CANONICAL EXTENSION OF ENDOMORPHISMS OF TYPE III FACTORS

MASAKI IZUMI

ABSTRACT. We extend the notion of the canonical extension of automorphisms of type III factors to the case of endomorphisms with finite statistical dimensions. Following the automorphism case, we introduce two notions for endomorphisms of type III factors: modular endomorphisms and Connes-Takesaki modules. Several applications to compact groups of automorphisms and subfactors of type III factors are given from the viewpoint of ergodic theory.

1. INTRODUCTION

The canonical extension of an automorphism of a type III factor gives rise to a natural isomorphism from the automorphism group of the type III factor into that of the crossed product by the modular automorphism group. Although the notion itself had been known to specialists before, (e.g. [12]), the first systematic analysis of the canonical extension was proposed and accomplished by U. Haagerup and E. Størmer. In [21], [22], they introduced several classes of automorphisms of von Neumann algebras and investigated their structure using the canonical extension.

One of the purposes of the present notes is to generalize the notion of the canonical extension to an endomorphism whose image has a finite index [38], [60], [41]. Our analysis is based on techniques of the common continuous decomposition of inclusions and endomorphisms of type III factors, mainly developed by Ph. Loi [47], T. Hamachi-H. Kosaki [25], [26], [27], R. Longo [49], [50], and H. Kosaki-R. Longo [44]. In [63], J. E. Roberts introduced an action of the dual object of a compact group on a von Neumann algebra, now called a Roberts action, which is a functor from the category of the finite dimensional unitary representations to that of the endomorphisms of the von Neumann algebra. As several invariants of discrete group actions on type III factors can be captured by the canonical extension [72], [65], it is natural to expect that the canonical extension of endomorphisms plays an important role in the analysis of compact group actions through the dual Roberts actions. Indeed, we obtain several new results on minimal actions of compact groups on type III factors. Study of minimal actions of compact groups was initiated by S. Popa and A. Wassermann using subfactor techniques developed in [61]. Interested readers are recommended to consult [62] for related topics.

We introduce two new notions for endomorphisms corresponding to extended modular automorphisms and the Connes-Takesaki module in the automorphism case [11]: modular endomorphisms and the Connes-Takesaki module of endomorphisms. As in

Work supported by Mathematical Sciences Research Institute.

the case of extended modular automorphisms, a modular endomorphism carries a unitary group valued cocycle of the flow of weights, and it gives rise to a Roberts action of the dual object of the essential range of the cocycle, the notion developed by G. Mackey and R. Zimmer [56], [80]. The dual actions of Roberts actions consisting of modular endomorphisms provide a large variety of new examples of minimal actions of compact groups on type III_0 factors. Also, we give a strong constraint for the possible types of an algebra-fixed point algebra pair for a minimal action of a compact semisimple Lie group using these two notions.

In [80], Zimmer introduced the notion of extensions of ergodic transformation groups with relatively discrete spectrum, which is a relative version of the notion of ergodic transformations with pure point spectrum. It turns out that a non-commutative generalization of this notion is closely related to modular endomorphisms (Theorem 5. 11), which may be expected because both objects are described in terms of cocycles of ergodic transformation groups with values in compact groups. In particular, we characterize a subfactor, not necessarily of a finite index, coming from an ergodic extension of the flow of weights in terms of a group-subgroup subfactor (c.f. [23], [24], [25], [26], [71]).

Our original purpose for introducing the canonical extension of endomorphisms is to settle a problem left unsolved in [31]. When a type III factor and its subfactor with a finite index have the common flow of weights, their "type II principal graph" makes sense via the common crossed product decomposition [47], [44]. In [31], we characterized the situation where the type II principal graph does not coincide with the principal graph of the original subfactor for the case of III_λ, $\lambda \neq 0$ factors. Namely, the two graphs do not coincide if and only if some power of the canonical endomorphism for the inclusion contains a non-trivial modular automorphism. It would be tempting to conjecture that the same statement should be true in the type III_0 case if the modular automorphism is replaced with an extended modular automorphism. However, it turns out that the right counterpart for our purpose is a modular endomorphism (Theorem 3.7). One can observe a similar phenomenon in the recent work of T. Masuda [57] (c.f. [43]).

Final part of this work was finished while the author stayed at MSRI, and he would like to thank them for their hospitality. He also would like to thank T. Hamachi and H. Kosaki for stimulating discussions. Some of the main results in this paper were announced in [32].

2. Canonical Extension

First, we briefly summarize notation used in this paper. Our basic references are [67] and [68] for Tomita-Takesaki theory, [41] for index theory of type III factors, [35] for sector theory and infinite index inclusions, and [80] for cocycles of ergodic transformation groups. Undefined terms and notations used in this paper should be found in these references. We always assume that von Neumann algebras have separable preduals, Hilbert spaces are separable, and locally compact groups are second countable. Automorphisms and endomorphisms of von Neumann algebras are always assumed to be $*$-preserving and unital. For a von Neumann algebra M, we denote by

$\mathrm{Aut}(M)$, $\mathrm{Int}(M)$, and $\mathrm{End}(M)$ the sets of automorphisms, inner automorphisms, and endomorphisms respectively. We denote by $\mathrm{Out}(M)$ and $\mathrm{Sect}(M)$ the sets of unitary equivalence classes of automorphisms and endomorphisms respectively. A member in $\mathrm{Sect}(M)$ is called a sector. For $\rho, \sigma \in \mathrm{End}(M)$, the intertwiner space from ρ to σ is defined by

$$(\rho, \sigma) = \{V \in M;\ V\rho(x) = \sigma(x)V\}.$$

When M is a factor and ρ is irreducible, that is, $\rho(M)' \cap M = \mathbb{C}1$, (ρ, σ) is a Hilbert space with the inner product

$$\langle V_1, V_2 \rangle 1 = V_2^* V_1.$$

Every action of a topological group on a von Neumann algebra M is assumed to be continuous with respect to the u-topology in $\mathrm{Aut}(M)$. For a group action α on M, M^α denotes the fixed point subalgebra of M under the action α. When M acts on a Hilbert space H, we denote by π_α the representation of M on $L^2(G, H)$ defined by

$$\pi_\alpha(x)\xi(g) = \alpha_{g^{-1}}(x)\xi(g), \quad \xi \in L^2(G, H).$$

We denote by $l(g)$ and $r(g)$ the left and right regular representations of G respectively. For a weight φ on M or an operator valued weight from M to a subalgebra, we denote by $\mathfrak{m}_\varphi$ and $\mathfrak{n}_\varphi$ the domain of φ and the left ideal corresponding to $\mathfrak{m}_\varphi$ as usual.

Let φ be a faithful normal semifinite weight on M. We denote by $\widetilde{M} = M \rtimes_{\sigma^\varphi} \mathbb{R}$ the crossed product of M by the modular automorphism group $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$, which is the von Neumann algebra generated by $\pi_{\sigma^\varphi}(M)$ and the implementing one-parameter unitary group $\{\lambda^\varphi(t)\}_{t\in\mathbb{R}}$, where $\lambda^\varphi(t) = 1 \otimes l(t)$. We often omit π_{σ^φ} when there is no possibility of confusion. We denote by θ and τ the dual action of σ^φ and the natural trace constructed from the dual weight of φ on $\widetilde{M}$ and the generator of $\{\lambda^\varphi(t)\}_{t\in\mathbb{R}}$. Then, the triple $(\widetilde{M}, \theta, \tau)$ does not depend on the choice of φ under the following identification

$$\lambda^\psi(t) = [D\psi : D\varphi]_t \lambda^\varphi(t),$$

where ψ is another faithful normal weight on M and $\{[D\psi : D\varphi]_t\}_{t\in\mathbb{R}}$ is the Connes cocycle derivative. As for the relationship between $\mathrm{Aut}(M)$ and $\mathrm{Aut}(\widetilde{M})$, we have the following [22, Proposition 12.1]:

Proposition 2.1. *Let M be a von Neumann algebra and φ be a faithful normal semifinite weight on M. Then, for every automorphism $\alpha \in \mathrm{Aut}(M)$, there exists a unique automorphism $\widetilde{\alpha} \in \mathrm{Aut}(M)$ satisfying*

$$\widetilde{\alpha}(x) = \alpha(x), \quad x \in M,$$

$$\widetilde{\alpha}(\lambda^\varphi(t)) = [D\varphi \cdot \alpha^{-1} : D\varphi]_t \lambda^\varphi(t).$$

Moreover, the map $\alpha \mapsto \widetilde{\alpha}$ is a homomorphism from $\mathrm{Aut}(M)$ into $\mathrm{Aut}(\widetilde{M})$.

$\widetilde{\alpha}$ is called the canonical extension of α. The goal of this section is to introduce the canonical extension of endomorphisms with finite statistical dimensions generalizing the automorphism case.

Let M be an infinite factor. We denote by $\mathrm{End}(M)_0$ the set of endomorphisms of M with finite statistical dimensions. For $\rho \in \mathrm{End}(M)_0$, we denote by E_ρ, ϕ_ρ, and $d(\rho)$, the minimal conditional expectation from M onto the image of ρ [29], the standard left inverse of ρ, and the statistical dimension of ρ, respectively: that is, $\phi_\rho = \rho^{-1} \cdot E_\rho$ and $d(\rho) = [M : \rho(M)]_0^{1/2}$, where $[M : \rho(M)]_0 = \text{Index } E_\rho$ is the minimum index of $M \supset \rho(M)$. We denote by $\text{Sect}(M)_0$ the unitary equivalence classes of the endomorphisms in $\text{End}(M)_0$. $\text{Sect}(M)$ has three natural operations, forming a direct sum, composing two endomorphisms (regarded as a product), and taking conjugation. The statistical dimensions are additive, multiplicative, and invariant under these three operations on $\text{Sect}(M)_0$ respectively [29], [50], [39], [53]. Let $M \supset N$ be an inclusion of type III factors with a finite index. Then, we regard $\widetilde{N}$ as a subalgebra of $\widetilde{M}$ through the identification $\lambda^{\omega \cdot E}(t) = \lambda^\omega(t)$, where E is the minimal expectation from M onto N and ω is a faithful normal semifinite weight on N. We denote by $\widetilde{E}$ the extension of E to $\widetilde{M}$ leaving $\lambda^{\varphi_0 \cdot E}(t)$ fixed, which is a normal conditional expectation from $\widetilde{M}$ onto $\widetilde{N}$ [44]. Even when $M \supset N$ has an infinite index, if E is a unique normal conditional expectation from M onto N, we use the same convention.

Before introducing the canonical extension, we need some preparation.

Definition 2.2. Let M be an infinite factor. A pair (φ, ρ) consisting of a faithful normal semifinite weight φ on M and $\rho \in \text{End}(M)_0$ is said to be an *invariant pair* if the following hold:

$$\varphi \cdot \rho = d(\rho)\varphi,$$

$$\varphi \cdot E_\rho = \varphi.$$

Note that the above conditions are equivalent to $d(\rho)\varphi \cdot \phi_\rho = \varphi$, thanks to $\rho \cdot \phi_\rho = E_\rho$ and $\phi_\rho \cdot \rho = \text{id}$.

Lemma 2.3. *Let M be a type* III *factor. Then, the following hold:*

1. *If φ is a faithful normal semifinite weight on M and $\rho \in \text{End}(M)_0$, then $\varphi \cdot \phi_\rho$ is a faithful normal semifinite weight on M.*
2. *If (φ, ρ) is an invariant pair, then ρ commutes with σ_t^φ for $t \in \mathbb{R}$.*
3. *Let ψ be a dominant weight (see [11] for the definition). Then, for every sector in $\text{Sect}(M)_0$, there exists a representative $\rho \in \text{End}(M)_0$ such that (ψ, ρ) is an invariant pair.*
4. *Let ψ be a dominant weight on M and $\rho \in \text{End}(M)_0$. If (ψ, ρ) is an invariant pair, then there exist irreducible $\rho_i \in \text{End}(M)_0$ and isometries $V_i \in (\rho_i, \rho)$ $i = 1, 2, \cdots, n$ such that each V_i belongs to the centralizer M_ψ and*

$$\rho(x) = \sum_{i=1}^{n} V_i \rho_i(x) V_i^*, \quad x \in M.$$

   Moreover, each (ψ, ρ_i) is an invariant pair.

5. *Let ψ be a dominant weight and $\rho_1, \rho_2 \in \text{End}(M)_0$. If (ψ, ρ_1) and (ψ, ρ_2) are invariant pairs, then (ρ_1, ρ_2) is in the centralizer M_ψ.*

Proof. (1). Since $\varphi \cdot \rho^{-1}$ is a faithful normal semifinite weight on $\rho(M)$, so is $\varphi \cdot \phi_\rho = \varphi \cdot \rho^{-1} \cdot E_\rho$ on M.

(2). Let $N=\rho(M)$ and φ_0 be the restriction of φ to N, which is a faithful normal semifinite weight on N by assumption. Then, we have

$$\varphi_0\cdot\rho=d(\rho)\varphi,$$
$$\varphi_0\cdot E_\rho=\varphi.$$

Regarding ρ as an isomorphism from M to N, we get

$$\rho\cdot\sigma_t^{\varphi}=\sigma_t^{\varphi_0}\cdot\rho=\sigma_t^{\varphi}\cdot\rho.$$

(3). This is [35, Lemma 2.12].

(4). Let $N=\rho(M)$ and ψ_0 be the restriction of ψ to N. Then, we have common continuous decomposition [47], [44]

$$M=M_\psi\rtimes_{\theta_0}\mathbb{R}\supset N=N_{\psi_0}\rtimes_{\theta_0}\mathbb{R},$$

such that

$$M\cap N'=(M_\psi\cap N'_{\psi_0})^{\theta_0}.$$

Therefore, each non-zero projection $p\in M\cap N'$ is equivalent to 1 in M_ψ (more strongly in $M_\psi^{\theta_0}$), and there exist isometries $V_i\in M_\psi$, $i=1,2,\cdots,n$ such that each $p_i=V_iV_i^*$ is a minimal projection in $M\cap N'$ and $\{p_i\}_{i=1}^n$ forms a partition of unity. Setting $\rho_i(x)=V_i^*\rho(x)V_i$, we get the first assertion. Since the minimal expectation for $p_iMp_i\supset Np_i$ is given by $E_\rho(x)p_i/E_\rho(p_i)$, we have

$$E_{\rho_i}(x)=\frac{d(\rho)}{d(\rho_i)}V_i^*\rho\cdot\phi_\rho(V_ixV_i^*)V_i.=\frac{d(\rho)}{d(\rho_i)}\rho_i\cdot\phi_\rho(V_ixV_i^*),\quad x\in M,$$

thanks to the local index formula for the minimal conditional expectation [29] and the fact that $\mathrm{Ad}(V_i^*)$ is an isomorphism from $p_iMp_i\supset N_{p_i}$ to $M\supset\rho_i(M)$. This implies

$$\phi_{\rho_i}(x)=\frac{d(\rho)}{d(\rho_i)}\phi_\rho(V_ixV_i^*),\quad x\in M.$$

Thus,

$$d(\rho_i)\psi\cdot\phi_{\rho_i}=d(\rho)\psi\cdot\phi_\rho(V_i\cdot V_i^*)=\varphi(V_i\cdot V_i^*).$$

Since V_i is in the centralizer of ψ, this implies that (ψ,ρ_i) is an invariant pair.

(5). Thanks to (4), the general case is reduced to the case where ρ_1 and ρ_2 are irreducible, and we make this assumption. If $(\rho_1,\rho_2)=0$, we have nothing to prove, and so we assume $\rho_1=\mathrm{Ad}(u)\cdot\rho_2$ for some unitary $u\in M$. Since σ_t^ψ commutes with ρ_1 and ρ_2, $\sigma_t^\psi(u)$ is proportional to u, and there exists a scalar $c>0$ such that $\sigma_t^\psi(u)=c^{it}u$. However, the KMS condition and $\psi\cdot\rho_1=\psi\cdot\rho_2$ imply $c=1$, which shows $u\in M_\psi$. □

Now we introduce the canonical extension of an endomorphism.

Theorem 2.4. *Let M be a type* III *factor and φ be a faithful normal semifinite weight on M. Then, for every $\rho\in\mathrm{End}(M)_0$, there exists a unique endomorphism $\widetilde{\rho}\in\mathrm{End}(\widetilde{M})$ satisfying*

$$\widetilde{\rho}(x)=\rho(x),\quad x\in M$$
$$\widetilde{\rho}(\lambda^\varphi(t))=d(\rho)^{it}[D\varphi\cdot\phi_\rho:D\varphi]_t\lambda^\varphi(t).$$

Under the identification $\lambda^\psi(t) = [D\psi : D\varphi]_t \lambda^\varphi(t)$, $\widetilde{\rho}$ does not depend on the choice of the weight φ.

Proof. Let ψ be a dominant weight. First we assume that (ψ, ρ) is an invariant pair, and N and ψ_0 are as in the proof of Lemma 2.3, (4). Thanks to Lemma 2.3, (2), we may consider ρ an isomorphism from M onto N intertwining σ^ψ and σ^{ψ_0}. Therefore, ρ extends to an isomorphism $\widetilde{\rho}$ from $\widetilde{M}$ onto $\widetilde{N}$ sending $\lambda^\psi(t)$ to $\lambda^{\psi_0}(t)$. Since ψ satisfies $\psi = \psi_0 \cdot E_\rho$, we may regard $\widetilde{N}$ as a subalgebra of $\widetilde{M}$ identifying $\lambda^{\psi_0}(t)$ with $\lambda^\psi(t)$. We show that $\widetilde{\rho}$ satisfies the required property. Indeed,

$$\widetilde{\rho}(\lambda^\varphi(t)) = \widetilde{\rho}([D\varphi : D\psi]_t \lambda^\psi(t)) = \rho([D\varphi : D\psi]_t)\lambda^\psi(t).$$

Regarding ρ as an isomorphism from M onto N again, we have

$$\rho([D\varphi : D\psi]_t) = [D\varphi \cdot \rho^{-1} : D\psi \cdot \rho^{-1}]_t = [D\varphi \cdot \rho^{-1} \cdot E_\rho : D\psi \cdot \rho^{-1} \cdot E_\rho]_t,$$

and so

$$\widetilde{\rho}(\lambda^\varphi(t)) = [D\varphi \cdot \phi_\rho : D\psi \cdot \phi_\rho]_t \lambda^\psi(t) = d(\rho)^{it}[D\varphi \cdot \phi_\rho : D\varphi]_t \lambda^\varphi(t).$$

Now, we treat the general case. Thanks to Lemma 2.3, (3), it suffices to show the statement for $\rho_1 = \mathrm{Ad}(u) \cdot \rho$, where u is a unitary in M, and ρ and ψ are as before. Indeed,

$$\begin{aligned} \mathrm{Ad}(u) \cdot \widetilde{\rho}(\lambda^\varphi(t)) &= d(\rho)^{it} u \sigma_t^{\varphi\cdot\phi_\rho}(u^*)\lambda^{\varphi\cdot\phi_\rho}(t) \\ &= d(\rho)^{it}[D\varphi \cdot \phi_\rho \cdot \mathrm{Ad}(u^*) : D\varphi \cdot \phi_\rho]_t \lambda^{\varphi\cdot\phi_\rho}(t) \\ &= d(\rho)^{it}[D\varphi \cdot \phi_{\rho_1} : D\varphi]_t \lambda^\varphi(t). \end{aligned}$$

Therefore, $\widetilde{\rho}_1 := \mathrm{Ad}(u) \cdot \widetilde{\rho}$ has the desired property. □

The same formula of the Connes cocycle as above also appears in [52].

