Conformal Subnets and Intermediate Subfactors

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Abstract

Given an irreducible local conformal net $\mathcal{A}$ of von Neumann algebras on S^1 and a finite-index conformal subnet $\mathcal{B} \subset \mathcal{A}$, we show that $\mathcal{A}$ is completely rational iff $\mathcal{B}$ is completely rational. In particular this extends a result of F. Xu for the orbifold construction. By applying previous results of Xu, many coset models turn out to be completely rational and the structure results in [14] hold. Our proofs are based on an analysis of the net inclusion $\mathcal{B} \subset \mathcal{A}$; among other things we show that, for a fixed interval I, every von Neumann algebra $\mathcal{R}$ intermediate between $\mathcal{B}(I)$ and $\mathcal{A}(I)$ comes from an intermediate conformal net $\mathcal{L}$ between $\mathcal{B}$ and $\mathcal{A}$ with $\mathcal{L}(I) = \mathcal{R}$. We make use of Watatani's result, extended for the purpose from the type II case to arbitrary factors, on the finiteness of the set $\mathfrak{I}(\mathcal{N}, \mathcal{M})$ of intermediate subfactors in an irreducible inclusion of factors $\mathcal{N} \subset \mathcal{M}$ with finite Jones index $[\mathcal{M} : \mathcal{N}]$. We provide an explicit bound for the cardinality of $\mathfrak{I}(\mathcal{N}, \mathcal{M})$ which depends only on $[\mathcal{M} : \mathcal{N}]$.

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1 Introduction

Operator algebraic methods have been used to good effect in Conformal Quantum Field Theory, in particular in understanding general model independent structure (e.g. [6, 14, 17, 23, 24, 38]), in the analysis of concrete models (e.g. [5, 42, 43, 45]) and for applications in different contexts (e.g. [33]). In most cases it seems to be impossible to proceed by different methods.

Because of their relevance in different areas, among others Topological QFT and 3-manifold invariants, conformal models with a rational and modular representation theory have been the subject of much attention, also in the physical literature (cf. [15]).

In [14] intrinsic, model independent conditions selecting a class of (local, irreducible) conformal nets $\mathcal{A}$ of von Neumann algebras on S^1 with the right rationality/modularity properties were given. $\mathcal{A}$ is *completely rational* if

1. $\mathcal{A}$ is split,
2. $\mathcal{A}$ is strongly additive,
3. the 2-interval inclusion of factors $\mathcal{A}(E) \subset \mathcal{A}(E')'$ has finite Jones index $\mu_{\mathcal{A}}$.

Here both $E \subset S^1$ and $E' \equiv S^1 \smallsetminus E$ are the union of two proper intervals. The split and strongly additivity properties are well-studied basic properties, see Section 3.5 for their definitions, and we do not dwell on them here, cf. [13, 7, 11] and [8, 24]. If $\mathcal{A}$ is completely rational, then $\mathcal{A}(E) \subset \mathcal{A}(E')'$ is obtained by a quantum double construction in [34], in particular

$$\mu_{\mathcal{A}} = \sum_i d(\rho_i)^2 \ ,$$

where the sum is taken over all the irreducible sectors of $\mathcal{A}$. Every representation of $\mathcal{A}$ (on a separable Hilbert space) is Möbius covariant and decomposes into the direct sum of irreducible representations with finite statistical dimension. There are only finitely many inequivalent irreducible representations, i.e. $\mathcal{A}$ is rational, and the associated braiding is non-degenerate, i.e. the representation tensor category is modular.

At this point the problem of verifying the complete rationality of known models arises. Certain examples were discussed in [14]. As an illustration from [14], consider the case of a non-trivial finite group G acting on a completely rational $\mathcal{A}$; if the fixed-point *orbifold* subnet $\mathcal{A}^G$ is also completely rational, then $\mu_{\mathcal{A}^G} \geq |G|^2$, while $\sum_{\pi\in\hat{G}} d(\rho_\pi)^2 = |G|$, where the the ρ_π's are the untwisted DHR sectors of $\mathcal{A}^G$ [12], and this shows that twisted sectors must appear. As $\mathcal{A}$ is the initial data, one would infer the complete rationality of $\mathcal{A}^G$ from that of $\mathcal{A}$. By [14] $\mathcal{A}^G$ inherits from $\mathcal{A}$ the split property and the finiteness of the μ-index. F. Xu [45] has then shown that $\mathcal{A}^G$ also inherits the strong additivity property and this has inspired our paper.

We shall now show that if $\mathcal{B}$ is any conformal subnet of $\mathcal{A}$ with finite index, then $\mathcal{B}$ is completely rational iff $\mathcal{A}$ is completely rational.

As a consequence, if $\mathcal{B}$ is a cofinite subnet of $\mathcal{A}$, namely $[\mathcal{A} : \mathcal{B} \vee \mathcal{B}^c] < \infty$, where $\mathcal{B} \vee \mathcal{B}^c$ is the subnet generated by $\mathcal{B}$ and its relative commutant $\mathcal{B}^c$ in $\mathcal{A}$, then $\mathcal{A}$ is completely rational iff both $\mathcal{B}$ and $\mathcal{B}^c$ are completely rational.

The subnet $\mathcal{B}^c$ is called the *coset* subnet associated with $\mathcal{B} \subset \mathcal{A}$, as it generalizes a coset construction that plays an important rôle in the theory of Kac-Moody Lie algebras, allowing one to construct of the minimal series representations of the Virasoro algebra [20].

Coset models have been intensively studied by Xu in [42, 43] by operator algebraic methods. In one approach he makes use of [14] too. Thanks to his work, coset models associated with many loop group inclusions are cofinite, rational and modular, see the list in Section 3.5.1. Property 3 holds, but the validity of strong additivity was left open.

By our work in all these examples $\mathcal{B}^c$ is strongly additive, thus $\mathcal{B}^c$ turns out to be completely rational and this completes the above discussion an explains the rationality/modularity structure better.

We now comment on our proof that the complete rationality property (and also the 'split & strongly additivity' property) for finite-index inclusions of conformal nets $\mathcal{B} \subset \mathcal{A}$ are hereditary. That 'Property 1 & 3' passes from $\mathcal{A}$ to $\mathcal{B}$ and viceversa is shown in [14]. The remaining more difficult point we have to prove is that $\mathcal{B}$ is strongly additive if $\mathcal{A}$ is split and strongly additive, see Sect. 3.5.

To this end, we have analyzed a finite-index inclusion of conformal nets $\mathcal{B} \subset \mathcal{A}$ by considering the relative superselection structure. In particular we show that, for a fixed interval I, every von Neumann algebra $\mathcal{R}$ intermediate between $\mathcal{B}(I)$ and $\mathcal{A}(I)$ comes from an intermediate conformal net $\mathcal{L}$ between $\mathcal{A}$ and $\mathcal{B}$ with $\mathcal{L}(I) = \mathcal{R}$.

Here we make use of a result of Watatani [41], following previous work by Popa [36], to the effect that the set $\mathfrak{I}(\mathcal{N}, \mathcal{M})$ of intermediate subfactors in an irreducible inclusion of II_1-factors $\mathcal{N} \subset \mathcal{M}$ with finite Jones index $[\mathcal{M} : \mathcal{N}]$ is finite. We extend this result to the case of arbitrary factors as we need it when $\mathcal{N}$ and $\mathcal{M}$ are type III factors. In the process, we provide an explicit bound for the cardinality $|\mathfrak{I}(\mathcal{N}, \mathcal{M})|$ of $\mathfrak{I}(\mathcal{N}, \mathcal{M})$ which depends only on $[\mathcal{M} : \mathcal{N}]$ and implies that

$$|\mathfrak{I}(\mathcal{N}, \mathcal{M})| \leq \ell^{\ell} \ ,$$

where $\ell = [\mathcal{M} : \mathcal{N}]^2$. There may be better bounds taking account of further structure associated with intermediate subfactors [3, 4], see the comments at the end of Section 2.2.

We conclude this introduction and include references to the books [1, 15, 25, 39] for basic facts on Operator Algebras and Quantum Field Theory, see also [28] for subfactors and sectors.

2 On subfactors and intermediate factors

The first part of this paper is devoted to an analysis of subfactors and intermediate factors, that will be used later on.

2.1 Some basic structure

Let $\mathcal{N} \subset \mathcal{M}$ be an irreducible inclusion of infinite factors with finite index $[\mathcal{M} : \mathcal{N}]$. We denote by γ the canonical endomorphism of $\mathcal{M}$ into $\mathcal{N}$ and by θ the dual canonical endomorphism $\theta \equiv \gamma\restriction_{\mathcal{N}}$. The Q-system associated with γ is denoted by (γ, T, S), namely $T \in \mathcal{M}$ and $S \in \mathcal{N}$ are the unique (up to a phase) isometries in (ι, γ) and (ι, θ), where ι always denotes the identity automorphism.

Let $\{[\rho_i], i = 0, \ldots N\}$ be the family of the irreducible sectors in the decomposition of $[\theta]$, namely

$$[\theta] = \bigoplus_{i=0}^{N} N_i[\rho_i] \ . \tag{1}$$

By Frobenious reciprocity for each i the Hilbert space of isometries in $\mathcal{M}$ (not necessarily with right support 1)

$$K_i \equiv \{R \in \mathcal{M} : Rx = \rho_i(x)R \ \forall x \in \mathcal{N}\} \tag{2}$$

has dimension N_i, indeed the map

$$v \in (\rho, \theta) \to v^*T \in K_i \tag{3}$$

is an anti-linear isomorphism of (ρ, θ) with K_i, whose inverse is given by

$$X \in K_i \to [\mathcal{M} : \mathcal{N}]\varepsilon(TX^*) \in (\rho, \theta) \ ,$$

where ε is the expectation of $\mathcal{M}$ onto $\mathcal{N}$. See [34, 26] for the following.

Lemma 1. *Let $\{R_{i,k}\}_{k=1}^{N_i}$ be an orthogonal basis of K_i with the normalization $R^*_{i,k}R_{i,k} = d(\rho_i)$. Then every $X \in \mathcal{M}$ has a unique Fourier expansion*

$$X = \sum_{i=0}^{N}\sum_{k=1}^{N_i} x_{i,k}R_{i,k} \tag{4}$$

where the coefficients $x_{i,k}$ belong to $\mathcal{N}$, indeed $x_{i,k} = d(\rho_i)\varepsilon(XR^*_{i,k})$.

Proof It is immediate that $\varepsilon(R_{i,k}R^*_{j,h}) \in (\rho_i, \rho_j)$, thus $\varepsilon(R_{i,k}R^*_{j,h}) = 0$ if $i \neq j$. Let $\{v_1, \ldots v_{N_i}\}$ be an orthonormal basis of isometries for K_i, then $R_{i,k} \equiv [\mathcal{M} : \mathcal{N}]^{\frac{1}{2}}v^*_kT$ satisfy

$$R^*_{i,k}R_{i,h} = T^*v_kv^*_hT = \delta_{hk}d(\rho_i)$$

because $v_k v_h^* \in (\theta, \theta)$ and $TT^* \in \mathcal{M}$ is the Jones projection for implementing the expectation $\varepsilon_1 : \mathcal{N} \to \theta(M)$ so that $\varepsilon_1 \restriction_{(\theta,\theta)}$ is the associated trace, see [30]. Therefore

$$\varepsilon(R_{i,k}R_{i,h}^*) = [\mathcal{M} : \mathcal{N}]\varepsilon(v_k^* TT^* v_h) = \varepsilon(v_k^* v_h) = \delta_{hk}. \tag{5}$$

Now $\mathcal{M} = \mathcal{N}T$, thus $\mathcal{M}$ is generated by $\mathcal{M} = \sum_{i,k} \mathcal{N}R_{i,k}$ because $\sum K_i$ has right support one. By the orthogonality relations (5) every $X \in \mathcal{M}$ has the expansion given by formula (4). □

Denote by $\mathfrak{I}(\mathcal{N}, \mathcal{M})$ the set of intermediate von Neumann algebras between $\mathcal{N}$ and $\mathcal{M}$. Clearly if $\mathcal{R} \in \mathfrak{I}(\mathcal{N}, \mathcal{M})$, then $\mathcal{R}$ is a factor and indeed both $\mathcal{N} \subset \mathcal{R}$ and $\mathcal{R} \subset \mathcal{M}$ are irreducible finite-index inclusions of factors.

