Prescribing scalar and boundary mean curvature on the three dimensional half sphere

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Abstract

We consider the problem of prescribing the scalar curvature and the boundary mean curvature of the standard half three sphere, by deforming conformally its standard metric. Using blow up analysis techniques, we prove some existence and compactness results.

1 Introduction

In this paper we study some equation arising in differential geometry, when the metric of a Riemannian manifold is conformally deformed. Precisely, is given a manifold with boundary (M, g) of dimension greater or equal than three; transforming the metric g into $g' = v^{\frac{4}{n-2}} g$, where v is a smooth positive function, the scalar curvatures $R_g, R_{g'}$ and the mean curvatures of the boundary $h_g, h_{g'}$, with respect to g and g' respectively, are related by the formulas

$$\begin{cases} -4\frac{n-1}{n-2}\Delta_g v + R_g v = R_{g'} v^{\frac{n+2}{n-2}}, & \text{in } M; \\ \frac{2}{n-2}\frac{\partial v}{\partial \nu} + h_g v = h_{g'} v^{\frac{n}{n-2}}, & \text{on } \partial M, \end{cases} \tag{1}$$

see e.g. [5]. In the above equation, ν denotes the outward unit vector perpendicular to ∂M, with respect to the metric g.

A problem arises naturally when looking at equation (1): assigned two functions $K : M \to \mathbb{R}$ and $H : \partial M \to \mathbb{R}$, does exists a metric g' conformally equivalent to g such that $R_{g'} \equiv K$ and $h_{g'} \equiv H$? From equation (1), the problem is equivalent to finding a smooth positive solution v of the equation

$$\begin{cases} -4\frac{n-1}{n-2}\Delta_g v + R_g v = K v^{\frac{n+2}{n-2}}, & \text{in } M; \\ \frac{2}{n-2}\frac{\partial v}{\partial \nu} + h_g v = H v^{\frac{n}{n-2}}, & \text{on } \partial M. \end{cases} \tag{2}$$

The requirement about the positivity of v is necessary for the metric g' to be Riemannian. For the two-dimensional case, there are analogous equations involving exponential nonlinearities.

We are mainly interested in the so-called *positive case*, see [21], when the quadratic part of the Euler functional associated to (2) is positive definite.

A first criterion for existence of solutions of (2), and also a proof of regularity, was given by P. Cherrier, [17], who discovered the second equation in (1). He proved that if the energy of some test function is below an explicit threshold, then problem (2) admits a solution as a mountain pass critical point. Combining this criterion with the Positive Mass Theorem by R. Schoen and S.T. Yau [35], J. Escobar proved some existence results in the interesting particular case of constant K and H, see [21], [22]. He proved that almost every compact manifold with boundary can be conformally deformed so that its scalar curvature is 1 and the boundary is minimal, i.e. the mean curvature is zero. He also gives some results when H is a constant close to zero. More recently Z.C. Han and Y.Y. Li, see [25], [26], extended most of the results of J. Escobar to the case in which $K \equiv 1$ and H is any constant. They also prove a compactness results in the locally conformally flat manifolds with umbilic boundary.

We consider here the case of the standard half sphere $S^n_+ = \{x \in \mathbb{R}^{n+1} : \|x\| = 1, x_{n+1} > 0\}$ endowed with its standard metric g_0, and in particular the case $n = 3$; the functions K and H are now non constant and K will always assumed to be positive. We are thus reduced to find positive solutions v of the problem

$$
\begin{cases} -4\frac{n-1}{n-2}\,\Delta v + n(n-1)\,v = K(x)\,v^{\frac{n+2}{n-2}}, & \text{in } S^n_+; \\ \frac{2}{n-2}\frac{\partial v}{\partial \nu} = H(x')\,v^{\frac{n}{n-2}}, & \text{on } \partial S^n_+. \end{cases} \tag{3}
$$

Problem (3) is in some sense related to the well-known *Scalar Curvature Problem* on S^n

$$
-4\frac{n-1}{n-2}\,\Delta v + n(n-1)\,v = K(x)\,v^{\frac{n+2}{n-2}}, \qquad \text{in } S^n, \tag{4}
$$

to which much work has been devoted, see [3], [7], [8], [9], [11], [12], [14], [15], [16], [27], [29], [30], [36] and references therein. As for (4), also for problem (3) there are topological obstructions for existence of solutions, based on Kazdan-Warner type conditions, see [10] and also the proof of Proposition 8.7. Hence it is not expectable to solve problem (3) for all the functions K and H, and it is natural to impose some conditions on them.

We would like to point out the following features of the scalar curvature problem on S^n in lower dimensions, referring to the above-mentioned papers. For $n = 2$, non-converging Palais Smale sequences are characterized by the presence of just one bubble. Under generic assumptions on K, it turns out that when $n = 3, 4$, solutions of (4), or of some subcritical approximation, possess only *isolated simple blow ups*, see Section 4 for the definition. When $n = 3$ there is indeed just one blow up point, while for $n = 4$ blow ups may as well occur at more points.

We now discuss problem (3). For $n = 2$ (about the corresponding equation with exponential nonlinearities), P.L. Li and J.Q. Liu proved in [28] that compactness is lost along one bubble only, as in the case of the problem on the sphere. The only difference is that blow up can only occur at the boundary of S^2_+. For the case $H \equiv 0$ and positive K, they prove existence results which are in some sense reminiscent of those of [11], see also [14], [15].

In [31] Y.Y. Li considered the case of $n = 3$ and $H \equiv 0$. Under generic assumptions on K, he proved that blow ups are isolated simple and at only the boundary. He also stated that, as for (4) when $n = 3$, blow ups may occur at most one point. Actually the last statement is not true: we correct this here and we prove that (also for non constant H) blow ups, which are always isolated simple at the boundary, can be multiple, see Section 6. Hence the situation could be considered similar to that of (4) for $n = 4$. This fact, in the case of $H \equiv 0$, could be roughly explained as follows. Reflecting both K and v evenly to the whole S^3, one could study symmetric solutions of (4). The blow up analysis for the three dimensional case strongly relies on the differentiability of K at blow up points. This implies that blow ups of symmetric functions outside ∂S^3_+, which are multiple, are ruled out. This argument does not apply when blow up points are on ∂S^3_+, since the symmetric extension of K is not regular there. See Remark 6.7 for a more quantitative explanation of this fact.

For the case of any n, some results are proved in [4] when K and H are close to some constants; here we are extending some of those results for $n = 3$ without the *close to constant* conditions, see also Remark 7.2. In the paper [13] the case $K \equiv 0$ and H close to a positive constant is considered, for $n \geq 3$. In the forthcoming paper [19] we will extend some of those results to the non-perturbative case.

Our first result is the following.

Theorem 1.1 *Assume $n = 3$, let $K : \overline{S^3_+} \to \mathbb{R}$ be a positive C^1 function and let $H : \partial S^3_+ \to \mathbb{R}$ be of class C^2. Let $\varphi : \partial S^3_+ \to \mathbb{R}$ be the function defined by*

$$
\varphi(x') = 4\pi \sqrt{\frac{6}{K(x')}} \left(\frac{\pi}{2} - \arctan \left(H(x') \sqrt{\frac{6}{K(x')}} \right) \right), \qquad x' \in \partial S^3_+. \tag{5}
$$

Suppose that for some point $q \in \partial S^3_+$ the following condition holds

$$
\varphi(q) = \min_{\partial S^3_+} \varphi; \qquad \frac{\partial K}{\partial \nu}(q) < 0. \tag{6}
$$

Then there exists a positive solution of problem (3).

The proof of Theorem 1.1 relies on the study of the following subcritical approximation of equation (3)

$$
\begin{cases} -4\frac{n-1}{n-2}\, \Delta u + n(n-1)\, u = K\, u^p, & \text{in } S^n_+; \\ \frac{2}{n-2} \frac{\partial u}{\partial \nu} = H\, u^{\frac{p+1}{2}}, & \text{on } \partial S^n_+. \end{cases} \tag{7}
$$

Here the exponent p is converging to $\frac{n+2}{n-2}$ from below. As mentioned before, blow ups of equation (7) can occur only at the boundary of S^3_+. Nevertheless, if v_p denotes a mountain pass solution of (7) for $p < \frac{n+2}{n-2}$, condition (6) implies that $\{v_p\}_p$ is uniformly bounded for $p \to \frac{n+2}{n-2}$, and hence converges to a solution of (3). The function $\varphi(x')$ represents the blow up energy at a point $x' \in \partial S^3_+$ and plays a crucial role in the blow up analysis. Indeed, see Section 6, blow ups can occur only at critical points of φ. We note that when H is a constant function, critical points of φ coincide with critical points of $K|_{\partial S^3_+}$, see also [31].

Under generic assumptions on K and H, ($(K, H) \in \mathcal{A}$ in the notation below), it is possible to give a complete description of the behavior of general solutions of (7) when p converges to $\frac{n+2}{n-2}$, and to deduce existence and compactness results for equation (3). The blow up analysis provides necessary conditions on these solutions, while the Implicit Function Theorem gives sufficient conditions for existence of solutions highly concentrating at some points of ∂S^3_+. In this way one can compute the total Leray Schauder degree of the solutions of (3) in the space $C^{2,\alpha}(\overline{S^3_+})$, for some $\alpha \in (0, 1)$. Such a method has been used in [36] and [30] for problem (4) in dimensions 3 and 4 respectively.

To state our next result we need to introduce some notation, which considerably simplifies in the case $H \equiv 0$, see Remark 1.3. Given $K \in C^2(\overline{S^3_+})$ and $H \in C^2(\partial S^3_+)$, let $\varphi \in C^2(\partial S^3_+)$ be defined by formula (5), and set

$$
\mathcal{F} = \left\{ q \in \partial S^3_+ \, : \, \nabla \varphi(q) = 0 \right\}; \qquad \mathcal{F}^{+(-)} = \left\{ q \in \partial S^3_+ \, : \, \nabla' \varphi(q) = 0, \frac{\partial K}{\partial \nu}(q) > 0\, (< 0) \right\};
$$

$$
\mathcal{M}_{K,H} = \left\{ v \in C^{2,\alpha}(\overline{S^3_+}) \, : \, v \text{ satisfies(3)} \right\}.
$$

Here ∇' denotes the gradient of functions defined on ∂S^n_+.

For $q \in \partial S^3_+$, let $\pi_q : S^3_+ \to \mathbb{R}^3_+$ denote the stereographic projection with pole $-q$. In π_q-stereographic coordinates, we define the function $G_q : S^3_+ \to \mathbb{R}$ by

$$
G_q(x) = \left(\frac{1 + |x|^2}{2} \right)^{\frac{1}{2}} \frac{1}{|x|}, \qquad x \in \mathbb{R}^3_+. \tag{8}
$$

The function G_q is the Green's function for the conformal laplacian $-8\Delta + 6$ on S^3 with pole q. Define also $\psi : \partial S^3_+ \to \mathbb{R}$ by

$$
\psi(x') = 1 + H(x') \sqrt{\frac{6}{K(x')}} \left(\arctan \left(H(x') \sqrt{\frac{6}{K(x')}} \right) - \frac{\pi}{2} \right) = 1 - \frac{H(x')\varphi(x')}{4\pi}. \tag{9}
$$

To each $\{q^1,\dots,q^N\}\subseteq \mathcal{F}\setminus\mathcal{F}^-$, $N\geq 1$, we associate an $N\times N$ symmetric matrix $M=M(q^1,\dots,q^N)$ defined by

$$
\begin{cases} M_{jj} = \frac{\partial K}{\partial \nu}(q^j)\,\frac{\psi(q^j)}{K(q^j)^{\frac{3}{2}}}, & j\in\{1,\dots,N\} \\ M_{lj} = -4\sqrt{2}\,\frac{G_{q^l}(q^j)}{K(q^l)^{\frac{1}{4}}K(q^j)^{\frac{1}{4}}}, & l,j\in\{1,\dots,N\},\quad l\neq j. \end{cases} \tag{10}
$$

Let $\rho=\rho(q^1,\dots,q^N)$ denote the least eigenvalue of M. It has been first pointed out by A. Bahri, [6], see also [8], that when the interaction between different bubbles is of the same order as the *self interaction*, the function ρ for a matrix as in (10) plays a fundamental role in the theory of the critical points at infinity. For problem (3), such kind of phenomenon appears when $n=3$.

Define the set $\mathcal{A}$ to be

$$
\begin{aligned}
\mathcal{A} \;=\; & \{(K,H)\in C^2(\overline{S^3_+})\times C^2(\partial S^3_+)\;:\;K>0,\varphi \text{ is a Morse function on } \partial S^3_+, \\
& \frac{\partial K}{\partial \nu}\neq 0 \text{ on } \mathcal{F}, \text{ and } \rho=\rho(q^1,\dots,q^N)\neq 0, \forall\, q^1,\dots,q^N\in\mathcal{F}^+\}.
\end{aligned}
$$

Let us observe that the condition $(K,H)\in\mathcal{A}$ is generic. We introduce an integer valued continuous function Index : $\mathcal{A}\to\mathbb{N}$ by the following formula

$$
\mathrm{Index}(K,H) = -1+\sum_{j=1}^{\ell}\;\sum_{\substack{\rho(q^{i_1},\dots,q^{i_j})>0,\\ 1\leq i_1<i_2\leq\cdots\leq i_j\leq\ell}} (-1)^{j+\sum_{l=1}^{j}(2-m(\varphi,q^j))},
$$

where $m(\varphi,q^{i_l})$ denotes the Morse index of φ at q^{i_l}, and $\ell = card|\mathcal{F}^+|$. Now we are able to state our next result, about existence and compactness of solutions of (3).

Theorem 1.2 *Let $n=3$ and suppose $(K,H)\in\mathcal{A}$. Then for all $\alpha\in(0,1)$, there exists some constant R depending only on $\min_{\overline{S^3_+}}K$, $\|K\|_{C^1(\overline{S^3_+})}$, $\|H\|_{C^2(S^3_+)}$, $\min\{|\rho(q^1,\dots,q^N)|\;:\;q^1,\dots,q^N\in\mathcal{F}^+, N\geq 2\}$ and α such that*

$$
\frac{1}{R}\leq v\leq R,\qquad \|v\|_{C^{2,\alpha}(\overline{S^3_+})}\leq R, \tag{11}
$$

for all positive solutions v of equation (3). Moreover (3) possesses a solution provided $\mathrm{Index}(K,H)\neq 0$.

Notice that only the least eigenvalue of $M(q^1,\dots,q^N)$ plays a role in counting the total degree of solutions of (3) and in the compactness. For instance, considering a continuous family of functions (K_t,H_t), the total degree changes when the least eigenvalue of $M_t(q^1,\dots,q^N)$ crosses zero, while it remains unchanged when other eigenvalues cross zero.

Remark 1.3 *(a) When the function H is identically equal to zero, the functions $\varphi(x')$ and $\psi(x')$ assume the simpler form*

$$
\varphi(x') = 2\pi^2\sqrt{\frac{6}{K(x')}};\qquad\qquad \psi(x')=1.
$$

In particular minima of φ coincide with maxima of K restricted to the boundary and viceversa. (b) We note that equation (3) for $n=3$ is invariant under the rescaling

$$
K\to\gamma K;\qquad H\to\gamma^{\frac{1}{2}}H;\qquad u\to\gamma^{-\frac{1}{4}}u,
$$

where γ is any positive number. The hypotheses involving K and H in Theorems 1.1 and 1.2 are both invariant under such a rescaling.

(c) While Theorem 1.2 is related to some known results for equation (4), Theorem 1.1 has no counterpart in the problem on the whole sphere. The existence argument is strictly related to the presence of the boundary.

The authors have been recently informed about some related results obtained in [23].

The paper is organized as follows. In Section 2 we collect some useful technical tools, while in Section 3 we compute the blow up energies, depending on the values of K and H at the blow up point. In section 4 we recall some known facts about blow up analysis of equations (3) and (4), and in section 5 we specialize to the case of boundary blow ups. Then in Section 6 we prove that blow ups are isolated simple, see Definition 4.5, and occur only at the boundary of ∂S^3_+. Finally, Sections 7 and 8 are devoted to the proofs of Theorems 1.1 and 1.2 respectively.

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2 Some preliminaries

We will use the notation x for variables belonging to the half sphere S^n_+, or to the half space $\mathbb{R}^n_+$, which is defined by $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_n > 0\}$; variables in both the boundaries will be denoted in general by x'.