We call the above $\widetilde{\rho}$ the *canonical extension* of ρ. When, α is an automorphism, the left inverse ϕ_α is nothing but the inverse of α. Therefore, our definition of the canonical extension generalizes that in the automorphism case.

Let $M \supset N$ be properly infinite von Neumann algebras. Then, the standard representation of M is automatically standard for N and the product of two modular conjugation $J_N J_M$ makes sense. The canonical endomorphism $\gamma \in \mathrm{End}(M)$ for the inclusion $M \supset N$ is defined by $\gamma(x) = J_N J_M x J_M J_N \in N$ [48]. In general, modular conjugations depend on the choice of natural cones. However, a different choice of modular conjugations amounts to only a perturbation of γ by an inner automorphism of N. Therefore, we call any endomorphism in $\mathrm{End}(M)$ of the form $\mathrm{Ad}(u) \cdot \gamma$ with a unitary $u \in N$ the canonical endomorphism as well.

Proposition 2.5. *Let M be a type* III *factor and $\rho, \mu \in \mathrm{End}(M)_0$. Then, the following hold:*

1. *$(\rho, \mu) \subset (\widetilde{\rho}, \widetilde{\mu})$.*
2. *$\widetilde{\rho \cdot \mu} = \widetilde{\rho} \cdot \widetilde{\mu}$.*
3. *Let N be a subfactor of M with a finite index and γ be the canonical endomorphism for $M \supset N$. Then, $\widetilde{\gamma}$ is the canonical endomorphism for $\widetilde{M} \supset \widetilde{N}$.*
4. *$\tau \cdot \widetilde{\rho} = d(\rho)\tau$.*

(5) $\widetilde{\rho}\cdot\theta_t=\theta_t\cdot\widetilde{\rho}$.

Proof. In the proof of Theorem 2.4, we have already shown that for $\rho\in\mathrm{End}(M)_0$ and a unitary $u\in M$, we have

$$\widetilde{(\mathrm{Ad}(u)\cdot\rho)}=\mathrm{Ad}(u)\cdot\widetilde{\rho}.$$

Thus, in order to prove the statements we may replace ρ and μ with unitary equivalent endomorphisms. Thanks to Lemma 2.3, (3), we may and do assume that (ψ,ρ) and (ψ,μ) are invariant pairs, where ψ is a dominant weight. Therefore, we have

$$\widetilde{\rho}(\lambda^{\psi}(t))=\widetilde{\mu}(\lambda^{\psi}(t))=\lambda^{\psi}(t).$$

In this situation, (2), (4), and (5) are obvious and (1) follows from Lemma 2.3, (5).

To prove (3), we consider the common continuous decomposition of $M\supset N$. We may and do assume that ψ is of the form $\psi_0\cdot E$, where ψ_0 is a dominant weight on N and E is the minimal expectation from M to N. Then, we have the common continuous decomposition

$$M=M_{\psi}\rtimes_{\theta_0}\mathbb{R}\supset N=N_{\psi_0}\rtimes_{\theta_0}\mathbb{R}.$$

Let τ_0 be the trace whose dual weight is ψ. Then, thanks to [49, Section 4], the canonical endomorphism of $M_{\psi}\supset N_{\psi_0}$ scales τ_0 by $[M:N]_0$. Let $J_{M_{\psi}}$ and $J_{N_{\psi_0}}$ be the modular conjugations of M_{ψ} and N_{ψ_0}, and γ_0 be the corresponding canonical endomorphism of $M_{\psi}\supset N_{\psi_0}$. To define the canonical endomorphism of $M\supset N$ and $\widetilde{M}\supset\widetilde{N}$, we utilize the modular objects of these algebras naturally coming from the modular objects of M_{ψ} and N_{ψ_0} through the crossed products as computed in [18, Lemma 2.8]. Let γ and γ_1 be such canonical endomorphisms of $M\supset N$ and $\widetilde{M}\supset\widetilde{N}$ respectively. Then, (ψ,γ) is an invariant pair [31, Section 3]. On the other hand, applying [31, Lemma 3.1] twice and using [18, Lemma 2.8] for the computation of Δ_{ψ} and Δ_{ψ_0}, we know that γ_1 is an extension of γ leaving $\lambda^{\psi}(t)$ invariant. Therefore, γ_1 coincides with the canonical extension of γ. $\square$

3. Modular endomorphisms

Let M be a type III factor and α be an automorphism of M. In [22, Proposition 5.4], it was shown that $\widetilde{\alpha}$ is inner if and only if α is the composition of an inner automorphism with an extended modular automorphism. We adopt this property as the definition of an endomorphism counterpart of an extended modular automorphism.

Definition 3.1. Let M be a type III factor and $\rho\in\mathrm{End}(M)_0$. We say that ρ is a *modular endomorphism* if $\widetilde{\rho}$ is an inner endomorphism, that is, there exist isometries $\{V_i\}_{i=1}^n\subset\widetilde{M}$ with mutually orthogonal ranges and $\sum_{i=1}^n V_iV_i^*=1$ such that

$$\widetilde{\rho}(x)=\sum_{i=1}^n V_ixV_i^*,\quad x\in\widetilde{M}.$$

The system of isometry $\{V_i\}_{i=1}^n$ satisfying the above condition is called an *implementing system* for $\widetilde{\rho}$.

We denote by $\mathrm{End}(M)_m$ the set of modular endomorphisms of M. Thanks to Proposition 2.5, (1), an endomorphism unitary equivalent to a modular endomorphism is again a modular endomorphism, and so it makes sense to call a sector modular if its representatives are modular endomorphisms. We denote by $\mathrm{Sect}(M)_m$ the set of sectors of modular endomorphisms. It is also easy to see from Proposition 2.5, (1), (2) that $\mathrm{End}(M)_m$ is closed under forming a direct sum and a product of finitely many elements. What is not clear for the moment is whether $\mathrm{End}(M)_m$ is closed under conjugation and irreducible decomposition, which will be shown later. The number n of the isometries of the implementing system in Definition 3.1 is nothing but $d(\rho)$, which also will be shown later.

Let M be as above and $Z(\widetilde{M})$ be the center of $\widetilde{M}$. Then, the restriction of θ to $Z(\widetilde{M})$ is an ergodic action of the real number group $\mathbb{R}$. Thus, by Mackey's point realization theorem [54], there exist a standard Borel space X_M, a probability measure μ_M on X_M, and an ergodic flow F^M on (X_M, μ_M) such that $Z(\widetilde{M}) = L^\infty(X_M, \mu_M)$ and $\theta_t(f)(\omega) = f(F^M_{-t}\omega)$ for $f \in L^\infty(X_M, \mu_M)$, $\omega \in X_M$, $t \in \mathbb{R}$. (For simplicity, we often omit F^M and denotes $F^M_{-t}\omega$ just by $\omega \cdot t$). F^M is called the smooth flow of weights [11].

Let K be a compact group. A Borel map $c : X_M \times \mathbb{R} \longrightarrow K$ is said to be a cocycle of F^M if for fixed $s, t \in \mathbb{R}$, the cocycle relation

$$c(\omega, s)c(\omega \cdot s, t) = c(\omega, s + t)$$

holds for almost all $\omega \in X_M$. Two cocycles agree on the outside of a null set are identified as usual. We denote by $Z^1(F^M, K)$ the set of all K-valued cocycles of F^M. Two cocycles c and c' are said to be equivalent (or cohomologous) if there exists a Borel map $a : X_M \longrightarrow K$ such that for fixed $t \in \mathbb{R}$

$$c(\omega, t) = a(\omega)c'(\omega, t)a(\omega \cdot t)^{-1}$$

holds for almost all $\omega \in X_M$. A cocycle equivalent to a constant function $e \in K$ is said to be a coboundary. We denote by $H^1(F^M, K)$ the set of equivalence classes of K-valued cocycles of F^M. For the significance of cocycles from the viewpoint of Mackey's notion of virtual groups, readers are refered to [56] and [80].

As Connes and Takesaki showed that the group of extended modular automorphisms divided by the inner automorphism group is isomorphic to $H^1(F^M, U(1))$ [11], we show that $\mathrm{Sect}(M)_m$ is "isomorphic" to the union $\bigcup_{n=1}^\infty H^1(F^M, U(n))$ of the unitary group $U(n)$-valued cohomology classes.

Let $\rho \in \mathrm{End}(M)_m$ and $\{V_i\}_{i=1}^n$ be an implementing system for $\widetilde{\rho}$. Since $(\mathrm{id}, \widetilde{\rho})$ is globally preserved by θ, $c(t)_{ij} := V_i^*\theta_t(V_j)$ belongs to $Z(\widetilde{M})$. $c(t) = (c(t)_{ij})$ is regarded as a $U(n)$-valued Borel function on $X_M \times \mathbb{R}$.

Lemma 3.2. *Let ρ and $c(t)$ be as above. Then,*

(1) *c is a $U(n)$-valued cocycle of F^M, whose cohomology class depends only on the sector of ρ.*
(2) *Let c' be a cocycle equivalent to c. Then, there exists an implementing system $\{V_i'\}_{i=1}^n$ for $\widetilde{\rho}$ such that $c'(t)_{ij} = V_i'^*\theta_t(V_j')$.*

Proof. (1). The cocycle relation of c follows from

$$\sum_{j=1}^{n} c(s)_{ij}\theta_s(c(t)_{jk}) = \sum_{j=1}^{n} V_i^*\theta_s(V_jV_j^*\theta_t(V_k)) = c(s+t)_{ik}.$$

Let $\{W_i\}_{i=1}^m \subset \widetilde{M}$ be another implementing system for $\widetilde{\rho}$, and set $a_{ij} = V_i^*W_j$. Since $a(t) = (a(t)_{ij})$ is a matrix valued function that is unitary, we have $n = m$. Setting $b(t)_{ij} = W_i^*\theta_t(W_j)$, we get

$$c(t)_{ij} = V_i^*\theta_t(V_j) = \sum_{k=1,l}^{n} V_i^*W_kW_k^*\theta_t(W_lW_l^*V_j) = \sum_{k=1,l}^{n} a_{ik}b(t)_{kl}\theta_t(a_{jl}^*),$$

which shows that b and c are equivalent. Let u be a unitary of M and $\rho_1 = \mathrm{Ad}(u)\cdot\rho$. Then, $\{uV_i\}_{i=1}^n$ is an implementing system for $\widetilde{\rho}_1$. Since u is fixed by θ_t, ρ_1 gives the same cohomology class as ρ.

(2). We take a $U(n)$-valued function f satisfying

$$c(t)_{ij} = \sum_{k,l=1}^{n} f_{ik}c'(t)_{kl}\theta_t(f_{jl}^*).$$

Then, $V_i' := \sum_{j=1}^n f_{ji}V_j$ has a desired property. □

We introduce a map

$$\delta_m : \mathrm{Sect}(M)_m \longrightarrow \bigcup_{n=1}^{\infty} H^1(F^M, U(n))$$

sending $[\rho]$ to $[c]$ in the above lemma.

As described in [80, Section 2], $Z^1(F^M, U(n))$ has natural three operations in analogous to the unitary representation theory of compact groups: direct sum $c \oplus c'$, tensor product $c \otimes c'$ and complex conjugate $\overline{c}$. $H^1(F^M, U(n))$ inherits these operations and we use the same notation for the cohomology classes as well.

Theorem 3.3. *Let δ_m be as above and $\rho_1, \rho_2 \in \mathrm{End}(M)_m$. Then,*

1. *δ_m is a bijection.*
2. *We have $\delta_m([\rho_1] \oplus [\rho_2]) = \delta_m([\rho_1]) \oplus \delta_m([\rho_2])$.*
3. *We have $\delta_m([\rho_1][\rho_2]) = \delta_m([\rho_1]) \otimes \delta_m([\rho_2])$.*
4. *$[\rho_1]$ is the conjugate sector of $[\rho_2]$ if and only if $\delta_m([\rho_1]) = \overline{\delta_m([\rho_2])}$. In particular, $\mathrm{Sect}(M)_m$ is closed under conjugation.*
5. *δ_m is grade preserving in the sense that $\delta_m([\rho]) \in H^1(F^M, U(n))$ if and only if $d(\rho) = n$.*

Proof. (1). First we show that δ_m is injective. Let $\rho_1, \rho_2 \in \mathrm{End}(M)_m$ satisfying $\delta_m([\rho_1]) = \delta_m([\rho_2])$. Then, thanks to Lemma 3.2, (2), there exist implementing systems $\{V_i^{(1)}\}_{i=1}^n$ and $\{V_i^{(2)}\}_{i=1}^n$ for $\widetilde{\rho}_1$ and $\widetilde{\rho}_2$ respectively such that they give the same cocycle. We set $u = \sum_{i=1}^n V_i^{(1)}V_i^{(2)*}$, which is a unitary in $\widetilde{M}^\theta = M$ satisfying $\rho_1 = \mathrm{Ad}(u)\cdot\rho_2$. This shows that δ_m is injective.

Next we show that δ_m is surjective. Let ψ be a dominant weight on M and $M = M_\psi \rtimes_{\theta_0} \mathbb{R}$ be the continuous decomposition. We take the trace τ_0 on M_ψ whose dual weight is ψ, and take the implementing one-parameter unitary group $\{u(s)\}_{s\in\mathbb{R}}$ for θ_0. We assume that M_ψ acts on a Hilbert space H. Then, the Takesaki duality theorem [58, Chapter I] implies that there exists an isomorphism Φ from $\widetilde{M}$ to $M_\psi \otimes B(L^2(\mathbb{R}))$ such that

$$\Phi(x) = \pi_{\theta_0}(x), \quad x \in M_\psi,$$
$$\Phi(u(s)) = 1 \otimes l(s),$$
$$\Phi(\lambda^\psi(t)) = 1 \otimes m(t),$$

where $l(s)$ is the left regular representation of $\mathbb{R}$ and $m(t)$ is the multiplication operator of e^{-its}. Moreover, θ_t is identified with $(\theta_{0,t} \otimes \mathrm{Ad}(r(t)))$ under Φ, or more precisely we have $\Phi \cdot \theta_t = (\theta_{0,t} \otimes \mathrm{Ad}(r(t))) \cdot \Phi$. Using Φ, we identify the flow $(Z(\widetilde{M}), \theta)$ with

$$(Z(M_\psi \otimes B(L^2(\mathbb{R}))), \theta_0 \otimes \mathrm{Ad}(r)) \cong (Z(M_\psi), \theta_0).$$

Let $c \in Z^1(F^M, U(n))$ be a given cocycle, where $L^\infty(X_M, \mu_M)$ is understood as the point realization of $Z(M_\psi)$. We take a system of isometries $\{W_i\}_{i=1}^n$ in M_ψ satisfying $\sum_{i=1}^n W_i W_i^* = 1$ and set

$$W(t) = \sum_{i,j=1}^n c(t)_{ij} W_i \theta_{0,t}(W_j^*) \in M_\psi.$$

Then, $W(t)$ is a unitary satisfying

$$\begin{aligned} W(s+t) &= \sum_{i,j,k=1}^n c(s)_{ik} \theta_{0,s}(c(t)_{kj}) W_i \theta_{0,s+t}(W_j^*) \\ &= \sum_{i,j,k=1}^n c(s)_{ik} W_i \theta_{0,s}(W_k^* c(t)_{lj} W_l \theta_{0,t}(W_j^*)) \\ &= W(s)\theta_{0,s}(W(t)), \end{aligned}$$

which means that $W(t)$ is a θ_0-cocycle. Since θ_0 is stable [11, Theorem III.5.1], every θ_0-cocycle is a coboundary and there exists a unitary w such that $W(t) = w^*\theta_{0,t}(w)$. Let $V_i = wW_i$. Then, V_i is an isometry in M_ψ satisfying $\sum_{i=1}^n V_i V_i^* = 1$ and

$$\theta_{0,t}(V_i) = \sum_{j=1}^n c(t)_{ji} V_j.$$

For $x \in M_\psi$ we set,

$$\rho_0(x) = \sum_{i=1}^n V_i x V_i^*,$$
$$\phi_0(x) = \frac{1}{n} \sum_{i=1}^n V_i^* x V_i,$$

and $E_0 = \rho_0 \cdot \phi_0$. Then, ρ_0 is an endomorphism of M_ψ commuting with $\theta_{0,t}$, ϕ_0 is a left inverse of ρ_0 satisfying $n\tau_0 \cdot \phi_0 = \tau_0$, and E_0 is a τ_0-preserving conditional expectation from M_ψ onto the image of ρ_0. We introduce $\rho \in \mathrm{End}(M)$ that is an extension of ρ_0 to M leaving $u(t)$ invariant. Indeed, such ρ exists because the following hold:

$$\Phi(\rho_0(x)) = \sum_{i=1}^{n} (V_i \otimes 1)\Phi(x)(V_i^* \otimes 1), \quad x \in M_\psi,$$

$$\Phi(u(t)) = \sum_{i=1}^{n} (V_i \otimes 1)\Phi(u(t))(V_i^* \otimes 1).$$

In a similar way, we define a left inverse ϕ of ρ and a conditional expectation E from M onto the image of ρ by the relations

$$\Phi(\phi(x)) = \frac{1}{n}\sum_{i=1}^{n} (V_i^* \otimes 1)\Phi(x)(V_i \otimes 1), \quad x \in M,$$

and $E = \rho \cdot \phi$. Note that ρ, ϕ, and E are extensions of ρ_0, ϕ_0, and E_0 respectively.

We claim that $[M : \rho(M)]_0 = n^2$, $\phi = \phi_\rho$, $E = E_\rho$, and (ψ, ρ) is an invariant pair. Indeed, it suffices to show Index $E = [M : \rho(M)]_0 = n^2$, and the other claims can be shown easily. We define $T \in M_\psi$ by

$$T = \sqrt{n} \sum_{i=1}^{n} V_i V_i.$$

Then, for $x \in M_\psi$ and $u(t)$, we have

$$\begin{aligned} T^* E(Txu(t)) &= T^* E_0(Tx)u(t) = \frac{1}{n}\sum_{i,j=1}^{n} T^* V_i V_j^* T x V_j V_i^* u(t) \\ &= \sum_{i,j=1}^{n} V_i^* V_j x V_j V_i^* u(t) = xu(t), \end{aligned}$$

which shows $\{T\}$ is a one-element Pimsner-Popa basis and Index $E = T^*T = n^2$ [60], [78]. To show that E is minimal, it suffices to show $T^*xT = n^2 E(x)$ for all $x \in M \cap \rho(M)'$ [78, Theorem 2.12.3]. Thanks to the Connes-Takesaki relative commutant theorem [11, Theorem II.5.1], for $x \in M \cap \rho(M_\psi)'$ we have

$$x_{ij} := V_i^* x V_j \in M \cap M_\psi' = Z(M_\psi),$$

and

$$x = \sum_{i,j=1}^{n} V_i x_{ij} V_j^*.$$

Thus,

$$T^* x T = n \sum_{i=1}^{n} x_{ii} = n^2 E(x),$$

which shows the claims.