Let $\mathcal{R} \in \mathfrak{I}(\mathcal{N}, \mathcal{M})$ and set $K_i' \equiv K_i \cap \mathcal{R}$. Then K_i' is a Hilbert subspace of K_i, so we may choose the $R_{i,k}$ so that $\{R_{i,k}\}_{k=1}^{N_i'}$ is a basis for K_i', where $N_i' = \mathrm{dim}K_i'$. We may also re-order the ρ_i's so that $N_i' > 0$ iff $i \leq N'$ where $N' \leq N$ is an integer.

Proposition 2. *With the above notations, $X \in \mathcal{M}$ belongs to $\mathcal{R}$ iff the in the expansion (4) $x_{i,k} = 0$ for all $k > N_i'$, namely*

$$X = \sum_{i=0}^{N'} \sum_{k=1}^{N_i'} x_{i,k} R_{i,k} \ . \tag{6}$$

In particular $\mathcal{R}$ is generated by $\mathcal{N}$ and the K_i''s.

Proof Recall that $[\mathcal{M} : \mathcal{R}] < \infty$, so there exists an expectation $\varepsilon_{\mathcal{R}} : \mathcal{M} \to \mathcal{R}$. From the definition (2) $\varepsilon_{\mathcal{R}}(K_i) \subset K_i$, thus $\varepsilon_{\mathcal{R}} \restriction_{K_i} \in B(K_i)$ is a norm one projection. Clearly $\varepsilon_{\mathcal{R}}(K_i) = K_i'$ thus if $X \in \mathcal{M}$ has the expansion (4) we have

$$\varepsilon_{\mathcal{R}}(X) = \sum_{i=0}^{N'} \sum_{k=1}^{N_i'} x_{i,k} R_{i,k} \ ,$$

which implies the statement in the proposition. □

The following theorem is due to Watatani [41] in the II_1-case, related results are contained in [36]. A proof for factors of arbitrary type will be given in next section, where we shall obtain a bound for the cardinality $|\mathfrak{I}(\mathcal{N}, \mathcal{M})|$ of $\mathfrak{I}(\mathcal{N}, \mathcal{M})$.

Theorem 3. $\mathfrak{I}(\mathcal{N}, \mathcal{M})$ *is a finite set.*

We note here a simple proof in the special case where $N_i = 1$, i.e. each irreducible sub-sector of θ is contained in θ with multiplicity one, and $\mathcal{M}$ is infinite. In this case the expansion (4) takes the form $X = \sum_{i=0}^{N} x_i R_i$. By re-ordering the R_i's if necessary we may assume that there is an integer $0 < N_1 \leq N$ such that $R_i \in \mathcal{N}$ iff $0 \leq i \leq N_1$.

With $\varepsilon_{\mathcal{L}}$ the expectation from $\mathcal{M}$ onto $\mathcal{L}$, $\varepsilon_{\mathcal{L}}(R_i)$ belongs to K_i, hence it must be a multiple of R_i by assumption. Therefore $\varepsilon_{\mathcal{L}}(R_i) = R_i$ if $0 \le i \le N_1$ and $\varepsilon_{\mathcal{L}}(R_i) = 0$ otherwise. Thus $X = \sum_{i=0}^{N} x_i R_i$ belongs to $\mathcal{N}$ iff $x_i = 0$, $N_1 < i \le N$. Therefore an intermediate subfactor is uniquely associated with a certain subset of $\{1, 2, \dots, N\}$, hence there are at most 2^N intermediate subfactors.

Corollary 4. *Let $\mathcal{L}$ be an intermediate subfactor between $\mathcal{N}$ and $\mathcal{M}$ and $\beta : \mathfrak{G} \to \mathrm{Aut}(\mathcal{M})$ a (pointwise weakly continuous) action of a connected topological group $\mathfrak{G}$ with $\beta_g(\mathcal{N}) = \mathcal{N}$, $g \in \mathfrak{G}$. Then $\beta_g(\mathcal{L}) = \mathcal{L}$, $g \in \mathfrak{G}$.*

Proof We consider on $\mathfrak{I}(\mathcal{M}, \mathcal{N})$ the topology of pointwise weak convergence of the associated conditional expectations ($\mathcal{L}_i \to \mathcal{L}$ iff $\varepsilon_{\mathcal{L}_i}(x) \to \varepsilon_{\mathcal{L}}(x)$ weakly for all $x \in \mathcal{M}$). Then β implements a continuous action of $\mathfrak{G}$ on $\mathfrak{I}(\mathcal{M}, \mathcal{N})$. The corollary is thus immediate because any continuous action of a connected group on a discrete set is trivial. $\square$

2.2 A bound for the number of intermediate subfactors

Let $\mathcal{N} \subset \mathcal{M}$ be an irreducible inclusion with finite index and denote by ε the conditional expectation from $\mathcal{M}$ to $\mathcal{N}$. We shall now determine a bound for $|\mathfrak{I}(\mathcal{M}, \mathcal{N})|$. Our proof is inspired by the papers [10, 41].

We assume that there exists a faithful normal state ω on $\mathcal{N}$ (otherwise replacing it by a weight). By considering the GNS representation of $\mathcal{M}$ associated with $\tilde{\omega} \equiv \omega \cdot \varepsilon$, we may assume that $\mathcal{M}$ acts on a Hilbert space $\mathcal{H}$ with cyclic and separating vector Ω so that $\tilde{\omega} = (\Omega, \cdot\Omega)$. Then $e \equiv [\mathcal{N}\Omega]$, equal to JeJ, is the Jones projection for $\mathcal{N} \subset \mathcal{M}$ and $\mathcal{M}_1 \equiv \langle \mathcal{M}, e \rangle = J\mathcal{N}'J$ is the Jones extension, where J is the modular conjugation of $\mathcal{M}$ associated with Ω. The projection $e \in \mathcal{N}' \cap \mathcal{M}_1$ and

$$\varepsilon(e) = \lambda \equiv [\mathcal{M} : \mathcal{N}]^{-1}.$$

Let $\mathcal{R}$, $\mathcal{S}$ be intermediate factors between $\mathcal{N}$ and $\mathcal{M}$ and denote by $p \equiv [\mathcal{R}\Omega]$ and $q \equiv [\mathcal{S}\Omega]$ their associated Jones projections. Note that $JpJ = p$, $JqJ = q$ and $\mathcal{R}_1 \equiv \langle \mathcal{R}, p \rangle = J\mathcal{R}'J$ and $\mathcal{S}_1 \equiv \langle \mathcal{S}, q \rangle = J\mathcal{S}'J$ are the corresponding Jones extensions and so there is a chain of inclusions

$$\mathcal{N} \subset \mathcal{R}, \mathcal{S} \subset \mathcal{M} \subset \mathcal{R}_1, \mathcal{S}_1 \subset \mathcal{M}_1 \ .$$

Clearly $p \in \mathcal{R}' \cap \mathcal{R}_1$ and $q \in \mathcal{S}' \cap \mathcal{S}_1$, thus p and q both belong to $\mathcal{N}' \cap \mathcal{M}_1$.

Proposition 5. *If $||p - q|| < \lambda/2$ then $\mathcal{R} = \mathcal{S}$.*

Proof We may assume $\mathcal{R} \neq \mathcal{N}$ as otherwise $p = e$, thus $q = e$ because $q \geq e$ and $\lambda \leq 1$.

As we have $[\mathcal{M}:\mathcal{N}]=[\mathcal{M}:\mathcal{R}][\mathcal{R}:\mathcal{N}]$, it follows that

$$[\mathcal{M}:\mathcal{N}]\geq 2[\mathcal{M}:\mathcal{R}] \tag{7}$$

because $[\mathcal{R}:\mathcal{N}]\geq 2$ [27]. In particular $\lambda\leq 1/2$. Let $\varepsilon_{\mathcal{R}_1}$ be the expectation from $\mathcal{M}_1$ onto $\mathcal{R}_1$ and set $q'\equiv\varepsilon_{\mathcal{R}_1}(q)$. Then $q'\in\mathcal{S}'\cap\mathcal{R}_1$ since obviously $q'\in\mathcal{R}_1$ and if $x\in\mathcal{S}$

$$xq'=x\varepsilon_{\mathcal{R}_1}(q)=\varepsilon_{\mathcal{R}_1}(xq)=\varepsilon_{\mathcal{R}_1}(qx)=q'x$$

because $\mathcal{S}\subset\mathcal{R}_1$. Moreover $0\leq q'\leq 1$ and $q'\neq 0$ because $\varepsilon_{\mathcal{R}_1}$ is positive and faithful. Setting $\delta=\lambda/2$ we have

$$||p-q'||=||p-\varepsilon_{\mathcal{R}_1}(q)||=||\varepsilon_{\mathcal{R}_1}(p-q)||\leq||p-q||<\delta\ . \tag{8}$$

Therefore the spectrum $\mathrm{sp}(q')\subset[0,\delta)\cup(1-\delta,1]$, see Lemma 8.

Thus the spectral projection $q''\equiv\chi_{(1-\delta,1]}(q')$ is a projection in $\mathcal{S}'\cap\mathcal{R}_1$ and $||q''-q'||<\delta$, thus

$$||p-q''||\leq||p-q'||+||q'-q''||<2\delta\leq 1\ ,$$

and this implies that p and q'' are equivalent projections of $\mathcal{N}'\cap\mathcal{R}_1$. Indeed the phase v in the polar decomposition of $t\equiv pq''$ is a partial isometry in $\mathcal{N}'\cap\mathcal{R}_1$ with $v^*v=q''$, $vv^*=p$ (see Lemma 9 below). Then we can define an isomorphism Φ of $\mathcal{S}$ into $\mathcal{R}$ by

$$\Phi(x)p\equiv vxv^*\ ,\ x\in\mathcal{S}\ ,$$

as $p\mathcal{R}_1p=\mathcal{R}p$ [27]. Moreover $\Phi(x)=x$ for all $x\in\mathcal{N}$ because $v\in\mathcal{N}'$.

We have the intertwining relation

$$\Phi(x)v=vx\ ,\ x\in\mathcal{S}\ .$$

With ε' the conditional expectation from $\mathcal{M}_1$ onto $\mathcal{M}$ we then have

$$\Phi(x)\varepsilon'(v)=\varepsilon'(v)x\ ,\ x\in\mathcal{S}\ ,$$

where $\varepsilon'(v)\in\mathcal{N}'\cap\mathcal{M}=\mathbb{C}$, thus $\varepsilon'(v)\neq 0$ would imply that Φ is the identity on $\mathcal{S}$ and $\mathcal{S}\subset\mathcal{R}$. Reversing the rôle of $\mathcal{R}$ and $\mathcal{S}$ also $\mathcal{R}\subset\mathcal{S}$, so $\mathcal{R}=\mathcal{S}$.

To show that indeed $\varepsilon'(v)\neq 0$ set $\lambda_0\equiv\varepsilon'(p)=[\mathcal{M}:\mathcal{R}]^{-1}$ and notice that, by using Lemma 9, we have

$$\begin{aligned}|\varepsilon'(v)-\lambda_0|&\leq||\varepsilon'(v)-\varepsilon'(t)||+||\varepsilon'(t)-\varepsilon'(p)||\leq||v-t||+||t-p||\\&=||v-t||+||p(q''-p)||\leq 2\delta+||q''-p||<4\delta=2\lambda\ ,\end{aligned}$$

thus $\varepsilon'(v)\neq 0$ because $2\lambda\leq\lambda_0$ by eq. (7). □

Corollary 6. *Let $\mathcal{N} \subset \mathcal{M}$ be an irreducible inclusion of factors. The cardinality of the set of intermediate factors between $\mathcal{N}$ and $\mathcal{M}$ is bounded by*

$$|\mathfrak{I}(\mathcal{N}, \mathcal{M})| \leq (4(n+2)\sqrt{n}+1)^{n^2} , \tag{9}$$

where n is the largest integer such that $n+1 \leq [\mathcal{M} : \mathcal{N}]$.