Solutions of problem (3) are critical points of the Euler functional $J_{K,H} : H^1(S^n_+) \to \mathbb{R}$ defined in the following way

$$J_{K,H}(v) = \frac{1}{2}\int_{S^n_+}\left(\frac{4(n-1)}{n-2}|\nabla v|^2 + n(n-1)\,v^2\right) - \frac{1}{2^*}\int_{S^n_+} K(x)\,|v|^{2^*} - (n-2)\int_{\partial S^n_+} H(x')\,|v|^{2\frac{n-1}{n-2}},$$

$$v \in H^1(S^n_+), \tag{12}$$

where $2^* = \frac{2n}{n-2}$. It will be convenient to perform some stereographic projection in order to reduce the above problem to $\mathbb{R}^n_+$. Let $\mathcal{D}^{1,2}(\mathbb{R}^n_+)$ denote the completion of $C^\infty_c\left(\overline{\mathbb{R}^n_+}\right)$ with respect to the Dirichlet norm. The stereographic projection π_q through a point $q \in \partial S^n_+$ induces an isometry $\iota : H^1(S^n_+) \to \mathcal{D}^{1,2}(\mathbb{R}^n_+)$ according to the following formula

$$(\iota\, v)(x) = \left(\frac{2}{1+|x|^2}\right)^{\frac{n-2}{2}} v(\pi_q^{-1}(x)), \qquad v \in H^1(S^n_+), \quad x \in \mathbb{R}^n_+. \tag{13}$$

In particular, one can check that the following relations hold true, for every $v \in H^1(S^n_+)$

$$\int_{S^n_+}\left(\frac{4(n-1)}{n-2}|\nabla v|^2 + n(n-1)\,v^2\right) = \int_{\mathbb{R}^n_+}\frac{4(n-1)}{n-2}|\nabla(\iota\, v)|^2;$$

$$\int_{S^n_+}|v|^{2^*} = \int_{\mathbb{R}^n_+}|\iota\, v|^{2^*}; \qquad \int_{\partial S^n_+}|v|^{2\frac{n-1}{n-2}} = \int_{\partial\mathbb{R}^n_+}|\iota\, v|^{2\frac{n-1}{n-2}}.$$

By means of these equations, the functional $J_{K,H}$ transforms into $I_{K,H} : \mathcal{D}^{1,2}(\mathbb{R}^n_+) \to \mathbb{R}$ given by

$$I_{K,H}(u) = \frac{1}{2}\int_{\mathbb{R}^n_+}\frac{4(n-1)}{n-2}|\nabla u|^2 - \frac{1}{2^*}\int_{\mathbb{R}^n_+} K(x)\,|u|^{2^*} - (n-2)\int_{\partial\mathbb{R}^n_+} H(x')\,|u|^{2\frac{n-1}{n-2}},$$

meaning that $J_{K,H}(v) = I_{K,H}(\iota v)$ for every $v \in H^1(S^n_+)$. Here we are identifying the functions K and H and their compositions with the stereographic projection π_q. This fact will be assumed as understood in the sequel.

Critical points of the functional $I_{K,H}$ are solutions of the following problem

$$\begin{cases} -4\frac{n-1}{n-2}\,\Delta u = K(x)\,u^{\frac{n+2}{n-2}}, & \text{in } \mathbb{R}^n_+; \\ -\frac{2}{n-2}\frac{\partial u}{\partial x_n} = H(x')\,u^{\frac{n}{n-2}}, & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{14}$$

Using similar arguments, one finds that the counterpart in $\mathbb{R}^n_+$ of equation (7) is given by

$$\begin{cases} -4\frac{n-1}{n-2}\,\Delta u = W(x)^\tau\,K(x)\,u^p, & \text{in } \mathbb{R}^n_+; \\ -\frac{2}{n-2}\frac{\partial u}{\partial x_n} = W(x')^{\frac{\tau}{2}}\,H(x')\,u^{\frac{p+1}{2}}, & \text{on } \partial\mathbb{R}^n_+, \end{cases} \tag{15}$$

where $W(x) = \left(\frac{2}{1+|x|^2}\right)^{\frac{n-2}{2}}$, and where $\tau = \frac{n+2}{n-2} - p$. The terms $W(x)^\tau$ and $W(x')^{\frac{\tau}{2}}$ in the equation above are corrections due to the non conformality of equation (7) when $p \neq \frac{n+2}{n-2}$.

As a typical feature of non compact variational problems like (14), it is fundamental to analyze the associated *problems at infinity*. Solutions of such problems describe the asymptotic profile of non-converging Palais Smale sequences. In the specific case of (14), these problems at infinity are of two kinds, namely

$$-4\frac{n-1}{n-2}\,\Delta u = K(\overline{x})\,u^{\frac{n+2}{n-2}}, \qquad \text{in } \mathbb{R}^n, \tag{16}$$

for some fixed $\overline{x} \in \overline{\mathbb{R}^n_+}$, and

$$\begin{cases} -4\frac{n-1}{n-2}\,\Delta u = K(\overline{x}')\,u^{\frac{n+2}{n-2}}, & \text{in } \mathbb{R}^n_+; \\ -\frac{2}{n-2}\frac{\partial u}{\partial x_n} = H(\overline{x}')\,u^{\frac{n}{n-2}}, & \text{on } \partial\mathbb{R}^n_+, \end{cases} \tag{17}$$

for some $\overline{x}' \in \partial\mathbb{R}^n_+$. Roughly, problem (16) corresponds to the case in which the functions are mostly concentrated in the interior of $\mathbb{R}^n_+$, while problem (17) corresponds to the case in which the functions are concentrated near the boundary. We note that solutions of problem (17) are critical points of the functional

$$I_{\overline{K},\overline{H}}(u) = \frac{1}{2}\int_{\mathbb{R}^n_+} \frac{4(n-1)}{n-2}|\nabla u|^2 - \frac{1}{2^*}\overline{K}\int_{\mathbb{R}^n_+}|u|^{2^*} - (n-2)\overline{H}\int_{\partial\mathbb{R}^n_+}|u|^{2\frac{n-1}{n-2}},$$

where $\overline{K} = K(\overline{x}')$ and $\overline{H} = H(\overline{x}')$.

Positive solutions of problems (16) and (17) have been completely classified in [24] and [33], see also [20]; we recall the results in the following Lemma.

Lemma 2.1 *The positive solutions u of problems* (16) *and* (17) *are, modulo translations in $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$, respectively of the form*

$$U_\lambda(x) = \left(\frac{\lambda}{1+\lambda^2 k|x|^2}\right)^{\frac{n-2}{2}}; \qquad \lambda > 0, \tag{18}$$

where $k = \frac{K(\overline{x})}{4n(n-1)}$, and

$$\overline{U}_\lambda(x) = \left(\frac{\lambda}{1+\lambda^2 k\left(|x'|^2 + (x_n + t_\lambda)^2\right)}\right)^{\frac{n-2}{2}}; \qquad \lambda > 0, \tag{19}$$

where $k = \frac{K(\overline{x}')}{4n(n-1)}$ and where t_λ is given by $2\,k\,t_\lambda\,\lambda = H(\overline{x}')$.

It will be convenient to consider the expression $\int_{\mathbb{R}^n_+} x_n \overline{U}_1^{\frac{2n}{n-2}}$ with $n = 3$. Using the formula

$$\int_0^\infty \frac{r^\alpha}{(1+r^2)^\beta} = \frac{\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\beta - \frac{\alpha+1}{2}\right)}{2\Gamma(\beta)},$$

and integrating first in the variable x' and then in the variable x_3, one obtains

$$\int_{\mathbb{R}^3_+} x_3 \overline{U}_1^6 = \frac{144\pi}{K(\overline{x}')^2}\left[1 + H(\overline{x}')\sqrt{\frac{6}{K(\overline{x}')}}\left(\arctan\left(H(\overline{x}')\sqrt{\frac{6}{K(\overline{x}')}} - \frac{\pi}{2}\right)\right)\right] = \frac{144\pi}{K^2(\overline{x}')}\,\psi(\overline{x}'). \tag{20}$$

The projection to $\mathbb{R}^n_+$ is not the only transformation we will perform. In the next section we will use a conformal transformation of S^n_+ onto some suitable spherical cap Σ_θ. In a similar way as before, this transformation induces an isometry between $H^1(S^n)$ and $H^1(\Sigma_\theta)$. We will not write the explicit formulas for this transformation, as in (13).

3 Study of the blow up energies

In this section we compute the energies of the solutions of problems (16) and (17) (i.e. of the functions given in (18) and (19)), highlighting the dependence on the values of K (and H) at $\overline{x}$ (resp. at $\overline{x}'$). It is well known that these energies are strongly related to those of non converging Palais Smale sequences of the functionals $J_{K,H}$ and $I_{K,H}$.

In order to simplify the computations it is convenient, using a suitable stereographic transformations, to reduce problems (16) and (17) to S^n and to some spherical cap respectively, see [26]. In the following ω_d denotes the volume of the unit d-dimensional sphere in $\mathbb{R}^{d+1}$.

Given $\theta \in (0, \pi)$, we define the spherical cap Σ_θ in the following way

$$\Sigma_\theta = \{x \in S^n : x_{n+1} \geq \cos\theta\}.$$

One can find with elementary computations that the mean curvature of $\partial\Sigma_\theta$ with respect to Σ_θ endowed with the standard metric g_0 is given by

$$h_{g_0}(\partial\Sigma_\theta, \Sigma_\theta)(x') = h_\theta := \frac{\cos\theta}{\sin\theta}, \qquad \text{for all } x \in \partial\Sigma_\theta.$$

We set for brevity $\overline{K} = K(\overline{x})$ (or $K(\overline{x}')$) and $\overline{H} = H(\overline{x}')$. We want to choose an appropriate θ in such a way that some solution of the problem

$$\begin{cases} -4\frac{n-1}{n-2}\Delta v + n(n-1)v = \overline{K} v^{\frac{n+2}{n-2}} & \text{in } \Sigma_\theta, \\ \frac{2}{n-2}\frac{\partial v}{\partial \nu} + h_\theta v = \overline{H}\, v^{\frac{n}{n-2}} & \text{in } \partial\Sigma_\theta. \end{cases} \tag{21}$$

can be chosen to be a constant v_θ. In this way the problem transforms into

$$\begin{cases} n(n-1)v_\theta = \overline{K} v_\theta^{\frac{n+2}{n-2}} & \text{in } \Sigma_\theta, \\ h_\theta v_\theta = \overline{H}\, v_\theta^{\frac{n}{n-2}} & \text{in } \partial\Sigma_\theta. \end{cases} \tag{22}$$

From equation (22) it follows that θ must satisfy the relation

$$\overline{H}\,\sin\theta = \left(\frac{\overline{K}}{n(n-1)}\right)^{\frac{1}{2}} \cos\theta, \tag{23}$$

and that v_θ solves

$$\overline{K}\, v_\theta^{\frac{4}{n-2}} = n(n-1). \tag{24}$$

As far as the interior blow up is concerned, we look for a constant function $\hat{v}$ on the whole sphere. Since $\hat{v}$ solves the equation

$$-4\frac{n-1}{n-2}\Delta\hat{v} + n(n-1)\hat{v} = \overline{K}\hat{v}^{\frac{n+2}{n-2}}, \qquad \text{in } S^n, \tag{25}$$

and is constant, it must also satisfy

$$\overline{K}\,\hat{v}^{\frac{4}{n-2}} = n(n-1). \tag{26}$$

Boundary blow up energy

We now compute the energy of a boundary blow up. Let $J_{\overline{K},\overline{H}}$ be the Euler functional as in (12) corresponding to $K \equiv K(\overline{x}')$ and $H \equiv H(\overline{x}')$; we have

$$J_{\overline{K},\overline{H}}(v_\theta) = \frac{1}{2}n(n-1)\int_{\Sigma_\theta} v_\theta^2 + \frac{(n-1)}{\tan\theta}\int_{\partial\Sigma_\theta} v_\theta^2 - \frac{\overline{K}}{2^*}\int_{\Sigma_\theta} v_\theta^{2^*} - \overline{H}\,(n-2)\int_{\partial\Sigma_\theta} v_\theta^{2\frac{n-1}{n-2}}.$$

Taking into account the fact that v_θ is a critical point of $J_{\overline{K},\overline{H}}$, and in particular that $J'_{\overline{K},\overline{H}}(v_\theta)[v_\theta] = 0$, it turns out that

$$n(n-1)\int_{\Sigma_\theta} v_\theta^2 + 2\frac{(n-1)}{\tan\theta}\int_{\partial\Sigma_\theta} v_\theta^2 - \overline{K}\int_{\Sigma_\theta} v_\theta^{2^*} - 2\overline{H}(n-1)\int_{\partial\Sigma_\theta} v_\theta^{2\frac{n-1}{n-2}} = 0.$$

Hence it follows that

$$J_{\overline{K},\overline{H}}(v_\theta) = \left(\frac{1}{2} - \frac{1}{2^*}\right)\overline{K}\int_{\Sigma_\theta} v_\theta^{2^*} + \overline{H}\int_{\partial\Sigma_\theta} v_\theta^{2\frac{n-1}{n-2}}.$$

Setting

$$F(\theta) = \int_0^\theta \sin^{n-1} s\, ds; \qquad \theta \in [0,\pi],$$

one immediately checks that

$$|\Sigma_\theta| = \omega_{n-1}\cdot F(\theta); \qquad |\partial\Sigma_\theta| = \frac{d}{d\theta}|\Sigma_\theta| = \omega_{n-1}\sin^{n-1}\theta.$$

So, taking into account formula (24), $J_{\overline{K},\overline{H}}(v_\theta)$ can be written as

$$J_{\overline{K},\overline{H}}(v_\theta) = \frac{\omega_{n-1}}{n}\overline{K}F(\theta)\cdot\left(\frac{n(n-1)}{\overline{K}}\right)^{\frac{n}{2}} + \overline{H}\,\omega_{n-1}\sin^{n-1}\theta\cdot\left(\frac{n(n-1)}{\overline{K}}\right)^{\frac{n-1}{2}}. \tag{27}$$

Finally, using equation (23) we deduce

$$J_{\overline{K},\overline{H}}(v_\theta) = \omega_{n-1}\left(\frac{n(n-1)}{\overline{K}}\right)^{\frac{n}{2}-1}\left[(n-1)\,F(\theta) + \cos\theta\sin^{n-2}\theta\right]. \tag{28}$$

Interior blow up energy

Solutions of (25) are critical points of the functional $J_{\overline{K}} : H^1(S^n) \to \mathbb{R}$ defined by

$$J_{\overline{K}}(v) = \frac{1}{2}\int_{S^n}\left(\frac{4(n-1)}{n-2}|\nabla v|^2 + n(n-1)v^2\right) - \frac{\overline{K}}{2^*}\int_{S^n}|v|^{2^*}, \qquad v \in H^1(S^n).$$

If $\hat{v}$ is the constant given by formula (26), then its energy is

$$J_{\overline{K}}(\hat{v}) = \frac{1}{2}n(n-1)\int_{S^n}\hat{v}^2 - \frac{1}{2^*}\int_{S^n}\overline{K}\hat{v}^{2^*}.$$

Since $\hat{v}$ is a critical point of $J_{\overline{K}}$ it turns out that the following relation must be also satisfied

$$n(n-1)\int_{S^n}\hat{v}^2 - \int_{S^n}\overline{K}\hat{v}^{2^*} = 0.$$

Hence it follows that

$$J_{\overline{K}}(\hat{v}) = \left(\frac{1}{2} - \frac{1}{2^*}\right)\int_{S^n}\overline{K}\hat{v}^{2^*} = \frac{\omega_n}{n}\left(n(n-1)\right)^{\frac{n}{2}}\left(\overline{K}\right)^{-\frac{n-2}{2}}. \tag{29}$$

Comparison of energies

We conclude this section by proving that, for the same value of $\overline{K}$, the interior blow up energy is always greater than the boundary blow up energy, namely we show that

$$J_{\overline{K},\overline{H}}(v_\theta) < J_{\overline{K}}(\hat{v}). \tag{30}$$

From equation (23) and from the obvious relation

$$\omega_n = \omega_{n-1}\,F(\pi),$$

one deduces that

$$J_{\overline{K}}(\hat{v}) = \omega_{n-1}\left[(n-1)\tan^{2-n}\theta\,F(\pi)\right]\overline{H}^{2-n}. \tag{31}$$

Taking into account (23) and (27), showing inequality (30) is equivalent to prove

$$G(\theta) := F(\theta) + \frac{1}{n-1}\sin^{n-2}\theta\cos\theta < F(\pi). \tag{32}$$

Since it is clearly $G(\pi) = F(\pi)$, we are done if we prove that $G'(\theta) > 0$ for all $\theta \in (0,\pi)$. There holds

$$G'(\theta) = \sin^{n-1}\theta + \frac{1}{n-1}\left((n-2)\sin^{n-3}\theta\cos^2\theta - \sin^{n-1}\theta\right).$$

With straightforward computations one finally finds that

$$G'(\theta) = \frac{n-2}{n-1}\sin^{n-3}\theta\cdot\left(\sin^2\theta + \cos^2\theta\right),$$

hence equation (32) is proved.