Since (ψ, ρ) is an invariant pair, $\widetilde{\rho}$ is an extension of ρ leaving $\lambda^{\psi}(t)$ invariant. Therefore, we have

$$\Phi(\widetilde{\rho}(x)) = \sum_{i=1}^{n} (V_i \otimes 1)\Phi(x)(V_i^* \otimes 1), \quad x \in \widetilde{M},$$

which shows that ρ is a modular endomorphism carrying c.

(2) and (3) are easy, and (5) follows from the above proof.

(4). Assume that $\delta_m([\rho_1]) = [c]$ and $\delta_m([\rho_2]) = [\overline{c}]$. Let $\{V_i\}_{i=1}^n$ and $\{W_i\}_{i=1}^n$ be implementing systems for $\widetilde{\rho}_1$ and $\widetilde{\rho}_2$ satisfying

$$\theta_t(V_i) = \sum_{j=1}^{n} c(t)_{ji} V_j,$$

$$\theta_t(W_i) = \sum_{j=1}^{n} \overline{c}(t)_{ji} W_j.$$

We define two isometries R and S by

$$R = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} V_i W_i,$$

$$S = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} W_i V_i.$$

R and S belong to $\widetilde{M}^{\theta} = M$ satisfying

$$R \in (\mathrm{id}, \rho_1 \cdot \rho_2)$$

$$S \in (\mathrm{id}, \rho_2 \cdot \rho_1)$$

$$S^* \rho_2(R) = R^* \rho_1(S) = \frac{1}{n}.$$

Thus, $[\rho_1] = \overline{[\rho_2]}$ [50]. Since δ_m is a bijection, this also shows that $[\rho] \in \mathrm{Sect}(M)_m$ implies $\overline{[\rho]} \in \mathrm{Sect}(M)_m$ and $\delta_m(\overline{[\rho]}) = \overline{\delta_m([\rho])}$. □

The following is a useful criterion for an irreducible endomorphism to be modular.

Proposition 3.4. *Let M be a type* III *factor.*

(1) *Let $\rho \in \mathrm{End}(M)_0$ be an irreducible endomorphism. If there exists a non-zero element in $(\mathrm{id}, \widetilde{\rho})$, ρ is a modular endomorphism.*
(2) *Every irreducible component of a modular endomorphism is again a modular endomorphism. In particular $\mathrm{Sect}(M)_m$ is closed under irreducible decomposition.*

Proof. (1). First, we show that $(\mathrm{id}, \widetilde{\rho})$ contains at least one isometry if $(\mathrm{id}, \widetilde{\rho}) \neq \{0\}$. Let $\mathcal{PI}(\mathrm{id}, \widetilde{\rho})$ be the set of partial isometries in $(\mathrm{id}, \widetilde{\rho})$. Using the polar decomposition, we know that $\mathcal{PI}(\mathrm{id}, \widetilde{\rho})$ is not empty. We introduce an order into $\mathcal{PI}(\mathrm{id}, \widetilde{\rho})$ as follows: For two element $v_1, v_2 \in \mathcal{PI}(\mathrm{id}, \widetilde{\rho})$, we say that v_2 dominates v_1 if the following holds

$$v_1 = v_2 v_1^* v_1.$$

Note that when this is the case, $v_1^*v_1 \in Z(\widetilde{M})$, and $v_2^*v_1 = v_1^*v_1$, $v_2v_1^* = v_1v_1^*$ hold. We show that $\mathcal{PI}(\mathrm{id},\widetilde{\rho})$ is an inductively ordered set with this order. Let $\mathcal{T}$ be a totally ordered subset of $\mathcal{PI}(\mathrm{id},\widetilde{\rho})$. Since $\{vv^*\}_{v\in\mathcal{T}}$ and $\{v^*v\}_{v\in\mathcal{T}}$ are increasing nets of projections in $Z(\widetilde{M})$ and $\widetilde{M}$, they converge to projections in $Z(\widetilde{M})$ and in $\widetilde{M}$ respectively in strong topology. We claim that the net $\{v\}_{v\in\mathcal{T}}$ converges in strong $*$ topology. Let $v_1, v_2 \in \mathcal{T}$ with v_2 dominating v_1, and ξ be a vector in the Hilbert space that $\widetilde{M}$ acts on. Then,

$$||(v_2 - v_1)\xi||^2 = \langle (v_2^*v_2 - v_1^*v_1)\xi, \xi\rangle,$$

$$||(v_2^* - v_1^*)\xi||^2 = \langle (v_2v_2^* - v_1v_1^*)\xi, \xi\rangle.$$

Therefore, the strong $*$ limit exists, and it gives a majorant of $\mathcal{T}$, which shows that $\mathcal{PI}(\mathrm{id},\widetilde{\rho})$ is inductively ordered. Now, we apply Zorn's lemma to $\mathcal{PI}(\mathrm{id},\widetilde{\rho})$ and take a maximal element $V \in \mathcal{PI}(\mathrm{id},\widetilde{\rho})$. We show that V is an isometry. Suppose $V^*V \neq 1$. Since $V^*V \in Z(\widetilde{M})$ and θ acts on $Z(\widetilde{M})$ ergodically, there exists some $t \in \mathbb{R}$ such that $\theta_t(V^*V)(1 - V^*V) \neq 0$. Therefore, $V_1 := V + \theta_t(V)(1 - V^*V) \in \mathcal{PI}(\mathrm{id},\widetilde{\rho})$ dominates V, which is contradiction. Thus, V is an isometry.

Let $\{V_\lambda\}_{\lambda\in\Lambda}$ be a maximal set of isometries in $(\mathrm{id},\widetilde{\rho})$ with mutually orthogonal ranges (such a set exists thanks to Zorn's lemma again.), and let $P = \sum_{\lambda\in\Lambda} V_\lambda V_\lambda^*$. If $P = 1$, we are done, and so we assume $P \neq 1$. Let $\mathcal{PI}_{P^\perp}(\mathrm{id},\widetilde{\rho})$ be the subset of elements in $\mathcal{PI}(\mathrm{id},\widetilde{\rho})$ with range projections orthogonal to P. We claim that there exists a non-zero element in $\mathcal{PI}_{P^\perp}(\mathrm{id},\widetilde{\rho})$. Indeed, since $(\widetilde{\rho},\widetilde{\rho})^\theta = (\rho,\rho) = \mathbb{C}1$, there exists some $t \in \mathbb{R}$ such that $(1-P)\theta_t(P) \neq 0$, which shows that there exists non-zero $x \in (\mathrm{id},\widetilde{\rho})$ with $(1 - P)x \neq 0$. Using the polar decomposition, we get the claim. We introduce an order into $\mathcal{PI}_{P^\perp}(\mathrm{id},\widetilde{\rho})$ as before and take a maximal element W. Since $\{V_\lambda\}_{\lambda\in\Lambda}$ is a maximal set, W is not an isometry, and we set $W^*W = z \neq 1$. We claim $(1-P)(1-z) = 0$. Indeed, by maximality of W, $(1-P)(1-z)x = (1-P)x(1-z) = 0$ holds for $x \in (\mathrm{id},\widetilde{\rho})$, and $\theta_t((1 - P)(1 - z))x = 0$ for $x \in (\mathrm{id},\widetilde{\rho})$ as well for all $t \in \mathbb{R}$ because $(\mathrm{id},\widetilde{\rho})$ is globally invariant under θ. Thus, we get

$$(\bigvee_{t\in\mathbb{R}} \theta_t((1 - P)(1 - z)))x = 0, \quad x \in (\mathrm{id},\widetilde{\rho}).$$

Since $(\widetilde{\rho},\widetilde{\rho})^\theta = \mathbb{C}1$, this implies $(1 - P)(1 - z) = 0$.

Let $V_\lambda' = V_\lambda(1 - z)$. Then, $\{V_\lambda'\}_{\lambda\in\Lambda}$ have mutually orthogonal ranges such that

$$\sum_{\lambda\in\Lambda} V_\lambda' V_\lambda'^* = P(1 - z) = (1 - z).$$

Thus, we get

$$(\widetilde{M}(1 - z) \supset \widetilde{\rho}(\widetilde{M})(1 - z)) \cong ((B(\ell^2(\Lambda)) \otimes \widetilde{M}(1 - z)) \supset (\mathbb{C}1 \otimes \widetilde{M}(1 - z))).$$

Since there exists a conditional expectation from $\widetilde{M}$ to $\widetilde{\rho}(\widetilde{M})$ satisfying the Pimsner-Popa inequality [27], [60], Λ is a finite set, and we identify Λ with $\{1, 2, \cdots, n\}$. We take $t \in \mathbb{R}$ satisfying $z_1 := \theta_t(z)(1 - z) \neq 0$, and set $\Omega \subset X_M$ to be the Borel subset corresponding z_1. Let $R_i = V_i' z_1$, $S_i = \theta_t(V_i) z_1$, $i = 1, 2, \cdots n$, and $S_{n+1} = \theta_t(W) z_1$. Then, $\{R_i\}_{i=1}^n \subset (\mathrm{id}, \widetilde{\rho})$ and $\{S_i\}_{i=1}^{n+1} \subset (\mathrm{id}, \widetilde{\rho})$ satisfy

$$R_i^* R_j = \delta_{i,j} z_1, \quad i, j = 1, 2, \cdots n,$$

$$\sum_{i=1}^n R_i R_i^* = z_1,$$

$$S_i^* S_j = \delta_{i,j} z_1, \quad i, j = 1, 2, \cdots, n+1,$$

$$\sum_{i=1}^{n+1} S_i S_i^* \leq z_1.$$

We set $f = (f_{ij}) = (R_i^* S_j)$, which is regarded as an n by $n+1$ matrix-valued function on Ω. However f satisfies

$$\sum_{i=1}^n f_{ij}^* f_{ik} = \delta_{j,k} z_1, \quad j, k = 1, 2, \cdots, n+1,$$

which is contradiction. Thus, we conclude $P = 1$, and

$$\widetilde{\rho}(x) = \sum_{\lambda \in \Lambda} V_\lambda x V_\lambda^*.$$

(2). Let ρ be a modular endomorphism with an implementing system $\{V_i\}_{i=1}^n$ for $\widetilde{\rho}$, and σ be an irreducible component of ρ. We take a non-zero element $T \in (\sigma, \rho)$. Thanks to Proposition 2.5, (1), $T^* V_i$ belongs to $(\mathrm{id}, \widetilde{\sigma})$, $i = 1, 2, \cdots, n$. Since $T^* = \sum_{i=1}^n T^* V_i V_i^*$, there exists some i such that $T^* V_i$ is non-zero. Thus, σ is a modular endomorphism thanks to (1). □

Though it is shown in the above that $\mathrm{Sect}(M)_m$ is closed under irreducible decomposition, we have not given a criterion for a modular endomorphism to be irreducible, or that of how to decompose it when reducible, in terms of ergodic theory yet. We discuss this issue now.

Let X be a standard Borel space, μ be a probability measure on X, and G be a locally compact group ergodically acting on (X, μ) as a non-singular transformation group. For a compact group K, we define a K-valued cocycle c as in the case of the flow. We denote by K_c the closed subgroup generated by the image of c. We collect necessary results on cocycles from R. Zimmer's fundamental paper [80, Part I. Section 3] in the next theorem in order to introduce a few notions for cocycles. Though Zimmer treated only the measure preserving case in [80], his argument works for the non-singular case with a formal modification (see also [15]):

Theorem 3.5 (R. Zimmer). *Let the notations be as above, and $c \in Z^1(G, K)$. Then, the following hold:*

(1) *There exists a closed subgroup $H \subset K$ such that c is equivalent to an H-valued cocycle c', and c' is never equivalent to c'' with $K_{c''}$ a proper subgroup of H. H is uniquely determined up to conjugacy. We call such c' a minimal cocycle and call H the minimal subgroup of c.*

(2) *Let K_1 be a compact group containing K. Then, if c is a minimal cocycle as a K-valued cocycle, it is the case as a K_1-valued cocycle.*
(3) *Let c be a minimal K-valued cocycle such that $K = K_c$, and π be a continuous homomorphism from K to a compact group K_1. Then, $\pi \cdot c$ is a minimal cocycle.*
(4) *Let c be a minimal K-valued cocycle with $K = K_c$, and (π_1, H_{π_1}), (π_2, H_{π_2}) be finite dimensional unitary representations of K. If a is a Borel map from X to $\mathrm{Hom}(H_1, H_2)$ such that for every fixed k,*

$$a(\omega)\pi_2(c(\omega, k)) = \pi_1(c(\omega, k))a(\omega \cdot k),$$

holds for almost all $\omega \in X$, then a is constant almost everywhere with the value in (π_1, π_2).

We say that a closed subgroup $K \subset U(n)$ is irreducible if the defining representation of K on $\mathbb{C}^n$ is irreducible.

Corollary 3.6. *Let M be a type* III *factor and ρ be a modular endomorphism with $\delta_m([\rho]) = [c]$. Then, ρ is irreducible if and only if the minimal subgroup of c is irreducible.*

Proof. Thanks to theorem 3.3, it suffices to show that the minimal subgroup of c is irreducible if and only if c is never equivalent to a direct sum of two cocycles. Assume that the minimal subgroup of c is irreducible. If c were equivalent to $c_1 \oplus c_2$, $c_i \in Z^1(G, U(n_i))$, $i = 1, 2$, with $n_1 + n_2 = n$, c could be considered as a $U(n_1) \times U(n_2)$-valued cocycle, and the minimal subgroup of c would be conjugate to a subgroup of $U(n_1) \times U(n_2)$ in $U(n)$ thanks to Theorem 3.5, (2). This is contradiction and c is never equivalent to a direct sum of two cocycles. The converse also follows from Theorem 3.5, (2). □

We end this section with giving an answer to the problem about the type II and type III principal graphs described in Introduction.

Let $M \supset N$ be an inclusion of type III factors of a finite index with the minimal expectation E. We say that M and N has the common flow of weights if $Z(\widetilde{N}) = Z(\widetilde{M})$. When this is the case, we have common central decomposition

$$(\widetilde{M} \supset \widetilde{N}) = \int_{X_M}^{\oplus} (\widetilde{M}(\omega) \supset \widetilde{N}(\omega)) d\mu_M(\omega),$$

and we could define the type II principal graph for that of $\widetilde{M}(\omega) \supset \widetilde{N}(\omega)$ for ω in a conull set. However, it would be cumbersome to treat "measurable field of subfactors" (though it should not be too hard to do so, c.f. [79].) Instead, we formulate the problem using a global term. Let

$$N \subset M \subset M_1 \subset \cdots M_n \subset \cdots$$

be the tower for $N \subset M$. Then, using the minimal conditional expectation in each step, we have the "tower for the core inclusion" [44]:

$$\widetilde{N} \subset \widetilde{M} \subset \widetilde{M_1} \subset \cdots \widetilde{M_n} \subset \cdots.$$

We still denote by θ the natural extension of θ to $\widetilde{M}_n$ leaving the Jones projections fixed. Then, we have

$$M_n \cap M' = (\widetilde{M}_n \cap \widetilde{M}')^\theta.$$

We say that *graph change occurs* if $Z(\widetilde{M}) \vee (M_n \cap M')$ does not coincides with $\widetilde{M}_n \cap \widetilde{M}'$ for some n. Let E_n be the minimal expectation from M_n onto M_{n-1} and $\widetilde{E}_n$ be the natural extension of E_n to $\widetilde{M}_n$, which is a conditional expectation from $\widetilde{M}_n$ onto $\widetilde{M}_{n-1}$. Then, using E_{n+1} and $\widetilde{E}_{n+1}$, we can show that $Z(\widetilde{M}) \vee (M_n \cap M') \neq \widetilde{M}_n \cap \widetilde{M}'$ implies $Z(\widetilde{M}) \vee (M_{n+1} \cap M') \neq \widetilde{M}_{n+1} \cap \widetilde{M}'$. In a similar way using downward basic construction and mirroring $J_M \cdot J_M$, $J_{\widetilde{M}} \cdot J_{\widetilde{M}}$, we can also show that graph change occurs for $M \supset N$ if and only if it occurs for $M_1 \supset M$.

A modular endomorphism is said to be non-trivial if it is irreducible and not equivalent to identity.

Theorem 3.7. *Let $M \supset N$ be an inclusion of type* III_0 *factors with a finite index and with the common flow of weights. Then, graph change occurs for $M \supset N$ if and only if a non-trivial modular endomorphism appears in some power of the canonical endomorphism γ for $M \supset N$.*

Proof. Assume that graph change occurs for $N \subset M$. Taking sufficiently large n, we may assume $Z(\widetilde{M}) \vee (M_{2n} \cap M') \neq \widetilde{M}_{2n} \cap \widetilde{M}'$. Let γ be the canonical endomorphism for $M \supset N$. Then, thanks to Proposition 2.5, this is equivalent to

$$(\widetilde{\gamma}^n, \widetilde{\gamma}^n) \neq (\gamma^n, \gamma^n) \vee Z(\widetilde{M}).$$

Therefore, either of the following two holds: (i) γ^n contains two irreducibles $\rho_1, \rho_2 \in \mathrm{End}(M)_0$ such that $[\rho_1] \neq [\rho_2]$ and $(\widetilde{\rho}_1, \widetilde{\rho}_2) \neq \{0\}$. (ii) γ^n contains an irreducible $\rho \in \mathrm{End}(M)_0$ such that $(\widetilde{\rho}, \widetilde{\rho}) \neq Z(\widetilde{M})$. Indeed, let $\{p_i\}_{i=1}^n$ be a partition of unity consisting of minimal projections in $M \cap \gamma^n(M)'$. If

$$X \in (\widetilde{\gamma}^n, \widetilde{\gamma}^n) \setminus (\gamma^n, \gamma^n) \vee Z(\widetilde{M}),$$

there exist some i and j such that

$$p_i X p_j \in (\widetilde{\gamma}^n, \widetilde{\gamma}^n) \setminus (\gamma^n, \gamma^n) \vee Z(\widetilde{M}).$$

If the irreducible components corresponding to p_i and p_j are different, (i) occurs. If they are the same, (ii) occurs.

In the case (i), the Frobenius reciprocity implies $(\mathrm{id}, \widetilde{\rho_2 \cdot \overline{\rho}_1}) \neq \{0\}$ (though $\widetilde{M}$ is not a factor, the argument in [33, Section 2] works). Thanks to Lemma 3.3, this shows that $\rho_2 \cdot \overline{\rho}_1$, which is contained in γ^{2n}, contains a modular endomorphism. Since $\rho_2 \cdot \overline{\rho}_1$ does not contain identity, it is indeed a non-trivial one.