Proof By the above proposition $|\mathfrak{I}(\mathcal{N}, \mathcal{M})|$ is dominated by the maximum number of projections $\geq e$ in $\mathcal{N}' \cap \mathcal{M}_1$ whose mutual distance is $\geq \lambda/2$. As e is a minimal central projection of $\mathcal{N}' \cap \mathcal{M}_1$, we can naturally embed $\mathcal{N}' \cap \mathcal{M}_1$ into $\mathbb{C} \oplus \mathrm{Mat}_m(\mathbb{C})$, where m an integer with $m+1 \leq [\mathcal{M} : \mathcal{N}]$ ($m = \sum_1^N N_i$ in eq. (1)). Indeed, as $J(\mathcal{N}' \cap \mathcal{M}_1)J = \mathcal{N}' \cap \mathcal{M}_1$, and $\mathrm{Ad}J$ implements an anti-automorphism of $\mathcal{N}' \cap \mathcal{M}_1$, we can assume this anti-automorphism to extend to an anti-automorphism of $\mathbb{C} \oplus \mathrm{Mat}_m(\mathbb{C})$ preserving the two components, in other words we may assume that the $\mathrm{Ad}J$-invariant part of $\mathcal{N}' \cap \mathcal{M}_1$ is contained in $\mathbb{R} \oplus \mathrm{Mat}_m(\mathbb{R})$.

Thus $|\mathfrak{I}(\mathcal{N}, \mathcal{M})|$ is dominated by the maximum number of projections in $\mathrm{Mat}_m(\mathbb{C})$ whose mutual distance is larger than $\frac{1}{2(n+2)}$ where n is the largest interger such that $n+1 \leq [\mathcal{M} : \mathcal{N}]$ (so $\lambda > \frac{1}{n+2}$). Moreover, as $Jp_iJ = p_i$, we may regard the p_i's as elements of $\mathrm{Mat}_m(\mathbb{R})$.

The following Lemma 7 with $\epsilon = \frac{1}{2(n+2)}$ then gives

$$|\mathfrak{I}(\mathcal{N}, \mathcal{M})| \leq (4(n+2)\sqrt{m}+1)^{m^2} \leq (4(n+2)\sqrt{n}+1)^{n^2} .$$

□

The following lemma slightly improves [7, Lemma 2.6].

Lemma 7. *Let $\epsilon > 0$ and $\{p_1, p_2, \ldots, p_k\}$ be elements in the unit ball of $\mathrm{Mat}_n(\mathbb{R})$ such that $||p_i - p_j|| \geq \epsilon$ if $i \neq j$. Then $k \leq (\frac{2\sqrt{n}}{\epsilon}+1)^{n^2}$.*

Proof As the uniform and Hilbert-Schmidt norms are related by $||X|| \leq ||X||_{HS} \leq \sqrt{n}||X||$, the p_i's give vectors of norm less $\sqrt{n}$ in the Euclidean space $\mathbb{R}^{n^2}$ (identified with $\mathrm{Mat}_n(\mathbb{R})$ with the Hilbert-Schmidt norm) with mutual distance larger than ϵ. Denoting by $B(r)$ the open ball of radius r in $\mathbb{R}^{n^2}$ we then have

$$k < \frac{\mathrm{Vol}(B(\sqrt{n}+\epsilon/2))}{\mathrm{Vol}(B(\epsilon/2))} = (\frac{2\sqrt{n}}{\epsilon}+1)^{n^2} .$$

□

The following lemmata are variations of known facts (cf. e.g. [39]) and are included for convenience.

Lemma 8. *Let x be a positive linear operator, $0 \leq x \leq 1$, and p a selfadjoint projection with $||x-p|| \leq \delta < 1/2$. Then $\mathrm{sp}(x) \subset [0, \delta] \cup [1-\delta, 1]$.*

Proof With $\ell \in (\delta, 1-\delta)$ we have $x - \ell = (p-\ell)(1 + (p-\ell)^{-1}(x-p))$, thus $x-\ell$ is invertible if $||(p-\ell)^{-1}(x-p)|| < 1$, which is the case if $\delta||(p-\ell)^{-1}|| < 1$. This holds because $||(p-\ell)^{-1}|| = \max\{\ell^{-1}, (1-\ell)^{-1}\} < \delta^{-1}$. $\square$

Lemma 9. *Let p and q be selfadjoint projections on a Hilbert space $\mathcal{H}$ and $t = vh$ be the polar decomposition of $t \equiv pq$. If $||p-q|| \leq \delta < 1$, then $v^*v = q$, $vv^* = p$ and $||v-t|| \leq \delta$.*

Proof As $||(p-q)^2|| \leq \delta^2 < 1$, the operator $1-(p-q)^2$ is invertible. Thus $s \equiv pq + (1-p)(1-q)$ is also invertible, indeed $s^{-1} = (1-(p-q)^2)^{-1}s^*$, and this implies that v is a partial isometry from q to p. Then we have

$$\begin{aligned} ||v-t|| = ||v-vh|| = ||v(1-h)|| \leq ||(1-h)\restriction_{q\mathcal{H}}|| = ||(1-\sqrt{qpq})\restriction_{q\mathcal{H}}|| \\ \leq ||(1-qpq)\restriction_{q\mathcal{H}}|| = ||q(q-p)q|| \leq ||q-p|| \leq \delta \ . \end{aligned}$$

$\square$

The bound (9) implies that

$$|\mathfrak{I}(\mathcal{N},\mathcal{M})| \leq \ell^\ell \ ,$$

where $\ell = [\mathcal{M}:\mathcal{N}]^2$. The arguments in this section can be improved, in particular taking into account that the p_i's are projections in Lemma 7, leading to a better bound $|\mathfrak{I}(\mathcal{N},\mathcal{M})| \leq \ell_1^{\ell_2}$ where however ℓ_2 is still quadratic in the index. It would be interesting to see if a bound $|\mathfrak{I}(\mathcal{N},\mathcal{M})| \leq [\mathcal{M}:\mathcal{N}]^{[\mathcal{M}:\mathcal{N}]}$ holds. This is the case of the example $\mathcal{N} = \mathcal{M}^G$ with G is a finite group where, because of the Galois correspondence (see e.g. [26]), $|\mathfrak{I}(\mathcal{N},\mathcal{M})| \leq |G|! = |\mathfrak{I}(\mathcal{N},\mathcal{M})|!$. We note that we have not made use of the specific form of the projections associated with intermediate subfactors and the canonical algebra they generate [3, 4].

3 Conformal nets and subnets

We now begin our study of conformal nets. Their subnets will be analyzed through the relative superselection structure.

3.1 Conformal nets

Let $\mathcal{I}$ denote the family of proper intervals of S^1, namely connected subsets of S^1 of positive measure (length) strictly less than 2π. The subnet structure is rather simple for net on the 4-dimensional Minkowski spacetime (see [9]), but this does not extend to the low-dimensional case.

A net (or precosheaf) $\mathcal{A}$ of von Neumann algebras on S^1 is a map

$$I \in \mathcal{I} \to \mathcal{A}(I) \subset B(\mathcal{H})$$

from $\mathcal{I}$ to von Neumann algebras on fixed a Hilbert space $\mathcal{H}$ that satisfies:

A. *Isotony.* If $I_1 \subset I_2$ belong to $\mathcal{I}$, then

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2).$$

The net $\mathcal{A}$ is called a (local) *conformal* net if in addition it satisfies the following properties:

B. *Locality.* If $I_1, I_2 \in \mathcal{I}$ and $I_1 \cap I_2 = \varnothing$ then

$$[\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\},$$

where brackets denote the commutator[1].

C. *Conformal invariance.* There exists a strongly continuous unitary representation U of $\mathrm{PSL}(2, \mathbb{R})$ on $\mathcal{H}$ such that

$$U(g)\mathcal{A}(I)U(g)^* \ = \ \mathcal{A}(gI), \quad g \in \mathrm{PSL}(2, \mathbb{R}), \ I \in \mathcal{I}.$$

Here $\mathrm{PSL}(2, \mathbb{R})$ acts on S^1 by Möbius transformations. We shall denote also by $\alpha_g = \mathrm{Ad}U(g)$ the adjoint action on $B(\mathcal{H})$.

D. *Positivity of the energy.* The generator of the one-parameter rotation subgroup of U (conformal Hamiltonian) is positive.

E. *Existence of the vacuum.* There exists a unit U-invariant vector $\Omega \in \mathcal{H}$ (vacuum vector).

We shall say that a conformal net is *irreducible* if $\vee_{I \in \mathcal{I}} \mathcal{A}(I) = B(\mathcal{H})$. Here the lattice symbol $\vee$ denotes the von Neumann algebra generated. We recall the following Lemma whose proof can be found in [23].

Lemma 10. *Let $\mathcal{A}$ be a conformal net. The following are equivalent:*

(i) $\mathcal{A}$ is irreducible;

(ii) Ω is cyclic for $\vee_{I \in \mathcal{I}} \mathcal{A}(I)$ and unique U-invariant;

(iii) Ω is cyclic for $\vee_{I \in \mathcal{I}} \mathcal{A}(I)$ and the local von Neumann algebras $\mathcal{A}(I)$ are factors. In this case they are III_1-factors (unless $\mathcal{A}(I) = \mathbb{C}$ identically).

Let $\mathcal{A}$ be an irreducible conformal net. By the Reeh-Schlieder theorem [18] the vacuum vector Ω is cyclic and separating for each $\mathcal{A}(I)$. The *Bisognano-Wichmann* property then holds [6, 19]: the Tomita-Takesaki modular operator Δ_I and conjugation J_I associated with $(\mathcal{A}(I), \Omega)$, $I \in \mathcal{I}$, are given by

$$U(\Lambda_I(2\pi t)) = \Delta_I^{it}, \ t \in \mathbb{R}, \tag{10}$$

$$U(r_I) = J_I, \tag{11}$$

[1]The locality condition will be always assumed in this paper, with the exception of Subsection 3.5.3

where Λ_I is the one-parameter subgroup of $\mathrm{PSL}(2,\mathbb{R})$ of special conformal transformations preserving I and $U(r_I)$ implements a geometric action on $\mathcal{A}$ corresponding, the Möbius reflection on S^1 mapping I onto I', i.e. fixing the boundary points of I, see [6].

This immediately implies Haag duality:

$$\mathcal{A}(I)' = \mathcal{A}(I'), \quad I \in \mathcal{I} ,$$

where $I' \equiv S^1 \smallsetminus I$.

3.2 Representations

Let $\mathcal{A}$ be an irreducible local conformal net. A *representation* π of $\mathcal{A}$ is a map

$$I \in \mathcal{I} \to \pi_I ,$$

where π_I is a representation of $\mathcal{A}(I)$ on a fixed Hilbert space $\mathcal{H}_\pi$ such that

$$\pi_{\tilde{I}} \restriction_{\mathcal{A}(I)} = \pi_I, \quad I \subset \tilde{I} ;$$

we shall always assume that π is locally normal, namely π_I is normal for all $I \in \mathcal{I}$, which is automatic if $\mathcal{H}_\pi$ is separable [39].

We shall say that a representation ρ is *localized* in a interval I_0 if $\mathcal{H}_\rho = \mathcal{H}$ and $\rho_{I_0'} = \mathrm{id}$. Given an interval I_0 and a representation π on a separable Hilbert space, there is a representation ρ unitarily equivalent to π and localized in I_0. This is due the type III factor property.

Let ρ be a representation of $\mathcal{A}$ localized in a given interval I_0. By Haag duality ρ satisfies the following properties

(*a*) *If $I \in \mathcal{I}$ and $I \supset I_0$ then ρ_I is an endomorphism of $\mathcal{A}(I)$, and $\rho_{\tilde{I}} \restriction_{\mathcal{A}(I)} = \rho_I$ for all $\tilde{I} \in \mathcal{I}$, $\tilde{I} \supset I$;*

(*b*) *If $I_1 \in \mathcal{I}$ and $I_1 \cap I_0 = \varnothing$, then ρ_{I_1} is the identity on $\mathcal{A}(I_1)$;*

(*c*) *If $I, I_1 \in \mathcal{I}$ and $I \supset I_0 \cup I_1$, there exists a unitary $u \in \mathcal{A}(I)$ such that the representation $I \to \rho'_I \equiv u\rho_I(\cdot)u^*$ is localized in I_1* (that is to say ρ'_{I_2} acts identically on $\mathcal{A}(I_2)$ if $I_1 \cap I_2 = \varnothing$ for all $I_2 \in \mathcal{I}$).