We also note that in the case when $\overline{H} = 0$, the boundary blow up energy is exactly one half of the interior blow up energy, see formulas (28) and (29).

Remark 3.1 *We are particularly interested in the boundary blow up case for $n = 3$. In this situation, using elementary trigonometric formulas, the explicit expression of $J_{\overline{K},\overline{H}}(v_\theta)$ becomes*

$$J_{\overline{K},\overline{H}}(v_\theta) = 4\pi\sqrt{\frac{6}{\overline{K}}}\left(\frac{\pi}{2} - \arctan\left(\overline{H}\sqrt{\frac{6}{\overline{K}}}\right)\right).$$

The above function will play a crucial role in the blow up analysis performed later.

4 Blow up analysis: definitions and preliminary results

In this section we recall the definition of isolated and isolated simple blow up due to R. Schoen, [34]; we also collect some useful tools and known results.

For a smooth bounded domain $\Omega \subseteq \mathbb{R}^n$ set $\Omega_+ = \Omega \cap \{x_n > 0\}$, $\partial_1 \Omega = \Omega \cap \partial \mathbb{R}^n_+$ and $\partial_2 \Omega = \partial \Omega \cap \mathbb{R}^n_+$, hence $\partial \Omega_+ = \overline{\partial_1 \Omega} \cup \partial_2 \Omega$. We also assume that $\partial \Omega$ and $\partial \mathbb{R}^n_+$ intersect transversally, so that $\partial \Omega \cap \partial \mathbb{R}^n_+$ is a smooth manifold of dimension $n-2$. Let ν denote the unit exterior normal to Ω, and let ν' denote the exterior unit normal of $\partial_1 \Omega$ in $\partial \mathbb{R}^n_+$. Given $w : \partial \mathbb{R}^n_+ \to \mathbb{R}$, the expression $\nabla' w$ stands for the gradient in $\mathbb{R}^{n-1}$. If w is defined on $\overline{\Omega_+}$, the same symbol will be used for the gradient of the restriction of w to $\partial_1 \Omega$. In the following $B_\sigma(x)$ denotes the open ball in $\mathbb{R}^n$ of radius σ centered at x; we just write B_σ if $x = 0$.

We will consider equation (14) restricted to Ω, or equation (15) when the exponent p is converging to $\frac{n+2}{n-2}$. For this reason we will not keep the functions K and H fixed, but we will allow them to vary; more precisely, we consider positive solutions u_i of the sequence of problems

$$\begin{cases} -\Delta u_i = \frac{n-2}{4(n-1)} K_i(x) u_i^{p_i}, & \text{in } \Omega_+; \\ -\frac{\partial u_i}{\partial x_n} = \frac{n-2}{2} H_i(x') u_i^{\frac{p_i+1}{2}}, & \text{on } \partial_1 \Omega. \end{cases} \tag{33}$$

We are interested in the case where the supremum of the functions u_i is tending to infinity, trying to give a precise characterization of the blow up phenomenon, as in [34] and [29]. A typical ingredient of blow up analysis of scalar curvature equations is a Pohozahev type identity, which we provide in the next Lemma.

Lemma 4.1 *Let $p \geq 1$, let $\Omega \subseteq \mathbb{R}^n$ be as above, and let $K \in C^1(\overline{\Omega_+})$, $H \in C^1(\partial_1 \Omega)$. Assume $u \in C^2(\overline{\Omega_+})$ is a positive solution of*

$$\begin{cases} -\Delta u = \frac{n-2}{4(n-1)} K(x)\, u^p, & \text{in } \Omega; \\ -\frac{\partial u}{\partial x_n} = \frac{n-2}{2} H(x')\, u^{\frac{p+1}{2}}, & \text{on } \partial_1 \Omega. \end{cases} \tag{34}$$

Then there holds

$$\begin{aligned} & \frac{n-2}{4(n-1)} \left(\frac{n-2}{2} - \frac{n}{p+1} \right) \int_{\Omega_+} K u^{p+1} + \frac{n-2}{2} \left(\frac{n-2}{2} - \frac{2(n-1)}{p+3} \right) \int_{\partial_1 \Omega} H\, u^{\frac{p+3}{2}} \\ = \; & \frac{n-2}{4(n-1)} \frac{1}{p+1} \int_{\Omega_+} (x \cdot \nabla K)\, u^{p+1} + \frac{n-2}{p+3} \int_{\partial_1 \Omega} (x' \cdot \nabla' H)\, u^{p+1} \\ & + \int_{\partial_2 \Omega} B - \frac{n-2}{4(n-1)} \int_{\partial_2 \Omega} K\, u^{p+1}\, x \cdot \nu - \frac{n-2}{p+3} \int_{\partial(\partial_1 \Omega)} H\, u^{\frac{p+3}{2}}\, x' \cdot \nu' \end{aligned} \tag{35}$$

where

$$B = B(x, u, \nabla u) = \frac{\partial u}{\partial \nu} x \cdot \nabla u + \frac{n-2}{2} u \frac{\partial u}{\partial \nu} - \frac{|\nabla u|^2}{2} x \cdot \nu.$$

Proof. Multiply the first equation in (34) by $\sum_{j=1}^n x_j \frac{\partial u}{\partial x_j}$ and integrate by parts: we obtain

$$\begin{aligned} \frac{n-2}{4(n-1)} \left(\frac{n-2}{2} - \frac{n}{p+1} \right) \int_{\Omega_+} K u^{p+1} \quad - \quad & \frac{n-2}{4(n-1)} \frac{1}{p+1} \int_{\Omega_+} (x \cdot \nabla K)\, u^{p+1} \\ = \quad & \int_{\partial_2 \Omega} B - \frac{n-2}{4(n-1)} \int_{\partial_2 \Omega} K\, u^{p+1}\, x \cdot \nu. \end{aligned}$$

Integrating by parts on $\partial_1 \Omega$, we deduce

$$\int_{\partial_1 \Omega} H\, u^{\frac{p+1}{2}}\, x' \cdot \nabla' u = -2 \frac{n-1}{p+1} \int_{\partial_1 \Omega} u^{\frac{p+3}{2}}\, H - \frac{2}{p+1} \int_{\partial_1 \Omega} u^{\frac{p+3}{2}} x' \cdot \nabla' H + \int_{\partial(\partial_1 \Omega)} x' \cdot \nu'\, H\, u^{\frac{p+3}{2}},$$

Using the second equation in (34), we easily reach the conclusion. ■

We have also the following Proposition, which proof is elementary.

Proposition 4.2 *Suppose the function $h : (B_\sigma)_+ \setminus \{0\} \to \mathbb{R}$ is of the form*

$$h(x) = a\, |x|^{2-n} + b(x),$$

with $a > 0$ and $b(x)$ of class C^1 on $\overline{(B_\sigma)_+}$. Then there holds

$$\lim_{\sigma \to 0} \int_{\partial_2 B_\sigma} B(x, h, \nabla h) = -\frac{(n-2)^2}{4}\, \omega_{n-1}\, a\, b(0).$$

Let $\Omega \subseteq \mathbb{R}^n$ be as above, let $1 < p_i \leq \frac{n+2}{n-2}$, $p_i \to \frac{n+2}{n-2}$, and let $\tau_i = \frac{n+2}{n-2} - p_i$, so that $\tau_i \to 0$. Let $\{K_i\}_i \subseteq C^1(\overline{\Omega_+})$, $\{H_i\}_i \subseteq C^1(\overline{\partial_1 \Omega})$ satisfy for some constant $A_1 > 0$

$$\frac{1}{A_1} \leq K_i(x) \leq A_1, \quad -A_1 \leq H_i(x') \leq A_1; \qquad \text{for all } x \in \overline{\Omega_+}, \text{ all } x' \in \overline{\partial_1 \Omega}, \text{ and all } i. \tag{36}$$

For every i, let also $u_i \in C^2(\overline{\Omega_+})$ be a positive solution of problem (33).

Definition 4.3 *The point $\overline{x} \in \Omega_+ \cup \partial_1 \Omega$ is called a* blow up point *for $\{u_i\}_i$ if there exists a sequence of points $x_i \in \Omega_+ \cup \partial_1 \Omega$ tending to $\overline{x}$ such that $u_i(x_i) \to +\infty$.*

Definition 4.4 *Let $\overline{x} \in \Omega_+ \cup \partial_1 \Omega$, and let $\{x_i\}$ be a sequence of local maxima of u_i such that $x_i \to \overline{x}$ and $u_i(x_i) \to +\infty$. The point $\overline{x}$ is called an* isolated blow up point *if there exist $0 < \overline{r} < dist(\overline{x}, \partial_2 \Omega)$ and $\overline{C} > 0$ such that*

$$u_i(x) \leq \overline{C}\, |x - x_i|^{-\frac{2}{p_i - 1}}, \qquad |x - x_i| \leq \overline{r}, x \in \Omega_+.$$

If $\overline{x}$ is a blow up point for $\{u_i\}_i$ we will write for brevity $x_i \to \overline{x}$ meaning that $\{x_i\}_i$ is a sequence of points as in Definition 4.4. It is possible to prove, using Proposition 5.1 and Lemma 4.6 below, that the points x_i having the properties in Definition 4.4 are uniquely determined, provided the functions K_i and H_i in (33) are uniformly bounded in C^1 and C^2 norm respectively, see [25].

If $x_i \to \overline{x}$ is a simple blow up for $\{u_i\}_i$ and if $\overline{r}$ is given by Definition 4.4 we define

$$\overline{u}_i(r) = \frac{1}{|\partial B_r(x_i) \cap \Omega_+|} \int_{\partial B_r(x_i) \cap \Omega_+} u_i, \qquad r \in (0, \overline{r}), \tag{37}$$

and

$$\tilde{u}_i(r) = r^{\frac{2}{p_i - 1}}\, \overline{u}_i(r), \qquad r \in (0, \overline{r}).$$

Definition 4.5 *The isolated blow up point $x_i \to \overline{x}$ is called* isolated simple *if there exists $\varrho \in (0, \overline{r})$ such that for large i there holds*

$$\tilde{u}_i \text{ has precisely one critical point in } (0, \varrho). \tag{38}$$

If $\overline{x}$ is a blow up point, we will call it *interior blow up point* if $\overline{x} \in \Omega_+$, or *boundary blow up point* if $\overline{x} \in \partial_1 \Omega$.

Another fundamental tool for the blow up analysis is the Harnack inequality; we recall the following version from [25], Appendix A.

Lemma 4.6 *(Harnack-type inequality) Let* $\{K_i\} \in L^\infty(\Omega_+)$ *and* $\{H_i\} \in L^\infty(\partial_1\Omega)$ *satisfy* (36). *Assume also that* $\{u_i\}_i$ *satisfy (33) with* $p_i \geq p_0 > 1$, *and let* $x_i \to \overline{x}$ *be an isolated blow up point. Then for every* $0 < r < \frac{1}{4}\overline{r}$ *the following Harnack-type inequality holds*

$$\sup_{x \in (B_{2r})_+(x_i) \setminus (B_{r/2})_+(x_i)} u_i(x) \leq C \inf_{x \in (B_{2r})_+(x_i) \setminus (B_{r/2})_+(x_i)} u_i(x),$$

where C *is a positive constant depending only on* n, A_1 *and* $\overline{C}$.

For the blow up analysis of the first equation in (33) we mainly refer to [29], where the following proposition regarding the interior blow up points is proved.

Proposition 4.7 *Assume* $\Omega \subseteq \mathbb{R}^3$ *and that* $\{K_i\}_i$ *is uniformly bounded in* $C^1(\overline{\Omega})$. *Assume that* $p_i \leq \frac{n+2}{n-2}$, $p_i \to \frac{n+2}{n-2}$, *and* $\{u_i\}_i$ *are solutions of*

$$-\Delta u_i = \frac{n-2}{4(n-1)} K_i u_i^{p_i}, \qquad u_i > 0 \ \text{in } \Omega.$$

Then, if $\overline{x} \in \Omega$ *is a blow up point for* u_i, *it is also an isolated simple blow up point. Moreover there exists an harmonic function* $b : B_{\varrho/2}(\overline{x}) \to \mathbb{R}$ *such that, passing to a subsequence*

$$u_i(x_i)\, u_i(x) \to a\, |x - \overline{x}|^{2-n} + b(x), \qquad \text{in } C^2_{loc}(B_{\varrho/2} \setminus \{\overline{x}\}), \tag{39}$$

where $a = (4n(n-1))^{\frac{n-2}{2}} (\lim_i K_i(x_i))^{\frac{2-n}{2}}$, *and where* ϱ *is given in Definition 4.5.*

5 Behavior of isolated simple blow ups

In this section we perform the study of isolated simple blow ups of equation (33). The situation of interior blow up has been treated in [29], hence we are reduced to consider the case in which the blow up point $\overline{x}$ is in $\partial_1\Omega$. We will refer sometimes to the paper [25], where it is studied equation (33) when K_i and H_i are converging to constant functions.

Proposition 5.1 *Assume* $\{K_i\}_i \subseteq C^1(\overline{\Omega_+})$ *and* $\{H_i\}_i \subseteq C^2(\overline{\partial_1\Omega})$ *satisfy (36) for some* $A_1 > 0$, *and satisfy also the condition*

$$\|\nabla K_i\|_{C(\overline{\Omega_+})} \leq A_2, \qquad \|\nabla' H_i\|_{C^1(\overline{\partial_1\Omega})} \leq A_2, \tag{40}$$

for some $A_2 > 0$. *For every* i, *let* u_i *be a positive solution of* (33), *and let* $x_i \to \overline{x}' \in \partial_1\Omega$ *be an isolated blow up point for* $\{u_i\}_i$. *Let also* x_i' *denote the projection of* x_i *onto* $\partial_1\Omega$. *Then, given* $R_i \to +\infty$ *and* $\varepsilon_i \to 0^+$, *after passing to a subsequence of* $\{u_i\}_i$ *(still denoted with* $\{u_i\}_i$*) we have*

$$\begin{cases} r_i := u_i(x_i)^{-\frac{p_i-1}{2}} R_i \to 0 \qquad \text{as } i \to +\infty, \\ \left\| u_i(x_i)^{-1} u_i(u_i(x_i)^{-\frac{p_i-1}{2}} x + x_i) - \left(\frac{\lambda}{(1+k\lambda^2(|x'|^2+|x_n+t|^2)} \right)^{\frac{n-2}{2}} \right\|_{C^2\left(\left(u_i(x_i)^{\frac{p_i-1}{2}}(\overline{\Omega_+}-x_i)\right) \cap \overline{B_{2R_i}}\right)} \leq \varepsilon_i, \end{cases}$$

where $k = (4n(n-1))^{-1} K_i(x_i')$, *while* λ, t *satisfy* $2k\lambda t = H_i(x_i')$ *and*

$$\lambda = \begin{cases} 1 + k\lambda^2 t^2 & \text{if } H_i(x_i') \geq 0, \\ 1 & \text{if } H_i(x_i') < 0. \end{cases} \tag{41}$$

PROOF. Consider the functions

$$w_i(x) = u_i(x_i)^{-1}\, u_i\left(u_i(x_i)^{-\frac{p_i-1}{2}}\, x + x_i\right), \qquad x \in u_i(x_i)^{\frac{p_i-1}{2}}(\Omega_+ - x_i).$$

It follows immediately that $w_i(0) = 1$ for all i and that 0 is a local maximum point for w_i. Moreover from the assumption of isolated blow up we have

$$w_i(x) \leq C\,|x|^{-\frac{p_i-1}{2}}, \qquad x \in u_i(x_i)^{\frac{p_i-1}{2}}(\Omega_+ - x_i) \cap \{|x| < u_i(x_i)^{\frac{p_i-1}{2}}\overline{r}\},$$

where $\overline{r}$ is given in Definition 4.4.

