]$. In conclusion, removing all redundant summands leads to a decomposition $\tilde{M} = \bigoplus_{i=1}^{l} v_i\, \mathbb{K}[Z_{v_i}]$ which proves $\mathit{Sdim}\, \tilde{M} \geq \min_{i=1}^{l} |Z_{v_i}| \geq \mathit{Sdim}\, M$. □

Of course, there are ideals I for which the inequality is proper. For instance, equality can never hold in the case that I has embedded components and $\mathit{Sdim}\, \tilde{M} =$

$\dim \tilde{M}$. The following example illustrates that we need not to have equality even in the pure case.

Example 5. *Consider the monomial ideal* $I = (x_1^2, x_2^2, x_3) \cap (x_2, x_3, x_4) \cap (x_3, x_4^2, x_5) = (x_3, x_2^2x_5, x_2^2x_4^2, x_1^2x_4x_5, x_1^2x_4^2, x_1^2x_2x_5)$ *and its radical* $\sqrt{I} = (x_1, x_2, x_3) \cap (x_2, x_3, x_4) \cap (x_3, x_4, x_5)$. *Application of Proposition 1 gives* $\tilde{M} = \mathbb{K}[x_4, x_5] \oplus x_1\ \mathbb{K}[x_1, x_5] \oplus x_2\ \mathbb{K}[x_1, x_2]$. *However,* $Sdim\, M < 2$ *since there is an unresolvable conflict between the elements* $x_1^2x_4$ *and* $x_2^2x_4$ *at the one hand side and the elements* x_4^2 *and* x_4x_5 *at the other hand side. While the first two elements require a direct summand containing* $x_4\ \mathbb{K}[x_1, x_2]$ *the other require* $x_4\ \mathbb{K}[x_4, x_5]$, *a contradiction. A Stanley decomposition of* M *is* $M = \mathbb{K}[x_4, x_5] \oplus x_1\ \mathbb{K}[x_1, x_5] \oplus x_2\ \mathbb{K}[x_1, x_2] \oplus x_1x_4\ \mathbb{K}[x_4, x_5] \oplus x_2x_4\ \mathbb{K}[x_4, x_5] \oplus x_2x_5\ \mathbb{K}[x_5] \oplus x_1x_2x_4\ \mathbb{K}[x_4, x_5] \oplus x_1x_2x_5\ \mathbb{K}[x_5] \oplus x_1^2x_4\ \mathbb{K}[x_1, x_2] \oplus x_2^2x_4\ \mathbb{K}[x_2] \oplus x_1x_2^2x_4\ \mathbb{K}[x_2]$.

The next example demonstrates a situation where the Stanley dimensions of M and $\tilde{M}$ coincide.

Example 6. *Let us consider the strongly generic monomial Cohen-Macaulay ideal* $I = (x, y^3z^2u) \subset \mathbb{K}[x, y, z, u]$. *I has the irredundant decomposition in irreducible components* $I = (x, y^3) \cap (x, z^2) \cap (x, u)$. $M = \mathbb{K}[z, u] \oplus y\ \mathbb{K}[z, u] \oplus y^2\ \mathbb{K}[z, u] \oplus y^3\ \mathbb{K}[y, u] \oplus y^3z\ \mathbb{K}[y, u] \oplus y^3z^2\ \mathbb{K}[y, z]$ *is a Stanley decomposition of* $M = R/I$ *(see our forthcoming Proposition 3). Application of Theorem 1 yields the decomposition* $\tilde{M} = \mathbb{K}[z, u] \oplus y\ \mathbb{K}[y, u] \oplus yz\ \mathbb{K}[y, z]$ *of the quotient modulo the radical* $\sqrt{I} = (x, yzu) = (x, y) \cap (x, z) \cap (x, u)$. *Since the Stanley dimension of* $\tilde{M}$ *cannot exceed the dimension of* $\tilde{M}$ *this is already a Stanley decomposition.*

In this example we can go also in the opposite direction, i.e. we can complete the above Stanley decomposition of $\tilde{M}$ *to a Stanley decomposition* $M = \mathbb{K}[z, u] \oplus y\ \mathbb{K}[y, u] \oplus yz\ \mathbb{K}[y, z] \oplus yzu\ \mathbb{K}[z, u] \oplus y^2zu\ \mathbb{K}[z, u]$ *of* M.

Let us discuss the abilities and the limits of the methods applied in the proof of Theorem 1. Given a Stanley decomposition of M we can decompose $\tilde{M}$ and we have some guaranteed quality of the resulting decomposition, i.e. it is not worse than this of M. However if $Sdim\, M < Sdim\, \tilde{M}$ then we will not obtain a Stanley decomposition of $\tilde{M}$, in general. For instance, replacing the first direct summand in the Stanley decomposition of M in Example 5 according to $\mathbb{K}[x_4, x_5] = \mathbb{K}[x_4] \oplus x_5\ \mathbb{K}[x_5] \oplus x_4x_5\ \mathbb{K}[x_4, x_5]$ will maintain the Stanley decomposition property for M but lifting the new decomposition to a decomposition of $\tilde{M}$ will not provide a Stanley decomposition anymore. This is of course not surprising but what one really would like to have is the opposite construction which could be applied successfully in both Examples 5 and 6, namely to complete a given Stanley decomposition of $\tilde{M}$ to a Stanley decomposition of M by adding some direct summands. There seems to be a good chance that this is always possible. However, the answer to this question remains open. It follows a generalization of Proposition 1 to the nonsquarefree case.

Proposition 3. *Let* $I \subset R$ *be a monomial ideal which possesses an irredundant primary decomposition* $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_k$ *such that* $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{i-1} \setminus \mathfrak{q}_i$, $i = 2, \ldots, k$, *contains exactly one minimal monomial* t_i. *Then* $Sdim\, M = \min_{\mathfrak{p}_i \in \operatorname{Ass} M} \dim R/\mathfrak{p}_i$, *where $M = R/I$. Moreover, a Stanley decomposition of M can be obtained by removing all redundant summands from the inner sums of*

$$M = \bigoplus_{i=1}^{k} \sum_{t \in \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{i-1} \setminus \mathfrak{q}_i} t\mathbb{K}[Y_i] \,, \tag{6}$$

where $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{i-1} := R$ for $i = 1$.