We now make a stereographic identification $\mathbb{R} = S^1 \smallsetminus \{\infty\}$ and denote by $\mathcal{I}_0 \subset \mathcal{I}$ the family of bounded intervals of $\mathbb{R}$, namely of the intervals of S^1 whose closure do not contain the point ∞ of S^1.

We denote by $\mathcal{A}_0$ the restriction of $\mathcal{A}$ to $\mathbb{R}$ (i.e. to $\mathcal{I}_0$) and by $\mathfrak{A}_0$ the associated quasi-local C*-algebra $\mathfrak{A}_0 \equiv \overline{\cup_{I \in \mathcal{I}_0} \mathcal{A}(I)}$ (norm closure). For a characterization of the so obtained net on $\mathbb{R}$, see [24].

Given $I_0 \in \mathcal{I}_0$ a *DHR endomorphism* ρ of $\mathcal{A}_0$ localized in I_0 is a map

$$I \supset I_0 \to \rho_I$$

that associates to each $I \in \mathcal{I}_0$ a representation ρ_I of $\mathcal{A}(I)$ on $\mathcal{H}$ such that the above conditions $(a),(b),(c)$ hold true with $\mathcal{I}$ replaced by $\mathcal{I}_0$.

Clearly a DHR endomorphism determines an endomorphism of $\mathfrak{A}_0$, still denoted by ρ, such that $\rho_I = \rho\restriction_{\mathcal{A}(I)}$, $I \subset \mathcal{I}_0$. The above properties $(a),(b),(c)$ are immediately expressed in terms of such endomorphism of $\mathfrak{A}_0$; we shall use the two descriptions interchangeably without further specifications.

Proposition 11. *Let ρ be a DHR endomorphism on $\mathcal{A}_0$ localized in an interval $I_0 \in \mathcal{I}_0$. There exists a unique representation $\tilde{\rho}$ of $\mathcal{A}$ extending ρ and localized I_0.*

Proof Our aim is to define consistently a representation ρ_I of $\mathcal{A}(I)$ for every $I \in \mathcal{I}$. To this end, given $I \in \mathcal{I}$, choose $I_1 \in \mathcal{I}_0$, $I_1 \subset I'$ and $L \in \mathcal{I}_0$ let be an interval with $L \supset I_0 \cup I_1$. Take then a DHR endomorphism of $\mathcal{A}_0$ equivalent to ρ and localized in I_1, $\rho' = \mathrm{Ad}u \cdot \rho$ for some unitary $u \in \mathcal{A}(L)$. We set

$$\tilde{\rho}_I(a) = u^* a u,\ \ a \in \mathcal{A}(I)\ ,$$

namely $\tilde{\rho}_I = \mathrm{Ad}u^* \cdot \rho'_I$. Clearly $\tilde{\rho}_I = \rho_I$ if $I \in \mathcal{I}_0$ and a routine checking shows that $I \in \mathcal{I} \to \tilde{\rho}_I$ is indeed a representation of $\mathcal{A}$. □

A representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\pi$ is *covariant* if there exists a unitary representation U_π of the universal covering group $\mathrm{PSL}(2,\mathbb{R})\tilde{}$ of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathcal{H}_\pi$ such that

$$\mathrm{Ad}U_\pi(g) \cdot \pi_I = \pi_{gI} \cdot \mathrm{Ad}U(g), \quad g \in \mathrm{PSL}(2,\mathbb{R})\tilde{},\ I \in \mathcal{I}_0.$$

Here U has been lifted to $\mathrm{PSL}(2,\mathbb{R})\tilde{}$. π is said to have positive energy if the generator of the rotation unitary subgroup of U_π is positive.

Let ρ be a representation of $\mathcal{A}$ localized in $I_0 \in \mathcal{I}$. By a *local cocycle* (w.r.t. to ρ) we shall mean the assignement of an interval $I \supset I_0$, a symmetric neighborhood $\mathcal{U}$ of the identity of $\mathrm{PSL}(2,\mathbb{R})\tilde{}$ such that $I_0 \cup gI_0 \subset I, \forall g \in \mathcal{U}$ and a strongly continuous unitary valued map $z : g \in \mathcal{U} \to z_\rho(g) \in \mathcal{A}(I)$ such that

$$z_\rho(g) \in \mathcal{A}(I) \tag{12}$$

$$z_\rho(gh) = z_\rho(g)\alpha_g(z_\rho(h)), \tag{13}$$

$$\mathrm{Ad}z_\rho(g)^* \cdot \rho_{\tilde{I}}(a) = \alpha_g \cdot \rho_{g^{-1}\tilde{I}} \cdot \alpha_{g^{-1}}(a),\ a \in \mathcal{A}(\tilde{I}), \tag{14}$$

for some open interval $\tilde{I}$ with $\bar{I} \subset \tilde{I}$ and all $g, h \in \mathcal{U}$ such that $I \cup gI \subset \tilde{I}$. We shall then say that z is localized in I.

If this holds, then eq. (14) is valid for all $L \in \mathcal{I}$:

$$\mathrm{Ad}z_\rho(g)^* \cdot \rho_L(a) = \alpha_g \cdot \rho_{g^{-1}L} \cdot \alpha_{g^{-1}}(a), \quad a \in \mathcal{A}(L). \tag{15}$$

Indeed, if $L \supset I$ then the above equation holds by additivity [18]. Thus it holds for sub-intervals $L_0 \subset L$. Again by additivity, the equation is then satisfied for all $L \in \mathcal{I}$.

If ρ is a covariant representation of $\mathcal{A}$ localized in I_0 then for any given interval $I \supset I_0$ there exists a local cocycle w.r.t. ρ localized in I. Indeed if $\mathcal{U}$ is a symmetric neighborhood of the identity of $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$ such that $I_0 \cup gI_0 \subset I, \forall g \in \mathcal{U}$, then by Haag duality the unitaries

$$z_\rho(g) \equiv U_\rho(g)U(g)^*$$

belong to $\mathcal{A}(I)$ for all $g \in \mathcal{U}$ and clearly verify the local cocycle property (13,14).

Notice now that, taking $I_0, I \in \mathcal{I}_0$, a local cocycle is expressed in terms of the DHR endomorphism of $\mathcal{A}_0$ associated with ρ. The converse construction is made in the following.

Proposition 12. *Let ρ be a DHR endomorphism of $\mathcal{A}_0$ localized in the interval $I_0 \in \mathcal{I}_0$ and $\tilde{\rho}$ the representation of $\mathcal{A}$ extending ρ given by Proposition 11. Then $\tilde{\rho}$ is covariant iff there exists a local cocycle z_ρ w.r.t. ρ (i.e. properties (12,13,14) hold with $\mathcal{I}$ replaced by $\mathcal{I}_0$).*

Proof We need only to show that $\tilde{\rho}$ is covariant if there exists a local cocycle z_ρ. By the above arguments eq. (15) holds. Now set $U_{\tilde{\rho}}(g) = z_\rho(g)U(g)$ for g in a suitable neighborhood of the identity of $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$. Then $U_{\tilde{\rho}}$ is a local representation of $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$, hence it extends to a unitary representation of $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$ because $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$ is simply connected. The local covariance then gives

$$U_\rho(g)\rho_L(a)U_\rho^*(g) = \rho_{gL}(\alpha_g(a)),\ a \in \mathcal{A}(L),$$

for any $L \in \mathcal{I}$. The covariance then follows by the group property of U_ρ, see also [21]. □

Before concluding this section we recall that, if ρ is a localizable representation of $\mathcal{A}$, the (statistical) dimension of ρ is $d(\rho) \equiv [\rho_{I'}(\mathcal{A}(I'))' : \rho_I(\mathcal{A}(I))]^{\frac{1}{2}}$, independently on $I \in \mathcal{I}$, and this clearly coincides with $[\mathcal{A}(I) : \rho_I(\mathcal{A}(I))]^{\frac{1}{2}}$ if ρ is localized in I [31].

If ρ is Möbius covariant and $d(\rho) < \infty$, then ρ has positive energy [2].

3.3 Subnets

Let $\mathcal{A}$ be a local irreducible conformal net of von Neumann algebras on S^1 as above and U the associated unitary positive energy representation of $\mathrm{PSL}(2,\mathbb{R})$ on the vacuum Hilbert space $\mathcal{H}$.

By a *conformal subnet* we shall mean a map

$$I \in \mathcal{I} \to \mathcal{B}(I) \subset \mathcal{A}(I)$$

that associates to each interval $I \in \mathcal{I}$ a von Neumann subalgebra $\mathcal{B}(I)$ of $\mathcal{A}(I)$, which is isotone

$$\mathcal{B}(I_1) \subset \mathcal{B}(I_2), \quad I_1 \subset I_2\ ,$$

and Möbius covariant w.r.t. the representation U, namely

$$U(g)\mathcal{B}(I)U(g)^{-1} = \mathcal{B}(gI)$$

for all $g \in \mathrm{PSL}(2,\mathbb{R})$ and $I \in \mathcal{I}$.

Let $\mathcal{H}_\mathcal{B}$ be the closure of $(\vee_{I\in\mathcal{I}}\mathcal{B}(I))\Omega$ and E the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_\mathcal{B}$. By the Reeh-Schlieder theorem $\overline{\mathcal{B}(I)\Omega} = \mathcal{H}_\mathcal{B}$ for each fixed $I \in \mathcal{I}$.

Clearly $\mathcal{H}_\mathcal{B}$ is U-invariant and Ω is unique $U\restriction_{\mathcal{H}_\mathcal{B}}$-invariant, thus by Lemma 10 the restriction of $\mathcal{B}$ to $\mathcal{H}_\mathcal{B}$ is an irreducible local conformal net on $\mathcal{H}_\mathcal{B}$ where $U\restriction_{\mathcal{H}_\mathcal{B}}$ is the associated unitary representation of $\mathrm{PSL}(2,\mathbb{R})$.

As Ω is separating for $\mathcal{A}(I)$, Ω is also separating for $\mathcal{B}(I)$, $I \in \mathcal{I}$. Thus the restriction map $b \in \mathcal{B}(I) \to b\restriction_{\mathcal{H}_\mathcal{B}}$ is is one-to-one, so we will often identify $\mathcal{B}$ with its restriction to $\mathcal{H}_\mathcal{B}$; should we need to specify, we shall talk on the net $\mathcal{B}$ on $\mathcal{H}$ or on $\mathcal{H}_\mathcal{B}$. Note that each $\mathcal{B}(I)$ is a factor.

Lemma 13. *For each $I \in \mathcal{I}$ there is a vacuum preserving conditional expectation $\varepsilon_I : \mathcal{A}(I) \to \mathcal{B}(I)$ such that $\varepsilon_{\tilde{I}}\restriction_{\mathcal{A}(I)} = \varepsilon_I$ if $I \subset \tilde{I}$. Thus $\mathcal{B}$ is a standard net of subfactors in the sense of [34].*

Proof By the Bisognano-Wichmann property $\mathcal{B}(I)$ is globally invariant under the modular group of $(\mathcal{A}(I),\Omega)$, hence by Takesaki's theorem there exists a conditional expectation $\varepsilon_I : \mathcal{A}(I) \to \mathcal{B}(I)$ given by

$$\varepsilon_I(a)E = EaE,\ a \in \mathcal{A}(I)\ .$$

As E is independent of I, we have that $\varepsilon_{\tilde{I}}\restriction_{\mathcal{A}(I)} = \varepsilon_I$ if $I \subset \tilde{I}$. □

By Möbius covariance the index $[\mathcal{A}(I) : \mathcal{B}(I)]$ is independent of the interval $I \in \mathcal{I}$ and will be denoted by $[\mathcal{A} : \mathcal{B}]$. The following lemma is contained in [5] with the strong additivity assumption and in [11] in the conformal case.