The function w_i is a solution of the problem

$$\begin{cases} -\Delta w_i(x) = \frac{n-2}{4(n-1)} K_i\left(u_i(x_i)^{\frac{p_i-1}{2}} x + x_i\right) w_i(x)^{p_i}, & \text{in } u_i(x_i)^{\frac{p_i-1}{2}}(\Omega_+ - x_i); \\ -\frac{\partial w_i}{\partial x_n}(x) = \frac{n-2}{2} H_i\left(u_i(x_i)^{\frac{p_i-1}{2}} x + x_i\right) u_i(x)^{\frac{p_i+1}{2}}, & \text{on } u_i(x_i)^{\frac{p_i-1}{2}}(\partial_1\Omega - x_i). \end{cases}$$

Denoting by $x_{i,n}$ the n-th component of x_i and setting $T_i = u_i(x_i)^{\frac{p_i-1}{2}}\, x_{i,n}$, two cases may occur, namely

$$T_i \to +\infty, \qquad \text{or} \qquad T_i \to T \in \mathbb{R}.$$

In the latter one, we can use (36), (40) and the results in [1] to prove that the functions w_i converge up to subsequence, and then one can conclude as in [25], Proposition 1.4.

Hence it is sufficient to exclude the first case. In order to do this, define the functions

$$\xi_i(x) = x_{i,n}^{\frac{2}{p_i-1}}\, u_i(x_i + x_{i,n}\, x), \qquad x \in T_i\,(\Omega_+ - x_i).$$

First, letting $\Omega_i = T_i\,(\Omega_+ - x_i)$, it is clear that Ω_i are relatively open sets which invade the half space $\mathbb{R}^n_1 := \{x \in \mathbb{R}^n \,:\, x_n > -1\}$. Then, since we are supposing by contradiction that $T_i \to +\infty$, 0 is an interior blow up point for the functions ξ_i, so from Proposition 4.7 it follows that 0 is an isolated simple blow up point. Using Lemma 4.6 and the inequality

$$\xi_i(x) \leq C\,|x|^{-\frac{p_i-1}{2}}, \qquad x \in T_i^{\frac{p_i-1}{2}}(\Omega_+ - x_i) \cap \{|x| < T_i^{\frac{p_i-1}{2}}\overline{r}\},$$

the convergence in (39) can be extended to the whole $\mathbb{R}^n_1 \setminus \{0\}$. Namely one has

$$\xi_i(0)\,\xi_i(x) \to h(x) \qquad \text{in } C^2_{loc}(\mathbb{R}^n_1 \setminus \{0\}),$$

where $h(x)$ is a non-negative harmonic function in $\mathbb{R}^n_1 \setminus \{0\}$ singular at 0 and satisfying

$$\frac{\partial h}{\partial x_n} = 0, \qquad \text{on } \partial\mathbb{R}^n_1. \tag{42}$$

By equation (42) and by the Schwartz's Reflection Principle, the function h possesses an harmonic extension to the set $\mathbb{R}^n \setminus \{0, \tilde{0}\}$, where $\tilde{0}$ is the symmetric point of 0 with respect to the plane $\partial\mathbb{R}^n_1$. By uniqueness of harmonic extensions, this must coincide with the symmetric prolongation of h through $\partial\mathbb{R}^n_1$. Hence the positivity of h implies that $h(x) = a|x|^{2-n} + A + o(|x|)$ for x close to 0 , where $a, A > 0$. Reasoning as in Proposition 3.1 of [29], one can reach a contradiction. ∎

Next, we establish the counterpart of Proposition 4.7 for blow up points in $\partial_1\Omega$.

Proposition 5.2 *Let $\Omega = B_2$, suppose $\{K_i\}_i \subseteq C^1(\overline{\Omega_+})$, $\{H_i\}_i \subseteq C^2(\overline{\partial_1\Omega})$ satisfy conditions (36) and (40) for some $A_1, A_2 > 0$. Suppose that for every i, u_i satisfies (33) and that $x_i \to 0$ is an isolated simple blow up with*

$$|x - x_i|^{\frac{2}{p_i-1}} u_i(x) \leq A_3, \qquad \text{for all } x \in \Omega_+.$$

Then there exists some positive constant $C = C(A_1, A_2, A_3, n, \rho)$ *such that*

$$u_i(x) \le C u_i(x_i)^{-1} |x - x_i|^{2-n} \qquad \text{for all } x \in (B_1(x_i))_+. \tag{43}$$

Furthermore, there exists $b : \overline{(B_1)_+}$ *satisfying*

$$\begin{cases} -\Delta b = 0 & \text{in } (B_1)_+; \\ -\frac{\partial b}{\partial y_n} = 0 & \text{on } B_1 \cap \partial \mathbb{R}^n_+, \end{cases}$$

such that

$$u_i(x_i) u_i(x) \to a\, |x|^{2-n} + b, \qquad \text{in } C^2_{loc}\left(\overline{(B_1)_+} \setminus \{0\}\right).$$

The coefficient a *is given by*

$$a = \begin{cases} (4n(n-1))^{\frac{n-2}{2}} (\lim_i K_i(x_i'))^{\frac{2-n}{2}} & \text{if } \lim_i H_i(x_i') < 0; \\ \lim_i \left(\frac{K_i(x_i')}{4n(n-1)} + \frac{H_i(x_i')^2}{4} \right)^{\frac{2-n}{2}} & \text{if } \lim_i H_i(x_i') \ge 0, \end{cases} \tag{44}$$

where x_i' *is the projection of* x_i *onto* $\partial_1 \Omega$.

Before proving Proposition 5.2, we need some preliminary Lemmas.

Lemma 5.3 *Under the hypotheses of Proposition 5.2, except for condition* (40)*, there exist* $\delta_i > 0$, $\delta_i = O(R_i^{-2+o(1)})$ *such that*

$$u_i(x) \le C\, u_i(x_i)^{-\lambda_i}\, |x - x_i|^{2-n+\delta_i}, \qquad \text{for } R_i\, u_i(x_i)^{-\frac{p_i-1}{2}} \le |x - x_i| \le 1,$$

where $\lambda_i = (n - 2 - \delta_i)\, \frac{p_i-1}{2} - 1$.

Proof. It follows from [25], pages 511-513. ∎

Lemma 5.4 *Under the hypotheses of Proposition 5.2 there holds*

$$\tau_i = O(u_i(x_i)^{-\frac{2}{n-2}+o(1)}), \qquad \text{as } i \to +\infty$$

and therefore

$$u_i(x_i)^{\tau_i} = 1 + o(1), \qquad \text{as } i \to +\infty.$$

Proof. Let $B(x, u, \nabla u)$ be the function defined in Lemma 4.1. By Lemma 5.3, Proposition 5.1 and standard elliptic theories we have

$$\int_{\partial_2 B_1} B(x, u_i, \nabla u_i) = O\left(u_i(x_i)^{-2+o(1)}\right); \qquad \int_{\partial_2 B_1} K_i\, |u_i|^{p_i+1} = O\left(u_i(x_i)^{-p_i-1+o(1)}\right);$$

$$\int_{\partial(\partial_1 B_1)} H_i\, u_i^{\frac{p_i+3}{2}}\, x' \cdot \nu' = O\left(u_i(x_i)^{-\frac{p_i+3}{2}+o(1)}\right).$$

Furthermore, since the gradients of K_i and H_i are uniformly bounded, one can deduce from Lemma 5.3, Proposition 5.1 and a rescaling argument that

$$\int_{(B_1)_+} u_i^{p_1+1}\, x \cdot \nabla K_i = O\left(u_i(x_i)^{-\frac{2}{n-2}+o(1)}\right); \qquad \int_{\partial_1 B_1} u_i^{\frac{p_i+3}{2}}\, x' \cdot \nabla' H_i = O\left(u_i(x_i)^{-\frac{2}{n-2}+o(1)}\right).$$

On the other hand, using again Lemma 5.3 and Proposition 5.1 we have also

$$
\begin{aligned}
&\frac{n-2}{4(n-1)}\left(\frac{n}{p_i+1}-\frac{n-2}{2}\right)\int_{(B_1)_+} K_i\, u_i^{p_i+1}+\frac{n-2}{2}\left(\frac{n-2}{2}-2\,\frac{n-1}{p_i+3}\right)\int_{\partial_1 B_1} u_i^{\frac{p_i+3}{2}} H_i \\
= \;& \tau_i\,\frac{(n-2)^3}{16(n-1)}\left(\frac{1}{n}K_i(x_i')\int_{\mathbb{R}^n_+}\overline{U}_\lambda^{2^*}+H_i(x_i')\int_{\partial\mathbb{R}^n_+}\overline{U}_\lambda^{2\frac{n-1}{n-2}}+o(1)\right), \qquad i\to+\infty.
\end{aligned}
$$

Here the function $\overline{U}_\lambda$ is given by formula (19) with $K(\overline{x}')$ (resp. $H(\overline{x}')$) replaced by $K_i(x_i')$ (resp. $H_i(x_i')$); we note that the values of the above integrals do not depend on the parameter λ.

Using the relation $I'_{K_i(x_i'),H_i(x_i')}(\overline{U}_\lambda)[\overline{U}_\lambda]=0$, it follows that

$$
\frac{1}{n}K_i(x_i')\int_{\mathbb{R}^n_+}\overline{U}_\lambda^{2^*}+H_i(x_i')\int_{\partial\mathbb{R}^n_+}\overline{U}_\lambda^{2\frac{n-1}{n-2}}=I_{\lim_i K_i(x_i'),\lim_i H_i(x_i')}(\overline{U}_\lambda)+o(1)>\alpha>0,
$$

where α is a positive constant depending only on n, A_1 and A_2. Then the conclusion follows from equation (35) and the above estimates. ■

Lemma 5.5 *Under the same assumptions of Proposition 5.2 there holds*

$$
\lim_i u_i(x_i)\int_{\partial_2 B_1}\frac{\partial u_i}{\partial\nu}<0.
$$

PROOF. Using the divergence formula and (33), we can write

$$
u_i(x_i)\int_{\partial_2 B_1}\frac{\partial u_i}{\partial\nu}=u_i(x_i)\left(\frac{n-2}{4(n-1)}\int_{(B_1)_+}K_i u_i^{p_i}-\frac{n-2}{2}\int_{\partial_1 B_1}H_i u_i^{\frac{p_i+1}{2}}\right). \tag{45}
$$

From Lemma 5.3 we deduce that

$$
\begin{aligned}
\int_{(B_1)_+\setminus(B_{r_i}(x_i))_+} u_i^{p_i} &\le C\int_{(B_1)_+\setminus(B_{r_i})_+}\left(u_i(x_i)^{-\lambda_i}\,|y-x_i|^{2-n+\delta_i}\right)^{p_i} \\
&\le C\,R_i^{n-p_i(n-2-\delta_i)}\,u_i(x_i)^{-1+O(\tau_i)}=o(1)\,u_i(x_i)^{-1}.
\end{aligned}
$$

In the same way we have that

$$
\int_{\partial_1 B_1\setminus\partial_1 B_{r_i}(x_i)} u_i^{\frac{p_i+1}{2}}=o(1)\,u_i(x_i)^{-1}.
$$

Hence, using Proposition 5.1 and a rescaling argument, choosing $\varepsilon_i\to 0$ sufficiently fast, formula (45) can be written as

$$
u_i(x_i)\int_{\partial_2 B_1}\frac{\partial u_i}{\partial\nu}=\frac{n-2}{4(n-1)}\lim_i K_i(x_i')\int_{(B_{R_i})_+}\overline{U}_\lambda^{\frac{n+2}{n-2}}-\frac{n-2}{2}\lim_i H_i(x_i')\int_{\partial_1 B_{R_i}}\overline{U}_\lambda^{\frac{n}{n-2}}+o(1),
$$

where λ satisfies equation (41). Again from the divergence theorem, we have

$$
\lim_i\left(\frac{n-2}{4(n-1)}K_i(x_i')\int_{(B_{R_i})_+}\overline{U}_\lambda^{\frac{n+2}{n-2}}-\frac{n-2}{2}H_i(x_i')\int_{\partial_1 B_{R_i}}\overline{U}_\lambda^{\frac{n}{n-2}}\right)=\lim_{R\to+\infty}\int_{\partial_2 B_R}\frac{\partial\overline{U}_\lambda}{\partial\nu}<0.
$$

This concludes the proof. ■

Proof of Proposition 5.2 Inequality (43) for $|x-x_i| \le r_i$ is an immediate consequence of Lemma 5.3 and Lemma 5.4. We now prove it for $r_i \le |x-x_i| \le 1$. Let $\overline{u}_i$ be given by formula (37) and set $\xi_i(x) = \overline{u}_i(1)\, u_i(x)$. It is easy to see that ξ_i satisfies

$$\begin{cases} -\Delta\, \xi_i = \frac{n-2}{4(n-1)}\, \overline{u}_i(1)^{p_i-1}\, K_i(x)\, \xi_i^{p_i}, & \text{in } (B_2)_+; \\ -\frac{\partial \xi_i}{\partial x_n} = \frac{n-2}{2}\, \overline{u}_i(1)^{\frac{p_i-1}{2}} H_i(x')\, \xi_i^{\frac{p_i+1}{2}}, & \text{on } \partial_1 B_2. \end{cases}$$

Reasoning as in the proof of Proposition 5.1 it follows that, passing to a subsequence, $\{\xi_i\}_i$ converges in $C^2_{loc}(\overline{(B_2)_+} \setminus 0)$ to some positive function $h \in C^2_{loc}(\overline{(B_2)_+} \setminus 0)$ satisfying

$$\begin{cases} -\Delta h = 0, & \text{in } (B_2)_+; \\ -\frac{\partial h}{\partial x_n} = 0, & \text{on } \partial_1 B_2. \end{cases}$$

Moreover, it follows from condition (38) that h must be singular at 0. Reflecting the function h evenly to B_2 and reasoning as above, we deduce that h must be of the form

$$h(x) = a_1\, |x|^{2-n} + b(x), \qquad x \in \overline{(B_2)_+} \setminus 0,$$

where $a_1 > 0$, and $b \in C^2(\overline{(B_2)_+})$ satisfies

$$\Delta b(x) = 0, \quad x \in (B_2)_+; \qquad\qquad \frac{\partial b}{\partial x_n}(x') = 0, \qquad x' \in \partial_1 B_2.$$

Now we can prove (43) for $|x - x_i| = 1$, namely

$$\overline{u}_i(1) \le C\, u_i(x_i)^{-1}. \tag{46}$$

To do this we observe that, by the harmonicity of $b(y)$ we have

$$\int_{\partial_2 B_1} \frac{\partial b}{\partial \nu} = \int_{\partial_1 B_1} \frac{\partial b}{\partial x_n} + \int_{(B_1)_+} (\Delta b) = 0,$$

and so we deduce that

$$\lim_i \overline{u}_1(1)^{-1} \int_{\partial_2 B_1} \frac{\partial u_i}{\partial \nu} = \int_{\partial_2 B_1} \frac{\partial h}{\partial \nu} = a_1 \int_{\partial_2 B_1} \frac{\partial |x|^{2-n}}{\partial \nu} < 0.$$

Hence formula (46) follows from Lemma 5.5. The inequality for a general x with $r_i \le |x-x_i| \le 1$ follows from a rescaling argument, as in [29] page 340. The value of the constant a in (44) can be computed multiplying the first equation in (33) by u_i, integrating by parts, and using Proposition 5.1. ■

We collect now a couple of technical lemmas which will be needed later.

Lemma 5.6 *Suppose that the hypotheses of Proposition 5.2 hold true. Then we have the following estimates*

$$\int_{(B_{r_i}(x_i))_+} |x-x_i|^s\, u_i(x)^{p_i+1} = O\left(u_i(x_i)^{-\frac{2s}{n-2}}\right), \qquad 0 < s < 2;$$

$$\int_{(B_1(x_i))_+ \setminus (B_{r_i}(x_i))_+} |x-x_i|^s\, u_i(x)^{p_i+1} = o\left(u_i(x_i)^{-\frac{2s}{n-2}}\right), \qquad 0 < s < 2.$$

$$\int_{\partial_1 B_{r_i}(x_i)} |x'-x_i'|^s\, u_i(x')^{\frac{p_i+3}{2}} = O\left(u_i(x_i)^{-\frac{2s}{n-2}}\right), \qquad 0 < s < 2;$$

$$\int_{\partial_1 B_1(x_i) \setminus \partial_1 B_{r_i}(x_i)} |x'-x_i'|^s\, u_i(x')^{\frac{p_i+3}{2}} = o\left(u_i(x_i')^{-\frac{2s}{n-2}}\right), \qquad 0 < s < 2;$$

$$\int_{\partial_1 B_1(x_i)} |x'-x_i'|^s\, u_i(x')^{p_i+1} = O\left(u_i(x_i)^{-\frac{4}{n-2}}\, \log u_i(x_i)\right), \qquad s = 2.$$

PROOF. The proof is a simple consequence of Proposition 5.1 and Proposition 5.2. ■

Lemma 5.7 *Suppose that $n = 3$ and that the hypotheses of Proposition 5.2 hold true. Then we have*

$$\tau_i = O(u_i(x_i)^{-2}).$$

PROOF. It is sufficient to use (35) and Lemma 5.6. ■

6 Blow up points are isolated simple and at the boundary

In this section we show that blow up points of equation (33) are isolated simple and that the case of interior blow up can be ruled out. We will use the same terminology about blow ups for describing both functions on S^n_+ or functions defined on some domain $\Omega_+ \subseteq \mathbb{R}^n_+$, having in mind the natural transformation (13) induced by the stereographic projection.