Proof: The equality in (6) is ensured by construction. That the first big sum is direct follows in a similar way as in the proof of Proposition 1. Formally, set $t_1 = 1$. It remains to show that for any $i \in \{1, \ldots, k\}$ and $s, t \in \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{i-1} \setminus \mathfrak{q}_i$ it holds $\gcd(s,t) \in \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{i-1} \setminus \mathfrak{q}_i$ and $s\ \mathbb{K}[Y_i] + t\ \mathbb{K}[Y_i] \subseteq \gcd(s,t)\ \mathbb{K}[Y_i]$ or the modules $s\ \mathbb{K}[Y_i]$ and $t\ \mathbb{K}[Y_i]$ intersect trivially. First observe $t_i \mid \gcd(s,t)$ and, hence, $\gcd(s,t) \in \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{i-1} \setminus \mathfrak{q}_i$. Suppose $\mathrm{lcm}(s,t) \in s\ \mathbb{K}[Y_i] \cap t\ \mathbb{K}[Y_i]$. Then it follows $\deg_x s = \deg_x t = \deg_x \gcd(s,t)$ for all $x \notin Y_i$ and, hence, $s\ \mathbb{K}[Y_i] + t\ \mathbb{K}[Y_i] \subseteq \gcd(s,t)\ \mathbb{K}[Y_i]$. □

Example 7. *Consider the strongly generic monomial Cohen-Macaulay ideal $I = (x^2, y^2z^2, yzw, w^2) = (x^2, y, w^2) \cap (x^2, z, w^2) \cap (x^2, y^2, w) \cap (x^2, z^2, w) \subset R = \mathbb{K}[x, y, z, w]$.*

With the notation of Proposition 3 the sequence $t_1 = 1, t_2 = y, t_3 = yz, t_4 = y^2z$ provides the Stanley decomposition $M = \mathbb{K}[z] \oplus y\ \mathbb{K}[y] \oplus yz\ \mathbb{K}[z] \oplus y^2z\ \mathbb{K}[y] \oplus x\ \mathbb{K}[z] \oplus w\ \mathbb{K}[z] \oplus xw\ \mathbb{K}[z] \oplus yw\ \mathbb{K}[y] \oplus xy\ \mathbb{K}[y] \oplus xyw\ \mathbb{K}[y] \oplus xyz\ \mathbb{K}[z] \oplus xy^2z\ \mathbb{K}[y]$.

However, the proposition will not be applicable to the irredundant decomposition $I = (x^2, y^2, w^2, yw) \cap (x^2, z^2, w^2, zw)$ in maximal primary ideals.

The last statement shows that it is sometimes advisable to work with irreducible rather than with maximal primary ideals and this will become even more apparent in the next sections. However, it should be mentioned that there are also situations were the use of maximal primary ideals is preferable. Consider, for instance, the simple case that I is a primary ideal. While application of Proposition 3 to the primary decomposition $I = \mathfrak{q}$ immediately yield $Sdim\, R/I = \dim R/I$ and a Stanley decomposition of R/I it may happen that no permutation of the irreducible components of I allows the application of the proposition, a simple example showing this behavior is $I = (x^2, y^2, z, u) \cap (x, y, z^2, u^2)$.

Next, we generalize Proposition 2 to the case of nonsquarefree ideals $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_k$. Instead of a sequence $t_1, \ldots, t_k$ of standard monomials modulo I we use a sequence $T_1, \ldots, T_k$ of finite sets of standard monomials modulo I for the definition of a descending sequence $J_1 \supseteq J_2 \supseteq \cdots \supseteq J_{k+1}$ of overideals of I by setting

$$J_1 := R \quad , \quad J_{i+1} := J_i \cap \mathfrak{q}_i + (a \in J_i \ :\ \forall t \in T_i \ :\ t \nmid a)R, \quad (i = 1, \ldots, k). \tag{7}$$

Theorem 2. *Let I be a monomial ideal with irredundant primary decomposition $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_k$. Furthermore, assume that there exists a sequence $T_1, \ldots, T_k$ of finite sets of standard monomials modulo I, where for all $i = 1, \ldots, k$ we have*

1. *$T_i \subset J_i \setminus \mathfrak{q}_i$, $T_i \cap (T_j\mathbb{K}[Y_j]) = \emptyset$ for all $j \in \{1, \ldots, i-1\}$,*
2. *for any two distinct elements $t, s \in T_i$ there exists a variable $x \in \mathfrak{p}_i$ such that $\deg_x t \neq \deg_x s$,*

3. *all* $t \in T_i$ *are minimal in the sense that for all* $x \in Y_i$ *dividing* t *it holds* $\frac{t}{x} \notin J_i$ *or* $\frac{t}{x} \in \bigcup_{j=1}^{i-1} T_j \mathbb{K}[Y_j]$,

such that the ideal sequence $J_1 \supseteq J_2 \supseteq \cdots \supseteq J_{k+1} \supseteq I$ *induced by (7) satisfies*

4. $J_{k+1} = I + UR$, *where* $U = \bigcup_{j=1}^{k-1} (T_j \mathbb{K}[Y_j] \cap J_{j+1})$, *and*
5. *for each monomial* $s \in J_{k+1} \setminus I$ *there exist a uniquely determined index* $i \in \{1, \ldots, k\}$ *and a uniquely determined monomial* $t \in T_i$ *such that* $s \in t\, \mathbb{K}[Y_i]$.

Then it holds $\mathit{Sdim}\, M = \min_{p_i \in \operatorname{Ass} M} \dim R/\mathfrak{p}_i$ *and* $M = \bigoplus_{i=1}^{k} \bigoplus_{t \in T_i} t\ \mathbb{K}[Y_i]$ *is a Stanley decomposition of* M.

Proof: The inclusion $M \supseteq \sum_{i=1}^{k} \sum_{t \in T_i} t\ \mathbb{K}[Y_i]$ is an immediate consequence of condition 1. Let s be a standard monomial modulo I. If there exists $i \in \{1, \ldots, k\}$ such that $s \in J_i \setminus J_{i+1}$ then $s \in t\mathbb{K}[Y_i]$ for some $t \in T_i$. Otherwise, $s \in J_{k+1} \setminus I$ and the existence of i and $t \in T_i$ such that $s \in t\mathbb{K}[Y_i]$ follows by condition 5. This proves also the other inclusion $M \subseteq \sum_{i=1}^{k} \sum_{t \in T_i} t\ \mathbb{K}[Y_i]$.

It remains to show that the sums are even direct. Each inner sum is direct according to conditions 2 and 3. That the outer sum is direct follows in full analogy to the proof of Proposition 2 from conditions 4 and 5. □

We have to leave open the question whether there is an analogy to the opposite direction of Proposition 2. Let us just remark why application of the same proof idea would not work here. No permutation of the sets $T_1, \ldots, T_k$ resulting from a given Stanley decomposition of M need to satisfy the assumptions of the theorem. Since the primary components were not assumed to be maximal, typically, we can form different set sequences $T_1, \ldots, T_k$ corresponding to a given Stanley decomposition of M, even the number of sets may vary. But also this additional freedom is still insufficient for saving our proof idea from the squarefree case.

Theorem 2 generalizes the previous Propositions 1, 2 (without its opposite direction) and 3. Later, in Section 9 we will see that each of the special cases considered in the propositions has a particular combinatorial meaning. There is one more particular case which deserves a separate presentation since all large classes of ideals for which we will prove Stanley's Conjecture in the subsequent sections will satisfy even its stronger assumptions.