Lemma 14. *If $[\mathcal{A} : \mathcal{B}] < \infty$ then $\mathcal{B}(I)' \cap \mathcal{A}(I) = \mathbb{C}$, $I \in \mathcal{I}$.*

Proof By the Bisognano-Wichmann property and the uniqueness of the vacuum the modular group of $\mathcal{A}(I)$ w.r.t. Ω acts ergodically on $\mathcal{B}(I)'\cap\mathcal{A}(I)$, hence $\mathcal{B}(I)'\cap\mathcal{A}(I) = \mathbb{C}$ because $\mathcal{B}(I)' \cap \mathcal{A}(I)$ is finite-dimensional. □

We shall make a variation of the analysis made in [34], which is needed because our nets are not directed.

Lemma 15. *Let $[\mathcal{A} : \mathcal{B}] < \infty$ and $I_0 \subset I$. There exists a canonical endomorphism $\gamma_I : \mathcal{A}(I) \to \mathcal{B}(I)$ with associated Q-system (γ_I, T, S) such that $T \in \mathcal{A}(I_0)$ and $S \in \mathcal{B}(I_0)$ and $\gamma_I\restriction_{\mathcal{A}(I_0)}$ is a canonical endomorphism of $\mathcal{A}(I_0)$ into $\mathcal{B}(I_0)$.*

Proof Let $\mathcal{C}(I_0) = \langle\mathcal{A}(I_0), E\rangle$ and $\mathcal{C}(I) = \langle\mathcal{A}(I), E\rangle$ be the Jones extensions and $\varepsilon'_{I_0} : \mathcal{C}(I_0) \to \mathcal{A}(I_0)$, $\varepsilon'_I : \mathcal{C}(I) \to \mathcal{A}(I)$ the dual expectations. Since every $X \in \mathcal{C}(I_0)$ can be written as $X = \sum_i x_i E y_i$ with $x_i, y_i \in \mathcal{A}(I_0)$ [27, 35], we then have

$$\varepsilon'_I(X) = \sum_i x_i \varepsilon'_I(E) y_i = \lambda \sum_i x_i y_i = \sum_i x_i \varepsilon'_{I_0}(E) y_i = \varepsilon'_{I_0}(X),\ X \in \mathcal{A}(I),$$

where $\lambda \equiv [\mathcal{A} : \mathcal{B}]^{-1}$, namely $\varepsilon'_I{\restriction}_{\mathcal{A}(I_0)} = \varepsilon'_{I_0}$.

Let $V \in \mathcal{C}(I_0)$ be an isometry $VV^* = E$. Then a canonical endomorphism $\gamma_I : \mathcal{A}(I) \to \mathcal{B}(I)$ is given by

$$\gamma_I(a)E = VaV^* \ . \tag{16}$$

Now

$$T = \lambda^{-1}\varepsilon'_I(V), \quad S = \lambda^{-1}\varepsilon_I(T),$$

are the isometries in the Q-system for γ_I and $T \in \mathcal{A}(I_0)$ and $S \in \mathcal{B}(I_0)$ by the compatibility of the expectations. $\square$

Notice the formula

$$\gamma_I(a) = \lambda^{-1}\varepsilon'_I(VaV^*), \quad a \in \mathcal{A}(I) \tag{17}$$

which is obtained applying ε'_I to both members of eq. (16).

Proposition 16. *Let $\mathcal{A}$ be a local irreducible conformal net on S_1 and $\mathcal{B} \subset \mathcal{A}$ a conformal subnet. Given an interval $I_0 \in \mathcal{I}$ the dual canonical endomorphism $\theta_{I_0} \equiv \gamma_{I_0}{\restriction}_{\mathcal{B}(I_0)}$ extends to a representation θ of $\mathcal{B}$ localized in I_0.*

Proof First we assume $[\mathcal{A} : \mathcal{B}] < \infty$. By Proposition 11 it is sufficient to show that θ_{I_0} extends to a DHR endomorphism of $\mathcal{B}$ localized in I_0 (properties $(a), (b), (c)$ with $\mathcal{I}$ replaced by $\mathcal{I}_0$). This is soon verified by applying Lemma 15. Indeed property (a) is an immediate consequence of this lemma. Concerning property (b) notice the formula [32]

$$\theta_I(b) = \lambda^{-1}\varepsilon_I(TbT^*),\ b \in \mathcal{B}(I),$$

with T as in Lemma 15, that follows similarly to the formula (17). Then (b) follows because $T \in \mathcal{A}(I_0)$.

Finally property (c) is immediate by the uniqueness up to inners of the canonical endomorphism [30].

The general case can be obtained along the same lines making use of [34, Theorem 3.2] instead of Lemma 15. $\square$

Proposition 17. *The representation θ of $\mathcal{B}$ on $\mathcal{H}_{\mathcal{B}}$ in Prop. 16 is unitarily equivalent to the identity representation of $\mathcal{B}$ on $\mathcal{H}$. In particular θ is covariant with positive energy.*

Proof Indeed the isometry $V \in \mathcal{C}(I)$ with $VV^* = E$ (as in Lemma 15) satisfies the equation

$$\theta_{\tilde{I}}(b)E = VbV^*, \ b \in \mathcal{B}(\tilde{I}) \ , \tag{18}$$

if $\tilde{I}$ is an interval containing I.

To show that V implements the desired unitary equivalence we need to further show that the above equation (18) holds true with $\tilde{I}$ replaced with an interval I_1 not containing I. This is certainly true if $I_1 \cap I = \varnothing$, because in this case θ acts trivially on $\mathcal{B}(I_1)$ and $V \in \mathcal{C}(I)$ commutes with $\mathcal{B}(I_1)$ because $\mathcal{C}(I) = \langle \mathcal{A}(I), E \rangle$ and $\mathcal{B}(I_1)$ commute.

So we may assume that $I_1 \supset I'$, extending I_1 if necessary. Choose then an interval $I_0 \subset I$ with $I_0 \cap I_1 = \varnothing$. By Lemma 15 we can find a canonical endomorphism $\mathcal{A}(I_0) \to \mathcal{B}(I_0)$ with dual canonical endomorphism extending to a representation θ' of $\mathcal{A}$ localized in I_0 with a unitary $u \in \mathcal{B}(I)$ such that $\theta' = \mathrm{Ad}u \cdot \theta$. Then the isometry V' associated with θ' belongs to $\mathcal{C}(I_0)$ and is given $V' = uV$. Therefore

$$\theta(b)E = u\theta'(b)u^* = uV'bV'^*u^* = VbV^*, \ b \in \mathcal{B}(I_1)$$

as desired. □

The following corollary is a consequence of the equivalence between local and global intertwiners for a finite-index covariant representation [23].

Corollary 18. *Assume $[\mathcal{A} : \mathcal{B}] < \infty$ and let θ be the representation of $\mathcal{B}$ on $\mathcal{H}_{\mathcal{B}}$ in Proposition 16. Then θ has a finite direct sum decomposition*

$$\theta = \bigoplus_{i=0}^{N} N_i \rho_i \ , \tag{19}$$

where the ρ_i's are positive-energy covariant irreducible representation of $\mathcal{B}$ on $\mathcal{H}_{\mathcal{B}}$ localized in I_0. Thus $\theta_I = \oplus_{i=0}^{N} N_i \rho_{iI}$ is a decomposition of the canonical endomorphism for any interval $I \supset I_0$.

Proof As $d(\theta) = [\mathcal{A} : \mathcal{B}] < \infty$ we may decompose θ into irreducible representations ρ_i as above localized in I_0; moreover the ρ_i are covariant because θ is covariant, see [23]. Then by [23, Theorem 2.3] each ρ_{iI} is an irreducible endomorphism of $\mathcal{B}(I)$ if $I \supset I_0$. □

Thus, if $[\mathcal{A} : \mathcal{B}] < \infty$ then the identity representation of $\mathcal{B}$ on $\mathcal{H}$ has finite statistical dimension. The converse is also true: as we have the inclusions $\mathcal{B}(I) \subset \mathcal{A}(I) \subset \mathcal{B}(I')'$, if the identity representation of $\mathcal{B}$ on $\mathcal{H}$ has finite statistical dimension, namely $[\mathcal{B}(I')' : \mathcal{B}(I)] < \infty$, then $[\mathcal{A}(I) : \mathcal{B}(I)] < \infty$.

Corollary 19. *Assume $[\mathcal{A} : \mathcal{B}] < \infty$. With the above notations, let $K_i \subset \mathcal{A}(I_0)$ be the Hilbert spaces of isometries corresponding to ρ_{iI_0} as in eq. (2) for the inclusion $\mathcal{B}(I_0) \subset \mathcal{A}(I_0)$. If I is an interval and $I \supset I_0$, then $\mathcal{A}(I)$ is generated by $\mathcal{B}(I)$ and K_i as in (4).*

If $R_i \in K_i$ then $\alpha_g(R_i) = z_{\rho_i}(g)^* R_i$, for all $g \in \mathrm{PSL}(2,\mathbb{R})\tilde{}$ such that $gI_0 \subset I$, where $z_{\rho_i}(g) \in \mathcal{B}(I)$ are unitaries in the local cocycle associated with ρ_i (12).

Proof By Corollary 18 we have $(\rho_{iI}, \theta_I) = (\rho_{iI_0}, \theta_{I_0}) \subset \mathcal{B}(I_0)$, therefore by formula (3) and Lemma 15 $K_i \subset \mathcal{A}(I_0)$ is also the Hilbert space associated with $\mathcal{B}(I) \subset \mathcal{A}(I)$ as in (4). By Lemma 1 $\mathcal{A}(I)$ is then generated by $\mathcal{B}(I)$ and the K_i's.

Let $g \in \mathrm{PSL}(2,\mathbb{R})\tilde{}$ such that $gI_0 \subset I$. Then $\alpha_g \cdot \rho_i \cdot \alpha_{g^{-1}}$ is localized in I and $\alpha_g \cdot \rho_i \cdot \alpha_{g^{-1}} = \mathrm{Ad} z_{\rho_i}(g)^* \cdot \rho_i$ by formula (14) and Cor. 18, where $z_{\rho_i}(g) \in \mathcal{B}(I)$. Therefore $\alpha_g(K_i) = z_{\rho_i}(g)^* K_i$, namely $\alpha_g(R_i) = z_{\rho_i}(g)^* D(g)^* R_i$ for all $R_i \in K_i$, where $D(g) \in B(K_i)$. It is immediate to check that D locally satisfies the cocycle property with respect to $\alpha_g^{\rho_i} \equiv \mathrm{Ad} z_{\rho_i}(g) \cdot \alpha_g$, namely $D(gh) = D(g)\alpha_g^{\rho_i}(D(h))$ for g, h in a suitable neighborhood of $\mathrm{PSL}(2,\mathbb{R})\tilde{}$. But $\alpha^{\rho_i}\restriction_{B(K_i)}$ is a finite-dimensional representation of $\mathrm{PSL}(2,\mathbb{R})\tilde{}$, thus it must be trivial because $\mathrm{PSL}(2,\mathbb{R})\tilde{}$ has no non-trivial unitary finite-dimensional representations. Thus D is a local finite-dimensional unitary representation of $\mathrm{PSL}(2,\mathbb{R})\tilde{}$ on K_i, so D is again trivial.

Therefore $\alpha_g(R_i) = z_{\rho_i}(g)^* R_i$, for all $g \in \mathrm{PSL}(2,\mathbb{R})\tilde{}$ such that $gI_0 \subset I$ as desired. □

If $\mathcal{A}$ is a conformal net on a Hilbert space $\mathcal{H}$ and $\mathcal{B}$ is a conformal subnet, we shall set $\mathcal{B}' \equiv (\vee_{L \in \mathcal{I}} \mathcal{B}(L))'$, where the commutants are taken on $\mathcal{H}$. For completeness we mention the following partial extension of Cor. 18 to the infinite index case, although it is not used in this paper.

Proposition 20. *Let $\mathcal{A}$ be a local irreducible conformal net on S^1 and $\mathcal{B} \subset \mathcal{A}$ a strongly additive conformal subnet. The following are equivalent:*

(i) The identity representation of $\mathcal{B}$ on $\mathcal{H}$ contains the vacuum representation of $\mathcal{B}$ with multiplicity one.