In the case (ii), we take $X \in (\widetilde{\rho}, \widetilde{\rho}) \setminus Z(\widetilde{M})$. Let $R \in (\mathrm{id}, \overline{\rho} \cdot \rho)$, $\overline{R} \in (\mathrm{id}, \rho \cdot \overline{\rho})$ be isometries satisfying the usual relation [50]

$$R^* \overline{\rho}(\overline{R}) = \overline{R}^* \rho(R) = \frac{1}{d(\rho)}.$$

Then, $X\overline{R} \in (\mathrm{id}, \widetilde{\rho \cdot \overline{\rho}})$. Let σ be an irreducible sector contained in $\rho \cdot \overline{\rho}$ that is not equivalent to identity, and $\{Y(\sigma)_i\}_{i=1}^k$ be a basis of $(\sigma, \rho \cdot \overline{\rho})$. If $Y(\sigma)_i^* X \overline{R} \in (\mathrm{id}, \widetilde{\sigma})$ is not zero for some i, we are done thanks to Lemma 3.3. Thus, we assume $Y(\sigma)_i^* X\overline{R} = 0$ for every σ not equivalent to identity. This would imply that $\overline{R}\overline{R}^* X\overline{R} = X\overline{R}$. However, since $\rho(R)$ commutes with X, we would get

$$X = d(\rho)\rho(R^*)X\overline{R} = d(\rho)\rho(R^*)\overline{R}\overline{R}^* X\overline{R} = \overline{R}^* X\overline{R} \in Z(\widetilde{M}),$$

which is contradiction. Thus, $\rho\cdot\overline{\rho}$ contains a non-trivial modular endomorphism, and so does γ^{2n}.

The converse can be shown easily. □

Remark 3.8. For a type III_λ, $0 < \lambda \leq 1$ factor, the flow is a transitive and every $U(n)$-valued cocycle is equivalent to a direct sum of $U(1)$-valued cocycles that come from homomorphisms from the stabilizer subgroup of a point [56], $\mathbb{Z}$ for $\lambda \neq 1$ and $\mathbb{R}$ for $\lambda = 1$. This means that every modular endomorphism is decomposed into usual modular automorphisms. Therefore, the above theorem generalizes the previous one obtained in [31, Theorem 3.5].

There exist plenty of examples in the type III_0 case, where higher dimensional non-trivial modular endomorphisms appear in the canonical endomorphisms (see Section 5 and Proposition A.5).

4. Connes-Takesaki modules

For a type III factor M, we denote by $\mathrm{Aut}(F^M)$ the set of automorphisms of $Z(\widetilde{M})$ that commute with the restriction of θ to $Z(\widetilde{M})$. For an automorphism α of M, the restriction of $\widetilde{\alpha}$ to $Z(\widetilde{M})$ belongs to $\mathrm{Aut}(F^M)$, which is called the Connes-Takesaki module of α. This correspondence gives a homomorphism

$$\mathrm{mod} : \mathrm{Aut}(M) \longrightarrow \mathrm{Aut}(F^M),$$

which is called the fundamental homomorphism. (Though the original definition of Connes and Takesaki looks different from ours [11], this is an equivalent description of it.) The purpose of this section is to introduce the Connes-Takesaki module for endomorphisms.

Definition 4.1. Let M be a type III factor and $\rho \in \mathrm{End}(M)_0$. We say that ρ *has a Connes-Takesaki module* if $M \supset \rho(M)$ has the common flow of weights. When ρ has a Connes-Takesaki module, we denote by $\mathrm{mod}(\rho)$ the restriction of $\widetilde{\rho}$ to $Z(\widetilde{M})$. $\mathrm{mod}(\rho) \in \mathrm{Aut}(F^M)$ is called the *Connes-Takesaki module of* ρ.

We denote by $\mathrm{End}(M)_{CT}$ (respectively $\mathrm{End}(M)_{tr}$) the set of endomorphisms in $\mathrm{End}(M)_0$ with Connes-Takesaki modules (respectively trivial Connes-Takesaki modules). Note that $\mathrm{mod}(\rho)$ depends only on the sector of ρ. We denote by $\mathrm{Sect}(M)_{CT}$ and $\mathrm{Sect}(M)_{tr}$ the corresponding subsets of $\mathrm{Sect}(M)_0$.

Proposition 4.2. *Let M be a type* III *factor. Then,*

(1) *If $\rho_1, \rho_2 \in \mathrm{End}(M)_0$ have Connes-Takesaki modules, then so does $\rho_1 \cdot \rho_2$ and* $\mathrm{mod}(\rho_1 \cdot \rho_2) = \mathrm{mod}(\rho_1) \cdot \mathrm{mod}(\rho_2)$.

(2) *Let $\rho, \rho_i \in \mathrm{End}(M)_0$, $i = 1, 2, \cdots, n$, satisfying $[\rho] = \oplus_{i=1}^n [\rho_i]$. Then, ρ has a Connes-Takesaki module if and only if each ρ_i does and $\mathrm{mod}(\rho_i)$ does not depend on i. When it is the case, $\mathrm{mod}(\rho) = \mathrm{mod}(\rho_i)$.*

(3) *$\rho \in \mathrm{End}(M)_0$ has a Connes-Takesaki module if and only if the conjugate $\overline{\rho}$ does. When it is the case, $\mathrm{mod}(\overline{\rho}) = \mathrm{mod}(\rho)^{-1}$.*

(4) *Every modular endomorphism has a trivial Connes-Takesaki module.*

(5) *Let $N \subset M$ be a subfactor with a finite index. If M and N have the common flow of weights, then the canonical endomorphism γ for $M \supset N$ has a trivial Connes-Takesaki module.*

Proof. (1). This is trivial.

(2). We take isometries $V_i \in (\rho_i, \rho)$ satisfying

$$\rho(x) = \sum_{i=1}^{n} V_i \rho_i(x) V_i^*, \quad x \in M.$$

Assume $\rho \in \mathrm{End}(M)_{CT}$. Thanks to Lemma 2.4, (1), we have

$$\widetilde{\rho}_i(z) = V_i^* \widetilde{\rho}(z) V_i = \widetilde{\rho}(z), \quad z \in Z(\widetilde{M}),$$

which shows that $\rho_i \in \mathrm{End}(M)_{CT}$ for every i and $\mathrm{mod}(\rho_i)$ does not depend on i. Next we assume that $\rho_i \in \mathrm{End}(M)_{CT}$ for every i and $\mathrm{mod}(\rho_i)$ does not depend on i, which is denoted by ν. Then,

$$\widetilde{\rho}(z) = \sum_{i=1}^{n} V_i \widetilde{\rho}_i(z) V_i^* = \sum_{i=1}^{n} \nu(z) V_i V_i^* = \nu(z), \quad z \in Z(\widetilde{M}),$$

which shows $\rho \in \mathrm{End}(M)_{CT}$.

(3). Since $M \supset \overline{\rho}(M)$ is the dual inclusion of $M \supset \rho(M)$, the first part is obvious. The second part follows from (1) and (2) because $\rho \cdot \overline{\rho}$ contains identity.

(4). This follows from the definition of modular endomorphisms.

(5). This follows from Proposition 2.5, (3). □

Let $M \supset N$ be an inclusion of type III_λ, $0 < \lambda < 1$ factors with a finite index, and E be the minimal expectation from M onto N. We say that $M \supset N$ has common discrete decomposition if there exist an inclusion of type II_∞ factors $P \supset Q$ with a finite index, a minimal expectation E_0 from P to Q preserving the trace, and a trace scaling automorphism θ_0 of P globally preserving Q such that

$$(M \supset N) \cong (P \rtimes_{\theta_0} \mathbb{Z} \supset Q \rtimes_{\theta_0} \mathbb{Z}).$$

Note that in [47], common discrete decomposition with respect to any faithful normal expectation is discussed, while here we consider the minimal expectation only.

The following lemma is standard and we omit the proof.

Lemma 4.3. *Let $M \supset N$ be an inclusion of type III_λ, $0 < \lambda < 1$ factors with a finite index, and E be the minimal expectation from M to N. Then, the following are equivalent:*

(1) *$M \supset N$ has common discrete decomposition.*

(2) *$M \supset N$ has the common flow of weights.*

(3) *For some (and any) generalized trace* ψ_0 *on* N *(see* [5] *for the definition, it is also called* λ*-trace in* [67]*),* $\psi_0 \cdot E$ *is a generalized trace on* M*.*

Corollary 4.4. *Let* M *be a type* III_λ*,* $0 < \lambda < 1$*, factor, and* $\rho \in \mathrm{End}(M)_0$*. Then,*

(1) *If* ρ *is irreducible,* ρ *has a Connes-Takesaki module.*

(2) $M \supset \rho(M)$ *has common discrete decomposition if and only if* ρ *has a Connes-Takesaki module, which is further equivalent to that every irreducible component of* ρ *has the same Connes-Takesaki module.*

Proof. (1). This follows from the condition (3) of Lemma 4.3. (2). This follows from Proposition 4.2, (2) and Lemma 4.3. □

Let M be a type III_λ factor, $0 < \lambda < 1$, and ψ be a generalized trace on M. We set $T = -2\pi/\log\lambda$. Then, the modular automorphism group $\{\sigma_t^\psi\}_{t\in\mathbb{R}}$ has period T and we consider σ^ψ an action of $\mathbb{T} := \mathbb{R}/T\mathbb{R}$. We denote by $\widehat{M}$ the crossed product $M \rtimes_{\sigma^\psi} \mathbb{T}$, which is a type II_∞ factor generated by a copy of M and the implementing unitary representation $\{\lambda_d^\psi(t)\}$ of $\mathbb{T}$. If φ is another weight on M satisfying $[D\varphi : D\psi]_T = 1$, $\{[D\varphi : D\psi]_t\}$ is a σ^ψ-cocycle as a $\mathbb{T}$ action, and we identify $[D\varphi : D\psi]_t\lambda_d^\psi(t)$ with $\lambda_d^\varphi(t)$. We denote by τ_d and θ_d the natural trace on $\widehat{M}$ and the dual action of σ^ψ respectively. In analogous to the canonical extension, we can construct a functor from $\mathrm{End}(M)_{tr}$ to $\mathrm{End}(\widehat{M})_0$. Note that thanks to Proposition 4.2, $\mathrm{End}(M)_{tr}$ is a full subcategory of $\mathrm{End}(M)_0$.

Proposition 4.5. *Let* M *be a type* III_λ *factor and* ψ *be a generalized trace. Then, for every* $\rho \in \mathrm{End}(M)_{tr}$*, there exists a unique endomorphism* $\widehat{\rho} \in \mathrm{End}(\widehat{M})$ *satisfying*

$$\widehat{\rho}(x) = \rho(x), \quad x \in M$$

$$\widehat{\rho}(\lambda_d^\psi(t)) = d(\rho)^{it}[D\psi \cdot \phi_\rho : D\psi]_t\lambda_d^\psi(t).$$

Moreover, for $\rho, \mu \in \mathrm{End}(M)_{tr}$*, the following hold:*

(1) $(\rho, \mu) \subset (\widehat{\rho}, \widehat{\mu})$.

(2) $\widehat{\rho\cdot\mu} = \widehat{\rho} \cdot \widehat{\mu}$.

(3) *Let* N *be a subfactor of* M *with a finite index and* γ *be the canonical endomorphism for* $M \supset N$*. If* $M \supset N$ *has common discrete decomposition, then,* $\widehat{\gamma}$ *is the canonical endomorphism for* $\widehat{M} \supset \widehat{N}$*.*

(4) $\tau_d \cdot \widehat{\rho} = d(\rho)\tau_d$.

(5) $\widehat{\rho} \cdot \theta_d = \theta_d \cdot \widehat{\rho}$.

Proof. Since the center of $\widetilde{M}$ is generated by $\lambda^\psi(T)$, we have $\widetilde{\rho}(\lambda^\psi(T)) = \lambda^\psi(T)$ by assumption, which implies

$$[Dd(\rho)\psi \cdot \phi_\rho : D\psi]_T = 1.$$

Therefore, there exists a unitary $u \in M$ such that $d(\rho)\psi \cdot \phi_\rho = \psi \cdot \mathrm{Ad}(u)$ thanks to [5, Théorème 4.3.2]. This means that $(\psi, \mathrm{Ad}(u) \cdot \rho)$ is an invariant pair. The rest of the proof is the same as those of Theorem 2.4 and Proposition 2.5 if the dominant weight is replaced with the generalized trace. □

We end this section with a remark on the type II_∞ case.

Remark 4.6. Let M be a type II_∞ factor and τ be a faithful normal semifinite trace on M. Then, the canonical extension and the Connes-Takesaki modules make sense for M as well. We introduce a scalar valued module $\mathrm{Mod}(\rho)$ of $\rho \in \mathrm{End}(M)_0$ by

$$\tau \cdot \rho = d(\rho)\mathrm{Mod}(\rho)\tau.$$

Note that every inclusion of type II_∞ factors with a finite index comes from the tensor product of a common type I factor and an inclusion of II_1 factors. Therefore, the restriction of τ to the image of ρ is a semifinite trace and the above definition makes sense. The center of $\widetilde{M}$ is generated by $\lambda^\tau(t)$, and we have

$$\widetilde{\rho}(\lambda^\tau(t)) = d(\rho)^{it}(\frac{d\tau \cdot \phi_\rho}{d\tau})^{it}\lambda^\tau(t) = \mathrm{Mod}(\rho)^{-it}(\frac{d\tau \cdot E_\rho}{d\tau})^{it}\lambda^\tau(t).$$

This means that ρ has a Connes-Takesaki module if and only if E_ρ is trace preserving. When this is the case, we have

$$\mathrm{mod}(\rho)(\lambda^\tau(t)) = \mathrm{Mod}(\rho)^{-it}\lambda^\tau(t).$$

Therefore, Proposition 4.2, (2) implies that $M \supset \rho(M)$ is an extremal inclusion (see [60] for the definition) if and only if each irreducible component of ρ has the same Mod.

5. Minimal Actions of Compact Groups

In this section, we investigate the structure of minimal actions of compact groups on type III factors applying our machinery to the corresponding Roberts actions. Modular endomorphisms and the Connes-Takesaki modules for endomorphisms provide new invariants for minimal actions of compact groups through Roberts actions.

5.1. **Minimal actions as dual actions.** Let K be a compact group and $\mathrm{Rep}(K)$ be the tensor category of finite dimensional unitary representations of K. Roughly speaking, a Roberts action of the dual of K on a von Neumann algebra N is a functor from a (sufficiently big) subcategory $\mathcal{R}$ of $\mathrm{Rep}(K)$, (called a ring in [58]), to $\mathrm{End}(N)_0$, where the set of arrows from $\rho_1 \in \mathrm{End}(N)_0$ to $\rho_2 \in \mathrm{End}(N)_0$ is (ρ_1, ρ_2). For the precise definition of the Roberts action and the crossed product by it, we refer to [63] and [58] (see also [13]). For each equivalence class of irreducible representations of K, we choose a representative (π, H_π), and denote by $\widehat{K}$ the collection of them. We always assume that $\mathcal{R}$ contains every member of $\widehat{K}$ and its complex conjugate representation. We often omit to specify $\mathcal{R}$ in a Roberts action, and say "a Roberts action of $\widehat{K}$" if there is no possibility of confusion (although it is a bit sloppy terminology). For $(\pi_1, H_{\pi_1}), (\pi_2, H_{\pi_2}) \in \mathcal{R}$, we denote by (π_1, π_2) the set of K-homomorphisms from H_{π_1} to H_{π_2}, that is, the set of arrows from π_1 to π_2.

Let M be a factor and α be an action of K on M. α is said to be *minimal* if α is faithful and the relative commutant $M \cap M^{\alpha\prime}$ of the fixed point subalgebra M^α is trivial [62]. Note that when α is minimal in our sense, the crossed product $M \rtimes_\alpha K$ is automatically a factor isomorphic to $M^\alpha \otimes B(L^2(K))$. Indeed, to show this, we may assume that M^α is infinite by taking tensor product with a type I factor if necessary. Then, a similar argument as in [2, Lemma III.3.4] implies that for each irreducible representation π of K, there exists a Hilbert space $\mathcal{H}_\pi$ in M that is globally invariant under α such that the restriction of α to $\mathcal{H}_\pi$ is equivalent to π. Therefore, [63, Proposition 6.9] implies the claim.

More strongly, [63, Theorem 6.5] shows that M is the crossed product $N \rtimes_\beta \widehat{K}$ by some Roberts action β, and that α is its dual action $\widehat{\beta}$, where $N = M^\alpha$. The relationship between β_π and $\mathcal{H}_\pi$ is as follows: Let $\{W(\pi)\}_{i=1}^{d(\pi)}$ be an orthonormal basis for $\mathcal{H}_\pi$, where $d(\pi)$ is the dimension of π. Then, $\beta_\pi \in \mathrm{End}(N)_0$ is given by

$$\beta_\pi(x) = \sum_{i=1}^{d(\pi)} W(\pi)_i x W(\pi)_i^*, \quad x \in N.$$

On the other hand, $\mathcal{H}_\pi$ is recovered from β_π via

$$\mathcal{H}_\pi = \{W \in M;\ Wx = \beta_\pi(x)W, \quad x \in N\}.$$

Note that β is uniquely determined up to equivalence defined in [63, Definition 5.6].

We will use these notations throughout this section whenever a minimal action α of a compact group K on a factor M is given.

Lemma 5.1. *Let N be an infinite factor, and β be a Roberts action of $\widehat{K}$ on N. Then, the dual action of β is minimal if and only if β_π is irreducible for every irreducible representation $(\pi, H_\pi) \in \widehat{K}$.*

Proof. We set $M = N \rtimes_\beta \widehat{K}$, $\alpha = \widehat{\beta}$, and

$$E(x) = \int_K \alpha_k(x) dm_K(k), \quad x \in M,$$

where m_K is the Haar measure of K. Let $\{W(\pi)_i\}_{i=1}^{d(\pi)}$ be as above. For $x \in M$, we define $x(\pi)_i \in N$ by

$$x(\pi)_i = d(\pi)E(W(\pi)_i x).$$

Then, x has the following formal expansion (see [35, Section 3]):

$$x \text{``}=\text{''} \sum_{(\pi,H_\pi)\in\widehat{K}} \sum_{i=1}^{d(\pi)} W(\pi)_i^* x(\pi)_i.$$

More precisely, the above summation converges in the GNS Hilbert space topology with respect to an α-invariant normal state, and $\{x(\pi)_i\}$ completely determines x. Using this expansion, we can see that $x \in M \cap N'$ if and only if $x(\pi)_i \in (\mathrm{id}, \beta_\pi)$. Therefore, α is minimal if and only if β_π does not contain identity for every non-trivial irreducible representation π. However, the latter is equivalent to the statement that β_π is irreducible for very irreducible π thanks to the Frobenius reciprocity for the usual compact group representations and that of sectors [34]. $\square$

Let $\{u(k)\}_{k\in K}$ be an α-cocycle. We denote by α^u the perturbation $\mathrm{Ad}(u(k)) \cdot \alpha_k$ of α by the cocycle u. α is called stable if every α-cocycle is a coboundary.