Corollary 1. *Let* I *be a monomial ideal with irredundant primary decomposition* $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_k$. *Further, let* $J_1 := R$ *and* $J_{i+1} := \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_i$ *for* $i \in \{1, \ldots, k\}$. *Finally, for each* $i \in \{1, \ldots, k\}$ *define* T_i *to be the set of all monomials* $t \in J_i \setminus \mathfrak{q}_i$ *such that* $\frac{t}{x} \notin J_i$ *for all variables* $x \in (\operatorname{supp} t) \cap Y_i$.

If any two distinct monomials belonging to the same set T_i *differ in the degree of at least one variable* $x \in \mathfrak{p}_i$, $i \in \{1, \ldots, k\}$, *then it holds* $\mathit{Sdim}\, M = \min_{p_i \in \operatorname{Ass} M} \dim R/\mathfrak{p}_i$ *and* $M = \bigoplus_{i=1}^{k} \bigoplus_{t \in T_i} t\ \mathbb{K}[Y_i]$ *is a Stanley decomposition of* M.

Proof: It is an easy excersise to show that the introduced sets T_i of standard monomials are finite and satisfy conditions 1-3 of Theorem 2. The ideals J_i introduced in the corollary are the same as induced from $T_1, \ldots, T_k$ by (7). Hence, also the conditions 4 and 5 hold. □

Corollary 2. *Let $I \subset R$ be a monomial ideal and $M = R/I$. If $\dim M \leq 1$ then it holds $Sdim\, M = \min_{\mathfrak{p} \in \operatorname{Ass} M} \dim R/\mathfrak{p}$.*

Proof: Let $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_k$ be an irredundant decomposition in irreducible components and consider an arbitrary fixed $l \in \{2, \ldots, k\}$. Since $\dim R/\mathfrak{q}_l \leq \dim R/I \leq 1$ any two monomials of $(\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{l-1}) \setminus \mathfrak{q}_l$ which coincide in the degree of all variables belonging to $\mathfrak{p}_l$ can differ only in the degree of at most one variable. Therefore, one is multiple of the other. Now, the assertion follows from Corollary 1. □

Example 8. *Let I be an arbitrary Borel-fixed ideal and consider $M = R/I$. If the irreducible components $\mathfrak{q}_1, \ldots, \mathfrak{q}_k$ are enumerated according to decreasing dimension of $R/\mathfrak{q}_i$ then for any $l \in \{2, \ldots, k\}$ we have the inclusion $\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_{l-1} \subseteq \mathfrak{p}_l$ for the associated prime ideals according to* [E, Corollary 15.25]. *Hence, we can apply Corollary 1.*

This example reflects the well-known fact that Conjecture 1 holds for $n = 1$ and, hence, also for higher n when I is in generic coordinates. In this situation such a decomposition was given first by Rees, whose decomposition satisfies both parts of Conjecture 1 in case $n = 1$ and $\mathbb{K}$ infinite (see [Ree]). Another decomposition satisfying the first part of Conjecture 1 for arbitrary $\mathbb{K}$ and $n = 1$ is due to Baclawski and Garsia (see [BG]).

Example 9 ([MSY] Example 4.18b). *Consider the monomial ideal $I = (a^4, b, c) \cap (a^2, b^4, d) \cap (a, b^3, e) \cap (a^3, b^2, e^2) \subset \mathbb{K}[a, b, c, d, e]$, which is cogeneric and Cohen-Macaulay.*

By Proposition 1 we obtain the Stanley decomposition $\tilde{M} = \mathbb{K}[d, e] \oplus c\ \mathbb{K}[c, e] \oplus cd\ \mathbb{K}[c, d]$ of the quotient $\tilde{M} = R/\sqrt{I}$. Proposition 3 is not applicable but Corollary 1 yields the Stanley decomposition $M = \mathbb{K}[d, e] \oplus a\ \mathbb{K}[d, e] \oplus a^2\ \mathbb{K}[d, e] \oplus a^3\ \mathbb{K}[d, e] \oplus c\ \mathbb{K}[c, e] \oplus ac\ \mathbb{K}[c, e] \oplus b\ \mathbb{K}[c, e] \oplus b^2\ \mathbb{K}[c, e] \oplus b^3\ \mathbb{K}[c, e] \oplus ab\ \mathbb{K}[c, e] \oplus ab^2\ \mathbb{K}[c, e] \oplus ab^3\ \mathbb{K}[c, e] \oplus cd\ \mathbb{K}[c, d] \oplus cde\ \mathbb{K}[c, d] \oplus bde\ \mathbb{K}[c, d] \oplus bd\ \mathbb{K}[c, d] \oplus b^2d\ \mathbb{K}[c, d] \oplus abd\ \mathbb{K}[c, d] \oplus abde\ \mathbb{K}[c, d] \oplus a^2b\ \mathbb{K}[c, d] \oplus a^2c\ \mathbb{K}[c, d] \oplus acd\ \mathbb{K}[c, d] \oplus a^2be\ \mathbb{K}[c, d] \oplus a^2ce\ \mathbb{K}[c, d] \oplus acde\ \mathbb{K}[c, d]$.

6. Generic Monomial Ideals

The central result of this section consists in proving the following theorem.

Theorem 3. *Let $I \subset R$ be a generic monomial ideal and $M = R/I$. Then $Sdim\, M = \min_{\mathfrak{p} \in \operatorname{Ass} M} \dim R/\mathfrak{p}$. Furthermore, let $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_k$ be an irredundant decomposition in irreducible components, where the irreducible components are ordered in such a way that $x_1^a \in \mathfrak{q}_i$ implies $x_1^a \in \mathfrak{q}_j$ for all positive integers a and all integers $1 \leq i < j \leq k$. Moreover, components with common minimal generator[2] in x_1 are ordered by recursive application of analogous conditions to $x_2, x_3, \ldots$ until the tie is broken. Then for each $l \in \{2, \ldots, k\}$ the set $(\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{l-1}) \setminus \mathfrak{q}_l$ contains exactly one minimal monomial.*

[2]All components containing no pure power of x_1 are also considered to have the minimal generator in x_1 in common.

For the proof we will need the following simple fact.