(ii) The identity representation of $\mathcal{B}$ on $\mathcal{H}$ contains the vacuum representation of $\mathcal{B}$ with finite multiplicity.

(iii) $\mathcal{B}^c = \mathbb{C}$, where $\mathcal{B}^c(I) \equiv \mathcal{B}' \cap \mathcal{A}(I)$.

Proof $(iii) \implies (i)$. By Proposition 17 we have to show that the intertwiner space between the representation θ on $\mathcal{H}_{\mathcal{B}}$ and the identity representation of $\mathcal{B}$ on $\mathcal{H}_{\mathcal{B}}$ is one-dimensional. If θ is localized in I then, by Haag duality, any such intertwiner belongs (θ_I, ι) namely it belongs to $\mathcal{B}(I)$ and intertwines θ_I and the identity automorphism of $\mathcal{B}(I)$. But (θ_I, ι) is one-dimensional because $\mathcal{B}(I) \subset \mathcal{A}(I)$ is an irreducible inclusion of infinite factors with a normal conditional expectation [31, 16].

$(i) \implies (ii)$ is obvious, we show $(ii) \implies (iii)$. Denote by π the subrepresentation of the identity representation of $\mathcal{B}$ on $\mathcal{H}$ corresponding to the vacuum representation and $\mathcal{K} \subset \mathcal{H}$ the corresponding subspace. Then we have a decomposition $\mathcal{K} = \mathcal{H}_0 \otimes \mathcal{H}'_0$ and $\pi = \pi_0 \otimes \mathrm{id}$ where π_0 is irreducible and $\dim\mathcal{H}'_0 < \infty$ The representation U of $\mathrm{PSL}(2,\mathbb{R})$ decomposes as $U = U_0 \otimes U'_0$ and, since $\mathrm{PSL}(2,\mathbb{R})$ has no non-trivial unitary representation, U'_0 is the identity on $\mathcal{H}'_0$. As Ω is unique U-invariant, we then have $\dim\mathcal{H}'_0 = 1$, namely (i) holds.

Thus $\mathcal{K} = \mathcal{H}_{\mathcal{B}}$ and the projection E onto $\mathcal{H}_{\mathcal{B}}$ belongs to the center of $\mathcal{B}'$. In particular if $b \in \mathcal{B}^c(I)$ then b commutes with E. But E implements the expectation ε_I and $b \in \mathcal{A}(I)$, so b belongs to the center of $\mathcal{B}(I)$, thus b is a scalar. $\square$

3.4 Intermediate subnets

Let $\mathcal{A}$ be a local irreducible conformal net and $\mathcal{B}$ a conformal subnet with finite index. We now show that there exists a one-to-one correspondence between $\mathfrak{I}(\mathcal{B}(I_0), \mathcal{A}(I_0))$ for a fixed interval I_0 and the set of intermediate conformal nets between $\mathcal{B}$ and $\mathcal{A}$.

Theorem 21. *Let I_0 be a fixed interval of S^1 and $\mathcal{R}$ be an intermediate subfactor between $\mathcal{B}(I_0)$ and $\mathcal{A}(I_0)$. There exists a unique conformal subnet $\mathcal{L}$ on $\mathcal{A}$ with $\mathcal{B}(I) \subset \mathcal{L}(I) \subset \mathcal{A}(I)$ and $\mathcal{L}(I_0) = \mathcal{R}$.*

Proof Let Λ_I denote as before the one-parameter group of special conformal transformations preserving I. As is easily seen $\Lambda_I(\mathbb{R})$ is exactly the subgroup of $\mathrm{PSL}(2,\mathbb{R})$ of those g with $gI = I$. Then $t \to \beta_t = \mathrm{Ad}U(\Lambda_{I_0}(t))\restriction_{\mathcal{A}(I_0)}$ is a one-parameter automorphism group of $\mathcal{A}(I_0)$ leaving $\mathcal{B}(I_0)$ globally invariant, hence $\beta_t(\mathcal{R}) = \mathcal{R}$ by Corollary 4.

Now, for $I \in \mathcal{I}$ we set

$$\mathcal{L}(I) \equiv \alpha_g(\mathcal{R}) \ ,$$

where $g \in \mathrm{PSL}(2,\mathbb{R})$ is a Möbius transformation such that $I_0 = gI$ and $\alpha_g \equiv \mathrm{Ad}U(g)$. $\mathcal{L}(I)$ is indeed well-defined because if $h \in \mathrm{PSL}(2,\mathbb{R})$ is any other element with $I_0 = hI$, then $h^{-1}gI_0 = I_0$, thus $h^{-1}g = \Lambda_{I_0}(t)$ for some $t \in \mathbb{R}$ and $\alpha_{h^{-1}g}(\mathcal{R}) = \beta_t(\mathcal{R}) = \mathcal{R}$, thus $\alpha_g(\mathcal{R}) = \alpha_h(\mathcal{R})$.

As $\mathcal{L}(I) \subset \mathcal{A}(I)$ and $\mathcal{A}$ is local, $\mathcal{L}(I_1)$ and $\mathcal{L}(I_2)$ clearly commute if $I_1 \cap I_2 = \varnothing$. To show that $I \in \mathcal{I} \to \mathcal{L}(I)$ is a conformal net we need to check the isotony property, namely that $\mathcal{L}(I_1) \subset \mathcal{L}(I)$ if $I_1 \subset I$ are intervals. By conformal invariance we may assume that $I_1 = I_0$ and that $I = g_0 I_0$ for some $g_0 \in \mathrm{PSL}(2,\mathbb{R})$ and then we need to show that $\alpha_{g_0}(\mathcal{R}) \supset \mathcal{R}$.

Now by Corollary 19 $\mathcal{A}(I)$ is generated by $\mathcal{B}(I)$ and Hilbert spaces of isometries $K_i \in \mathcal{A}(I_0)$ corresponding to the expansion (4). Moreover

$$\alpha_{g_0}(R_i) = z_{\rho_i}(g_0)^* R_i, \quad \forall R_i \in K_i \ ,$$

where $z_{\rho_i}(g_0) \in \mathcal{B}(I)$ by Corollary 19. With $K_i' \equiv K_i \cap \mathcal{R}$, by Lemma 19 $\mathcal{R}$ is generated by $\mathcal{B}(I_0)$ and the K_i''s, thus

$$\begin{aligned}\alpha_{g_0}(\mathcal{R}) = \{\alpha_{g_0}(\mathcal{B}(I_0)), \alpha_{g_0}(K_i')\}'' = \{\mathcal{B}(I), z_{\rho_i}(g_0)^* K_i'\}''& \\ = \{\mathcal{B}(I), K_i'\}'' \supset \{\mathcal{B}(I_0), K_i'\}'' = \mathcal{R}&\end{aligned}$$

as desired. □

3.5 Complete rationality is hereditary

To simplify notations, given two different points $\mathbf{a}, \mathbf{b}$ of S^1, we shall write $[\mathbf{a}, \mathbf{b}]$ for the closure of the set of all $z \in S^1$ that follow $\mathbf{a}$ and precede $\mathbf{b}$ in the counterclockwise order, and by $(\mathbf{a}, \mathbf{b})$ the interior of $[\mathbf{a}, \mathbf{b}]$. Two intervals I_1, I_2 will be called *adjacent* if there are three different points $\mathbf{a}, \mathbf{b}, \mathbf{c} \in S^1$ such that $\bar{I}_1 = [\mathbf{a}, \mathbf{b}]$, $\bar{I}_2 = [\mathbf{b}, \mathbf{c}]$ and $\bar{I}_1 \cup \bar{I}_2$ belongs to $\mathcal{I}$.

Let $\mathcal{A}$ be a conformal net on S^1. Recall that $\mathcal{A}$ is *strongly additive* if

$$\mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I) \ ,$$

if I_1, I_2 are adjacent intervals and $I = \bar{I}_1 \cup \bar{I}_2 \in \mathcal{I}$.

$\mathcal{A}$ is *split* if $\mathcal{A}(I_0) \subset \mathcal{A}(I)$ is a split inclusion of von Neumann algebras, namely $\mathcal{A}(I_0) \vee \mathcal{A}(I)'$ is naturally isomorphic to $\mathcal{A}(I_0) \otimes \mathcal{A}(I)'$, if $I_0, I \in \mathcal{I}$ and $\bar{I}_0$ contained in the interior of I.

If $\mathcal{A}$ is split and I_1, I_2 are intervals with disjoint closures, then $\mathcal{A}(I_1) \vee \mathcal{A}(I_2) \simeq \mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$ is a factor and we shall denote by $\mu_{\mathcal{A}}$ the index of the 2-interval inclusion $\mathcal{A}(I_1) \vee \mathcal{A}(I_2) \subset (\mathcal{A}(I_3) \vee \mathcal{A}(I_4))'$ where I_3, I_4 are the two connected components of $S^1 \smallsetminus (I_1 \cup I_2)$.

We shall say that $\mathcal{A}$ is *completely rational* if $\mathcal{A}$ is split, strongly additive and the index $\mu_{\mathcal{A}} < \infty$ where the I_i's are intervals as above.

Lemma 22. *Let $\mathcal{A}$ be an irreducible local conformal net on S^1 and $\mathcal{B} \subset \mathcal{A}$ a finite-index conformal subnet. Then $\mathcal{A}$ is split and $\mu_{\mathcal{A}} < \infty$ iff $\mathcal{B}$ is split and $\mu_{\mathcal{B}} < \infty$.*

In this case the relation $\mu_{\mathcal{B}} = [\mathcal{A} : \mathcal{B}]^2 \mu_{\mathcal{A}}$ holds.

Proof The implication $\mathcal{A}$ split $\implies$ $\mathcal{B}$ split is rather immediate. The rest is proved in [14, Prop. 24]. □

Lemma 23. *Let $\mathcal{A}$ be an irreducible local conformal net on S^1 and $\mathcal{B} \subset \mathcal{A}$ a finite-index conformal subnet. Then $\mathcal{A}$ is strongly additive if $\mathcal{B}$ is strongly additive.*

Proof Let I_1, I_2 be adjacent intervals with $I \equiv \bar{I}_1 \cup \bar{I}_2$ and let $T \in \mathcal{A}(I_1)$ be the isometry in the Q-system for γ_{I_1} as in Lemma 15. Then, by applying Lemma 15, T is also the isometry in the Q-system associated with γ_I. In particular $\mathcal{A}(I_1) = \mathcal{B}(I_1)T$ and $\mathcal{A}(I) = \mathcal{B}(I)T$, thus

$$\mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{B}(I_1)T \vee \mathcal{A}(I_2) \supset \{\mathcal{B}(I_1) \vee \mathcal{B}(I_2), T\}'' = \{\mathcal{B}(I), T\}'' = \mathcal{A}(I) \ .$$

This concludes the proof. □

Theorem 24. *Let $\mathcal{A}$ be an irreducible local conformal net on S^1 and $\mathcal{B} \subset \mathcal{A}$ a conformal subnet with $[\mathcal{A} : \mathcal{B}] < \infty$. Then $\mathcal{A}$ is completely rational iff $\mathcal{B}$ is completely rational.*

We postpone the proof of this theorem to Subsection 3.5.2.

3.5.1 Complete rationality of coset models

We begin with the following simple lemma.

Lemma 25. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be irreducible local conformal nets on S^1. Then $\mathcal{A}_1 \otimes \mathcal{A}_2$ is split (resp. strongly additive; completely rational) iff both $\mathcal{A}_1$ and $\mathcal{A}_2$ are split (resp. strongly additive; completely rational).*

Proof All this can be checked directly, see [13, 14]. □

Let $\mathcal{A}$ be an irreducible local conformal net on S^1 and $\mathcal{B} \subset \mathcal{A}$ a conformal subnet. Then

$$\mathcal{B}^c : I \in \mathcal{I} \to \mathcal{B}^c(I) \equiv \mathcal{B}' \cap \mathcal{A}(I)$$

is clearly a conformal subnet of $\mathcal{A}$ and is called the *coset* net associated with $\mathcal{B} \subset \mathcal{A}$, cf. [42]. Also $\mathcal{B} \vee \mathcal{B}^c : I \in \mathcal{I} \to \mathcal{B}(I) \vee \mathcal{B}^c(I)$ is then a conformal subnet of $\mathcal{A}$.