The following statements are probably well-known among specialists, and actually some of them already exist in the literature. However, since we cannot find complete proofs in the literature, we provide them here for convenience of readers.

Proposition 5.2. *Let α be a minimal action of a compact group K on a factor M. We denote by E the unique normal conditional expectation from M onto M^α obtained by taking average over K. Then, the following hold:*

1. *Every cocycle perturbation of α is again minimal.*
2. *If M is of type II_1, so is M^α. α is stable in this case* [76, Theorem 12].
3. *If M is of type II_∞, so is M^α. α is stable in this case.*
4. *If M is of type III_0, M^α is of type III. α is stable in this case (more generally, α is stable whenever M^α is of type III).*
5. *M is of type III_λ, $\lambda \neq 0$ and M^α is of type II, if and only if the center of α_K contains the modular automorphism group $\{\sigma_t^{\varphi\cdot E}\}_{t\in\mathbb{R}}$, for some faithful normal semifinite weight φ on M^α (which is actually a trace if this is the case). α is not stable in this case.*

Proof. (1). First we claim that the second dual action $\alpha\otimes\mathrm{Ad}(r)$ on $M\otimes B(L^2(K))$ is minimal whenever α is so. To show the claim, we may assume that M^α is infinite as usual. Thus, we have a Hilbert space $\mathcal{H}_\pi$ in M for each $(\pi, H_\pi)\in\widehat{K}$ as before. This implies that there exists a Hilbert space $\mathcal{H}$ in M such that the restriction of α to $\mathcal{H}$ is equivalent to the regular representation. Therefore, (M,α) is conjugate to $(M\otimes B(L^2(K)), \alpha\otimes\mathrm{Ad}(r))$, which shows the claim. Let α^u be a cocycle perturbation of α. Since $M\rtimes_\alpha K\subset M\otimes B(L^2(K))$ is an irreducible inclusion and there exists a projection $p\in M\rtimes_\alpha K$ such that

$$(M^{\alpha^u}\subset M)\cong (p(M\rtimes_\alpha K)p\subset p(M\otimes B(L^2(K)))p),$$

which shows that α^u is minimal.

(2),(3). The first part is easy. Stability in these two cases (and the case (4) as well) follows from (1) and Connes' 2 by 2 matrix trick [9].

(4). We show that if M^α is of type II, M is either of type II or of type III_λ, $\lambda\neq 0$. Let τ be a faithful normal semifinite trace on M^α. Then, the centralizer $M_{\tau\cdot E}$ is an intermediate subfactor between M^α and M, and in particular $M_{\tau\cdot E}$ is a factor. Therefore, the Connes spectrum $\Gamma(\sigma^{\tau\cdot E})$ coincides with the Arveson spectrum $\mathrm{Sp}(\sigma^{\tau\cdot E})$ [5], [67]. Thus, M is either of type II or type III_λ, $\lambda\neq 0$ depending on the period of the modular automorphism group $\{\sigma_t^{\tau\cdot E}\}_{t\in\mathbb{R}}$.

(5). Note that part of the proof has been already done in the above. Assume that α_K contains $\{\sigma_t^{\varphi\cdot E}\}_{t\in\mathbb{R}}$, for some faithful normal semifinite weight φ on M^α. Then, the centralizer $M_{\varphi\cdot E}$ is an intermediate subfactor of $M^\alpha\subset M$. Let F be the restriction of E to $M_{\varphi\cdot E}$, which is a normal conditional expectation from $M_{\varphi\cdot E}$ to M^α. Since the restriction of $\varphi\cdot E$ to $M_{\varphi\cdot E}$ is nothing but $\varphi\cdot F$ and it is semifinite, $M_{\varphi\cdot E}$ is of type II, and so is M^α as well because $\sigma^{\varphi\cdot F}$ is trivial. This finishes the proof of the first statement.

Now, we show that α is not stable in the above situation with type III M . First we assume that M^α is of type II_∞. Let $p\in M^\alpha$ be a non-zero finite projection, and $V\in M$ be an isometry satisfying $VV^*=p$. We set $u(k)=V^*\alpha_k(V)$, which is an α-cocycle. It is easy to show that M^{α^u} is finite and u is not a coboundary. When M^α is of type II_1, a similar construction identifying M with a corner of $M\otimes B(H)$ works. ∎

Remark 5.3. In the above situation, assume that M^α is of type II_∞ with a faithful normal trace τ. Then, the type of M (or the period of $\{\sigma^{\tau\cdot E}\}_{t\in\mathbb{R}}$) is completely determined by $\mathrm{Mod}(\beta_\pi)$ for the Roberts action β satisfying $M = M^\alpha \rtimes_\beta \widehat{K}$, where Mod is the scalar Connes-Takesaki module introduced in Remark 4.6. Indeed, as in [35, Section 3], we can obtain the action of the modular automorphism group on $\mathcal{H}_\pi$ as follows:

$$\sigma_t^{\tau\cdot E}(W) = \mathrm{Mod}(\beta_\pi)^{it}W, \quad W \in \mathcal{H}_\pi.$$

It is possible to show that the canonical extension gives continuous homomorphism from $\mathrm{Aut}(M)$ to $\mathrm{Aut}(\widetilde{M})$ in the u-topologies in the same spirit of the Cones-Takesaki's proof of continuity of the fundamental homomorphism [11]. However, we just mention here that when α is an action of a locally compact group G on M with a faithful normal invariant state, then it is particularly easy to show that $G \ni g \mapsto \widetilde{\alpha}_g$ is continuous. On the other hand, when β is a Roberts action of $\widehat{K}$ on a factor N, Proposition 2.5 shows that β extends to $\widetilde{\beta}$ on $\widetilde{N}$ via the canonical extension.

Lemma 5.4. *Let α be an action of a compact group K on a factor M, and E be the normal conditional expectation from M onto $N := M^\alpha$ given by the average of α_k over K. We regard $\widetilde{N}$ as a subalgebra of $\widetilde{M}$ identifying $\lambda^{\varphi_0}(t)$ with $\lambda^{\varphi_0\cdot E}(t)$, where φ_0 is a faithful normal semifinite weight on $\widetilde{N}$. Then,*

1. *The fixed point subalgebra of $\widetilde{M}$ under $\widetilde{\alpha}$ coincides with $\widetilde{N}$.*
2. *When α is a minimal action that is the dual action of a Roberts action β of $\widehat{K}$ on N. Then, $\widetilde{M}$ is naturally isomorphic to $\widetilde{N} \rtimes_{\widetilde{\beta}} \widehat{K}$.*

Proof. (1). We set $\varphi = \varphi_0 \cdot E$. Let $\widetilde{E}$ be the normal conditional expectation from $\widetilde{M}$ onto the fixed point subalgebra of $\widetilde{M}$ under $\widetilde{\alpha}$ obtained by average of $\widetilde{\alpha}_k$ over K. Since $\widetilde{\alpha}_k$ leaves $\lambda^\varphi(t)$ fixed for all $k \in K$ and $t \in \mathbb{R}$, the image of $\widetilde{E}$ coincides with $\widetilde{N}$.

(2). Let ψ_0 be a dominant weight on N, and $\psi = \psi_0 \cdot E$. For each β_π, $(\pi, H_\pi) \in \widehat{K}$, we may and do assume that (ψ_0, β_π) is an invariant pair thanks to Lemma 2.3, (3). Then, the modular automorphism group $\{\sigma_t^\psi\}_{t\in\mathbb{R}}$ acts on $\mathcal{H}_\pi$ trivially [35, Section 3] and $\widetilde{\beta}_\pi$ leaves $\lambda^{\psi_0}(t)$ invariant. Therefore, $\widetilde{\beta}_\pi$ is given by

$$\widetilde{\beta}_\pi(x) = \sum_{i=1}^{d(\pi)} W(\pi)_i x W(\pi)_i^*, \quad x \in \widetilde{N}.$$

Thus, (1) and [63, Theorem 6.5] imply that $\widetilde{M} = \widetilde{N} \rtimes_{\widetilde{\beta}} \widehat{K}$ and $\widetilde{\alpha}$ is the dual action of $\widetilde{\beta}$. □

5.2. **A Roberts action of modular endomorphisms.** The purpose of this subsection is to show how a system of modular endomorphisms gives rise to a Roberts action, and to see the structure of its dual action. There are two options to do so, (of course they are essentially the same), and we start with the one using the minimal subgroup of a cocycle.

Let N be a type III factor and $\mathcal{D} \subset \mathrm{End}(N)_m$ be a countable family of modular endomorphisms of N. By adding new endomorphisms to $\mathcal{D}$ if necessary, we may and do assume the following conditions:

(i) $\mathcal{D}$ is closed under taking a product of any two members in $\mathcal{D}$.
(ii) For every pair $\rho_1, \rho_2 \in \mathcal{D}$ there exists $\rho \in \mathcal{D}$ such that $[\rho_1] \oplus [\rho_2] = [\rho]$.
(iii) If $\sigma \in \mathrm{End}(N)_0$ is contained in $\rho \in \mathcal{D}$, there exists an endomorphism in $\mathcal{D}$ that is equivalent to σ.
(iv) Every $\rho \in \mathcal{D}$ has conjugate (up to equivalence) in $\mathcal{D}$.

For each $\rho \in \mathcal{D}$ we choose $c_\rho \in Z^1(F^N, U(d(\rho)))$ satisfying $\delta_m([\rho]) = [c_\rho]$, and set

$$K_1 = \prod_{\rho \in \mathcal{D}} U(d(\rho)),$$

$$c = (c_\rho)_{\rho \in \mathcal{D}} \in Z^1(F^N, K_1),$$

where $U(n)$ is the unitary group. Thanks to Theorem 3.5, we may and do assume that c is already a minimal cocycle. We denote by K the minimal subgroup of c.

Let (π_ρ, H_{π_ρ}) be the representation of K on $H_{\pi_\rho} = \mathbb{C}^{d(\rho)}$ that is the projection onto the ρ-component. Then, we have $c_\rho = \pi_\rho \cdot c$. We denote by $\mathcal{R}$ the collection of (π_ρ, H_{π_ρ}), $\rho \in \mathcal{D}$.

Now, we construct a Roberts action β of $\mathcal{R}$ on N sending π_ρ to ρ. Thanks to Lemma 3.2, (2), we can choose an implementing system $\{V(\rho)_i\}_{i=1}^{d(\rho)}$ for $\widetilde{\rho}$ giving c_ρ. Let $\mathcal{K}_\rho$ be the linear span of $\{V(\rho)_i\}_{i=1}^{d(\rho)}$, which is a Hilbert space in $\widetilde{N}$. For $\rho_1, \rho_2 \in \mathcal{D}$, we denote by $\mathcal{K}_{\rho_2}\mathcal{K}_{\rho_1}^*$ the linear span of elements of the form $V_2V_1^*$, $V_1 \in \mathcal{K}_1$, $V_2 \in \mathcal{K}_2$, which is naturally identified with $\mathrm{Hom}(\mathcal{K}_{\rho_1}, \mathcal{K}_{\rho_2})$ and $\mathrm{Hom}(H_{\pi_{\rho_1}}, H_{\pi_{\rho_2}})$. Since (ρ_1, ρ_2) is the fixed point subset of $\mathcal{K}_{\rho_2}\mathcal{K}_{\rho_1}^*$ under θ, Theorem 3.5, (4) implies that the above identification carries $(\rho_1, \rho_2) \subset (\widetilde{\rho}_1, \widetilde{\rho}_2)$ onto $(\pi_{\rho_1}, \pi_{\rho_2}) \subset \mathrm{Hom}(H_{\pi_{\rho_1}}, H_{\pi_{\rho_2}})$. Therefore, we define a functor β setting β_T, $T \in (\pi_{\rho_1}, \pi_{\rho_2})$ to be the corresponding element in (ρ_1, ρ_2). Note that since β maps every irreducible object to an irreducible object, Lemma 5.1 implies that the resulting crossed product gives irreducible inclusion of factors $N \subset N \rtimes_\beta \widehat{K} =: M$, and the dual action $\alpha := \widehat{\beta}$ is minimal.

Another way to construct (M, α, K) from $\mathcal{D}$ is to apply the Doplicher-Roberts duality theorem [13, Theorem 7.1] to $\mathcal{D}$ (see also [14]). For this purpose, we have to specify the permutation symmetry $\mathcal{E}(\rho_1, \rho_2)$ for every pair $\rho_1, \rho_2 \in \mathcal{D}$. Indeed, we set

$$\mathcal{E}(\rho_1, \rho_2) := \sum_{i=1}^{d(\rho_1)} \sum_{j=1}^{d(\rho_2)} V(\rho_2)_j V(\rho_1)_i V(\rho_2)_j^* V(\rho_1)_i^*.$$

Note that $\mathcal{E}(\rho_1, \rho_2)$ belongs to $\widetilde{N}^\theta = N$ and it does not depend on the choice of implementing systems. It is a routine work to show that $\mathcal{E}(\rho_1, \rho_2)$ is a permutation symmetry for $\mathcal{D}$, and that the resulting crossed product by $\mathcal{D}$ and the dual action constructed in [13] are the same as M and α above.

In general, $\mathcal{D}$ does not uniquely determine either the compact group K or the Roberts action β [36]. (When K is abelian, the obstruction is nothing but the usual $H^2(\widehat{K}, \mathbb{T})$ obstruction for uniqueness of liftings of the $\widehat{K}$-kernel with a trivial 3 cocycle

[37], [70].) Whenever we deal with Roberts actions consisting of modular endomorphisms in the rest of this paper, we always assume that we make the above choice of β on the arrows (or equivalently, the above choice of the permutation symmetries).

Theorem 5.5. *Let the notations be as above. Then,*

(1) *$\widetilde{M}$ is generated by $\widetilde{N}$ and the relative commutant $\widetilde{M}\cap\widetilde{N}'$, and $\widetilde{M}\cap\widetilde{N}'$ coincides with the center $Z(\widetilde{M})$ of M. The smooth flow of weights F^M of M is given by the skew product $(X_N\times_c K, \mu_N\times m_K)$, that is: $X_M = X_N\times K$, $\mu_M=\mu_N\times m_K$, and the flow is given by*

$$(\omega,k)\cdot t = (\omega\cdot t, kc(\omega,t)),\quad t\in\mathbb{R},\ k\in K,\ \omega\in X_N.$$

The factor map from X_M onto X_N corresponding to the inclusion $Z(\widetilde{N})\subset Z(\widetilde{M})$ is the projection onto the first component.

(2) *The Connes-Takesaki module* $\mathrm{mod}(\alpha_k)$ *of α_k, $k\in K$ is given by*

$$\mathrm{mod}(\alpha_k)(f)(\omega,l) = f(\omega,k^{-1}l),\quad f\in L^\infty(X_M,\mu_M),\ (\omega,l)\in X_M.$$

Proof. Before starting the proof, we simplify the notation a little. We may and do assume $\widehat{K}\subset\mathcal{R}$ thanks to the assumption (iii). For $\rho=\beta_\pi$, $(\pi,H_\pi)\in\widehat{K}$, we simply denote $\mathcal{K}_\pi=\mathcal{K}_\rho$ and $V(\pi)_i=V(\rho)_i$.

(1) We take $\mathcal{H}_\pi\in M$ and an orthonormal basis $\{W(\pi)_i\}_{i=1}^{d(\pi)}$ as before, where $\{W(\pi)_i\}_{i=1}^{d(\pi)}$ is transformed by α in the same way as the canonical basis of $H_\pi=\mathbb{C}^{d(\pi)}$. Let E be the unique normal conditional expectation from M onto N, and $\widetilde{E}$ be the natural extension of E to $\widetilde{M}$ that is a normal conditional expectation from $\widetilde{M}$ onto $\widetilde{N}$. As in the proof of Lemma 5.1, for $x\in\widetilde{M}$ we set

$$x(\pi)_i = d(\pi)\widetilde{E}(W(\pi)_i^*x).$$

Then, x is in $\widetilde{M}\cap\widetilde{N}'$ if and only if $x(\pi)_i\in(\mathrm{id},\widetilde{\beta}_\pi)$ for every $(\pi,H_\pi)\in\widehat{K}$ and every $i=1,2,\cdots,d(\pi)$. This is further equivalent to that there exists $z(\pi)_{ij}\in Z(\widetilde{N})$ such that

$$x(\pi)_i=\sum_{j=1}^{d(\pi)} z(\pi)_{ij}V(\pi)_j,$$

for every $(\pi,H_\pi)\in\widehat{K}$ and every $i=1,2,\cdots,d(\pi)$. Note that thanks to Proposition 4.2, (4) and Lemma 5.4, $W(\pi)_i$ commutes with every element in $Z(\widetilde{N})$. Therefore, in the same way as in [35, Lemma 3.8], we can conclude that $\widetilde{M}\cap\widetilde{N}'$ is generated by $Z(\widetilde{N})$ and $\{f(\pi_{ij})\}_{\pi,i,j}$, where $f(\pi_{ij})=W(\pi)_i^*V(\pi)_j$. Since $\widetilde{M}$ is generated by $\widetilde{N}$ and $\{W(\pi)_i\}_{\pi,i}$, it is also generated by $\widetilde{N}$ and $\{f(\pi_{ij})\}_{\pi,i,j}$ as well. Thus, to prove $Z(\widetilde{M})=\widetilde{M}\cap\widetilde{N}'$, it suffices to show that $\{f(\pi_{ij})\}_{\pi,i,j}$ generate a commutative von Neumann algebra.