Remark 1. *Let $I \subset \mathbb{K}[x_1, \ldots, x_n]$ be a monomial ideal and $m_1, \ldots, m_l$ its minimal generators ordered in such a way that $\deg_{x_1} m_i \le \deg_{x_1} m_{i+1}$ for all $i \in \{1, \ldots, l-1\}$. Furthermore, let $r \in \{1, \ldots, l\}$ be such that $\deg_{x_1} m_{r-1} < \deg_{x_1} m_r$ and set $d := \deg_{x_1} m_r$. Then the overideal $I_d \supseteq I$ generated by $\left\{x_1^d, m_1|_{x_1=1}, \ldots, m_{r-1}|_{x_1=1}\right\}$, where $m_i|_{x_1=1}$ is obtained from m_i by setting $x_1 = 1$ $(1 \le i < r)$, has the following properties:*

1. *if I is (strongly) generic, then so is I_d,*
2. *each irreducible component of I which has x_1^d as a minimal generator is also an irreducible component of I_d,*
3. *x_1^d is a minimal generator of each irreducible component of I_d.*

The given generating set of I_d needs not to be minimal, nevertheless, the first and the third property are obvious. The second property follows immediately by repeated application of the well-known formula

$$(ts)R + J = (tR + J) \cap (sR + J), \tag{8}$$

which is valid for arbitrary monomial ideals J and arbitrary monomials t and s such that $\gcd(t, s) = 1$. The statement of the remark extends to the ideal $I_\infty := (m_1|_{x_1=1}, \ldots, m_l|_{x_1=1})$. In this case the second condition has to be understood as: each irreducible component of I whose associated prime ideal does not contain x_1 is an irreducible component of I_∞. The third condition reads as $x_1 \notin \mathfrak{p}$ for all associated primes of R/I_∞. Obviously, I_∞ can be identified with $I_{(x_2,\ldots,x_n)}$, the localization of I at the prime ideal $(x_2, \ldots, x_n)$, in a natural way. In this sense Remark 1 generalizes Remark 2.1 from [MSY].

Lemma 3. *Let the monomial ideal $I \subset \mathbb{K}[x_1, \ldots, x_n]$ be generic with irredundant decomposition $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_k$ in irreducible components, where the components are ordered as in Theorem 3. Furthermore, let d be a positive integer such $x_1^d \notin I$ but x_1^d is a minimal generator for some irreducible component of I and let $l \in \{2, \ldots, k\}$ be minimal such that $x_1^d \in \mathfrak{q}_l$. Define $\tilde{I} := \bigcap_{i=1}^{l-1} \mathfrak{q}_i$ as the intersection of all irreducible components which do not contain x_1^d. Then the set $\tilde{I} \setminus \mathfrak{q}$, where $\mathfrak{q}$ is an arbitrary irreducible component of I having x_1^d as a minimal generator, contains exactly one minimal monomial t. Moreover, $x_1^d t$ is a minimal generator of I.*

Proof: Recall by construction $\mathfrak{q} = \mathfrak{q}_l$ is one possible choice, but there may be others.

Consider two minimal monomials t and t' of $\tilde{I} \setminus \mathfrak{q}$. By construction it follows immediately $\tau, \tau' \in I$, where $\tau := \mathrm{lcm}(x_1^d, t) = x_1^d t$ and $\tau' := \mathrm{lcm}(x_1^d, t') = x_1^d t'$. Let m and m' be minimal generators of I such that $m \mid \tau$ and $m' \mid \tau'$. Since $\frac{\tau}{x_1}, \frac{\tau'}{x_1} \notin \mathfrak{q}$ we have $\deg_{x_1} m = \deg_{x_1} m' = d$. By minimality of t and t' it follows $\frac{\tau}{x_i} \notin I$ for all $x_i \in \mathrm{supp}\, t$ and $\frac{\tau'}{x_i} \notin I$ for all $x_i \in \mathrm{supp}\, t'$. Hence, $\tau = m$ and $\tau' = m'$.

If $m = m'$ then it will follow also $t = t'$ since two minimal monomials of $\tilde{I} \setminus \mathfrak{q}$ cannot differ only in the degree of x_1. So let us assume $m \ne m'$. Then by genericity of I there exists a minimal generator m'' of I which satisfies $\deg_{x_i} m'' < \max(\deg_{x_i} m, \deg_{x_i} m')$

for all $x_i \in \operatorname{supp} m''$. Hence, $\deg_{x_1} m'' < d$ and, consequently, $m'' \mid_{x_1=1} \in \mathfrak{q}$. But since m'' divides $\operatorname{lcm}(\tau, \tau')$ we have also that $m'' \mid_{x_1=1}$ divides $\operatorname{lcm}(t, t')$. This implies $\operatorname{lcm}(t, t') \in \mathfrak{q}$ and since $\mathfrak{q}$ is irreducible we must have $t \in \mathfrak{q}$ or $t' \in \mathfrak{q}$, a contradiction.

In summary, we observed that $\tilde{I} \setminus \mathfrak{q}$ contains exactly one minimal monomial t and that $x_1^d t$ is a minimal generator of I. □

Proof of Theorem 3: We proceed by induction on the number of variables n. The initial case $n = 1$ is trivial. So let us assume that the statement holds for $(n-1)$ variables. In particular, the statement holds for the localization I_∞ and for each ideal I_d introduced in Remark 1, in the latter case just consider the quotient $I_d/(x_1)$ and lift the result.

Fix an arbitrary $l \in \{2, \dots, k\}$ and consider the set $D = (\mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_{l-1}) \setminus \mathfrak{q}_l$. If $x_1 \notin \mathfrak{p}_l$ then the assertion follows immediately by the corresponding property of I_∞. So, consider the case that x_1^d is a minimal generator of $\mathfrak{q}_l$ for some positive integer d. If $x_1^d \notin \mathfrak{q}_{l-1}$ the assertion follows from Lemma 3. So, finally, it remains to consider the case $x_1^d \in \mathfrak{q}_{l-1}$. Let $r \in \{1, \dots, l-1\}$ be the minimal index such that $x_1^d \in \mathfrak{q}_r$. If $r = 1$ then the assertion follows immediately by the corresponding property of I_d. Suppose, $1 < r < l$. Then it follows the existence of a uniquely determined minimal monomial $t \in (\mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_{r-1}) \setminus \mathfrak{q}_l$ from Lemma 3. According to Remark 1 I_d has the form $I_d = \tilde{I} \cap \mathfrak{q}_r \cap \dots \cap \mathfrak{q}_{l'}$, where $l' \in \{l, \dots, k\}$ is the maximal index such that $x_1^{d-1} \notin \mathfrak{q}_{l'}$ and $\tilde{I}$ is a certain monomial overideal of $\mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_{r-1}$ with the property that x_1^d is a minimal generator for all its irreducible components. Applying the induction assumption to I_d it follows the existence of a uniquely determined minimal monomial $s \in (\tilde{I} \cap \mathfrak{q}_r \cap \dots \cap \mathfrak{q}_{l-1}) \setminus \mathfrak{q}_l$. Since $\mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_{r-1} \cap I_d = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_{l-1}$ any element of D must be a multiple of $\operatorname{lcm}(t, s)$. The simple observation $\operatorname{lcm}(t, s) \in D$ finishes the proof of the second part of Theorem 3 and the first part follows immediately in view of Proposition 3. □

7. The 3-variate Case

Another class of modules $M = R/I$ for which the validity of Stanley's Conjecture can be proved using the results presented in Section 5 are the quotients modulo monomials ideals in at most three variables.