Proposition 6.1 *Let $n = 3$, let Ω be as above, suppose that $\{K_i\}$ and $\{H_i\}$ satisfy conditions (36) and (40), that for every i u_i is a positive solution of (33) and that $\overline{x}' \in \partial_1 \Omega$ is an isolated blow up point for $\{u_i\}$. Then $\overline{x}'$ is also an isolated simple blow up point.*

PROOF. The proof follows that of Proposition 3.1 in [25], combined with some argument in [29] page 353, and Lemma 5.6. We omit the details. ■

Proposition 6.2 *Let Ω, $\{K_i\}$, $\{H_i\}$, $\{u_i\}$ be as in Proposition 6.1, and let $\overline{x}'$ be an isolated simple blow up point for $\{u_i\}$. Suppose also that $\{K_i\}$ are uniformly bounded in $C^2(\overline{\Omega_+})$. Let $\varphi_i : \partial_1 B_1 \to \mathbb{R}$ be the sequence of functions defined by*

$$\varphi_i(x') = 4\pi \sqrt{\frac{6}{K_i(x')}} \left(\frac{\pi}{2} - \arctan \left(H_i(x') \sqrt{\frac{6}{K_i(x')}} \right) \right).$$

Let also x'_i denote the projection of x_i onto $\partial_1 B_1$. Then there holds

$$|\nabla' \varphi_i(x'_i)| \leq O(u_i(x_i)^{-2}), \qquad \text{as } i \to +\infty.$$

PROOF. Choose a test function $\eta \in C^\infty(B_1)$ which satisfies

$$\eta(x) = 1, \quad x \in B_{1/4}; \qquad\qquad \eta(x) = 0, \quad x \in B_1 \setminus B_{1/2}.$$

Multiplying equation (33) by $\eta \frac{\partial u_i}{\partial x_j}$, $j = 1, 2$, we obtain

$$\int_{(B_1)_+} (-\Delta u_i) \eta \frac{\partial u_i}{\partial x_j} = \frac{1}{8} \int_{(B_1)_+} K_i \, u_i^{p_i} \eta \frac{\partial u_i}{\partial x_j}.$$

Integrating by parts we deduce

$$\int_{(B_1)_+} K_i \, u_i^{p_i} \eta \frac{\partial u_i}{\partial x_j} = -\frac{1}{p_i + 1} \int_{(B_1)_+} u_i^{p_i+1} \left(\eta \frac{\partial K_i}{\partial x_j} + K_i \frac{\partial \eta}{\partial x_j} \right).$$

and also

$$\begin{aligned} \int_{(B_1)_+} (-\Delta u_i) \, \eta \frac{\partial u_i}{\partial x_j} \quad &= \quad \frac{1}{p_i + 3} \int_{\partial_1 B_1} u_i^{\frac{p_i+3}{2}} \left(\eta \frac{\partial H_i}{\partial x_j} + H_i \frac{\partial \eta}{\partial x_j} \right) - \frac{1}{2} \int_{(B_1)_+} |\nabla u_i|^2 \frac{\partial \eta}{\partial x_j} \\ &+ \int_{(B_1)_+} \nabla u_i \frac{\partial u_i}{\partial x_j} \nabla \eta. \end{aligned}$$

From the above equations, Proposition 5.2, and the fact that $\nabla\eta$ has support in $\overline{(B_{1/2})_+} \setminus (B_{1/4})_+$, we obtain

$$\frac{1}{p_i+1}\int_{(B_1)_+} u_i^{p_i+1}\,\eta\,\frac{\partial K_i}{\partial x_j} + \frac{1}{p_i+3}\int_{\partial_1 B_1} u_i^{\frac{p_i+3}{2}}\,\eta\,\frac{\partial H_i}{\partial x_j} = O(u_i(x_i)^{-2}).$$

Using the uniform bounds on the second derivatives of K_i and H_i, and taking into account Lemmas 5.6 and 5.7 we deduce

$$\frac{1}{6}\frac{\partial K_i}{\partial x_j}(x_i')\int_{(B_1)_+} u_i^{p_i+1}\,\eta + \frac{1}{8}\frac{\partial H_i}{\partial x_j}(x_i')\int_{\partial_1 B_1} u_i^{\frac{p_i+3}{2}}\,\eta = O(u_i(x_i)^{-2}). \tag{47}$$

Let $\overline{U}_\lambda$ be the function given in formula (19) with $K(\overline{x}')$ replaced by $K_i(x_i')$ and $H(\overline{x}')$ replaced by $H_i(x_i')$. Using Proposition 5.1 and Lemma 5.7, equation (47) becomes

$$\frac{1}{6}\frac{\partial K_i}{\partial x_j}(x_i')\int_{\mathbb{R}^n_+} \overline{U}_\lambda^6 + \frac{1}{8}\frac{\partial H_i}{\partial x_j}(x_i')\int_{\partial\mathbb{R}^n_+} \overline{U}_\lambda^4 = O(u_i(x_i)^{-2}). \tag{48}$$

By Remark 3.1, it turns out that

$$\varphi_i(x_i') = I_{K_i(x_i'),H_i(x_i')}(\overline{U}_\lambda) = 4\int_{\mathbb{R}^n_+} |\nabla\overline{U}_\lambda|^2 - \frac{1}{6}K_i(x_i')\int_{\mathbb{R}^n_+} \overline{U}_\lambda^6 - H_i(x_i')\int_{\partial\mathbb{R}^n_+} \overline{U}_\lambda^4.$$

Differentiating with respect to x_i', and taking into account that $I'_{K_i(x_i'),H_i(x_i')}(\overline{U}_\lambda) = 0$, we deduce that

$$\begin{aligned}
\frac{\partial\varphi_i}{\partial y_j'}(x_i') &= \frac{\partial I_{K_i(\cdot),H_i(\cdot)}}{\partial y_j'}|_{x_i'}(\overline{U}_\lambda) + I_{K_i(\cdot),H_i(\cdot)}\left(\frac{\partial\overline{U}_{\lambda(\cdot)}(x,\cdot)}{\partial y_j'}|_{x_i'}\right)\\
&= \frac{1}{6}\frac{\partial K_i}{\partial x_j}(x_i')\int_{\mathbb{R}^n_+} \overline{U}_\lambda^6 + \frac{1}{8}\frac{\partial H_i}{\partial x_j}(x_i')\int_{\partial\mathbb{R}^n_+} \overline{U}_\lambda^4.
\end{aligned}$$

In the above formula the boundary point y_i' is considered as a parameter on which I_{K_i,H_i} and $\overline{U}_\lambda$ depend, through the functions K_i and H_i. The conclusion then follows from equation (48) and the last expression.
■

Remark 6.3 *We note that if $\{K_i\}$ is just bounded in C^1 norm, the above proof yields anyway $\nabla'\varphi_i(x_i') \to 0$ as $i \to +\infty$.*

Now the local blow up analysis will be applied to equation (7) on the whole half sphere. We begin with the following Proposition which can be proved as in [25] pages 499-502, with minor modifications.

Proposition 6.4 *Assume $K \in C^1(\overline{S^n_+})$ and $H \in C^2(\partial S^n_+)$ satisfy*

$$\frac{1}{A_1} \le K(x) \le A_1, \quad \forall x \in S^n_+; \qquad \|K\|_{C^1(\overline{S^n_+})} \le A_2; \tag{49}$$

$$-A_1 \le H(x') \le A_1, \quad \forall x' \in \partial S^n_+; \qquad \|K\|_{C^2(\partial S^n_+)} \le A_2. \tag{50}$$

Then, given any $R \ge 1$ and any $0 < \varepsilon < 1$, there exist positive constants δ_0, C_0, C_1, depending only on n, ε, R, A_1 and A_2 such that, for all $\tau \le \delta_0$, and for all the solutions v of equation (33) with $\sup_{S^n_+} v \ge C_0$, the following properties hold true. There exist $\{q^1,\dots,q^N\} \subseteq S^n_+$, with $N \ge 1$, such that *i) each q^j is a local maximum for v and*

$$\overline{B_{\overline{r}_j}(q^j)} \cap \overline{B_{\overline{r}_l}(q^l)} = \emptyset, \qquad \text{for } j \neq l,$$

where $\overline{r}_j = R\, v(q^j)^{-\frac{p-1}{2}}$;

ii) either $dist(q^j, \partial S^n_+) > \overline{r}_j$ *and*

$$(a) \qquad \left\| v(q^j)^{-1}\, v\left(v(q^j)^{-\frac{p-1}{2}}\, x\right) - \left(\frac{1}{1 + k_j\, |x|^2}\right)^{\frac{n-2}{2}} \right\|_{C^2(B_{2R})} < \varepsilon,$$

or $dist(q^j, \partial S^n_+) < \overline{r}_j$ *and*

$$(b) \qquad \left\| v(q_i)^{-1}\, v\left(v(q^j)^{-\frac{p-1}{2}}\, x\right) - \left(\frac{\lambda_j}{1 + k_j \lambda_j^2 (|x'|^2 + |x_n + t_j|^2)}\right)^{\frac{n-2}{2}} \right\|_{C^2((B_{2R})_+)} < \varepsilon,$$

In the above two formulas it is $k_j = (4n(n-1))^{-1} K(q^j)$, *while* λ_j *and* t_j *satisfy* $2k_j \lambda_j t_j = H(q^j)$, *with*

$$\lambda_j = \begin{cases} 1 + k_j \lambda_j^2 t_j^2, & \text{if } H(q^j) \geq 0; \\ 1 & \text{if } H(q^j) < 0. \end{cases}$$

The function v is identified with its image through the map ι, the projection being suitably chosen depending on the point q^j.

iii) $|q^j - q^l|^{\frac{2}{p-1}}\, v(q^l) \geq C_0$, *for* $j < l$, *while* $v(q) \leq C_1\, dist\left(q, \{q^1, \dots, q^N\}\right)^{-\frac{2}{p-1}}$ *for all* $q \in S^n_+$.

Properties (a) and (b) in assertion $ii)$ above distinguish, roughly, the cases of interior and boundary blow ups. Property $iii)$ implies that, if the mutual distance of the points $\{q^j\}$ is bounded from below along some sequence of solutions, then blow ups are isolated. This fact is the content of the next Proposition.

Proposition 6.5 *Suppose that $K \in C^1(\overline{S^n_+})$ and $H \in C^2(\partial S^n_+)$ satisfy conditions* (49) *and* (50) *respectively. Then, given any $R \geq 1$, and any $0 < \varepsilon < 1$, there exist positive constants δ_0, δ_1 and C_0 such that, for all $\tau \leq \delta_0$, and for all the solutions v of equation* (7) *with $\sup_{S^n_+} v \geq C_0$ the following property holds true. If $\{q^1, \dots, q^N\} \subseteq S^n_+$ are the points given by Proposition 6.4, then there holds*

$$\min_{l \neq j} |q^l - q^j| \geq \delta_1.$$

PROOF. The proof is very similar to that of Proposition 1.2 in [25], and is based on the use of formula (35) and on a rescaling argument. The only difference is that K and H here are non constant, but one can use conditions (49), (50), Proposition 5.1 and Lemma 5.6. ■

Proposition 6.6 *Suppose that $\{K_i\}_i \subseteq C^1(\overline{S^3_+})$ and $\{H_i\}_i \subseteq C^2(\partial S^3_+)$ satisfy assumptions* (49) *and* (50) *uniformly in i. Suppose that $\{v_i\}_i$ is a sequence of positive solutions of* (7)*; then there are no interior blow-ups for $\{v_i\}_i$.*

PROOF. By Propositions 4.7, 6.1, 6.4, 6.5 we know that both interior and boundary blow ups are isolated simple and hence isolated. As a consequence, by Definition 4.4, the number of blow up points is bounded above by a constant depending on A_1 and A_2 only.

Suppose by contradiction that $\overline{x} \in S^3_+$ is an interior blow-up point for v_i. Then, it follows from the Harnack inequality, the fact that there is just a finite number of blow-up points and Propositions 4.7, 5.2

that for some finite set $\{q^1,\dots,q^N\} \subseteq \overline{\mathbb{R}^3_+}$, with $q^1 \in \mathbb{R}^3_+$, and some harmonic function $b : \mathbb{R}^3_+ \to \mathbb{R}$, the following holds

$$u_i(x_i)\, u_i(x) \to a_1\, |x-q^1|^{-1} + \sum_{j=2}^{N} a_j |x-q^j|^{-1} + b(x), \qquad \text{in } C^2_{loc}(\overline{R}^3_+ \setminus \{q^1,\dots,q^N\}).$$

Here $a_j > 0$ for all j, $u_i = \iota\, v_i$, and x_i is the local maximum point of u_i converging to q^1 with $u_i(x_i) \to +\infty$; for simplicity we can suppose that the pole of the stereographic projection is not a blow up point for v_i.

It follows from the Liouville Theorem and from $\frac{\partial b}{\partial x_3} = 0$ on $\partial\mathbb{R}^3_+$ that b is constant on $\mathbb{R}^3_+$; reasoning as above we deduce

$$h(x) = a_0\, |x-\overline{x}|^{-1} + \overline{b} + O(|x-q^1|), \qquad \text{for } x \text{ close to } q^1,$$

where $\overline{b} > 0$. Let $\sigma > 0$ be such that $B_\sigma(q^1) \subseteq \mathbb{R}^3_+ \setminus \{q^2,\dots,q^N\}$; as for (35), the function u_i satisfies

$$\begin{aligned} \frac{1}{8}\left(\frac{1}{2} - \frac{3}{p_i+1}\right) \int_{B_\sigma(q^1)} K_i u_i^{p_i+1} \quad &- \quad \frac{1}{8(p_i+1)} \int_{B_\sigma(q^1)} x \cdot \nabla K_i\, u_i^{p_i+1} + \frac{1}{8} \int_{\partial B_\sigma(q^1)} K_i\, u_i^{p_i+1}\, x \cdot \nu \\ &= \int_{\partial B_\sigma(q^1)} B(x, u_i, \nabla u_i). \end{aligned} \tag{51}$$

The estimates of the above terms are completely analogous to the corresponding ones in boundary blow up analysis, see [29]. Hence, using also Proposition 4.2, one deduces that

$$\left(\frac{1}{2} - \frac{3}{p_i+1}\right) \int_{B_\sigma(q^1)} K_i u_i^{p_i+1} = \frac{\tau_i}{4}\left(\lim_i K_i(x_i) \int_{\mathbb{R}^3} U_1^6 + o(1)\right);$$

$$\int_{B_\sigma(q^1)} x \cdot \nabla K_i(x)\, u_i^{p_i+1} = \int_{B_\sigma(q^1)} x \cdot \nabla K_i(x_i)\, u_i^{p_i+1} + \int_{B_\sigma(q^1)} x \cdot \nabla (K_i(x) - K_i(x_i))\, u_i^{p_i+1} = o\left(u_i(x_i)^{-2}\right);$$

$$\int_{\partial B_\sigma(q^1)} K_i\, u_i^{p_i+1} = o\left(u_i(x_i)^{-2}\right); \qquad \int_{\partial B_\sigma(q^1)} B(x, u_i, \nabla u_i) = -\pi\, u_i(x_i)^{-2} a_1\, \overline{b} + o\left(u_i(x_i)^{-2}\right),$$

for σ small. Using the last estimates and (51) we reach a contradiction. This concludes the proof. ■

Remark 6.7 *As anticipated in the Introduction, the fact that there are no interior blow ups for equation (7) is strongly related to the fact that there are no multiple blow ups for the scalar curvature equation on S^3. The proof relies on the above estimate*

$$\int_{B_\sigma} x \cdot \nabla K_i u_i^{\frac{p+1}{2}} = o\left(u_i(x_i)^{-2}\right).$$

In our case, the corresponding term $\int_{(B_\sigma)_+} x \cdot \nabla K_i u_i^{\frac{p+1}{2}}$ may be of order $u_i(x_i)^{-2}$ (indeed this always happens if $(K_i, H_i) \equiv (K, H) \in \mathcal{A}$, see the proof of Theorem 1.2) and the above proof breaks down; this is the reason of the possible presence of multiple blow ups. See Proposition 8.5.