Let B be the von Neumann algebra generated by $\{f(\pi_{ij})\}_{\pi,i,j}$. We show that there exists a $*$ isomorphism from $L^\infty(K)$ onto B sending π_{ij} to $f(\pi_{ij})$. For this, it suffices to check that both π_{ij} and $f(\pi_{ij})$ have the same algebraic relations because we have

$\widetilde{E}(f(\pi_{ij}))=m_K(\pi_{ij})=\delta_{\pi,1}$, where

$$m_K(\xi)=\int_K\xi(k)dm_K(k),\quad \xi\in L^\infty(K).$$

First, we show $f(\pi_{ij})^*=f(\overline{\pi}_{ij})$. Let $R_\pi\in(\mathrm{id},\beta_{\overline{\pi}}\cdot\beta_\pi)$ and $R_{\overline{\pi}}\in(\mathrm{id},\beta_\pi\cdot\beta_{\overline{\pi}})$ be isometries defined by

$$R_\pi=\frac{1}{\sqrt{d(\pi)}}\sum_{i=1}^{d(\pi)}V(\overline{\pi})_iV(\pi)_i,$$

$$R_{\overline{\pi}}=\frac{1}{\sqrt{d(\pi)}}\sum_{i=1}^{d(\pi)}V(\pi)_iV(\overline{\pi})_i.$$

Then, the definition of the crossed product by a Roberts action implies that we also have

$$R_\pi=\frac{1}{\sqrt{d(\pi)}}\sum_{i=1}^{d(\pi)}W(\overline{\pi})_iW(\pi)_i,$$

$$R_{\overline{\pi}}=\frac{1}{\sqrt{d(\pi)}}\sum_{i=1}^{d(\pi)}W(\pi)_iW(\overline{\pi})_i.$$

Thus, we get

$$\begin{aligned}f(\pi_{ij})^* &= V(\pi)_j^*W(\pi)_i=\sqrt{d(\pi)}V(\pi)_j^*W(\overline{\pi})_i^*R_\pi=\sqrt{d(\pi)}W(\overline{\pi})_i^*\widetilde{\beta}_{\overline{\pi}}(V(\pi)_j^*)R_\pi\\ &= W(\overline{\pi})_i^*V(\overline{\pi})_j=f(\overline{\pi})_{ij}.\end{aligned}$$

For the same reason, we have

$$\mathcal{E}(\pi,\sigma)=\sum_{i=1}^{d(\pi)}\sum_{a=1}^{d(\sigma)}V(\sigma)_aV(\pi)_iV(\sigma)_a^*V(\pi)_i^*=\sum_{i=1}^{d(\pi)}\sum_{a=1}^{d(\sigma)}W(\sigma)_aW(\pi)_iW(\sigma)_a^*W(\pi)_i^*,$$

which implies

$$\begin{aligned}f(\pi_{ij})f(\sigma_{ab}) &= W(\pi)_i^*V(\pi)_jW(\sigma)_a^*V(\sigma)_b=W(\pi)_i^*W(\sigma)_a^*\widetilde{\beta}_\sigma(V(\pi)_j)V(\sigma)_b\\ &= W(\pi)_i^*W(\sigma)_a^*V(\sigma)_bV(\pi)_j=W(\pi)_i^*W(\sigma)_a^*\mathcal{E}(\pi,\sigma)V(\pi)_jV(\sigma)_b\\ &= W(\sigma)_a^*W(\pi)_i^*V(\pi)_jV(\sigma)_b=f(\sigma_{ab})f(\pi_{ij}).\end{aligned}$$

Thus, B is abelian. For the same reason again, there exist $T^{(\nu,s)}_{(\pi,i),(\sigma,a)}\in(\beta_\nu,\beta_\pi\cdot\beta_\sigma)$ such that

$$V(\pi)_iV(\sigma)_a=\sum_{(\nu,H_\nu)\in\widehat{K}}\sum_{s=1}^{d(\nu)}T^{(\nu,s)}_{(\pi,i),(\sigma,a)}V(\nu)_s,$$

$$W(\pi)_iW(\sigma)_a=\sum_{(\nu,H_\nu)\in\widehat{K}}\sum_{s=1}^{d(\nu)}T^{(\nu,s)}_{(\pi,i),(\sigma,a)}W(\nu)_s,$$

which implies

$$f(\pi_{ij})f(\sigma_{ab}) = \sum_{(\nu,H_\nu)\in\widehat{K}} \sum_{s,t=1}^{d(\nu)} \langle T^{(\nu,t)}_{(\pi,j),(\sigma,b)}, T^{(\nu,s)}_{(\pi,i),(\sigma,a)} \rangle f(\nu_{st}).$$

Note that thanks to the Peter-Weyl theorem, $T^{(\nu,s)}_{(\pi,i),(\sigma,a)}$ is obtained by

$$T^{(\nu,s)}_{(\pi,i),(\sigma,a)} = d(\nu)E(W(\pi)_i W(\sigma)_a W(\nu)^*_s).$$

Thus, we have

$$\begin{aligned}
&\langle T^{(\nu,t)}_{(\pi,j),(\sigma,b)}, T^{(\nu,s)}_{(\pi,i),(\sigma,a)} \rangle \\
&= d(\nu)^2 \int_K E(W(\nu)_s W(\sigma)^*_a W(\pi)^*_i \alpha_g(W(\pi)_j W(\sigma)_b W(\nu)^*_t)) dm_K(g) \\
&= d(\nu)^2 \sum_{r=1}^{d(\nu)} \int_K \pi_{ij}(g)\sigma_{ab}(g)\overline{\nu_{rt}(g)} E(W(\nu)_s W(\nu)^*_r) dm_K(g) \\
&= d(\nu) \int_K \pi_{ij}(g)\sigma_{ab}(g)\overline{\nu_{st}(g)} dm_K(g).
\end{aligned}$$

This shows that

$$\pi_{ij}\sigma_{ab} = \sum_{(\nu,H_\nu)\in\widehat{K}} \sum_{s,t=1}^{d(\nu)} \langle T^{(\nu,t)}_{(\pi,j),(\sigma,b)}, T^{(\nu,s)}_{(\pi,i),(\sigma,a)} \rangle \nu_{st}$$

holds as well thanks to the Peter-Weyl theorem again. Therefore, there exists a desired isomorphism from $L^\infty(K)$ onto B.

Let φ_0 be a faithful normal state of $\widetilde{N}$ whose restriction is given by the measure μ_N, and we set $\varphi = \varphi_0 \cdot \widetilde{E}$. Then, for $z \in Z(\widetilde{N})$, $f \in B$, we have

$$\varphi(zf) = \varphi_0(z\widetilde{E}(f)) = \varphi(z)\varphi(f).$$

This shows that $Z(\widetilde{M}) = Z(\widetilde{N}) \otimes B$. θ acts on $f(\pi_{ij})$ as

$$\theta_t(f(\pi_{ij})) = W(\pi)^*_i \theta_t(V(\pi)_j) = \sum_{l=1}^{d(\pi)} W(\pi)^*_i V(\pi)_l \pi_{lj}(c(t)) = \sum_{l=1}^{d(\pi)} f(\pi_{il})\pi_{lj}(c(t)),$$

which shows that the flow is given by the skew product.

(2) Since $\widetilde{\alpha}$ acts on $V(\pi)_i$ trivially, we have

$$\widetilde{\alpha}_k(f(\pi_{ij})) = \alpha_k(W(\pi)^*_i)V(\pi)_j = \sum_{l=1}^{d(\pi)} \overline{\pi_{li}(k)} f(\pi_{lj}) = \sum_{l=1}^{d(\pi)} \pi_{il}(k^{-1}) f(\pi_{lj}).$$

This means that the Connes-Takesaki module of α_k is given as in the statement. □

Definition 5.6. Let N be a type III factor, K be a compact group, and $c \in Z^1(F^N, K)$ be a minimal cocycle with $K_c = K$. For each $(\pi, H_\pi) \in \widehat{K}$, we choose $\rho_\pi \in \mathrm{End}(N)_m$ such that $\delta_m([\rho_\pi]) = [\pi \cdot c]$. Let $\mathcal{D}$ be the subset of $\mathrm{End}(N)_m$ generated by $\{\rho_\pi\}_{(\pi,H_\pi)\in\widehat{K}}$ and satisfying the above conditions (i)-(iv). We denote by

$N \otimes_c L^\infty(K)$ the above crossed product $N \rtimes_\beta \widehat{K}$ and call it the *skew product of* N *and* $L^\infty(K)$ *by the cocycle* c. We call $\widehat{\beta}$ the *dual action* of the skew product.

Note that $(N \otimes_c L^\infty(K), \widehat{\beta})$ depends only on the class $[c] \in H^1(F^N, K)$.

Our next task is to show the "converse" of the above theorem, and we need some preparation.

Lemma 5.7. *Let α be an action of a locally compact group G on a von Neumann algebra M, and H be the standard Hilbert space of M. Let $\widetilde{J}$ be the modular conjugation of $M \rtimes_\alpha G := \pi_\alpha(M) \vee (\mathbb{C} \otimes l(G)'')$ acting on $L^2(G, H)$ as given by* [18, Lemma 2.8]. *Then, we have*

$$\widetilde{J}((M \rtimes_\alpha G) \cap \pi_\alpha(M)')\widetilde{J} = (M \otimes B(L^2(G))) \cap (M \rtimes_\alpha G)'.$$

Proof. Let J be the modular conjugation of M on H. Then, $\widetilde{J}$ is given by

$$(\widetilde{J}\xi)(g) = \Delta_G(g)^{-1/2} u(g)^* J\xi(g), \quad \xi \in L^2(G, H),$$

where Δ_G is the modular function of G and $u(g)$ is the canonical implementation of G on H. Direct computation yields

$$\widetilde{J}\pi_\alpha(x)\widetilde{J} = JxJ \otimes \mathbb{C}, \quad x \in M,$$
$$\widetilde{J}(u(g) \otimes r(g))\widetilde{J} = 1 \otimes l(g), \quad g \in G.$$

Thus, we get

$$\begin{aligned}
\widetilde{J}((M \rtimes_\alpha G) \cap \pi_\alpha(M)')\widetilde{J} &= \widetilde{J}((M \otimes B(L^2(G))) \cap (u \otimes r)(G)' \cap \pi_\alpha(M)')\widetilde{J} \\
&= (\widetilde{J}((M' \otimes \mathbb{C}) \vee (u \otimes r)(G)'' \vee \pi_\alpha(M))\widetilde{J})' \\
&= (\pi_\alpha(M) \vee (\mathbb{C} \otimes l(G)'') \vee (M' \otimes \mathbb{C}))' \\
&= (M \rtimes_\alpha G)' \cap (M \otimes B(L^2(G))).
\end{aligned}$$

□

Let $M \supset N$ be an irreducible inclusion of factors with a normal conditional expectation E. We assume that $\widetilde{M}$ is generated by $\widetilde{N}$ and $\widetilde{M} \cap \widetilde{N}'$. Then, the restriction of $\widetilde{E}$ to $\widetilde{M} \cap \widetilde{N}'$ is a conditional expectation onto $Z(\widetilde{N})$. For $a, b \in \widetilde{M} \cap \widetilde{N}'$, $\widetilde{E}$ satisfies the trace property $\widetilde{E}(ab) = \widetilde{E}(ba)$. Indeed, for $x \in \widetilde{N}$ in the domain of τ, we have

$$\tau(\widetilde{E}(ab)x) = \tau(abx) = \tau(axb) = \tau(bax) = \tau(\widetilde{E}(ba)x),$$

which shows $\widetilde{E}(ab) = \widetilde{E}(ba)$.

Lemma 5.8. *Let N be a type* III *factor and A be a von Neumann algebra whose center includes $Z(\widetilde{N})$. We assume that there exist an ergodic $\mathbb{R}$-action ϑ on A extending the flow of $Z(\widetilde{N})$, and a faithful normal conditional expectation ϵ from A onto $Z(\widetilde{N})$ such that $\theta_t \cdot \epsilon = \epsilon \cdot \vartheta_t$, $t \in \mathbb{R}$ and $\epsilon(xy) = \epsilon(yx)$ for $x, y \in A$. Then, there exists a unique irreducible inclusion $M \supset N$ of type* III *factors with a normal conditional expectation E such that $\widetilde{M} = (\widetilde{M} \cap \widetilde{N}') \vee \widetilde{N}$ and*

$$(A \supset Z(\widetilde{N}), \vartheta, \epsilon) \cong (\widetilde{M} \cap \widetilde{N}' \supset Z(\widetilde{N}), \theta|_{\widetilde{M} \cap \widetilde{N}'}, \widetilde{E}|_{\widetilde{M} \cap \widetilde{N}'}).$$

Proof. First, we show uniqueness of M. Let M be a von Neumann algebra including N with a unique conditional expectation E from M onto N. We assume that $\widetilde{M}$ is generated by $\widetilde{M}\cap\widetilde{N}'$ and $\widetilde{N}$. Then, the linear span of $(\widetilde{M}\cap\widetilde{N}')\widetilde{N}$ is dense in $\widetilde{M}$ by assumption. For a faithful normal state φ on $\widetilde{N}$ and $a\in\widetilde{M}\cap\widetilde{N}'$, $x\in\widetilde{N}$, we have

$$\varphi\cdot\widetilde{E}(ax)=\varphi(\widetilde{E}(a)x).$$

This means that the structure of $(\widetilde{M}\supset\widetilde{N},\widetilde{E},\theta)$ is uniquely determined by $(\widetilde{N},\theta|_{\widetilde{N}})$ and

$$(\widetilde{M}\cap\widetilde{N}'\supset Z(\widetilde{N}),\theta|_{\widetilde{M}\cap\widetilde{N}'},\widetilde{E}|_{\widetilde{M}\cap\widetilde{N}'}).$$

On the other hand, $(\widetilde{M}\supset\widetilde{N},\widetilde{E},\theta)$ completely determine the original inclusion via the Takesaki duality theorem. Therefore, such M as in the statement is unique if it exists.

We set $L=\widetilde{N}\otimes_{Z(\widetilde{N})}A$ as in [64], [69]. More precisely, let H be the standard Hilbert space of $\widetilde{N}$. We introduce an inner product into the algebraic tensor product $H\odot_{Z(\widetilde{N})}A$ by

$$\langle\xi\odot a,\eta\odot b\rangle=\langle\epsilon(b^*a)\xi,\eta\rangle,\quad a,b\in A,\ \xi,\eta\in H.$$

Let H_1 be the completion of $H\odot_{Z(\widetilde{N})}A$, and Λ be the natural map from $H\odot_{Z(\widetilde{N})}A$ to H_1. $\widetilde{N}$ and A naturally act on the first and the second tensor components of H_1, and we denote by L the von Neumann algebra generated by $\widetilde{N}$ and A in $B(H_1)$. In terms of direct integral, L can be expressed as follows (though we do not use it in the proof): Let

$$\widetilde{N}=\int_{X_N}^{\oplus}\widetilde{N}(\omega)d\mu_N(\omega),$$

$$A=\int_{X_N}^{\oplus}A(\omega)d\mu_N(\omega),$$

be disintegration of $\widetilde{N}$ and A over $Z(\widetilde{N})$. Then L is given by

$$L=\int_{X_N}^{\oplus}\widetilde{N}(\omega)\otimes A(\omega)d\mu_N(\omega).$$

Thanks to the Connes-Takesaki relative commutant theorem and Lemma 5.7, we have $\widetilde{N}\cap N'=Z(\widetilde{N})$. We show $L\cap N'=A$ using this and applying [69, Theorem 2.3] (or alternatively, using the above direct integral expression). To do so, we need to separate A and N by a type I von Neumann algebra. Indeed, let F be the commutant of $Z(\widetilde{N})$ in $B(H)$. Then, F naturally acts on H_1 and we have $N\subset F$ and $A\subset F'$, where the commutant F' is considered in $B(H_1)$. Applying [69, Theorem 2.3] twice, we get

$$\begin{aligned} L\cap N' &= (N\vee Z(\widetilde{N}))'\cap(\widetilde{N}\vee A) = (N\vee Z(\widetilde{N})\vee(\widetilde{N}\vee A)')' \\ &= (N\vee Z(\widetilde{N})\vee(F\cap\widetilde{N}')\vee(F'\cap A'))' \\ &= ((F\cap(\widetilde{N}\vee N')')\vee(F'\cap A'))' \\ &= ((F\cap Z(\widetilde{N})')\vee(F'\cap A'))' \\ &= (F\vee(F'\cap A'))' \\ &= Z(\widetilde{N})\vee A = A. \end{aligned}$$

We introduce a 1-parameter automorphism group Θ on L extending both θ and ϑ. Let $u(t)$ be the canonical implementation of θ_t on H. We set

$$U(t)\Lambda(\xi\odot a) = \Lambda(u(t)\xi\odot\vartheta_t(a)), \quad \xi\in H,\ a\in A.$$

Then, $U(t)$ extends to a 1-parameter unitary group on H_1 satisfying

$$U(t)xaU(t)^* = \theta_t(x)\vartheta(a), \quad x\in\widetilde{N},\ a\in A.$$

We define Θ_t to be the restriction of $\mathrm{Ad}(U(t))$ to L.

Next, we show that there exists a faithful normal conditional expectation $\widetilde{\epsilon}$ from L onto $\widetilde{N}$ satisfying

$$\widetilde{\epsilon}(a) = \epsilon(a), \quad a\in A.$$

Let φ be a faithful normal state on $\widetilde{N}$, and $\xi_\varphi\in H$ be the cyclic and separating vector of $\widetilde{N}$ giving φ. We set $\Omega = \Lambda(\xi_\varphi\odot 1)$, and claim that Ω is cyclic and separating vector for L. Indeed, Ω is clearly a cyclic vector. To show that it is separating, we introduce a conjugation J on H_1 by

$$J\Lambda(\xi\odot a) = \Lambda(J_{\widetilde{N}}\xi\odot a^*), \quad \xi\in H,\ a\in A$$

where $J_{\widetilde{N}}$ is the modular conjugation of $\widetilde{N}$. It is easy to show that J extends to a conjugation such that $JLJ\subset L'$. Since JLJ is cyclic for Ω, Ω is a separating for L. We denote by ω the state of L defined by $\omega(x) = \langle x\Omega,\Omega\rangle$, $x\in L$. Then, it is a routine work to show that J is the modular conjugation for ω, and the modular automorphism group $\{\sigma_t^\omega\}_{t\in\mathbb{R}}$ is given by

$$\sigma_t^\omega(xa) = \sigma_t^\varphi(x)a, \quad x\in\widetilde{N},\ a\in A.$$

Thanks to Takesaki's theorem [74], there exists a ω-preserving normal conditional expectation $\widetilde{\epsilon}$ from L onto $\widetilde{N}$. Since $\omega(xa) = \omega(x\epsilon(a))$ for $x\in\widetilde{N}$, $a\in A$, $\widetilde{\epsilon}$ satisfies $\widetilde{\epsilon}(a) = \epsilon(a)$.

We set $M = L^\Theta$ and E to be the restriction of $\widetilde{\epsilon}$ to M. Since $\widetilde{\epsilon}$ satisfies $\widetilde{\epsilon}\cdot\Theta = \theta\cdot\widetilde{\epsilon}$, E is a faithful normal conditional expectation from M onto N. Therefore, M is of type III [67]. Let ν be the restriction of $\mathrm{Ad}(\lambda^\psi(t))$ to M, where ψ is a faithful normal semifinite weight on N. Then, thanks to the Landstad theorem [58, Chapter II, Section 2], $L = M\rtimes_\nu\mathbb{R}$ and Θ is the dual action of ν. We claim that $M\cap N' = \mathbb{C}$, and in particular M is a factor. Indeed, we have

$$M\cap N'\subset L\cap N' = A.$$

Since Θ acts on A ergodically, we get $M \cap N' = \mathbb{C}$.