Theorem 4. *Let $I \subset R = \mathbb{K}[x_1, x_2, x_3]$ be a monomial ideal and $M = R/I$. Then it holds $Sdim\, M = \min_{\mathfrak{p} \in \operatorname{Ass} M} \dim R/\mathfrak{p}$. Moreover, let $I = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_k$ be an irredundant decomposition in irreducible components, where $\dim R/\mathfrak{q}_i \geq \dim R/\mathfrak{q}_j$ for all integers i and j such that $1 \leq i < j \leq k$. For all $l \in \{2, \dots, k\}$ set $D_l := (\mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_{l-1}) \setminus \mathfrak{q}_l$ and define $T_l := \left\{ t \in D_l \ : \ t \text{ is a monomial and } \forall x_i \in \operatorname{supp} t \setminus \mathfrak{p}_l : \frac{t}{x_i} \notin D_l \right\}$. Then for any $l \in \{2, \dots, k\}$ and any $t, t' \in T_l$ either it holds $t = t'$ or there exists $x_i \in \mathfrak{p}_l$ such that $\deg_{x_i} t \neq \deg_{x_i} t'$.*

Proof: Again it suffices to prove the second part of the statement. Then the first part is an immediate consequence of Corollary 1.

Assume, there exists an ideal $I = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_k \in \mathbb{K}[x_1, x_2, x_3]$ which violates the assertion. Let $l \in \{2, \dots, k\}$ be minimal such that T_l contains two distinct monomials

t and t' which coincide in the degree in all variables belonging to $\mathfrak{p}_l$. Such monomials must differ in the degrees in at least two variables belonging to Y_l, hence, it follows $\dim R/\mathfrak{q}_l \geq 2$. Taking into account the order of irreducible components and the fact that we are in the 3-variate case we can deduce $\dim R/\mathfrak{q}_1 = \cdots = \dim R/\mathfrak{q}_l = 2$. Hence, $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{l-1}$ is a principal ideal, obviously this refutes the existence of monomials t and t' with the assumed properties. □

At the end of this section we present an example which demonstrates that already in four variables we may have $Sdim\, R/I < \dim R/I$ even for pure dimensional monomial ideals which are connected in codimension 1.

Example 10. *Consider $I = (x, y^3) \cap (u^2, y^2) \cap (y, z) \cap (u, z^2) \subset \mathbb{K}[x, y, z, u]$. The two monomials y^2u, y^2z^2 belong to all but the first irreducible component, hence, they require a direct summand containing $y^2\mathbb{K}[z, u]$. A similar argument shows that y^3, xy^2 require a direct summand containing $y^2\mathbb{K}[x, y]$. Obviously, both direct summands must be equal and since any module $t\mathbb{K}[Y]$ satisfying $y^2\mathbb{K}[z, u] + y^2\mathbb{K}[x, y] \subseteq t\mathbb{K}[Y]$ will have a nontrivial intersection with I, we have an unresolvable conflict proving $Sdim\, R/I < \dim R/I$.*

8. Cogeneric Cohen-Macaulay Monomial Ideals

Example 10 shows that there is no hope to prove a result for cogeneric monomial ideals which is similarly nice as Theorem 3. However, at least in the Cohen-Macaulay case we will be successful.

Theorem 5. *For any cogeneric Cohen-Macaulay monomial ideal $I \subset R$ it holds $Sdim\, R/I = \dim R/I$. Further, let $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_k$ be an irredundant decomposition in irreducible components, where the components are ordered in the same was as in Theorem 3. For all $l \in \{2, \ldots, k\}$ set $D_l := (\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{l-1}) \setminus \mathfrak{q}_l$ and define $T_l := \left\{ t \in D_l \ : \ t \text{ is a monomial and } \forall x_i \in \operatorname{supp} t \setminus \mathfrak{p}_l : \frac{t}{x_i} \notin D_l \right\}$. Then for any $l \in \{2, \ldots, k\}$ and any $t, t' \in T_l$ either it holds $t = t'$ or there exists $x_i \in \mathfrak{p}_l$ such that $\deg_{x_i} t \neq \deg_{x_i} t'$.*

For the proof we will need the following corollary to Lemma 1.

Corollary 3. *Let I be a cogeneric Cohen-Macaulay monomial ideal I with irredundant decomposition $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_k$ in irreducible components. Furthermore, let $\mathfrak{q}_i$ and $\mathfrak{q}_j$ be irreducible components such that $\mathrm{codim}(\mathfrak{p}_i + \mathfrak{p}_j) > \mathrm{codim} I + 1$. Then there exists an irreducible component $\mathfrak{q}_m$ of I such that*

1. $\mathfrak{q}_m \subset \mathfrak{q}_i + \mathfrak{q}_j$,
2. $\mathrm{codim}(\mathfrak{p}_i + \mathfrak{p}_m) \leq \mathrm{codim} I + 1$,
3. *$\mathfrak{q}_i$ and $\mathfrak{q}_m$ have no minimal generator in common,*
4. *$x_r^d \notin \mathfrak{q}_m$ for all minimal generators x_r^d of $\mathfrak{q}_i$ such that $x_r^{d-1} \notin \mathfrak{q}_j$.*

Proof: Lemma 1 yields the existence of $\mathfrak{q}_{l'}$ such that $\mathfrak{q}_{l'} \subset \mathfrak{q}_i + \mathfrak{q}_j$. Moreover, taking into account the cogenerity of I we can arrange that $\mathfrak{q}_{l'}$ does not contain any of the common minimal generators of $\mathfrak{q}_j$ and $\mathfrak{q}_i + \mathfrak{q}_j$. If $\mathrm{codim}(\mathfrak{p}_i + \mathfrak{p}_{l'}) > \mathrm{codim} I + 1$ we repeat this process and after finitely many iterations we obtain an irreducible component $\mathfrak{q}_l$ such that $\mathfrak{q}_l \subset \mathfrak{q}_i + \mathfrak{q}_j$ and $\operatorname{codim}(\mathfrak{p}_i + \mathfrak{p}_l) < \operatorname{codim}(\mathfrak{p}_i + \mathfrak{p}_j)$. By induction on $\operatorname{codim}(\mathfrak{p}_i + \mathfrak{p}_j)$ we can even assume $\operatorname{codim}(\mathfrak{p}_i + \mathfrak{p}_l) \leq \operatorname{codim} I + 1$. Now, suppose $\mathfrak{q}_i$ and $\mathfrak{q}_l$ have a minimal generator in common. Then by definition of cogeneric monomial ideals there exists $\mathfrak{q}_{m'} \subset \mathfrak{q}_i + \mathfrak{q}_l \subseteq \mathfrak{q}_i + \mathfrak{q}_j$ having no minimal generator in common with $\mathfrak{q}_i + \mathfrak{q}_l$. If $\mathfrak{q}_i$ and $\mathfrak{q}_{m'}$ still have common minimal generators we repeat the argument with $l = m'$. Since $\mathfrak{q}_i$ cannot share any of its minimal generators with more than one of the intermediate components appearing in this process, eventually, we obtain $\mathfrak{q}_m$ satisfying the first three conditions of the corollary. Finally, the validity of the fourth condition is an immediate consequence of the validity of the first and the third condition. □

Proof of Theorem 5: In analogy to the proof of Theorem 3 we proceed by induction on the number of variables n. Again the initial case $n = 1$ is trivial. So let us assume that the statement holds for $(n-1)$ variables.