Following F. Xu [42], we call $\mathcal{B}$ *cofinite* in $\mathcal{A}$ if $[\mathcal{A} : \mathcal{B} \vee \mathcal{B}^c] < \infty$. Indeed in [42] $\mathcal{B}$ is strongly additive so $\mathcal{B}^c(I) = \mathcal{B}(I)' \cap \mathcal{A}(I)$ is the relative commutant of $\mathcal{B}(I)$ in $\mathcal{A}(I)$.

By Takesaki theorem $\mathcal{B}(I) \vee \mathcal{B}^c(I)$ is naturally isomorphic to the von Neumann tensor product $\mathcal{B}(I) \otimes \mathcal{B}^c(I)$.

Corollary 26. *Let $\mathcal{A}$ be an irreducible local conformal net on S^1 and $\mathcal{B} \subset \mathcal{A}$ a cofinite conformal subnet. With the above notations, $\mathcal{A}$ is completely rational iff both $\mathcal{B}$ and $\mathcal{B}^c$ are completely rational.*

If $\mathcal{A}$ is 'split & strongly additive', so are $\mathcal{B}$ and $\mathcal{B}^c$.

Proof It is enough to apply Theorem 24 and Lemma 25 and Proposition 34 below. □

To give a first application, suppose now that the net $\mathcal{A}$ is diffeomorphism invariant. Then one can consider the conformal subnet $\mathcal{A}_{\mathrm{Vir}(c)} \subset \mathcal{A}$, which is associated with the vacuum representation of the Virasoro algebra with central charge $c > 0$ see e.g. [8].

Corollary 27. *Let $\mathcal{A}$ be split, strongly additive and diffeomorphism invariant. If $\mathcal{A}_{\mathrm{Vir}(c)}$ is cofinite in $\mathcal{A}$, then $c \leq 1$.*

Proof If $\mathcal{A}_{\mathrm{Vir}(c)}$ is cofinite in $\mathcal{A}$ then by Cor. 26 also $\mathcal{A}_{\mathrm{Vir}(c)}$ is strongly additive, which is not the case if $c > 1$ [8]. □

We now turn our attention to coset models. Let G be simply connected semisimple compact Lie group of type A, i.e. $G = SU(N_1) \times SU(N_2) \times \cdots \times SU(N_n)$. If $H \subset G$ is a closed subgroup, there is a corresponding inclusion of loop groups $LH \subset LG$. Then the vacuum representation of LG at level k (see [37]) gives an inclusion of conformal nets denoted by $H \subset G_k$ (where H may also have a suffix denoting the appearing level). Thanks to results of Xu [42, Corollary 3.1], see also the correction in [43], the inclusions of conformal nets associated with

(*i*) $G_{k_1+k_2+\cdots+k_m} \subset G_{k_1} \times G_{k_2} \times \cdots G_{k_m}$ where the inclusion is diagonal, $k_i \in \mathbb{N}$, $i = 1, \ldots, m$ and $G = SU(n)$,

(*ii*) $H_{\ell k} \subset G_\ell$, if $H_k \subset G_1$ is a conformal inclusion, where k is the Dynkin index, $\ell \in \mathbb{N}$, H is simple and of type A and G is simple,

(*iii*) $H \subset G_m$, where H is the Cartan subgroup of G,

are all cofinite. So we have the following corollary.

Corollary 28. *The coset subnets corresponding to the inclusions of nets in* (*i*), (*ii*), (*iii*) *of the above Xu's list are completely rational.*

Proof As the conformal net $SU(N)_k$ is completely rational [44] (the proof of the strong additivity is contained in [40]), and the subnets in Xu's list are cofinite [42, 43], it is then enough to apply Theorem 24. □

It then follows from [14] that for the above coset nets the tensor category of all represenations is rational and modular, as shown in [43], and the results in [14] apply.

3.5.2 Proof of Theorem 24

The remaining and more difficult part to prove in Theorem 24 is that $\mathcal{A}$ split and strongly additive implies that $\mathcal{B}$ is strongly additive. In the following we thus assume that $\mathcal{A}$ is split and strongly additive and prove that $\mathcal{B}$ is strongly additive in a series of Lemmata. The starting argument is similar to the one in [45].

Lemma 29. *Let $\mathcal{A}$ be an irreducible, split and strongly additive, local conformal net on S^1. If $\mathcal{B} \subset \mathcal{A}$ a conformal subnet with $[\mathcal{A} : \mathcal{B}] < \infty$ and I_1 and I_2 are adjacent intervals, $I = \bar{I}_1 \cup \bar{I}_2$, then $\mathcal{A}(I_1) \vee \mathcal{B}(I_2) \subset \mathcal{A}(I)$ is a finite-index irreducible inclusion of factors.*

Proof First notice that

$$(\mathcal{A}(I_1) \vee \mathcal{B}(I_2))' \cap \mathcal{A}(I) = \mathcal{A}(I_1)' \cap \mathcal{A}(I) \cap \mathcal{B}(I_2)' = \mathcal{A}(I_2) \cap \mathcal{B}(I_2)' = \mathbb{C} \ ,$$

where $\mathcal{A}(I_1)' \cap \mathcal{A}(I) = \mathcal{A}(I_2)$ because $\mathcal{A}$ is strongly additive. Thus $\mathcal{A}(I_1) \vee \mathcal{B}(I_2) \subset \mathcal{A}(I)$ is an irreducible inclusion of factors.

To show that $[\mathcal{A}(I) : \mathcal{A}(I_1) \vee \mathcal{B}(I_2)] < \infty$ we set $I_1 = [\mathbf{a}, \mathbf{b}]$, $I_2 = [\mathbf{b}, \mathbf{c}]$ and take intervals $I_2^n = [\mathbf{b}_n, \mathbf{c}]$, where $\text{lenght} I_2^n \nearrow \text{lenght} I_2$. Then, by the split property, $\mathcal{N}_n \equiv \mathcal{A}(I_1) \vee \mathcal{B}(I_2^n) \subset \mathcal{M}_n \equiv \mathcal{A}(I_1) \vee \mathcal{A}(I_2^n)$ is isomorphic to $\mathcal{A}(I_1) \otimes \mathcal{B}(I_2^n) \subset \mathcal{A}(I_1) \otimes \mathcal{A}(I_2^n)$ and thus $[\mathcal{M}_n : \mathcal{N}_n] = \lambda^{-1}$. As

$$\mathcal{N}_n \nearrow \mathcal{A}(I_1) \vee \mathcal{B}(I_2), \qquad \mathcal{M}_n \nearrow \mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I),$$

we have $[\mathcal{A}(I) : \mathcal{A}(I_1) \vee \mathcal{B}(I_2)] \leq \lambda^{-1}$ by [14, Proposition 3]. □

As in Xu's proof for the group case, we consider two adjacent intervals I_1, I_2 and set $I = \bar{I}_1 \cup \bar{I}_2$. Then $[\mathcal{A}(I) : \mathcal{A}(I_1) \vee \mathcal{B}(I_2)] < \infty$, and we consider an expectation $\mu : \mathcal{A}(I) \to \mathcal{A}(I_1) \vee \mathcal{B}(I_2)$.

Then $\mathcal{R}(I_2) \equiv \mu(\mathcal{A}(I_2))$ is contained in $\mathcal{A}(I_1)' \cap \mathcal{A}(I)$ and the latter coincides with $\mathcal{A}(I_2)$ because $\mathcal{A}$ is strongly additive [24]. Hence $\mathcal{R}(I_2)$ is a von Neumann algebra and $\mathcal{B}(I_2) \subset \mathcal{R}(I_2) \subset \mathcal{A}(I_2)$. The following lemma is contained in [45].

Lemma 30. *If $\mathcal{R}(I_2) = \mathcal{A}(I_2)$ then $\mathcal{B}$ is strongly additive.*

Proof If $\mathcal{R}(I_2) = \mathcal{A}(I_2)$ then

$$\mu(\mathcal{A}(I)) \supset \mathcal{A}(I_1) \vee \mathcal{R}(I_2) = \mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I) \ ,$$

thus μ is the identity and $\mathcal{A}(I_1) \vee \mathcal{B}(I_2) = \mathcal{A}(I)$. We then have

$$\mathcal{B}(I) = \varepsilon_I(\mathcal{A}(I)) = \varepsilon_I(\mathcal{A}(I_1) \vee \mathcal{B}(I_2)) = \varepsilon_I(\mathcal{A}(I_1)) \vee \mathcal{B}(I_2) = \mathcal{B}(I_1) \vee \mathcal{B}(I_2) \ .$$

□

Lemma 31. *Assume $\mathcal{R}(I_2) = \mathcal{B}(I_2)$. Given intervals $L_0 \subset L$, $\bar{L}_0 \neq \bar{L}$, and $\epsilon > 0$ there exists a projection $e \in \mathcal{A}(L)$ such that*

$$\varepsilon_{L_0}(a)e = eae, \forall a \in \mathcal{A}(L_0) \ , \tag{20}$$

$$(e\Omega, \Omega) > 1 - \epsilon \ . \tag{21}$$

Proof As $\mathcal{B}(I_2)' \cap \mathcal{A}(I_2) = \mathbb{C}$, there exists a unique expectation of $\mathcal{A}(I_2)$ onto $\mathcal{B}(I_2)$, thus

$$\mu\restriction_{\mathcal{A}(I_2)} = \varepsilon_{I_2}$$

is the vacuum preserving conditional expectation.

In order to show the Lemma we can clearly assume that L_0 and L have one common boundary point. As the Möbius group acts transitively on the family of three different points of S^1, property (20,21) does not depend on the choice of the pair $L_0 \subset L$.

Let $e \neq 0$ be a projection in $B(\mathcal{H})$ implementing μ namely

$$\mu(a)e = eae, \ a \in \mathcal{A}(I).$$

As μ acts identically on $\mathcal{A}(I_1)$, we have $e \in \mathcal{A}(I_1)' = \mathcal{A}(I_1')$.

Setting $L = I_1'$, $L_0 = I_2$ we then have: $L \supset L_0$ are intervals with one common boundary point and there exists a non-zero projection $e \in \mathcal{A}(L)$, such that the property in formula (20) holds, i.e.

$$e \neq 0 \ \& \ \varepsilon_{L_0}(a)e = eae, \ \forall a \in \mathcal{A}(L_0) \ . \tag{22}$$

Clearly the above property (22) is a fortiori true if we replace L with a larger interval and L_0 with a smaller interval.

Set $L_0 = [\mathbf{a}, \mathbf{b}]$, $L = [\mathbf{a}, \mathbf{c}]$ and choose sequences of points $\mathbf{b}_n \in (\mathbf{a}, \mathbf{b})$ and $\mathbf{c}_n \in (\mathbf{c}, \mathbf{a})$ in S^1, such that lenght$[\mathbf{a}, \mathbf{b}_n] \searrow 0$ and lenght$[\mathbf{a}, \mathbf{c}_n] \nearrow 2\pi$.

As $[\mathbf{b}_n, \mathbf{c}_n]$ is an increasing sequence of intervals and $\cup_n [\mathbf{b}_n, \mathbf{c}_n]$ is dense in S^1, it follows that $\cup \mathcal{A}([\mathbf{b}_n, \mathbf{c}_n])$ is strongly dense in $B(\mathcal{H})$ (this is a consequence of Haag duality and the factoriality of the local von Neumann algebras). Therefore the unitaries of $\cup_n \mathcal{A}([\mathbf{b}_n, \mathbf{c}_n])$ are strongly dense in the unitaries of $B(\mathcal{H})$. Given $\epsilon > 0$, then there exists an integer n and a unitary $u \in \mathcal{A}([\mathbf{b}_n, \mathbf{c}_n])$ such that

$$(eu\Omega, u\Omega) > 1 - \epsilon \ .$$

Replacing L with $[\mathbf{a}, \mathbf{c}_n]$, L_0 with $[\mathbf{b}_n, \mathbf{c}_n]$ and e with $e' \equiv u^* e u$, equation (21) clearly holds. But also equation (20) is satisfied because

$$\varepsilon_{L_0}(a)e' = u^* \varepsilon_{L_0}(a) e u = u^* eaeu = u^* euau^* eu = e'ae', \quad a \in \mathcal{A}(L_0) \ ,$$

as u commutes with $\mathcal{A}(L_0)$. □

Lemma 32. *$\mathcal{R}(I_2) \neq \mathcal{B}(I_2)$ unless $\mathcal{B} = \mathcal{A}$.*

Proof Let's assume $\mathcal{R}(I_2) = \mathcal{B}(I_2)$. Note that by Möbius covariance the equality $\mathcal{R}(I_2) = \mathcal{B}(I_2)$ is independent of the choice of I_1, I_2.