Instead of showing that ν comes from the modular automorphism group directly, we show it for the dual action of Θ. We define $\tau_1 = \tau \cdot \widetilde{\epsilon}$, which is a faithful normal semifinite trace on L satisfying $\tau_1 \cdot \Theta_t = e^{-t}\tau_1$. Indeed, since τ is a trace, $\sigma_t^{\tau_1}$ acts on $\widetilde{N}$ trivially. On the other hand, since $\widetilde{\epsilon}$ is a faithful normal conditional expectation from L onto $\widetilde{N}$ and $L \cap \widetilde{N}' = A$, the restriction of $\sigma_t^{\tau_1}$ to A is the same as the restriction of σ_t^{ω} to A [10], which is trivial as we saw before. Thus, τ_1 is a trace with the scaling property

$$\tau_1 \cdot \Theta_t = \tau \cdot \theta_t \cdot \widetilde{\epsilon} = e^{-t}\tau_1.$$

This means that the dual action of Θ is the modular automorphism group for the dual weight $\widehat{\tau}_1$. Thus, identifying $L \rtimes_\Theta \mathbb{R}$ with $M \otimes B(L^2(R))$ using the Takesaki duality theorem, we get

$$\widehat{\tau}_1 = \widehat{\tau} \cdot (E \otimes id),$$
$$\sigma_t^{\widehat{\tau}_1} = (\nu_t \otimes \mathrm{Ad}(r(t))).$$

Let h be the generator of $r(t)$. Then, we have $\widehat{\tau} = \psi \otimes \mathrm{Tr}(h\cdot)$, which implies $\widehat{\tau}_1 = \psi \cdot E \otimes \mathrm{Tr}(h\cdot)$. Thus, we get $\nu_t = \sigma_t^{\psi \cdot E}$, which implies $L = \widetilde{M}$ and $\widetilde{\epsilon} = \widetilde{E}$. $\square$

Theorem 5.9 (Automatic Minimality). *Let α be an action of a compact group K on a type* III *factor M. If the kernel of* $\mathrm{mod} \cdot \alpha$ *is trivial, then α is minimal. The smooth flow of weight of the fixed point algebra M^α is the factor flow of F^M by* $\mathrm{mod} \cdot \alpha_K$. *There exists a minimal cocycle $c \in Z^1(F^{M^\alpha}, K)$ with $K_c = K$, unique up to equivalence, such that M is the skew product $M^\alpha \otimes_c L^\infty(K)$, and α is the dual action of the skew product.*

Proof. We use the same notation as in Lemma 5.4. Lemma A.1 implies that the smooth flow of weights of M is of the form $(X_M, \mu_M) = (Y \times K, \nu \times m_K)$ with the flow and $\mathrm{mod}(\alpha_k)$ given by

$$(\omega, l) \cdot t = (\omega \cdot t, lc(\omega, t)), \quad (\omega, l) \in X_M,\ t \in \mathbb{R},$$
$$(\mathrm{mod}(\alpha_k)f)(\omega, l) = (\omega, k^{-1}l), \quad k \in K,\ f \in L^\infty(X_M, \mu_M),$$

where c is a minimal cocycle with $K_c = K$. According to the above splitting of (X_M, μ_X), we set $C = L^\infty(K) \subset Z(\widetilde{M})$ and $Z = L^\infty(Y, \nu) \subset Z(\widetilde{M})$ that is the fixed point subalgebra of $Z(\widetilde{M})$ under $\widetilde{\alpha}$. Note that the factor flow $(Z, \theta|_Z)$ and the class of c are uniquely determined by F^M and $\mathrm{mod} \cdot \alpha$. Since C is an equivariant copy of $L^\infty(K)$, the Landstad's type theorem for coactions [58, Chapter II, Section 2] implies that $\widetilde{M}$ is the crossed product $\widetilde{N} \rtimes \widehat{K}$ by a coaction of K, where we use the fact that $\widetilde{N}$ is the fixed point subalgebra of $\widetilde{M}$ under $\widetilde{\alpha}$. Moreover, since $\widetilde{N}$ commutes with C, the coaction is trivial and we actually have $\widetilde{M} \cong \widetilde{N} \otimes C$, where $\widetilde{N}$ and C are identified with $\widetilde{N} \otimes \mathbb{C}$ and $\mathbb{C} \otimes C$ in the right-hand side respectively. This implies $Z = Z(\widetilde{N})$, $\widetilde{M} \cap \widetilde{N}' = Z(\widetilde{M})$ and $\widetilde{M} = (\widetilde{M} \cap \widetilde{N}') \vee \widetilde{N}$. Since θ acts on Z ergodically, N is a factor. If N were semifinite, c would be a coboundary, which would imply that K is trivial and $M = N$. This is contradiction because M is of type III. Therefore, N is a type III factor. Now, Theorem 5.5 and Lemma 5.8 show that M is the skew product $N \otimes_c L^\infty(K)$. Since the Galois group for $M \supset N$ coincides with α_K [63],[2] and every member α_k in the Galois group is specified by $\mathrm{mod}(\alpha_k)$, we conclude that α is the dual action of the skew product. □

Corollary 5.10. *If M is an AFD factor, then for every compact subgroup K of $\mathrm{Aut}(F^M)$, there exists a minimal action α of K on M such that $\mathrm{mod}\cdot\alpha_k = k$ for all $k \in K$. α is unique up to conjugacy: namely if α^1 also satisfies the condition, then there exists an automorphism $\nu \in \mathrm{Aut}(M)$ with a trivial Connes-Takesaki module such that*

$$\alpha_k = \nu \cdot \alpha^1_k \cdot \nu^{-1}, \quad k \in K.$$

Proof. This is an easy consequence of Theorem 5.5, Theorem 5.9, and the fact that AFD type III factors are determined by their flows [6], [7], [8], [40], [20]. □

As for the existence of a homomorphic lifting in the above corollary, a more general statement is known: there always exists a homomorphic lifting α from $\mathrm{Aut}(F^M)$ to $\mathrm{Aut}(M)$ for every AFD type III factor M [73].

Now we characterize (not necessarily finite index) subfactors whose canonical endomorphism is decomposed into modular endomorphisms. Let $M \supset N$ be an irreducible inclusion of factors with a unique normal conditional expectation from M onto N. We denote by M_1 and $\widehat{E}$ the basic extension and the dual operator valued weight from M_1 to M respectively [19], [41], [35, Section 2]. We say that the inclusion $M \supset N$ is *discrete* if the restriction of $E \cdot \widehat{E}$ on the relative commutant $M_1 \cap N'$ is semifinite. We say that a discrete inclusion is *unimodular* if $E \cdot \widehat{E}|_{M_1\cap N'}$ is invariant under $j(x) := J_M x^* J_M$, where J_M is the modular conjugation of M (see [35, Section 3] for conditions equivalent to these.)

Theorem 5.11. *Let $M \supset N$ be an inclusion of type* III *factors with a unique normal conditional expectation E from M onto N. Then, the following conditions are equivalent:*

1. *$M \supset N$ is a unimodular inclusion and the restriction of the canonical endomorphism $\gamma : M \longrightarrow N$ to N is decomposed into modular endomorphisms.*
2. *$M \supset N$ is a discrete inclusion and $\widetilde{M}$ is generated by $\widetilde{N}$ and $\widetilde{M} \cap \widetilde{N}'$.*
3. *$(\widetilde{M} \cap \widetilde{N}', \theta)$ is an extension of the flow $(Z(\widetilde{N}), \theta)$ with relatively discrete spectrum in the sense of Definition A.2, and $\widetilde{M}$ is generated by $\widetilde{N}$ and $\widetilde{M} \cap \widetilde{N}'$.*
4. *There exists a compact group K, a faithful ergodic action Ψ of K on a von Neumann algebra B, a minimal action α of K on a factor P such that the kernel of $\mathrm{mod}\cdot\alpha$ is trivial and*

$$(M \supset N) \cong ((P \otimes B)^{\alpha\otimes\Psi} \supset P^\alpha \otimes \mathbb{C}).$$

In the case (4), K is determined by the cocycle c corresponding to the system of modular endomorphisms appearing in the decomposition of $\gamma|_N$, P is the skew product $N \otimes_c L^\infty(K)$, and α is the dual action of the skew product. B and Ψ are uniquely determined from $(\widetilde{M} \cap \widetilde{N}' \supset Z(\widetilde{N}), \theta)$ by Theorem A.4.

When one (and all) of the above conditions holds, N is AFD if and only if M is AFD.

Proof. (1) $\Rightarrow$ (2). We assume that (1) holds and the irreducible decomposition of $\gamma|_N$ is given by

$$[\gamma|_N] = \bigoplus_{\xi\in\Xi} n_\xi[\rho_\xi].$$

Thanks to [35, pp. 39], the multiplicity n_ξ of ρ_ξ is always finite. We set

$$\mathcal{L}_\xi = \{X \in M;\ Xx = \rho_\xi(x)X,\ \forall x \in N\}.$$

Then, the dimension of $\mathcal{L}_\xi$ is n_ξ and N is generated by N and $\bigcup_{\xi\in\Xi}\mathcal{L}_\xi$ [35, Theorem 3.3]. Let $\{X(\xi)_i\}_{i=1}^{n_\xi}$ be an orthonormal basis of $\mathcal{L}_\xi$. We choose a dominant weight ψ_0 on N and assume that (ψ_0, ρ_ξ) is an invariant pair for every $\xi \in \Xi$ as before. Then, since $M \supset N$ is unimodular, [35, Theorem 3.3, Remark 3.4] imply that $\mathcal{L}_\xi$ is in the centralizer of $\psi = \psi_0 \cdot E$. By assumption, there exists an implementing system $\{V(\xi)_i\}_{i=1}^{d(\xi)} \subset \widetilde{N}$ for $\widetilde{\rho}_\xi$. As $X(\xi)_i$ is in the centralizer of ψ, it commutes with $\lambda^\psi(t)$, and so

$$V(\xi)_i^* X(\xi)_j \in \widetilde{M} \cap \widetilde{N}', \quad \xi \in \Xi,\ 1 \le i \le d(\rho_\xi),\ 1 \le j \le n_\xi.$$

Since $\widetilde{M}$ is generated $\widetilde{N}$ and $\cup_{\xi\in\Xi}\mathcal{L}_\xi$, it is also generated by $\widetilde{N}$ and $\widetilde{M} \cap \widetilde{N}'$.

(2) $\Rightarrow$ (3). We assume (2), and set $A = \widetilde{M}\cap\widetilde{N}'$, $Z = Z(\widetilde{N})$, $\epsilon = \widetilde{E}|_A$, and $\vartheta_t = \theta_t|_A$. Then, ϵ is a normal conditional expectation with the trace property $\epsilon(ab) = \epsilon(ba)$, $a, b \in A$. Let M_1 be the basic extension of $M \supset N$. As before, we regard $\widetilde{M}$ as a subalgebra of $\widetilde{M_1}$ identifying $\lambda^\psi(t)$ with $\lambda^{\psi\cdot\widehat{E}}(t)$. Then, $\widetilde{M_1}$ is the basic extension of $\widetilde{M} \supset \widetilde{N}$ with respect to $\widetilde{E}$ [35, Lemma 2.4], [17, Theorem 2.1].

On the other hand, we consider the inclusion $Z \subset A$ acting on the standard Hilbert space $L^2(A)$ of A. Then, the basic extension of this inclusion is $Z_1 = Z' \subset B(L^2(A))$. Let u be the canonical implementation of ϑ, and J_A be the modular conjugation of A. We use the same notation as in the proof of Lemma 5.8, and identify $\widetilde{M}$ with L. Note that Z_1 naturally acts on H_1 through the action on the second tensor component. In the same way as before, we get the following:

$$\widetilde{M_1} = J\widetilde{N}'J = J_{\widetilde{N}}FJ_{\widetilde{N}} \vee J_A Z_1 J_A = \widetilde{N} \vee Z_1,$$

where everything takes place in $B(H_1)$. Thus, $\widetilde{M_1} \cap \widetilde{N}' = Z_1$, and

$$(\widetilde{M_1} \cap \widetilde{N}', \theta, E \cdot \widehat{E}) = (Z_1, \mathrm{Ad}(u(t)), \epsilon \cdot \widehat{\epsilon}),$$

$$((M_1 \cap N')_{E\cdot\widehat{E}}, E \cdot \widehat{E}) = (Z_1 \cap u(\mathbb{R})', \epsilon \cdot \widehat{\epsilon}),$$

where $(M_1\cap N')_{E\cdot\widehat{E}}$ is the fixed point subalgebra of $M_1\cap N'$ under the relative modular automorphism group $\{\sigma_t^{E\cdot\widehat{E}}\}_{t\in\mathbb{R}}$ [19]. Since $M \supset N$ is discrete, [35, Theorem 3.3] implies that $(M_1 \cap N')_{E\cdot\widehat{E}}$ is a direct sum of matrix algebras and the restriction of $E\cdot\widehat{E}$ to $(M_1\cap N')_{E\cdot\widehat{E}}$ is still semifinite. Therefore, if $\epsilon\cdot\widehat{\epsilon}$ is the center valued trace $\natural$ of Z_1, (A, ϑ) is an extension of the flow F^N with relatively discrete spectrum. Indeed, let μ be a faithful normal state on $Z(\widetilde{N})$. Then using the spatial derivative, we can show that the modular automorphism group of $\mu\cdot\epsilon\cdot\widehat{\epsilon}$ is implemented by the modular operator $\Delta_{\mu\cdot\epsilon}^{it}$ of $\mu\cdot\epsilon$ on $L^2(A)$ [41]. However, since $\mu\cdot\epsilon$ is a trace and $\Delta_{\mu\cdot\epsilon}$ is trivial, $\mu\cdot\epsilon\cdot\widehat{\epsilon}$ is a trace as well. This means that $\epsilon\cdot\widehat{\epsilon} = \natural(z\cdot)$ holds with some density z

affiliated with $Z(\widetilde{N})$. Let $e_{Z(\widetilde{N})}$ be the Jones projection for ϵ, and $a \in A$. Then, $ae_A a^*$ is considered as a field of scalar multiples of rank one projections in the central decomposition of Z_1. Since

$$\epsilon \cdot \widehat{\epsilon}(ae_A a^*) = \epsilon(aa^*) = \epsilon(a^* a),$$

we actually have $z = 1$.

(3) $\Rightarrow$ (4). We assume (3). Thanks to Theorem A.4, there exists a compact group K, a minimal cocycle $c \in Z^1(F^N, K)$ with $K_c = K$, and a faithful ergodic action Ψ of K on a von Neumann algebra B such that

$$(\widetilde{M} \cap \widetilde{N}' \supset Z(\widetilde{N}), \theta) \cong ((Z(\widetilde{N}) \otimes B) \supset (Z(\widetilde{N}) \otimes \mathbb{C}), \Psi^c),$$

(see Theorem A.4 for the notations). Let P be the skew product $N \otimes_c L^\infty(K)$ and α be the dual action. We set $Q = (P \otimes B)^{\alpha \otimes \Psi}$. Then, thanks to ergodicity of Ψ, $Q \supset N$ is an irreducible inclusion of factors. Let τ_B be the unique Ψ-invariant trace [30]. Then, the restriction of $(\mathrm{id} \otimes \tau_B)$ to Q gives a normal conditional expectation from Q onto N. Thanks to Lemma 5.8, to prove the statement it suffices to show that $\widetilde{Q}$ is generated by $\widetilde{N}$ and $\widetilde{Q} \cap \widetilde{N}'$, and

$$(\widetilde{Q} \cap \widetilde{N}' \supset Z(\widetilde{N}), \theta) \cong (Z(\widetilde{N}) \otimes B \supset Z(\widetilde{N}) \otimes \mathbb{C}, \Psi^c).$$

Lemma 5.4 and Theorem 5.9 show that $\widetilde{Q}$ is actually given by

$$\widetilde{Q} = (\widetilde{P} \otimes B)^{\widetilde{\alpha} \otimes \Psi} = \widetilde{N} \otimes (L^\infty(K) \otimes B)^{Ad(l) \otimes \Psi},$$

where l is the left regular representation. Thus, we have

$$\widetilde{Q} \cap \widetilde{N}' = L^\infty(X_N, \mu_N) \otimes (L^\infty(K) \otimes B)^{\mathrm{Ad}(l) \otimes \Psi},$$

which shows $\widetilde{Q} = \widetilde{N} \vee (\widetilde{Q} \cap \widetilde{N}')$. Note that θ acts on B trivially and it acts on $Z(\widetilde{P}) = L^\infty(X_N, \mu_N) \otimes L^\infty(K)$ as in Theorem 5.9.

Now we assume that $Z(\widetilde{P}) \otimes B$ acts on $L^2((X_P, \mu_P), L^2(B))$, where $L^2(B)$ is the standard Hilbert space of B and $(X_P, \mu_P) = (X_N \times K, \mu_N \times m_K)$. Let u and v be the canonical implementations of F^P on $L^2(X_P, \mu_P)$ and Ψ on $L^2(B)$ respectively. Then, u is given by

$$(u(t)\xi)(\omega, k) = \frac{dt \cdot \mu_P}{d\mu_P}(\omega)^{1/2} \xi(\omega \cdot t, kc(\omega, t)), \quad \xi \in L^2(X_P, \mu_P).$$

We introduce a unitary operator W on $L^2(K, L^2(B))$ by

$$(W\eta)(k) = v(k)\eta(k), \quad \eta \in L^2(K, L^2(B)),$$

which satisfies $W(l(k) \otimes 1)W^* = l(k) \otimes v(k)$. Let Π be the restriction of $\mathrm{Ad}(W)$ on $L^\infty(K) \otimes B$, which is an automorphism of $L^\infty(K) \otimes B$. Then, the above relation implies

$$(L^\infty(K) \otimes B)^{\mathrm{Ad}(l) \otimes \Psi} = \Pi(\mathbb{C} \otimes B),$$

$$(r(k) \otimes 1)\Pi(1 \otimes b)(r(k)^* \otimes 1) = \Pi(1 \otimes \Psi_k(b)), \quad b \in B.$$

For $f \in L^\infty(X_N, \mu_N)$, $b \in B$, and $\xi \in L^2((X_N, \mu_N), L^2(K) \otimes L^2(B))$ we have

$$\begin{aligned}
&(u(t)(f \otimes \Pi(1 \otimes b))u(t)^*)\xi(\omega) \\
&= f(\omega \cdot t)(r(c(\omega, t)) \otimes 1)\Pi(1 \otimes b)(r(c(\omega, t))^* \otimes 1)\xi(\omega) \\
&= f(\omega \cdot t)\Pi(1 \otimes \Psi_{c(\omega,t)}(b))\xi(\omega).
\end{aligned}$$

Thus, we get

$$(\widetilde{Q} \cap \widetilde{N}' \supset Z(\widetilde{N}), \theta) \cong (Z(\widetilde{N}) \otimes B \supset Z(\widetilde{N}) \otimes \mathbb{C}, \Psi^c).$$

(4) $\Rightarrow$ (1). We assume (4) and set $N = P^\alpha$, $M = (P \otimes B)^{\alpha \otimes \Phi}$. Let τ_B be the unique Ψ-invariant trace on B, $L^2(B)$ be the GNS Hilbert space of τ_B, and Λ_{τ_B} be the natural map from B into $L^2(B)$. We denote by v the canonical implementation of Ψ on $L^2(B)$. For each $(\pi, H_\pi) \in \widehat{K}$, we denote by B_π the spectral subspace corresponding to π, which is the image of the map

$$E_\pi(x) = d(\pi) \int_K \mathrm{Tr}(\overline{\pi}(k))\Psi_k(x) dm_K(k), \quad x \in B.$$

It is known that the multiplicity of B_π is finite [30] and there exists an orthonormal basis $\{b(\pi, s)_i\} \subset B_\pi$, $1 \le s \le n_\pi$, $1 \le i \le d(\pi)$ with respect to the inner product of $L^2(B)$ such that

$$\Psi_k(b(\pi, s)_i) = \sum_{j=1}^{d(\pi)} \pi_{ji}(k) b(\pi, s)_j, \quad k \in K.$$

Since τ_B is given by the average of Ψ_k over K, the Peter-Weyl theorem implies

$$\sum_{i=1}^{d(\pi)} b(\pi, s)_i^* b(\pi, t)_i = \sum_{i=1}^{d(\pi)} b(\pi, t)_i b(\pi, s)_i^* = d(\pi)\delta_{s,t}.$$

Note that $\{b(\pi, s)_i\}_{\pi,s,i}$ form an orthonormal basis of $L^2(B)$. Though expansion of $x \in B$ with this basis does not converge in weak topology in general, we can still show that $\{b(\pi, s)_i\}_{\pi,s,i}$ generate B using the same argument as in [35, Lemma 3.8]. The point of the argument is that when a von Neumann subalgebra is globally invariant under the modular automorphism group of a state, the corresponding subspace in the GNS subspace determines the subalgebra. We use this principle in the rest of the proof without mentioning it.