Suppose, there exists $l \in \{2, \ldots, k\}$ such that T_l contains two distinct elements t and t' which have the same degree in any variable belonging to $\mathfrak{p}_l$. Let $\hat{t}$ and $\hat{t}'$ denote the monomials obtained by setting all variables of $\mathfrak{p}_l$ equal to 1 in t or t', respectively. By construction we have $\operatorname{lcm}(s, \hat{t}) = t$ and $\operatorname{lcm}(s, \hat{t}') = t'$, where $s := \gcd(t, t')$ is the greatest common divisor of t and t'. From $t, t' \in \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{l-1}$ we conclude that any irreducible component $\mathfrak{q}_j$, $1 \leq j < l$, which does not contain s has to contain both $\hat{t}$ and $\hat{t}'$ but cannot contain $\gcd(\hat{t}, \hat{t}')$. In conclusion,

$$(9) \qquad \forall j \in \{1, \ldots, l-1\} \; : \; s \notin \mathfrak{q}_j \Longrightarrow \operatorname{codim}(\mathfrak{p}_j + \mathfrak{p}_l) \geq \operatorname{codim} I + 2 \, .$$

$s \in \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{l-1}$ is impossible by construction of T_l. Hence, there exists $j \in \{1, \ldots, l-1\}$ such that $s \notin \mathfrak{q}_j$. Application of Corollary 3 yields the existence of an irreducible component $\mathfrak{q}_m$ with the following three properties: (i) $\mathfrak{q}_m \subset \mathfrak{q}_l + \mathfrak{q}_j$, (ii) $\operatorname{codim}(\mathfrak{p}_m + \mathfrak{p}_l) \leq \operatorname{codim} I + 1$, and (iii) $x_r^a \notin \mathfrak{q}_m$ for all minimal generators x_r^a of $\mathfrak{q}_l$ such that $x_r^{a-1} \notin \mathfrak{q}_j$. I_∞ as a localization of I at a monomial prime ideal is cogeneric. Hence, application of the induction assumption to I_∞ shows $x_1 \in \mathfrak{p}_l$. Therefore, $\mathfrak{q}_l$ possesses a minimal generator of the form x_1^d and by the assumption on the order of the irreducible components of I this implies $x_1^{d-1} \notin \mathfrak{q}_j$ and, therefore, $x_1^d \notin \mathfrak{q}_m$ according to (iii). Consequently, $m < l$ and in view of (9) and (ii) we obtain $s \in \mathfrak{q}_m$. Application of (i) yields $s \in \mathfrak{q}_m \subset \mathfrak{q}_j + \mathfrak{q}_l$, in contradiction to $s \notin \mathfrak{q}_j$ and $s \notin \mathfrak{q}_l$. In summary, the assumption that T_l contains distinct elements t, t' which have the same degree in all variables belonging to $\mathfrak{p}_l$ must be wrong. This proves the second part of the theorem and the validity of the first statement follows by Corollary 1. □

9. Connections to Properties of Related Complexes

Proposition 1 is strongly related to the notion of shellable nonpure simplicial complexes due to Björner and Wachs (see [BW]). In fact, the assumptions of our proposition turn out to be an algebraic translation of Björner's and Wachs' notion in terms of the corresponding Stanley-Reisner ring. Our decompositions are directly related to [BW, Formula 2.2] rather than to the decompositions of Stanley-Reisner rings investigated by Björner and Wachs in Section 12 contained in the second part of [BW]. In the pure case the connection can be found also in [St79], where shellable pure complexes were generalized to partitionable pure complexes. In fact, our Proposition 2 is the algebraic counter part to partitionable nonpure complexes. Let us illustrate this fact by application of the opposite direction of Proposition 2 to the example presented at the top of page 149 in [St79].

Example 11 (Stanley). *The upper partition* $[\emptyset, ab], [c, cd], [d, de], [e, ce]$ *of the simplicial complex* $\Delta = \langle ab, cd, ce, de\rangle$ *corresponds to the direct sum decomposition* $\mathbb{K}[\Delta] = \mathbb{K}[a,b] \oplus c\mathbb{K}[c,d] \oplus d\mathbb{K}[d,e] \oplus e\mathbb{K}[c,e]$ *of the Stanley-Reisner ring of* Δ.

In view of Proposition 2 we obtain $I = (c,d,e) \cap (a,b,e) \cap (a,b,c) \cap (a,b,d) \subset \mathbb{K}[a,b,c,d,e]$ *and the sequence* $1, c, d, e$ *of standard monomials modulo* I. *Further, we compute* $J_1 = (1)$, $J_2 = (c,d,e)$, $J_3 = (d,e,ac,bc)$, $J_4 = (e,ac,ad,bc,bd,cd)$, $J_5 = (ac,ad,ae,bc,bd,be,cd,de)$. *This sequence satisfies the condition of Proposition 2, in particular, the critical monomials* $cd, de \in J_5 \setminus I$ *are contained in the direct sum decomposition.*

Stanley's example illustrates that $Sdim\, R/I = \dim R/I$ can hold even for pure monomial ideals which are not connected in codimension 1. Let us remark another immediate consequence of Stanley's paper. Shellability implies the Cohen-Macaulay property of the Stanley-Reisner ring independent on the field characteristic. Therefore, the attempt to treat Reisner's Example 2 by Proposition 1 must fail.

The next remark concerns the combinatorial properties of nonsquarefree monomial ideals. Miller, Sturmfels and Yanagawa proved that both the Scarf complex and the Stanley-Reisner complex corresponding to a pure generic monomial ideal are shellable (see [MSY, Theorem 2.5]). The generic monomial ideals are exactly this large class of ideals for which we were able to construct a Stanley decomposition by means of Proposition 3, which is the natural generalization of Proposition 1 to the nonsquarefree case. This motivates to study the relationship between the algebraic properties expressed in the assumptions of Proposition 3 and the shellability of complexes related to a nonsquarefree monomial ideal. There are indications that this connection will be not as close as in the squarefree case. At the one hand side there exist generic monomial ideals which are nonpure and whose Scarf complex is not shellable in the nonpure sense. But in view of Proposition 3 nonpure generic ideals show the same nice behavior as pure generic monomial ideals.