We can summarize the above results with the following proposition.

Proposition 6.8 *Suppose that $\{K_i\}_i \subseteq C^1(S^3_+)$ and that $\{H_i\}_i \subseteq C^2(\partial S^3_+)$ satisfy (49) and (50) uniformly in i. Suppose that $\{v_i\}_i$ is a sequence of positive solutions of*

$$\begin{cases} -8\Delta v_i + 6 v_i = K_i(x) v_i^{p_i}, & \text{in } S^3_+; \\ 2\frac{\partial v_i}{\partial \nu} = H_i(x') v_i^{\frac{p_i+1}{2}}, & \text{on } \partial S^3_+. \end{cases}$$

with $\sup_i v_i = +\infty$. *Then the functions* $\{v_i\}$ *blow up at a finite number of points of* ∂S^3_+. *These blow ups are isolated simple and their distance is bounded below by a positive constant depending on* $\min_i K_i$, *the* C^1 *bounds of* $\{K_i\}_i$ *and the* C^2 *bounds of* $\{H_i\}_i$. *As a consequence the number of blow ups is bounded above by a constant depending only on these numbers. If* $(K_i, H_i) = (K, H)$ *for some fixed functions* K *and* H, *then the blow up points are critical for* φ.

7 Proof of Theorem 1.1

Consider the subcritical approximation (7) of equation (3), with $p < 5$. From the discussion in Section 2, we are reduced to find solutions of the equivalent problem (15) in the half-space. We can choose as pole of the projection the point $-q$, where q is a global minimum point of φ, as in the statement of Theorem 1.1. In this way the image of q under the projection is the origin in $\mathbb{R}^3$.

Solutions of (15) can be found as critical points of the Euler functional $I_\tau : \mathcal{D}^{1,2}(\mathbb{R}^3_+) \to \mathbb{R}$ defined as

$$I_\tau(u) = \frac{1}{2} \int_{\mathbb{R}^3_+} |\nabla u|^2 - \frac{1}{6-\tau} \int_{\mathbb{R}^3_+} W^\tau(x) K(x) |u|^{6-\tau} - \frac{4}{4-\tau} \int_{\partial \mathbb{R}^n_+} W^{\frac{\tau}{2}}(x') H(x') |u|^{4-\tau}, \qquad u \in \mathcal{D}^{1,2}(\mathbb{R}^3_+).$$

Let also J_τ denote the corresponding functional on $H^1(S^3_+)$. Since the standard half sphere is of *positive type* (see the Introduction), it is clear that the functional I_τ possesses a mountain pass structure; we denote by $\overline{I}_\tau$ the mountain pass level of I_τ. When $\tau = 0$, namely when the problem is purely critical, the functional I_τ is simply $I_{K,H}$, see the notation in Section 2. We first give an estimate from above of $\overline{I}_\tau$.

Lemma 7.1 *There exist* δ_0 *and* τ_0, *depending on* K *and* H, *such that*

$$\overline{I}_\tau \leq \varphi(q) - \delta_0, \qquad \text{for all } \tau \in (0, \tau_0).$$

PROOF. For $\lambda > 0$, let $\overline{U}_\lambda$ be the function defined in formula (19), with $k = \frac{K(0)}{4n(n-1)}$ and with t satisfying $2k\lambda t = H(0)$. Using a rescaling, it is easy to prove that

$$(52) \quad \int_{\mathbb{R}^3_+} |\nabla \overline{U}_\lambda|^2 = \int_{\mathbb{R}^3_+} |\nabla \overline{U}_1|^2; \qquad \int_{\partial \mathbb{R}^3_+} H(x') \overline{U}_\lambda^4 = H(0) \int_{\partial \mathbb{R}^3_+} \overline{U}_1^4 + O\left(\lambda^{-2} \log \lambda\right), \qquad \text{for } \lambda \text{ large},$$

and

$$(53) \qquad \int_{\mathbb{R}^3_+} K(x) \overline{U}_\lambda^6 = K(0) \int_{\mathbb{R}^3_+} \overline{U}_1^6 + \lambda^{-1} \frac{\partial K}{\partial x_3}(0) \int_{\mathbb{R}^3_+} x_3 \overline{U}_1^6 + o(\lambda^{-1}), \qquad \text{for } \lambda \text{ large}.$$

Using equations (52), (53) and some simple computations one finds

$$\sup_{t \geq 0} I_{K,H}(t\overline{U}_\lambda) = \varphi(q) - \frac{1}{6} \lambda^{-1} \frac{\partial K}{\partial x_3}(0) \int_{\mathbb{R}^3_+} x_3 |\overline{U}_1|^{2^*} + o(\lambda^{-1}), \qquad \text{for } \lambda \text{ large}.$$

We note that the condition $\frac{\partial K}{\partial \nu}(q) < 0$ implies $\frac{\partial K}{\partial x_3}(0) > 0$ in $\mathbb{R}^3_+$. Hence, choosing λ_0 sufficiently large, we find the existence of δ_0, depending on K and H, such that

$$\sup_{t \geq 0} I_{K,H}(t\overline{U}_{\lambda_0}) \leq \varphi(q) - 2\delta_0.$$

By continuity, choosing $\tau_0 > 0$ sufficiently small we deduce that

$$\overline{I}_\tau \leq \sup_{t \geq 0} I_\tau(t\overline{U}_{\lambda_0}) \leq \varphi(q) - \delta_0, \qquad \tau \in (0, \tau_0).$$

The proof is thereby completed. ∎

PROOF OF THEOREM 1.1 CONCLUDED. For $\tau > 0$ small, let v_τ be the mountain pass solution of (7). We claim that the functions $\{v_\tau\}_\tau$ remain bounded in $L^\infty(S^3_+)$ as $\tau \to 0$. In fact, supposing by contradiction that $\{v_\tau\}_\tau$ blows up, by Proposition 6.8 blow ups of $\{v_\tau\}_\tau$ occur at the boundary of S^3_+ only; let $q^1, \dots, q^N$ be the blow up points. It follows from Proposition 5.1 and Lemma 5.6 that

$$\lim_{\tau\to 0} J_\tau(v_\tau) = \sum_{j=1}^{N} \varphi(q^j). \tag{54}$$

On the other hand, since $\varphi(q) = \min_{\partial S^3_+} \varphi$, and since $\varphi > 0$, Lemma 7.1 contradicts equation (54).

Hence, the functions $\{v_\tau\}$ converge to a solution v of (7). We note that the function v is non-zero and strictly positive: this follows from the fact that v_τ is uniformly away from zero in $H^1(S^3_+)$, or also from the Harnack inequality. This concludes the proof. ∎

Remark 7.2 *Using the Mountain Pass scheme and the standard analysis of Palais Smale sequences for the functional $I_{K,H}$, one can prove some existence results of problem* (3) *for any n with more restrictive hypotheses. Let $\tilde\varphi(x')$, $x' \in \partial S^n_+$, denote the blow up energy corresponding to $K(x')$ and $H(x')$, computable for example by formula* (28) *(note that for $n = 3$ $\tilde\varphi$ is nothing but φ). The assumptions on K and H are the following. There exists $q \in \partial S^n_+$ with*

$$\tilde\varphi(q) = \min\{\tilde\varphi(x') \,:\, x' \in \partial S^n_+\}; \qquad \frac{\partial K}{\partial \nu}(q) < 0; \qquad \tilde\varphi(q) \le \frac{\omega_n}{n}(n(n-1))^{\frac{n}{2}} \left(\sup_{S^n_+} K\right)^{-\frac{n-2}{2}}. \tag{55}$$

By formula (29)*, the last inequality asserts that the interior blow up has energy larger that the boundary blow up. We also note that, by* (30)*, this assumption is non empty.*

When $n = 3$, the first two conditions in (55) *coincide with* (6)*, and the last condition can be completely removed.*

Remark 7.3 *For the case $H \equiv 0$, Theorem 1.1 could be proved also using the observations in Remark 7.2 and the minimization technique in [27].*

In fact, if $\sup_{S^3_+} K \le 4 \sup_{\partial S^3_+} K$, condition (55) *is satisfied, see Remark 1.3* (a)*.*

On the other hand, if $\sup_{S^3_+} K > \sup_{\partial S^3_+} K$, we can reflect K evenly on all S^3 and look for symmetric solutions of (4)*, see the discussion in the Introduction. Then, using the condition $\sup_{S^3_+} K > 4 \sup_{\partial S^3_+} K$, one can reason as in the proof of Theorem 4 in [27], ruling out concentration of mountain pass Palais Smale sequences outside ∂S^3_+.*

8 Proof of Theorem 1.2

In this section we prove Theorem 1.2. We start by giving some further characterizations of blow up point for solutions of (33). We recall the definition of the matrix M_{lj} given in formula (10) and its least eigenvalue ρ.

Proposition 8.1 *Let $K \in C^1(\overline{S^3_+})$ be a positive function, and let $H \in C^2(\partial S^3_+)$. Then there exists some number $\delta^* > 0$, depending only on $\min_{\overline{S^3_+}} K$, $\|K\|_{C^1(\overline{S^3_+})}$ and $\|H\|_{C^2(\partial S^3_+)}$, with the following properties.*

Let $\{p_i\}$ be such that $p_i \le 5$, $p_i \to 5$, let $K_i \to K$ in $C^1(\overline{S^3_+})$, $H_i \to H$ in $C^2(\partial S^3_+)$, and let $v_i > 0$ satisfy

$$\begin{cases} -4\frac{n-1}{n-2}\Delta v_i = K_i(x)\, v_i^{p_i} & \text{in } \partial S^3_+, \\ \frac{2}{n-2}\frac{\partial v_i}{\partial \nu} = H_i(x') v_i^{\frac{p_i+1}{2}} & \text{on } \partial S^3_+. \end{cases} \tag{56}$$

with $\max_{\overline{S^3_+}} v_i \to +\infty$ *as* $i \to +\infty$. *Then, after passing to a subsequence, the following properties hold true*

i) $\{v_i\}$ *has only isolated simple blow up points* $(q^1, \dots, q^N) \in \mathcal{F} \setminus \mathcal{F}^-$ $(N \geq 1)$, *with* $|q^j - q^l| \geq \delta^*$ $\forall j \neq l$, *and* $\rho(q^1, \dots, q^N) \geq 0$. *Furthermore* $q^1, \dots, q^N \in \mathcal{F}^+$ *if* $N \geq 2$.

ii) Setting

$$\mu_j = 2\left(\frac{K(q^j)}{6} + H^+(q^j)^2\right)^{-\frac{1}{2}} K(q^j)^{\frac{1}{4}} \lim_i \frac{v_i(q_i^1)}{v_i(q_i^l)}$$

$$\lambda_j = \frac{1}{16\pi}\left(\frac{K(q^j)}{6} + H^+(q^j)^2\right) \frac{\varphi(q^j)}{K(q^j)^{\frac{1}{2}}} \lim_i \tau_i \, v_i(q_i^j)^2,$$

where H^+ *is the positive part of* H *and* $q_i^j \to q^j$ *is the local maximum of* v_i, *there holds*

$$\mu_j \in (0, +\infty), \qquad \lambda_j \in [0, +\infty), \qquad \forall j = 1, \dots, N.$$

iii) When $N = 1$

$$\lambda_1 = \frac{\partial K}{\partial \nu}(q^1) \frac{\psi(q^1)}{K(q^1)^{\frac{3}{2}}};$$

when $N \geq 2$

$$\sum_{l=1}^{N} M_{lj} \mu_l = \lambda_j \, \mu_j, \qquad \forall j = 1, \dots, N. \tag{57}$$

iv) $\lambda_j \in (0, +\infty)$ $\forall j = 1, \dots, N$ *if and only if* $\rho(q^1, \dots, q^N) > 0$.

Proof. Assertion *ii)* follows from Proposition 5.2 and Lemmas 4.6, 5.7. From another part, it follows from Propositions 6.1, 6.5 and Remark 6.3 that v_i has only isolated simple blow up points $q^1, \dots, q^N \in \mathcal{F}$ $(N \geq 1)$ with $|q^j - q^l| \geq \delta^*$ $(j \neq l)$ for a fixed $\delta^* > 0$ depending only on the above quantities.

Let $q_i^1 \to q^1$ be the local maximum of v_i for which $v_i(q_i^1) \to +\infty$; performing a stereographic projection through the point $-q^1$, equation (56) is transformed into

$$\begin{cases} -\Delta u_i(x) = K_i(y)\, W(x)^{\tau_i}\, u_i^{p_i} & \text{in } \mathbb{R}^3_+, \\ -\frac{2}{n-2}\frac{\partial u_i}{\partial x_n} = H_i(x') W(x')^{\frac{\tau_i}{2}}\, u_i^{\frac{p_i+1}{2}} & \text{on } \partial\mathbb{R}^3_+. \end{cases}$$

By our choice of the projection, it is clear that 0 is also an isolated simple blow up point for $\{u_i\}$. We can also suppose that none of the points $\{q^1, \dots, q^N\}$ is mapped to $+\infty$ by the stereographic projection, and we still denote their images by $q^1, \dots, q^N$ (in particular we have $q^1 = 0$). It follows from Proposition 5.2 that

$$u_i(q_i^1)\, u_i(x) \to h_1(x) := a(q^1)\, |x|^{-1} + b_1(x) \qquad \text{in } C^2_{loc}(\overline{\mathbb{R}^3_+} \setminus \{q^1, \dots, q^N\}), \tag{58}$$

where $a(q^1)$ is the coefficient in (44) with $\lim_i K_i(x_i)$ replaced by $K(q^1)$ and $\lim_i H_i(x_i)$ replaced by $K(q^1)$. The function $b_1(x)$ is harmonic in $\overline{\mathbb{R}^3_+} \setminus \{q^2, \dots, q^N\}$, and we have still used the notation q_i^j for the local maxima of u_i converging to q^j.

Coming back to v_i on S^3_+ we have

$$\lim_i v_i(q_i^1)\, v_i(x) = \frac{1}{2} a(q^1)\, G_{q^1}(x) + \tilde{b}_1(x), \qquad \text{in } C^2_{loc}(\overline{S^3_+} \setminus \{q^2, \dots, q^N\}),$$

where $\tilde{b}_1$ is some regular function on $\overline{S^3_+} \setminus \{q^2, \dots, q^N\}$ satisfying $(-8\Delta + 6)\, \tilde{b}_1 = 0$ with $\frac{\partial \tilde{b}_1}{\partial \nu} = 0$ on ∂S^3_+.

If $N = 1$, then $\tilde{b}_1 = 0$ by the maximum principle, while for $N \geq 2$, taking into account the contribution of all the poles, we deduce that

$$\lim_i v_i(q_i^j)\, v_i(x) = \frac{1}{\sqrt{2}} a(q^1)\, G_{q^1}(x) + \frac{1}{\sqrt{2}} \sum_{l \neq 1} a(q^l)\, G_{q^l}(x) \, \lim_i \frac{v_i(q_i^1)}{v_i(q_i^l)}.$$

In fact, subtracting all the poles from the limit function, we obtain a regular function $r : S^3_+ \to \mathbb{R}$ such that $(-8\Delta + 6)r = 0$ and $\frac{\partial r}{\partial \nu} = 0$ on ∂S^3_+, so it must be $r \equiv 0$. In the above formula, $G_q(x)$ is the function defined in the Introduction, and the convergence is in $C^2_{loc}(\overline{S^3_+} \setminus \{q^1, \dots, q^N\})$.