Let I be a fixed interval and I_n a decreasing sequence of intervals with a common boundary point with I such that $\cap_n I_n = I$ and choose a projection $e_n \in \mathcal{A}(I_n)$ such that

$$\varepsilon_I(a)e_n = e_n a e_n, \forall a \in \mathcal{A}(I) \ \&\ (e_n\Omega, \Omega) > 1 - \frac{1}{n} \ . \tag{23}$$

Let e be a weak limit point of $\{e_n\}$. Then $e \in \cap\mathcal{A}(I_n) = \mathcal{A}(I)$ and $e \in \mathcal{B}(I)'$, thus e is a scalar, $0 \leq e \leq 1$. As $(e_n\Omega, \Omega) > 1 - \frac{1}{n}$, we have $e = 1$. Thus $e_n \to 1$ weakly. As the weak and strong topologies coincide on the set of selfadjoint projections, $e_n \to 1$ strongly. Going to the limit in eq. (23) we then have $\varepsilon_I(a) = a$, $a \in \mathcal{A}(I)$, namely $\mathcal{B}(I) = \mathcal{A}(I)$. □

Lemma 33. *Let $\mathcal{A}$ be a local irreducible conformal net and $\mathcal{B}$ a finite-index subnet. Suppose $\mathcal{A}$ is split and strongly additive and $\mathcal{B}$ is not strongly additive. Then there exists an intermediate conformal net $\mathcal{B} \subset \mathcal{L} \subset \mathcal{A}$ such that $\mathcal{L}$ is split and strongly additive and $\mathcal{L} \neq \mathcal{A}$.*

Proof We use the above notations. By Lemma 32 $\mathcal{R}(I_2) \neq \mathcal{B}(I_2)$ and Lemma 30 $\mathcal{R}(I_2) \neq \mathcal{A}(I_2)$. By Theorem 21 there exists a conformal subnet $\mathcal{R}$ intermediate between $\mathcal{B}$ and $\mathcal{A}$ such that the associated local von Neumann algebra $\mathcal{R}(I_2)$ is such a factor. Set $\mathcal{R}_1 \equiv \mathcal{R}$. Replacing $\mathcal{B}$ by $\mathcal{R}_1$ and repeating the construction we find a factor $\mathcal{R}_2(I_2)$ between $\mathcal{R}_1(I_2)$ and $\mathcal{A}(I_2)$. Iterating the procedure we get a sequence of factors $\mathcal{R}_n(I_2)$, coming from a conformal subnets $\mathcal{R}_n$, such that

$$\mathcal{B}(I_2) \subset \mathcal{R}_1(I_2) \subset \mathcal{R}_2(I_2) \subset \cdots \subset \mathcal{A}(I_2) \ .$$

As $[\mathcal{A}(I_2) : \mathcal{B}(I_2)] < \infty$, after finitely many steps the iteration stabilizes, so let n be the smallest integer such that $\mathcal{R}_n(I_2) = \mathcal{R}_{n+1}(I_2)$. Then $n \geq 2$ by the above discussion.

By Lemma 32 we then have $\mathcal{R}_n(I_2) = \mathcal{A}(I_2)$. Thus $\mathcal{L} \equiv \mathcal{R}_{n-1}$ is strongly additive by Lemma 30. $\mathcal{L}$ is clearly split and, by construction, properly between $\mathcal{B}$ and $\mathcal{A}$. □

Proposition 34. *Let $\mathcal{A}$ be a local irreducible conformal net and $\mathcal{B}$ a finite-index subnet. If $\mathcal{A}$ is split and strongly additive then $\mathcal{B}$ is split and strongly additive.*

Proof Clearly $\mathcal{B}$ is split. Suppose that $\mathcal{B}$ is not strongly additive. Then by Lemma 33 there exists a strongly additive conformal subnet $\mathcal{L}_1$ intermediate between $\mathcal{B}$ and $\mathcal{A}$ such that $\mathcal{L}_1 \neq \mathcal{A}$. Again by Lemma 33 there exists a strongly additive conformal subnet $\mathcal{L}_2$ intermediate between $\mathcal{B}$ and $\mathcal{L}_2$ such that $\mathcal{L}_2 \neq \mathcal{L}_1$. By iteration we find a sequence of subnets

$$\mathcal{A} \supset \mathcal{L}_1 \supset \mathcal{L}_2 \supset \cdots \supset \mathcal{B} \ ,$$

where all inclusions are proper, thus $[\mathcal{L}_{n+1} : \mathcal{L}_n] \geq 2$ by Jones' theorem [27]. So $[\mathcal{A} : \mathcal{B}] = \infty$ by the multiplicativity of the index [29], contrary to our assumptions. $\square$

Proof of Theorem 24 It is now sufficient to apply Lemma 22, Lemma 23 and Proposition 34. $\square$

3.5.3 Case of a Fermi net

Most of our analysis extends to the case of non-local Fermi conformal nets. As there are several examples of local conformal finite-index subnets of Fermi nets, we sketch here how to modify our arguments and reduce to the local situation.

In this subsection $\mathcal{A}$ is a *twisted local* irreducible net of von Neumann algebras on S^1, namely $\mathcal{A}$ an irreducible net satisfying all properties **A** to **E** in Section 3.1, except **B** which is now replaced by

B$'$. *Twisted locality.* There exists a unitary Z commuting with the unitary representation U such that $Z\Omega = \Omega$ and

$$Z\mathcal{A}(I')Z^* \subset \mathcal{A}(I)'$$

for all intervals I.

The basic results for local nets (modular structure, duality, etc.) have a version for twisted local nets, see [11].

We shall say that a conformal net $\mathcal{A}$ is a *Fermi net* if there exists a self-adjoint unitary V on $\mathcal{H}$ such that $V\Omega = \Omega$ and $\beta(\mathcal{A}(I)) = \mathcal{A}(I)$, $I \in \mathcal{I}$, where $\beta \equiv \mathrm{Ad}V$, with canonical commutation relations: if I_1, I_2 are disjoint intervals then the commutator or anti-commutator

$$[a_1, a_2]_\pm = 0, \quad a_i \in \mathcal{A}(I_i) \ ,$$

if $\beta(a_i) = \pm a_i$; the commutator vanishes if one of the a_i's is a Bose operator ($\beta(a_i) = a_i$) and the anti-commutator vanishes if both the a_i are Fermi operators ($\beta(a_i) = -a_i$).

A conformal Fermi net satisfies twisted locality, hence twisted duality, where the unitary Z is given by

$$Z = \frac{1+iV}{1+i} \ ,$$

in particular $ZbZ^* = b$, for all $b \in \vee_I \mathcal{B}_b(I)$, see [22], where we denote by $\mathcal{B}_b$ the Bose subnet $\mathcal{A}^\beta$ of $\mathcal{A}$. Note that $[\mathcal{A} : \mathcal{B}_b] = 2$ and $\mathcal{B}_b$ is a local conformal net.

Lemma 35. *Let $\mathcal{B}$ be a local subnet of $\mathcal{A}$. Then $\mathcal{B} \subset \mathcal{B}_b$.*

Proof This is obvious since otherwise each $\mathcal{B}(I)$ would contain non-zero Fermi operators and these do not commute if they are localized in disjoint intervals. $\square$

Due to the above lemma, the results in the previous sections extend to the case of a local finite-index subnet $\mathcal{B}$ of $\mathcal{A}$ once we show them in the particular case $\mathcal{B} = \mathcal{B}_b$. We give here explicitly the extension of Theorem 24

Proposition 36. *Let $\mathcal{A}$ be an irreducible Fermi conformal net on S^1 and $\mathcal{B} \subset \mathcal{A}$ a local conformal subnet with $[\mathcal{A} : \mathcal{B}] < \infty$. Then $\mathcal{A}$ is split and strongly additive iff $\mathcal{B}$ is split and strongly additive.*

Proof By the above discussion we may assume that $\mathcal{B}$ is the Bose subnet. We assume that $\mathcal{A}$ is split and strong additive and show that $\mathcal{B}$ strong additive, the other implications are obtained essentially as in the local case.

It is enough to show that $\mathcal{A}(I_1) \vee \mathcal{B}(I_2) = \mathcal{A}(I)$ if I_1, I_2 are adjacent intervals and $I = \bar{I}_1 \cup \bar{I}_2$. The inclusion $\mathcal{A}(I_1) \vee \mathcal{B}(I_2) \subset \mathcal{A}(I)$ has finite index as in the local case and we assume $\mathcal{A}(I_1) \vee \mathcal{B}(I_2) \neq \mathcal{A}(I)$. We consider an expectation $\mu : \mathcal{A}(I) \to \mathcal{A}(I_1) \vee \mathcal{B}(I_2)$. Let $u_i \in \mathcal{A}(I_i)$ be Fermi unitaries. Then $\mathrm{Ad}u_2$ implements an automorphism of $\mathcal{B}(I_2)$, acts trivially on $\mathcal{B}(I_1)$ and $\mathrm{Ad}u_2(u_1) = -u_1$, therefore $\mathrm{Ad}u_2$ implements an automorphism of $\mathcal{A}(I_1) \vee \mathcal{B}(I_2)$. As $\mathcal{A}(I)$ is generated by $\mathcal{A}(I_1) \vee \mathcal{B}(I_2)$ and u_2, it follows that $\mathcal{A}(I)$ is the crossed product of $\mathcal{A}(I_1) \vee \mathcal{B}(I_2)$ by $\mathrm{Ad}u_2$, thus μ acts trivially on $\mathcal{A}(I_1)$ and $\mu(u_2) = 0$, so $\mu(\mathcal{A}(I_2)) = \mathcal{B}(I_2)$.

Let $e \neq 0$ be a projection implementing μ. Then $e \in Z\mathcal{A}(I_1')Z^*$. Arguing as in the local case $e \in \mathcal{A}(I_2)' \cap Z\mathcal{A}(I_2)Z^*$, but

$$\begin{aligned} \mathcal{A}(I_2)' \cap Z\mathcal{A}(I_2)Z^* &= Z(Z^*\mathcal{A}(I_2)'Z \cap \mathcal{A}(I_2))Z^* \\ &\subset Z(Z^*\mathcal{B}(I_2)'Z \cap \mathcal{A}(I_2))Z^* = Z(\mathcal{B}(I_2)' \cap \mathcal{A}(I_2))Z^* = \mathbb{C} \ . \end{aligned}$$

so $e = 1$ and μ is trivial, which contradicts the assumption $\mathcal{A}(I_1) \vee \mathcal{B}(I_2) \neq \mathcal{A}(I)$. $\square$

Remark. In this paper the positivity of the energy has been used only indirectly, essentially to entail the Reeh-Schlieder property and the factoriality of the local algebras. Thus our results extend to the case of conformal nets on S^1 with the above properties, without assuming the positivity of the conformal Hamiltonian. We encounter this situation if we consider a local conformal net $\mathcal{A}$ on the 1+1-dimensional Minkowski spacetime and look at the corresponding time zero net $\mathcal{A}_{t=0}$. In particular, if $\mathcal{A}$ is a finite-index local extension of a $1+1$-dimensional chiral net $\mathcal{A}_1 \otimes \mathcal{A}_2$, as discussed in [38], then $\mathcal{A}_{t=0}$ is split and strongly additive iff both $\mathcal{A}_1$ and $\mathcal{A}_2$ are split and strongly additive.

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