Example 12 ([BPS]Example 4.2). *Bayer, Peeva and Sturmfels showed that the Scarf complex of* $I = (xyz, x^4y^3, x^3y^5, y^4z^3, y^2z^4, x^2z^2) = (x,y^2) \cap (x^2,y) \cap (x,y^4,z^4) \cap (x,z^3) \cap (y,z^2) \cap (x^3,z) \cap (x^4,y^5,z) \cap (y^3,z) \subset \mathbb{K}[x,y,z]$ *is not shellable even in the nonpure sense. But the above order of irreducible components together with the sequence* $1, x, y^2, y^4, x^2, xy, x^3y, x^4y$ *of standard monomials provides an "algebraic shelling" of* R/I.

At the other hand side the ideal I considered in Example 1 is pure and satisfies $Sdim\, R/I < \dim R/I$ but its Scarf complex is a square and, hence, shellable.

However, the Stanley-Reisner complex of the ideal from Example 1 is not even partitionable and, hence, there is still a good chance to find a connection for pure ideals which makes use of the shellings of both related complexes.

10. Outlook to Graded Polynomial Modules

Finally, let us generalize our investigations in two directions. As before, let $R = \mathbb{K}[X]$ be a polynomial ring but equipped with a $\mathbb{N}^\nu$-grading for some $\nu \in \{1, \dots, n\}$. Furthermore, let F be a free R-module with basis $\{e_1, \dots, e_m\}$ and equipped with a $\mathbb{Z}^\nu$-grading which extends the $\mathbb{N}^\nu$-grading of R in the sense that $\deg_{\mathbb{Z}^\nu} ue_j = \deg_{\mathbb{N}^\nu} u + \deg_{\mathbb{Z}^\nu} e_j$ for arbitrary homogeneous elements $u \in R$ and $1 \le j \le m$. Finally, assume $\sum_{i=1}^{\nu} d_i = 1$ for all $x \in X$, where $\deg_{\mathbb{N}^\nu} x = (d_1, \dots, d_\nu)$. Throughout this section we will always assume that the field $\mathbb{K}$ is infinite.

We consider modules $M = F/N$, where N is a $\mathbb{Z}^\nu$-graded submodule of F. In the particular case that N is monomial, what here should mean, that N is generated by elements of the form ue_j, where $u \in R$ is a monomial and $1 \le j \le m$, the previous results extend easily. We subdivide the set B of minimal generators of N in m pairwise disjoint (not necessarily nonempty) sets $B_1, \dots, B_m$, where B_i consists of all minimal generators of the form ue_i, $1 \le i \le m$. Let I_i, $i \in \{1, \dots, m\}$, denote the monomial ideal of R generated by the set $\{u : ue_i \in B_i\}$. Since, $\mathit{depth}(A \oplus B) = \min(\mathit{depth}\, A, \mathit{depth}\, B)$ for arbitrary R-modules A and B we can construct a Stanley decomposition for M by plugging in Stanley decompositions for each module R/I_i in the direct sum decomposition $M = \bigoplus_{i=1}^{m} (R/I_i)\, e_i$.

Now, let N be an arbitrary $\mathbb{Z}^\nu$-graded submodule of F and consider an admissible term order $\prec$ of F refining the $\mathbb{Z}^\nu$-grading, i.e. for any monic $\mathbb{Z}^\nu$-homogeneous elements $a, b \in F$ such that $\deg_{\mathbb{Z}^\nu} a < \deg_{\mathbb{Z}^\nu} b$ it holds $\mathrm{in}_\prec a \prec \mathrm{in}_\prec b$. As usual, $\mathrm{in}_\prec a$ denotes the initial term of $a \in F$ with respect to $\prec$. The modules $F/\mathrm{in}_\prec N$ and F/N are isomorphic graded $\mathbb{K}$-spaces via the linear mapping induced by the assignment $a + \mathrm{in}_\prec N \mapsto a + N$ for all module monomials $a = ue_i$. In view of this isomorphism any Stanley decomposition of $\mathrm{in}_\prec M := F/\mathrm{in}_\prec N$ is also a Stanley decomposition of $M = F/N$ whenever it holds $\mathit{depth}\, M \le \mathit{depth}\ \mathrm{in}_\prec M$. Gräbe proved the inequality $\mathit{depth}\, M \ge \mathit{depth}\ \mathrm{in}_\prec M$ for arbitrary homogeneous M and there are examples where the inequality is proper (see [G, Theorem 4.1]). But we have the additional freedom of applying two preparatory transformations before we fix the order $\prec$ and the corresponding initial module $\mathrm{in}_\prec M$. First, we can apply a variable transformation which has to respect the $\mathbb{N}^\nu$-grading of R. Second, we can transform the basis of F in a $\mathbb{Z}^\nu$-grading respecting way. There are examples where both transformations are necessary, but it is open if they are also sufficient.

Remark 2. *Let $\mathbb{K}$ be infinite. Suppose, that the following two questions have a positive answer:*

1. *Does Conjecture 2 hold for arbitrary monomial ideals $I \subset R$?*
2. *Given an arbitrary $\mathbb{Z}^\nu$-graded module $M = F/N$. Does there exist grading respecting transformations of R and F and an admissible term order $\prec$ of the transformed module monomials such that $\mathit{depth}\, M = \mathit{depth}\ \mathrm{in}_\prec M$?*

Then Stanley's Conjecture 1 holds for arbitrary $\mathbb{Z}^\nu$-graded polynomial modules M.

The proof is an immediate consequence of our above observations. This article is a step towards the solution of the first question. In the case $\nu = 1$ and $F = R$ a positive answer to the second question was given by Bayer and Stillman who proved that the equality $depth\, M = depth\ \mathrm{in}_\prec M$ holds when $\prec$ is a reverse lexicographical term order and M is in generic coordinates ([BS, Theorem 2.4], see also [E, Section 15.7]). In the extremal case $F = R$ and $\nu = n$ the second question has obviously a positive answer since $M = \mathrm{in}_\prec M$ even without application of a preparatory transformation. This gives reason to the hope that question 2 has a general positive answer in the case $F = R$, when each group of variables of some fixed $\mathbb{N}^\nu$-degree is transformed generically and an admissible term order $\prec$ is used which acts as a reverse lexicographical term order on each set of transformed variables of a fixed $\mathbb{N}^\nu$-degree. Note, if the variables are enumerated in such a way that variables of the same $\mathbb{N}^\nu$-degree have consecutive numbers then the variable transformations which respect the $\mathbb{N}^\nu$-grading of R correspond to the subgroup of $GL(n, \mathbb{K})$ formed by all block-diagonal matrices with blocks corresponding to groups of variables of common $\mathbb{N}^\nu$-degree. In the 3-variate case it is easy to generalize Theorem 4 in this way.