Using the last expression, we can compute the value of $b_1(0)$ in (58), which is

$$b_1(0) = \sqrt{2}\, a(q^1) \sum_{l \neq 1} a(q^l)\, G_{q^l}(x) \, \lim_i \frac{v_i(q_i^1)}{v_i(q_i^l)}. \tag{59}$$

Hence, using (59) and Proposition 4.2, we deduce that

$$\lim_{\sigma \to 0} \int_{\partial_2 B_\sigma} B(x, h_1, \nabla h_1) = -\sqrt{2}\, \pi\, a(q^1) \sum_{l \neq 1} a(q^l)\, G_{q^l}(q^1) \, \lim_i \frac{v_i(q_i^1)}{v_i(q_i^l)}.$$

From another part, it follows from Lemma 4.1, Proposition 5.1, Lemma 5.7 and some computations as in Lemma 5.5 that

$$\int_{\partial_2 B_\sigma} B(x, h_1, \nabla h_1) = \frac{1}{16} \varphi(q^1) \, \lim_i \tau_i v_i(q_i^1)^2 - \frac{1}{48} \lim_i u_i(q_i^1)^2 \int_{B_\sigma} x \cdot \nabla K_i \, u_i^{p_i+1}.$$

By Proposition 5.1, Lemma 5.7 and equation (20) it follows that

$$\frac{1}{48} \lim_i u_i(q_i^1)^2 \int_{B_\sigma} x \cdot \nabla K_i \, u_i^{p_i+1} = 3\pi\, \psi(q^1) \left(1 + 6 \frac{H^+(q^1)^2}{K(q^1)}\right)^{-1} \frac{\partial K}{\partial x_3}(q^1),$$

where ψ is defined in (9). The tangent map of the stereographic projection π, calculated in q^1, is $\frac{1}{2} Id$, hence it turns out that $\frac{\partial (K \circ \pi^{-1})}{\partial x_3}(q^1) = -2 \frac{\partial K}{\partial \nu}(q^1)$. Then, always identifying K with $K \circ \pi^{-1}$, from the last two formulas we obtain

$$\frac{1}{16} \varphi(q^1) \, \lim_i \tau_i v_i(q_i^1)^2 = -\sqrt{2}\, \pi\, a(q^1) \sum_{l \neq 1} a(q^l)\, G_{q^l}(q^1) \, \lim_i \frac{v_i(q_i^1)}{v_i(q_i^l)} + \frac{1}{4} \pi \, \frac{\partial K}{\partial \nu}(q^1)\, a(q^1)^2 \frac{\psi(q^1)}{K(q^1)}.$$

Finally, using the expression of $\{\mu_l\}$ and λ_1 we get

$$\frac{\psi(q^1)}{K(q^1)^{\frac{3}{2}}} \frac{\partial K}{\partial \nu}(q^1)\, \mu_1 - 4\sqrt{2}\, \frac{G_{q^l}(q^1)}{K(q^1)^{\frac{1}{4}} K(q^l)^{\frac{1}{4}}} \mu_l = \lambda_1 \, \mu_1.$$

Of course a similar formula holds for every q^j with $j \neq 1$. We have thus established (57) and completed the proof of *iii*).

From the last formula it follows that $q^j \in \mathcal{F} \setminus \mathcal{F}^-$, $\forall j = 1, \dots, N$, and when $N \geq 2$, $q^j \in \mathcal{F}^+$. Furthermore, since $M_{jj} \geq 0$ for every j, and $M_{lj} < 0$ for $l \neq j$, it follows from linear algebra and the variational characterization of the least eigenvalue that there exists some $y = (y_1, \dots, y_N) \neq 0$, $y_l \geq 0 \, \forall l$, such that $\sum_{j=1}^N M_{lj}\, y_j = \rho\, y_l$.

Multiplying (57) by y_j and summing over j, we have

$$\rho \sum_l \mu_l \, y_l = \sum_{l,j} M_{lj}\, y_j \, \mu_l = \sum_j \lambda_j \, \mu_j \, y_j \geq 0.$$

It follows that $\rho \geq 0$, so we have verified part i). Part iv) follows from i)-iii). ■

As a consequence of i), iii) and iv) in Proposition 8.1 we deduce the following corollary. It asserts that, in the *nondegenerate* situation $(K, H) \in \mathcal{A}$, one can obtain a-priori estimates for positive solutions of the subcritical problem (7), provided p is sufficiently close to $\frac{n+2}{n-2}$.

Corollary 8.2 *Suppose $(K, H) \in \mathcal{A}$. Then there exist constants τ_* and C_*, depending only on $\min_{\overline{S^3_+}} K$, $\|K\|_{C^1(\overline{S^3_+})}$ and $\|H\|_{C^2(\partial S^3_+)}$ and $\min\{|\rho(q^1, \ldots, q^N)| \,:\, q^1, \ldots, q^N \in \mathcal{F}, N \geq 1\}$ such that*

$$\max_{S^3_+} v \leq \frac{C_*}{\sqrt{\tau}}$$

for every positive solution v of (7) with $0 < \tau = \frac{n+2}{n-2} - p < \tau_*$.

For $\overline{x}' \in \partial S^3_+$ and $\gamma > 0$ large, let $\delta_{\overline{x}',\gamma} : S^3_+ \to \mathbb{R}$ be the function defined in the following way

$$\delta_{\overline{x}',\gamma}(x) = \left(\frac{24}{K(\overline{x}')}\right)^{\frac{1}{4}} \left(\frac{\gamma}{\gamma^2 + 1 + (1-\gamma^2)\cos d(\tilde{x}, x)}\right)^{\frac{1}{2}},$$

where $\tilde{x} = (\tilde{x}', \tilde{x}_4) \in S^3$ is given by

$$\frac{\tilde{x}'}{|\tilde{x}'|} = \overline{x}'; \qquad \tilde{x}_4 = -\sqrt{24}\,\frac{\gamma}{\gamma^2 - 1}\,\frac{H(\overline{x}')}{\sqrt{K(\overline{x}')}},$$

and where $d(\tilde{x}, x)$ denotes the geodesic distance in S^3.

For all the choices of $\overline{x}'$ and γ, $\delta_{\overline{x}',\gamma}$ satisfies the equation $(-8\Delta + 6)\delta_{\overline{x}',\gamma} = K(\overline{x}')\delta^5_{\overline{x}',\gamma}$ with the boundary condition $2\frac{\partial \delta_{\overline{x}',\gamma}}{\partial \nu} = H(\overline{x}')\delta^3_{\overline{x}',\gamma}$. The functions $\{\delta_{\overline{x}',\gamma}\}_{\overline{x}',\gamma}$, restricted to the half sphere S^3_+, are nothing but the pre-images under the map ι of the family $\{\overline{U}_\lambda\}_\lambda$ defined in (19), or of some of their translations in $\mathbb{R}^{n-1}$.

For $\tau = p - \frac{n+2}{n-2}$, $\tau > 0$, let J_τ denote the Euler functional corresponding to problem (7), namely

$$J_\tau(v) = 4\int_{S^3_+} |\nabla v|^2 + 3\int_{S^3_+} v^2 - \frac{1}{6-\tau}\int_{S^3_+} K(x)\,|v|^{6-\tau} - \frac{4}{4-\frac{\tau}{2}}\int_{\partial S^3_+} H(x')\,|v|^{4-\frac{\tau}{2}}, \qquad v \in H^1(S^3_+).$$

Let $q^1, \ldots, q^N \in \mathcal{F}^+$ be critical points of φ with $\rho(q^1, \ldots, q^N) > 0$. For ε small, define the set $V_\varepsilon = V_\varepsilon(\tau, q^1, \ldots, q^N) \subseteq H^1(S^3_+)$ as

$$V_\varepsilon = \left\{\sum_{i=1}^N \delta_{a_i,\gamma_i} \,:\, (\gamma, a) \in \mathbb{R}^N_+ \times (\partial S^3_+)^N, \quad |a_i - q^i| < \varepsilon, \quad \varepsilon < \tau\,\gamma_i < \frac{1}{\varepsilon}, \quad i = 1, \ldots, N\right\}.$$

We also define $\mathcal{U}_\varepsilon = \mathcal{U}_\varepsilon(\tau, q^1, \ldots, q^N)$ to be the ε-tubular neighborhood of V_ε, namely

$$\mathcal{U}_\varepsilon = \left\{v + z \,:\, v \in V_\varepsilon, z \in (T_v V_\varepsilon)^\perp, \|z\| < \varepsilon\right\},$$

where $(T_v V_\varepsilon)^\perp$ denotes the subspace of $H^1(S^3_+)$ orthogonal to $T_v V_\varepsilon$.

For $R > 0$, set

$$\mathcal{O}_R = \left\{v \in C^{2,\alpha}(\overline{S^3_+}) \,|\, \frac{1}{R} \leq v \leq R, \|v\|_{C^{2,\alpha}(\overline{S^3_+})} \leq R\right\}.$$

Using the last definitions and standard regularity results, Proposition 8.1 can be reformulated as follows.

Proposition 8.3 *Let $(K,H)\in\mathcal{A}$ and let $\alpha\in]0,1[$. Then there exist a small positive constant ε, and a large positive constant R such that, when $\tau>0$ is sufficiently small, there holds*

$$v\in\mathcal{O}_R\cup\left\{\mathcal{U}_\varepsilon(\tau,q^1,\dots,q^N)\,:\,q^1,\dots,q^N\in\mathcal{F}^+,\rho(q^1,\dots q^N)>0,N\geq 1\right\}$$

for all $v\in H^1(S^3_+)$ satisfying $v\geq 0$ a.e. and $J'_\tau(v)=0$.

Using blow up analysis, we gave necessary conditions on blowing up solutions of (7) when p tends to $\frac{n+2}{n-2}$ from below. Now we are going to show that if $(K,H)\in\mathcal{A}$, one can construct solutions highly concentrating at any N points $q^1,\dots,q^N\in\mathcal{F}^+$ provided $\rho(q^1,\dots,q^N)>0$, see Proposition 8.5 below. The main tool is Implicit Function Theorem. Since the procedure is well-known, see [30], [36], we just give a general idea of the proof omitting some details.

We begin with the following technical Lemma, which proof is a consequence of standard estimates, see [6].

Lemma 8.4 *Let ε and $\tilde\delta$ be fixed positive numbers, let $a_i\in\partial S^3_+$, $i=1,2$ be such that $d(a_1,a_2)\geq\tilde\delta$, and let $\gamma_i\in(0,+\infty)$ be such that $\varepsilon<\tau\gamma_i<\frac{1}{\varepsilon}$, $i=1,2$. Then there exist a positive constant C such that for τ sufficiently small, the following estimates hold*

$$\|\delta^{4-\tau}_{a_i,\gamma_i}\delta_{a_j,\gamma_j}\|_{L^{\frac{6}{5}}(S^3_+)}\leq C\,\tau,\quad i\neq j;\qquad\qquad \|\delta^{5-\tau}_{a_i,\gamma_i}-\delta^5_{a_i,\gamma_i}\|_{L^{\frac{6}{5}}(S^3_+)}\leq C\,\tau|\log\tau|;$$

$$\|d(\cdot,a_i)\delta^5_{a_i,\gamma_i}\|_{L^{\frac{6}{5}}(S^3_+)}\leq C\,\tau;\qquad\qquad \|\delta^{3-\tau}_{a_i,\gamma_i}\delta_{a_j,\gamma_j}\|_{L^{\frac{4}{3}}(\partial S^3_+)}\leq C\,\tau,\quad i\neq j;$$

$$\|\delta^{3-\tau}_{a_i,\gamma_i}-\delta^3_{a_i,\gamma_i}\|_{L^{\frac{4}{3}}(\partial S^3_+)}\leq C\,\tau|\log\tau|;\qquad\qquad \|d(\cdot,a_i)\delta^3_{a_i,\gamma_i}\|_{L^{\frac{4}{3}}(\partial S^3_+)}\leq C\,\tau.$$

Following the original arguments in [6], [30], and using the estimates in Lemma 8.4, one can prove that for τ sufficiently small

$$\|J'_\tau(v)\|\leq O(\tau\,|\log\tau|),\qquad\text{for }\tau\text{ small and }v\in V_\varepsilon.$$

Moreover, from Proposition 3.2 in [26] and standard computations, it follows that, for τ small, $I''_\tau(u)$ is invertible in $(T_vV_\varepsilon)^\perp$, uniformly with respect to τ and $v\in V_\varepsilon$. Hence by the local inversion theorem, see [2], there exists $\varepsilon>0$ small (independent of τ) with the following property. For any $v\in V_\varepsilon$, there exists a unique $w(v,\tau)$ such that

$$w(v,\tau)\in(T_vV_\varepsilon)^\perp;\qquad\qquad J'_\tau(v+w(v,\tau))\in T_vV_\varepsilon. \tag{60}$$

Furthermore, the norm of $w(v,\tau)$ can be estimated as

$$\|w(v,\tau)\|\leq C\|J'_\tau(v)\|\leq C'\,\tau\,|\log\tau|, \tag{61}$$

where C and C' are fixed constants. As a consequence of the above discussion and of some computations, one finds

$$\begin{aligned} J_\tau(v+w(v,\tau)) &= J_\tau(v)+J'_\tau(v)[w(v,\tau)]+O\left(\|w(v,\tau)\|^2\right)=J_\tau(v)+O\left(\tau\,|\log\tau|\right)^2\\ &= \sum_{i=1}^N\varphi(a_i)-\frac{1}{6}\sum_{i=1}^N\left(\gamma_i^{-\frac{\tau}{2}}-1\right)K(a_i)\int_{S^3_+}\delta^6_{a_i,\gamma_i}-\sum_{i=1}^N\left(\gamma_i^{-\frac{\tau}{4}}-1\right)H(a_i)\int_{\partial S^3_+}\delta^4_{a_i,\gamma_i}\\ &\quad+4\,\pi\sqrt{6}\sum_{i=1}^N\frac{\partial K}{\partial\nu}(a_i)\frac{\psi(a_i)}{K(a_i)^{\frac{3}{2}}}\,\frac{1}{\gamma_i}-64\,\pi\sqrt{3}\sum_{l\neq j}\frac{1}{\sqrt{\gamma_j}}\frac{1}{\sqrt{\gamma_j}}\frac{G_{a_j}(a_l)}{K(a_j)^{\frac{1}{4}}K(a_j)^{\frac{1}{4}}}+o(\tau), \end{aligned}\tag{62}$$

as $\tau \to 0$. By means of equation (60), the manifold

$$\tilde{V}_\varepsilon = \{v + w(v,\tau), \, : \, v \in V_\varepsilon\}$$

is a *natural constraint* for J_τ, namely a point u which is critical for $J_\tau|_{\tilde{V}_\varepsilon}$ is also critical for J_τ. In order to find critical points of $J_\tau|_{\tilde{V}_\varepsilon}$, we differentiate $J_\tau(v + w(v,\tau))$ with respect to the parameters a_i, γ_i. Using standard estimates we obtain

$$\frac{\partial}{\partial a_i} J_\tau(v + w(v,\tau)) = \frac{\partial \varphi}{\partial a_i} + o(1), \qquad v \in V_\varepsilon, \tau \to 0; \tag{63}$$

$$\begin{aligned} \frac{\partial}{\partial \gamma_j} J_\tau(v + w(v,\tau)) \quad = \quad & \frac{1}{12}\frac{\tau}{\gamma_j} K(a_j) \int_{S^3_+} \delta^6_{a_j,\gamma_j} + \frac{1}{4}\frac{\tau}{\gamma_j} H(a_j) \int_{\partial S^3_+} \delta^4_{a_j,\gamma_j} \\ & -4\,\pi\sqrt{6} \sum_{i=1}^N \frac{\partial K}{\partial \nu}(a_j) \frac{\psi(a_j)}{K(a_j)^{\frac{3}{2}}} \frac{1}{\gamma_j^2} + 32\,\pi\sqrt{3} \sum_{l \neq j} \frac{1}{\gamma_j^{\frac{3}{2}}} \frac{1}{\gamma_l^{\frac{1}{2}}} \frac{G_{a_j}(a_l)}{K(a_j)^{\frac{1}{4}} K(a_j)^{\frac{1}{4}}} \\ & + o(\tau^2), \qquad\qquad v \in V_\varepsilon, \tau \to 0. \end{aligned} \tag{64}$$

Let us point out that the coefficients of $\frac{1}{\gamma_j^2}$ and of $\frac{1}{\gamma_j^{\frac{3}{2}}} \frac{1}{\gamma_l^{\frac{1}{2}}}$ in formula (64) coincide, when $\{a_j\} \equiv \{q^j\}$, with a constant multiple of the numbers M_{jj} and M_{lj} given in (10). As a consequence, since we are assuming that the least eigenvalue ρ of (M_{lj}) is positive, the above coefficients form a positive definite and invertible matrix. Using this condition, equation (63) and the fact that φ is Morse, one can prove that

$$deg_{H^1(S^3_+)} \left(J'_\tau|_{\tilde{V}_\varepsilon}, \tilde{V}_\varepsilon, 0 \right) = (-1)^{\sum_{j=1}^N (2 - m(\varphi, q^j))}. \tag{65}$$

By the invertibility of J''_τ in the normal direction to V_ε, and by the fact that the functions δ_{a_i,γ_i} have Morse index 1, it follows from (65) that

$$deg_{H^1(S^3_+)} \left(J'_\tau, \mathcal{U}_\varepsilon, 0 \right) = (-1)^{N + \sum_{j=1}^N (2 - m(\varphi, q^j))}. \tag{66}$$

Since the above degree is always different from zero, J_τ has at least one critical point in $\mathcal{U}_\varepsilon$; moreover it is standard to prove that critical points of J_τ in $\mathcal{U}_\varepsilon$ are non-negative functions when τ is sufficiently small. From [1] and [17] then it follows that these solutions are also regular and strictly positive.