Theorem 6. *Let $\mathbb{K}$ be an infinite field and $R = \mathbb{K}[x_1, x_2, x_3]$ a $\mathbb{N}^\nu$-graded polynomial ring over $\mathbb{K}$, where $\nu \in \{1, 2, 3\}$. Assume that for each of the variables x_1, x_2, x_3 the sum of the coordinates of its $\mathbb{N}^\nu$-degree is 1.*

Then $Sdim\, R/I = \min_{\mathfrak{p} \in \mathrm{Ass}\, R/I} \dim R/\mathfrak{p}$ for any $\mathbb{N}^\nu$-homogeneous ideal I of R.

Proof: The assertion follows from [BS, Theorem 2.4] in case $\nu = 1$ and from Theorem 4 in case $\nu = 3$. So it remains to consider only the case $\nu = 2$, where we can assume without loss of generality that $\deg_{\mathbb{N}^\nu} x_1 = (1, 0)$ and $\deg_{\mathbb{N}^\nu} x_2 = \deg_{\mathbb{N}^\nu} x_3 = (0, 1)$. Using exactly the same arguments as Eisenbud in [E, Chapter 15] one can show that there is a linear variable transformation fixing x_1 and mapping x_2 and x_3 to linearly independent linear combinations of them such that the initial ideal of the transformed ideal with respect to an arbitrary degree compatible admissible term order satisfying $x_3 \prec x_2$ possesses a minimal generator $x_1^\alpha x_2^\beta$ which has strictly larger x_2-degree than any other minimal generator and which has minimal x_1-degree among all minimal generators. In fact, concentrating only on the transformation of x_2 and x_3 there is a Zariski open subset of $GL(2, \mathbb{K})$ describing suitable transformations. Further, according to [E, Proposition 15.15] it suffices to check the equality $depth\, R/I = depth\, R/\mathrm{in}_\prec I$ for the case that each variable occurs in some minimal generator of $\mathrm{in}_\prec I$. Let G denote the reduced Gröbner basis of I with respect to $\prec$. By homogeneity of I each element of G can be written in the form $x_1^{\alpha'}(x_2^{\beta'} + p(x_2, x_3))x_3^{\gamma'}$, where $p(x_2, x_3)$ is a homogeneous polynomial in x_2 and x_3 of $\mathbb{N}$-degree β' without $x_2^{\beta'}$-term.

One easily observes $depth\, R/I \leq 1$. Hence, it remains to prove that whenever $\mathrm{in}_\prec I$ possesses a trivial irreducible component then also I possesses a trivial primary component (i.e. $(x_1, x_2, x_3) \in \mathrm{Ass}\ R/I$). Now, assume that (x_1^a, x_2^b, x_3^c) is an irreducible component of $\mathrm{in}_\prec I$ and consider the ideal J generated by all elements $g\,|_{x_1=1}$ which are obtained by subsituting 1 for x_1 in the Gröbner basis elements $g \in G$ which satisfy $\deg_{x_1} \mathrm{in}_\prec g < a$ and $\deg_{x_3} \mathrm{in}_\prec g < c$. The preparatory transformation ensures that J contains a homogeneous element whose initial term is a power of x_2. Moreover, for any homogeneous element $h \in J$ we have $\deg_{x_2} \mathrm{in}_\prec h \geq b$ or $\deg_{x_3} \mathrm{in}_\prec h \geq c$ since, otherwise, there must be a nonnegative integer $a' < a$ such that $x^{a'} h \in I$ which is impossible because of $x^{a'} \mathrm{in}_\prec h \notin \mathrm{in}_\prec I \subseteq (x_1^a, x_2^b, x_3^c)$. $(x_1^a, x_3^c) + J$ is a (x_1, x_2, x_3)-primary ideal containing I and by the above observations I is not contained in any higher dimensional primary subideal of $(x_1^a, x_3^c) + J$. Hence, $(x_1, x_2, x_3) \in$ Ass R/I and we are through. □

Finally, let us demonstrate the additional problems arising in the *rank* $F > 1$ case by two examples.

Example 13. *Consider* $R = \mathbb{K}[x_1, x_2, x_3]$ *with an arbitrary* $\mathbb{N}^\nu$*-grading. Further, let rank* $F = 3$ *and* $\deg_{\mathbb{Z}^\nu} e_1 = \deg_{\mathbb{Z}^\nu} e_2 = \deg_{\mathbb{Z}^\nu} e_3 = (0, \ldots, 0)$. *Assume char* $\mathbb{K} \neq 2$ *and let* $M = F/N$, *where* $N = (x_1e_1 + x_1e_2, x_2e_1 + x_2e_3, x_3e_2 + x_3e_3)$. *Then the elements* $x_1e_1, x_2e_1, x_3e_2, x_1x_2e_2, x_1x_2x_3e_3$ *generate the initial module* $\mathrm{in}_\prec M = F/\mathrm{in}_\prec N$, *where* $\prec$ *is an arbitrary admissible term order such that* $e_3 \prec e_2 \prec e_1$. *Hence, depth* $\mathrm{in}_\prec M = 1 <$ *depth* M *and by symmetry reasons the same is true for any admissible term order* $\prec$. *However, after an obvious basis transformation we obtain* $N = (x_1\hat{e}_1, x_2\hat{e}_2, x_3\hat{e}_3)$ *and, hence, depth* $\mathrm{in}_\prec M =$ *depth* $M = 2$.

Example 14. *Consider* $R = \mathbb{K}[x_1, \ldots, x_n]$ *with respect to an arbitrary* $\mathbb{N}^\nu$*-grading. Furthermore, let rank* $F = n + 1$ *and* $\deg_{\mathbb{Z}^\nu} e_1 = (0, \ldots, 0)$, $\deg_{\mathbb{Z}^\nu} e_i = \deg_{\mathbb{N}^\nu} x_{i-1}$ $(i = 2, \ldots, n+1)$ *the degrees of the free generators of* F. *Then the module* $M = F/N$, *where* N *is generated by the elements* $x_ie_1 + e_{i+1}$, $i = 1, \ldots, n$, *is free of rank* 1 *and has depth* n.

If two module monomials ue_i *and* ve_j *have the same* $\mathbb{Z}^\nu$*-degree we have to break ties by the order* $\prec$. *If ties are broken by* $ue_i \prec ve_j \Leftrightarrow i < j$ *then we obtain* $\mathrm{in}_\prec N = (e_2, \ldots, e_{n+1})$ *and depth* $\mathrm{in}_\prec M = n =$ *depth* M. *However, depth* $\mathrm{in}_\prec M = 0 <$ *depth* M *if we break ties according to* $ue_i \prec ve_j \Leftrightarrow i > j$.

Acknowledgment

I thank Bernd Sturmfels for drawing my attention to this subject and Hans-Gert Gräbe and Jürgen Stückrad for helpful comments. I am thankful to the Mathematical Sciences Research Institute and the Mathematics Department of the University of California at Berkeley for their hospitality during the one year research visit in which the work on this project was done. Last but not least I am grateful to the Deutsche Forschungsgemeinschaft for supporting the visit by a research grant.

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