We collect the above discussion in the following Proposition.

Proposition 8.5 *Let $(K,H) \in \mathcal{A}$, and let $\varepsilon > 0$ be small enough. Then, if $q^1, \dots, q^N \in \mathcal{F}^+$ with $\rho(q^1, \dots, q^N) > 0$, and if $\tau > 0$ is sufficiently small, the functional J_τ possesses a critical point in $\mathcal{U}_\varepsilon(\tau, q^1, \dots, q^N)$. Moreover, formula (66) holds true and all the critical points of J_τ are strictly positive functions on S^n.*

We need now the following lemma which, combined with Corollary 8.2, will be useful to obtain a-priori estimates for the computation of some degree formula, see Proposition 8.7 below.

Lemma 8.6 *Suppose $(K,H) \in \mathcal{A}$. Then there exists an homotopy $(K_t, H_t) : [0,1] \to \mathcal{A}$ with the following properties.*

(j) $H_t = tH$ *for all* $t \in [0,1]$*; moreover* $4\pi\sqrt{\frac{6}{K_0}} \equiv \varphi$ *and* $K_1 = K$.

(jj) Setting

$$\varphi_t(x') = 4\pi \sqrt{\frac{6}{K_t(x')}} \left(\frac{\pi}{2} - \arctan\left(H_t(x') \sqrt{\frac{6}{K_t(x')}} \right) \right), \qquad x' \in \partial S^3_+,$$

then there holds $\varphi_t \equiv \varphi$ *for all* $t \in [0,1]$.

(*jjj*) *Let $\mathcal{F}_t$, $\mathcal{F}_t^{\pm}$, ρ_t denote the counterparts of $\mathcal{F}$, $\mathcal{F}^{\pm}$, ρ corresponding to the functions (K_t, H_t). Then $\mathcal{F}_t \equiv \mathcal{F}$ and $\mathcal{F}_t^{\pm} \equiv \mathcal{F}^{\pm}$ for all $t \in [0,1]$. Moreover there exists a positive constant $\overline{C}$, depending only on $\min_{\overline{S^3_+}} K$, $\|K\|_{C^2(\overline{S^3_+})}$, $\|H\|_{C^2(\partial S^3_+)}$ and $\min\{|\rho(q^1,\dots,q^N)| : q^1,\dots,q^N \in \mathcal{F}, N \geq 1\}$ such that*

$$\min_{S^3_+} K_t \geq \frac{1}{\overline{C}}, \qquad \|K\|_{C^2(\overline{S^3_+})} \leq \overline{C}, \qquad \textit{for all } t \in [0,1]; \tag{67}$$

$$\min\{|\rho(q^1,\dots,q^N)| : q^1,\dots,q^N \in \mathcal{F}_t, N \geq 2\} \geq \frac{1}{\overline{C}}, \qquad \textit{for all } t \in [0,1]. \tag{68}$$

PROOF. First we note that, for a fixed valued of $H(x')$ we have

$$\lim_{K(x') \to +\infty} \varphi(x') = 0; \qquad \lim_{K(x') \to 0} \varphi(x') = \begin{cases} +\infty & \text{if } H(x') \leq 0, \\ \frac{4\pi}{H(x')} & \text{if } H(x') > 0. \end{cases} \tag{69}$$

Moreover, using some simple computations, one finds

$$\frac{\partial \varphi(x')}{\partial K(x')} = -4\pi \frac{\sqrt{6}}{K(x')^{\frac{3}{2}}} \left[\frac{\pi}{2} - \arctan\left(H(x') \sqrt{\frac{6}{K(x')}} \right) - \frac{H(x')\sqrt{6K(x')}}{K(x') + 6H(x')^2} \right] < 0. \tag{70}$$

As a consequence of (69), (70) and the implicit function theorem, one finds a unique positive function $K_t(x')$, $x' \in \partial S^3_+$, for which

$$4\pi \sqrt{\frac{6}{K_t(x')}} \left(\frac{\pi}{2} - \arctan\left(tH(x') \sqrt{\frac{6}{K_t(x')}} \right) \right) = \varphi(x'), \qquad \text{for all } x' \in \partial S^3_+.$$

We point out that, since φ is of class C^2, also K_t is of class C^2 on ∂S^3_+. With such a choice of K_t, properties (j) and (jj) are clearly satisfied.

We are now going prove (jjj), finding a suitable extension of K_t in the interior of S^3_+. Note that by (jj), $\mathcal{F}_t$ coincides with $\mathcal{F}$ for all t. For $q^j \in \mathcal{F}$, choose $\frac{\partial K_t}{\partial \nu}(q^j)$ satisfying

$$\frac{\partial K_t}{\partial \nu}(q^j) = \frac{K_t}{K}(q^j) \, \frac{4\pi - H(q^j)\varphi(q^j)}{4\pi - tH(q^j)\varphi(q^j)} \, \frac{\partial K}{\partial \nu}(q^j), \qquad q^j \in \mathcal{F}. \tag{71}$$

Let $(M_t)_{lj}$ be the counterpart of the matrix M_{lj} defined in (10) for the functions (K_t, H_t). It is clear from (9) and (71) that

$$(M_t)_{lj} = \frac{K(q^l)^{\frac{1}{4}} K(q^j)^{\frac{1}{4}}}{K_t(q^l)^{\frac{1}{4}} K_t(q^j)^{\frac{1}{4}}} M_{lj}, \qquad q^j \in \mathcal{F}.$$

As a consequence, from the multi-linearity of the determinant one deduces that

$$\det M_t(q^1,\dots,q^N) = \prod_{j=1}^N \frac{K_t(q^j)^{\frac{1}{2}}}{K(q^j)^{\frac{1}{2}}} \det M(q^1,\dots,q^N),$$

and hence it follows that $\rho_t(q^1,\dots,q^N) \neq 0$ whenever $\rho(q^1,\dots,q^N) \neq 0$. This implies that $(K_t, H_t) \in \mathcal{A}$ for all t, that $\mathcal{F}_t^{\pm} \equiv \mathcal{F}^{\pm}$ for all t, and that (68) is satisfied. Then it is easy to extend K_t in the interior of S^3_+ so that also (67) holds true. This concludes the proof. ■

Consider the following problem in S^3_+

$$
\begin{cases} -8\Delta v + 6v = f_1 & \text{in } S^3_+; \\ 2\frac{\partial v}{\partial \nu} = f_2 & \text{on } \partial S^3_+. \end{cases} \tag{72}
$$

It is standard, see e.g. [1], that if $f_1 \in C^\alpha(\overline{S^3_+})$ and if $f_2 \in C^{1,\alpha}(\partial S^3_+)$ for some $\alpha \in (0,1)$, then there exists a solution $v \in C^{2,\alpha}$ of (72). We denote by Ξ the operator which associates to (f_1, f_2) the solution v of (72), and we extend the definition of Ξ also to the case of weak solutions of (72).

When $(K,H) \in \mathcal{A}$ and the number τ is bounded from below, we have compactness result for positive solutions of (7) and we can compute their total degree. We recall the above definition of the set $\mathcal{O}_R$.

Proposition 8.7 *Suppose $(K,H) \in \mathcal{A}$. Then there exist constants τ_0, C_0 and δ_0, depending only on $\min_{S^3_+} K$, $\|K\|_{C^1(\overline{S^3_+})}$, $\|H\|_{C^2(\partial S^3_+)}$ and $\min\{|\rho(q^1, \dots, q^N)| : q^1, \dots, q^N \in \mathcal{F}, N \geq 1\}$ with the following properties*

i) $\left\{v \in H^1(S^3_+) : v \geq 0 \text{ a.e.}, J'_{\tau_0}(v) = 0\right\} \subseteq \mathcal{O}_{C_0}$;

ii) for $C, \delta > 0$ set $\mathcal{O}_{C,\delta} = \{u \in H^1(S^3_+) : \exists v \in \mathcal{O}_C$ such that $\|u - v\|_{H^1(S^3_+)} < \delta\}$: then $J'_{\tau_0} \neq 0$ on $\partial \mathcal{O}_{C_0,\delta_0}$, and

$$
deg_{H^1(S^3_+)}(u - \Xi(K\,|u|^{4-\tau_0}u, H\,|u|^{2-\frac{\tau_0}{2}}u), \mathcal{O}_{C_0,\delta_0}, 0) = -1. \tag{73}
$$

PROOF. Let $\tilde{K} : \overline{S^3_+} \to \mathbb{R}$ be the function defined in the following way, using stereographic coordinates

$$
\tilde{K}(x) = 2 + \frac{|x|^2 - 1}{x_1^2 + x_2^2 + (x_3+1)^2}, \qquad x \in \overline{\mathbb{R}^3_+}.
$$

We point out that $\tilde{K}$ is smooth and strictly positive on $\overline{S^3_+}$ and satisfies

$$
x \cdot \nabla \tilde{K}(x) \geq 0, \qquad \text{for all } x \in \overline{\mathbb{R}^3_+}. \tag{74}
$$

As a consequence, by equation (35), there is no solution of (3) with $(K,H) = (\tilde{K}, 0)$.

Consider the following homotopy from $[0,2]$ into $C^2(\overline{S^3_+}) \times C^2(\partial S^3_+)$

$$
s \to (\mathcal{K}_s, \mathcal{H}_s) = \begin{cases} \left((1-s)\tilde{K} + sK_0, 0\right) & \text{for } s \in [0,1]; \\ (K_{s-1}, H_{s-1}) & \text{for } s \in [1,2], \end{cases}
$$

where K_t and H_t are the functions given in Lemma 8.6. This homotopy connects $(\tilde{K}, 0)$ to (K,H), when the parameter goes from 0 to 2.

Define $J_{s,\tau} : H^1(S^3_+) \to \mathbb{R}$ to be the Euler functional corresponding to (7) for $(K,H) = (\mathcal{K}_s, \mathcal{H}_s)$. We claim that for C_0 sufficiently large and τ_0 sufficiently small there holds

$$
\left\{v \in H^1(S^3_+) : v \geq 0 \text{ a.e.}, J'_{s,\tau_0}(v) = 0\right\} \subseteq \mathcal{O}_{C_0}, \qquad \text{for all } s \in [0,2]. \tag{75}
$$

Of course, by the above discussion, all these weak solutions are of class $C^{2,\alpha}$ and positive.

For $s \in [0,1]$, upper bounds follow from standard blow up arguments and from non-existence results for the problems

$$
-\Delta u = u^p \qquad \text{in } \mathbb{R}^n, \qquad u > 0, \tag{76}
$$

and

$$(77)\qquad \begin{cases} -\Delta u = u^p & \text{in } \mathbb{R}^n_+, \qquad u > 0; \\ \frac{\partial u}{\partial x_n} = 0 & \text{on } \partial \mathbb{R}^n_+. \end{cases}$$

The non-existence result for (76) has been proved in [24] while that for (77) is a consequence of the former.

For $s \in [1,2]$, upper bounds follow from Corollary 8.2 and Lemma 8.6, choosing τ_0 sufficiently small.

Once upper bounds are achieved, lower bounds follow from the Harnack inequality, see Lemma 4.6. This proves (75) and hence property $i)$ in the statement.

Using (75), it is standard to prove that $J'_{s,\tau_0} \neq 0$ on $\partial \mathcal{O}_{C_0,\delta_0}$ for δ_0 sufficiently small and for all $s \in [0,2]$; this simply follows by testing J'_{s,τ_0} on the positive parts of the solutions.

Therefore, for the homotopy property of the degree, we only need to establish (73) for $(K,H) = (\tilde{K},0)$. In this case the formula follows from Propositions 8.1, 8.3 and 8.5, since there are no solutions of (3) with $(K,H) = (\tilde{K},0)$ and since $\tilde{K}|_{\partial S^3_+}$ possesses just one critical point with $\frac{\partial \tilde{K}}{\partial \nu} > 0$. This concludes the proof. ■

Remark 8.8 *In the case in which $H \geq 0$ on ∂S^3_+, the a-priori estimates in the previous proof could be obtained from the non existence results for* (76) *and for the problem*

$$(78)\qquad \begin{cases} -\Delta u = u^p & \textit{in } \mathbb{R}^n_+, \qquad u > 0; \\ \frac{\partial u}{\partial \nu} = a\, u^q & \textit{on } \partial \mathbb{R}^n_+, \end{cases}$$

where $1 < p < \frac{n+2}{n-2}$, $1 < q < \frac{n}{n-2}$ and $a \leq 0$, see [18] and [32]. As far as our knowledge, existence or non-existence of solutions is not known for $a > 0$ and general subcritical exponents p and q.

Proof of Theorem 1.2 The norm inequality in (11) follows from Propositions 5.1, 6.8 and some standard estimates. Suppose by contradiction that the second inequality in (11) is not true; then there exist a sequence of solutions $\{v_i\}$ of (3) blowing up at $q^1, \dots, q^N \in \partial S^3_+$, and these blow ups are isolated simple. Taking into account that $(K,H) \in \mathcal{A}$ and that $\lambda_j = 0$ for all j (since $\tau_i = 0$ for all i), we get a contradiction from Proposition 8.1 $iv)$. Hence (11) is proved.

Using Proposition 8.3, (11) and the homotopy invariance of the Leray-Schauder degree, we have

$$(79)\quad \deg_{C^{2,\alpha}(\overline{S^3_+})}(u - \Xi(K\,|u|^4 u, H\,|u|^2 u)), \mathcal{O}_R, 0) = \deg_{C^{2,\alpha}(\overline{S^3_+})}(u - \Xi(K\,|u|^{4-\tau} u, H\,|u|^{2-\frac{\tau}{2}} u), \mathcal{O}_R, 0),$$

for τ sufficiently small. By Propositions 8.3 and 8.5, for suitable value of ε, and for τ small, we know that the non-negative solutions of $J'_\tau = 0$ are either in $\mathcal{O}_R$ or in some $\mathcal{U}_\varepsilon(\tau, q^1, \dots, q^N)$; viceversa for all $q^1, \dots, q^N \in \mathcal{F}_+$ with $\rho(q^1, \dots, q^k) > 0$, there are (positive) solutions of $J'_\tau = 0$ in $\mathcal{U}_\varepsilon$, and degree of J'_τ on $\mathcal{U}_\varepsilon$ is given by (66).

Let $C_0 >> R$, τ_0 and δ_0 be given by Proposition 8.7; take also $\delta_1 << \delta_0$. By Proposition 8.5, (73) and by the excision property of the degree, we have

$$(80)\qquad \deg_{H^1(S^3_+)}(u - \Xi(K\,|u|^{4-\tau_0} u, H\,|u|^{2-\frac{\tau_0}{2}} u), \mathcal{O}_{R,\delta_1}, 0) = \mathrm{Index}(K,H).$$

As in the proof of Proposition 8.7, one can check that there are no critical points of J_{τ_0} in $\overline{\mathcal{O}_{R,\delta_1}} \setminus \mathcal{O}_R$, hence Theorem B.2 of [29] applies and yields

(81)

$$\deg_{H^1(S^3_+)}(u - \Xi(K\,|u|^{4-\tau_0} u, H\,|u|^{2-\frac{\tau_0}{2}} u), \mathcal{O}_{R,\delta_1}, 0) = \deg_{C^{2,\alpha}(\overline{S^3_+})}(u - \Xi(K\,|u|^{4-\tau_0} u, H\,|u|^{2-\frac{\tau_0}{2}} u), \mathcal{O}_R, 0).$$

Then the conclusion follows from (79), (80) and (81). The proof of Theorem 1.2 is thereby completed. ■

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