Let E_0 be the unique normal conditional expectation from P onto N, φ_0 be a faithful normal state on N, and $\varphi = \varphi_0 \cdot E_0$. We denote by $L^2(P)$, Λ_φ, and w the GNS Hilbert space of φ, the natural map from P into $L^2(P)$, and the canonical implementation of α on $L^2(P)$ respectively. We regard the subspace of $w \otimes v$-invariant vectors in $L^2(P) \otimes L^2(B)$ as the GNS Hilbert space $L^2(M)$ of the restriction of $\varphi \otimes \tau_B$ to M. Let β be the Roberts action of $\widehat{K}$ on N whose dual action is α, and $\{W(\pi)_i\}_{i=1}^{d(\pi)} \subset P$, $(\pi, H_\pi) \in \widehat{K}$ be an orthonormal basis of $\mathcal{H}_\pi \subset P$ as before. Thanks to Theorem 5.9, β_π is a modular endomorphism for all $(\pi, H_\pi) \in \widehat{K}$. Note that $N\mathcal{H}_\pi$ is globally invariant under the modular automorphism group $\{\sigma_t^\varphi\}_{t \in \mathbb{R}}$ [35, Lemma 3.8]. Let

$$\mathcal{L}_\pi = \{X \in M;\ Xx = \beta_\pi(x)X,\ \forall x \in N\}.$$

Thanks to [35, Theorem 3.3], if M is generated by N and $\bigcup_{(\pi,H_\pi)\in\widehat{K}}\mathcal{L}_\pi$, $M\supset N$ is a discrete inclusion.

Now we set

$$X(\pi)_s=\frac{1}{\sqrt{d(\pi)}}\sum_{i=1}^{d(\pi)}W(\pi)_i\otimes b(\pi,s)_i^*\in\mathcal{L}_\pi.$$

Then, $\{X(\pi)_s\}_{s=1}^{n_\pi}$ are isometries with mutually orthogonal ranges. The definition of the crossed product by a Roberts action implies

$$L^2(P)=\bigoplus_{(\pi,H_\pi)\in\widehat{K}}\overline{\Lambda_\varphi(\mathcal{H}_\pi^*N)}^{||\cdot||}.$$

Therefore, direct computation using the orthonormal basis $\{b(\pi,s)_i\}_{\pi,s,i}$ yields

$$L^2(M)=\bigoplus_{(\pi,H_\pi)\in\widehat{K}}\overline{\Lambda_{\varphi\otimes\tau_B}(\mathcal{L}_\pi^*N)}^{||\cdot||}.$$

Thus, M is generated by N and $\bigcup_{(\pi,H_\pi)\in\widehat{K}}\mathcal{L}_\pi$, and so $M\supset N$ is a discrete inclusion.

Let E be the restriction of $\mathrm{id}\otimes\tau_B$ to M, which is a conditional expectation from M onto N. Then, we have

$$\begin{aligned} d(\pi)E(X(\pi)_sX(\pi)_t^*) &= \sum_{i,j=1}^{d(\pi)}\tau_B(b(\pi,s)_i^*b(\pi,t)_j)W(\pi)_iW(\pi)_j^* \\ &= \delta_{s,t}\sum_{i=1}^{d(\pi)}W(\pi)_iW(\pi)_i^* \\ &= \langle X(\pi)_s,X(\pi)_t\rangle. \end{aligned}$$

This shows that the inclusion is unimodular [35, Theorem 3.3].

When N is AFD, $\widetilde{N}$ is AFD and $\widetilde{M}$ is AFD thanks to the proof of Lemma 5.8 and Theorem A.4. Therefore, M is AFD. Since there exists a conditional expectation from M onto N, the converse also holds [7]. □

Remark 5.12. In the above, if $\widetilde{M}\cap\widetilde{N}'$ is commutative, B in (4) is commutative, and there exists a unique closed subgroup $L\subset K$ such that $B=L^\infty(K/L)$ where K action is given by the right translation. Thus,

$$(M\supset N)\cong(P^{\alpha|_L}\supset P^\alpha).$$

This characterizes an inclusion coming from a (commutative) ergodic extension of the flow of weights. For related topics, see [25], [26], [33], [27], [42], [45], [71], [66], [77]. Under the condition that the kernel of $\mathrm{mod}\cdot\alpha$ is trivial, (and L does not contain non-trivial normal subgroups, which is regarded as a trivial constraint), $L\subset K$ are unique up to conjugacy. However, if we drop this condition, there are essentially different description of $M\supset N$ in terms of a group-subgroup subfactor [46], [36].

Another easy but interesting case is when B is a matrix algebra, that is, $M\supset N$ is a Wassermann's type subfactor. In this case, M and N have the common flow and graph change occurs (c.f. [57]).

5.3. **Connes-Takesaki modules for Roberts actions.** In this subsection, we show how the Connes-Takesaki modules of endomorphisms give constraints to minimal actions of compact groups, in particular semisimple compact Lie groups.

Let K be a compact group and β be a Roberts action of $\widehat{K}$ on a factor N such that β_π is irreducible and has a Connes-Takesaki module for every $(\pi, H_\pi) \in \widehat{K}$. We denote $m_\beta(\pi) = \mathrm{mod}(\beta_\pi)$ for simplicity. Then, thanks to Proposition 4.2, $\{m_\beta(\pi)\}_{(\pi,H_\pi)\in\widehat{K}}$ form a group, which is denoted by $m_\beta(\widehat{K})$. We treat $m_\beta(\widehat{K})$ as a discrete group, and we denote by $\widehat{m_\beta}$ its dual group.

Theorem 5.13. *Let K be a compact group and β be a Roberts action of $\widehat{K}$ on a factor N such that β_π is irreducible for every $(\pi, H_\pi) \in \widehat{K}$. Then,*

1. *If K is connected, β_π has a Connes-Takesaki module for every $(\pi, H_\pi) \in \widehat{K}$.*
2. *If β_π has a Connes-Takesaki module for every $(\pi, H_\pi) \in \widehat{K}$, then there exists a continuous isomorphism ν from $\widehat{m_\beta}$ into the center $Z(K)$ of K satisfying*

$$\chi(m_\beta(\pi))1_\pi = \pi(\nu(\chi)), \quad \chi \in \widehat{m_\beta},\ (\pi, H_\pi) \in \widehat{K}.$$

Proof. (1). Suppose that there exists $(\pi, H_\pi) \in \widehat{K}$ such that β_π does not have a Connes-Takesaki module. Let $N = \beta_\pi(M)$, and we consider the Loi-Hamachi-Kosaki decomposition of $M \supset N$ [47], [25], [26] as follows:

$$M \supset P \supset Q \supset N,$$

where P and Q are determined by the conditions

$$\widetilde{Q} = Z(\widetilde{M} \cap \widetilde{N}') \vee \widetilde{N},$$

$$\widetilde{P} = \widetilde{M} \cap Z(\widetilde{M} \cap \widetilde{N}')'.$$

Since β_π does not have a Connes-Takesaki module, either $M \neq P$ or $Q \neq N$ occurs. We may assume $M \neq P$ by considering $\beta_{\overline{\pi}}$ if necessary. Then, $\beta_\pi \cdot \beta_{\overline{\pi}}$ contains the the canonical endomorphism γ for $M \supset P$. Note that the dual inclusion $P_1 \supset M$ satisfies the condition of Theorem 5.11 and $(Z(\widetilde{P}_1 \cap \widetilde{M}') = Z(\widetilde{P}_1), \theta)$ is a finite to one extension of the flow F^M [25], [26]. Thus, Theorem 5.11 implies that γ generates a Roberts action of the dual $\widehat{G}$ of a finite group G, which is a subset of $\{\beta_\pi\}_{(\pi,H_\pi)\in\widehat{K}}$. We regard $\widehat{G}$ as a subset of $\widehat{K}$. We define a closed normal subgroup $L \subset K$ by

$$L = \bigcap_{(\pi,H_\pi)\in\widehat{G}} \mathrm{Ker}(\pi).$$

Since $\widehat{G}$ is a finite set of irreducible representations closed under complex conjugate and irreducible decomposition of products of any members in $\widehat{G}$, the quotient group K/L is finite [28, Section 30]. However, this is contradiction because K is connected. Therefore, β_π has a Connes-Takesaki module for every $(\pi, H_\pi) \in \widehat{K}$.

(2). For $\chi \in \widehat{m_\beta}$, we consider a family of unitary operators

$$\{\chi(m_\beta(\pi))1_\pi\}_{(\pi,H_\pi)\in\widehat{K}}.$$

Then, thanks to Theorem 4.2 and the Tannaka duality theorem [28, Section 30], there exists a unique element $\nu(\chi) \in K$ such that $\chi(m_\beta(\pi))1_\pi = \pi(\nu(\chi))$ holds. Since $\pi(\nu(\chi))$ is a scalar for every irreducible π, $\nu(\chi)$ is in the center $Z(K)$. ν is clearly a homomorphism from $\widehat{m_\beta}$ into $Z(K)$, which is injective as the set of irreducible representations separates points of K. ν is continuous because the topology of K is the same as that induced by the weak topology of the image of K by the direct sum representation of all members in $\widehat{K}$. $\square$

Corollary 5.14. *Let α be a minimal action of a compact connected semisimple Lie group K on a type* III *factor M. Then,*

(1) *If M is of type* III_1, *so is M^α.*
(2) *If M is of type* III_λ, $0 < \lambda < 1$, *there exists a positive integer n such that M^α is of type* III_{λ^n}. *Let β be the Roberts action of $\widehat{K}$ on M^α whose dual action is α, then $m_\beta(\widehat{K}) \cong \widehat{m_\beta} \cong \mathbb{Z}/n\mathbb{Z}$.*
(3) *If M is of type* III_0, *so is M^α. (The flow of M and M^α could be very much different in this case.)*

Proof. Since the center of a semisimple Lie group is a finite group, Proposition 5.2, (2) implies that $N := M^\alpha$ is of type III. As before, we regard $\widetilde{N}$ as a subalgebra of $\widetilde{M}$ using a unique conditional expectation E. Let β be the Roberts action of $\widehat{K}$ on $N = M^\alpha$ whose dual action is α.

We first assume that N is of type III_λ, $0 < \lambda \leq 1$. Since every non-trivial modular endomorphism of N is a composition of an inner automorphism and a modular automorphism in this case, no non-trivial modular endomorphisms appear in $\{\beta_\pi\}_{(\pi,H_\pi)\in\widehat{K}}$ because a connected semisimple Lie group has no non-trivial 1 dimensional representation. Thus, the same computation as in the proof of Theorem 5.5 implies

$$\widetilde{M} \cap \widetilde{N}' = Z(\widetilde{M}).$$

Thanks to Theorem 5.13, β_π has a Connes-Takesaki module for every $(\pi, H_\pi) \in \widehat{K}$, which implies

$$Z(\widetilde{M}) = Z(\widetilde{N})^{m_\beta(\widehat{K})}.$$

This shows that if N is of type III_1, so is M. When N is of type III_λ, $0 < \lambda < 1$, we have $\mathrm{Aut}(F^N) \cong \mathbb{T}$. Since $Z(K)$ is a finite group, $\widehat{m_\beta}$ is finite and its dual group $m_\beta(\widehat{K})$ is a finite subgroup of $\mathbb{T}$. Thus, $m_\beta(\widehat{K})$ is a finite cyclic group and so is $\widehat{m_\beta}$.

Now we assume that N is of type III_0. Then, in the same way as above we can show

$$Z(\widetilde{M}) \supset Z(\widetilde{N})^{m_\beta(\widehat{K})}.$$

Since $m_\beta(\widehat{K})$ is a finite group, $(Z(\widetilde{N})^{m_\beta(\widehat{K})}, \theta)$ is a non-transitive ergodic flow and so is F^M. Therefore, M is of type III_0. $\square$

We give a simple example of a minimal action of a compact group with non-trivial $m_\beta(\widehat{K})$. Let λ be a positive number with $0 < \lambda < 1$, and set $T = -2\pi/\log\lambda$. We define a state of the 3 by 3 matrix algebra $M(3, \mathbb{C})$ by

$$\varphi_0(x) = \mathrm{Tr}(\mathrm{Diag}(\frac{1}{1+2\lambda}, \frac{\lambda}{1+2\lambda}, \frac{\lambda}{1+2\lambda})x),$$

where Diag means a diagonal matrix. We set

$$(M,\varphi)=\bigotimes_{n=1}^{\infty}(M(3,\mathbb{C}),\varphi_0),$$

which is a Powers factor of type III_λ. For $g\in SU(2)$, we define

$$\alpha_g=\bigotimes_{n=1}^{\infty}(\mathrm{Ad}(\mathrm{Diag}(1,g))).$$

Then, α is a minimal action of $SU(2)$. It is easy to show $\sigma^{\varphi}_{T/2}=\alpha_{-1}$, which implies that $P=M^{\alpha_{-1}}$ is of type III_{λ^2}. Note that the restriction of α to P induces a minimal action of $SO(3)$, whose center is trivial. Thus, Corollary 5.14 implies that M^α and P has the common flow and M^α is of type III_{λ^2}. Corollary 5.14 again implies that the endomorphism in the Roberts action corresponding to the spin $1/2$ representation has a non-trivial Connes-Takesaki module.

Appendix A. Miscellaneous results in ergodic theory

In this appendix, we collect some results from ergodic theory, (or its simple non-commutative generalization), used in the main body of the present notes. We provide proofs here because we could not find appropriate references, though some (or all) of them are probably well-known to specialists.

For a locally compact group G, we say that (X,μ) is a G-space if X is a standard Borel space, μ is a probability measure on X, and G acts on (X,μ) as a non-singular Borel transformation group. We say that a G-action is faithful if the corresponding action on $L^\infty(X,\mu)$ is faithful. We say that two G-spaces (X_1,μ_1) and (X_2,μ_2) are isomorphic if the corresponding G-actions on the von Neumann algebras (or equivalently, measure algebras) $L^\infty(X_1,\mu_1)$ and $L^\infty(X_2,\mu_2)$ are conjugate. For $g\in G$ and a Borel subset $E\subset X$, we use the notation $g\cdot\mu(E):=\mu(E\cdot g)$.

A.1. **Compact automorphism groups of ergodic transformation groups.**

Lemma A.1. *Let G be a locally compact group and (X,μ) be an ergodic G-space. We assume that a compact group K faithfully acts on (X,μ) commuting with G. We use the notation with G acting on X from right and K acting on X from left. Let (Y,ν) be the factor space of (X,μ) by K. Then, there exists a minimal K-valued cocycle $c: Y\times G\longrightarrow K$ with $K_c=K$ such that (X,μ) is isomorphic to the skew product $(Y\times_c K,\nu\times m_K)$, that is, under this isomorphism, the factor map from X onto Y corresponds to the projection onto the first component, and the $K\times G$-action corresponds to the following action:*

$$k\cdot(\omega,l)=(\omega,kl),\quad \omega\in Y,\ k,l\in K,$$

$$(\omega,l)\cdot g=(\omega\cdot g,lc(\omega,g)),\quad g\in G.$$

c is unique up to equivalence.

Proof. Considering the Gelfand spectrum of an appropriate weakly dense separable C^*-subalgebra of $L^\infty(X,\mu)$, we may and do assume that X is a compact metric space, on which $G\times K$ acts continuously, and that μ is invariant under K. Since K is compact, every K-orbit is closed and there exists a Borel subset $Y\subset X$ that meets each K-orbit exactly once (see, for example, [3, Section 3.4]). Let $\mathcal{K}$ be the set of all closed subgroups of K. Then, $\mathcal{K}$ has a Polish space structure such that K continuously acts on $\mathcal{K}$ by $k\cdot H=kHk^{-1}$, $H\in\mathcal{K}$, $k\in K$ [16]. We denote by $[H]$ the class of H in the quotient space of $\mathcal{K}/K$. For each $\omega\in X$, let $K_\omega\subset K$ be the stabilizer subgroup of ω. Then, thanks to [4, Chapter II, Proposition 2.3], the map

$$X\ni\omega\mapsto[K_\omega]\in\mathcal{K}/K$$

is a Borel map satisfying $[K_{\omega\cdot g}]=[K_\omega]$ for $g\in G$. Thanks to the ergodicity of the G-action, there exist $H\in\mathcal{K}$ and $K\times G$-invariant Borel null set $E\subset X$ such that for every $\omega\in X_0:=X\setminus E$, we have $[K_\omega]=[H]$. We set $Y_0=Y\cap X_0$.

Let $N(H)$ be the normalizer subgroup of H in K. Choosing a Borel cross section from $N(H)\backslash K$ to K, we have a Borel map $\varphi:Y_0\longrightarrow K$ such that $\varphi(\omega)\cdot H=K_\omega$. We set

$$Y_1=\{\varphi(\omega)^{-1}\cdot\omega\in X_0;\omega\in Y_0\}.$$

Then, Y_1 is a Borel subset of X_0 that meets each K-orbit exactly once and $K_\omega=H$ for all $\omega\in Y_1$. Choosing a Borel cross section $\psi:H\backslash K\longrightarrow K$, we set

$$\Phi(\omega,kH)=\psi(kH)\cdot\omega,\quad \omega\in Y_1,\ kH\in H\backslash K.$$

Thanks to [3, pp. 72, Corollary], Φ is a Borel isomorphism from $Y_1\times H\backslash K$ onto X_0. Since μ is K-invariant, $\mu\cdot\Phi$ is of the form $\nu\times m_{H\backslash K}$, where $m_{H\backslash K}$ is the (normalized) Haar measure of $H\backslash K$. We introduce a $K\times G$-action on $Y_1\times H\backslash K$ from the action on X_0 through Φ.

When we regard (Y_1,ν) as the factor space of X_0 by the K-action, we denote the action of $g\in G$ on Y_1 by $\omega\circ g$, $\omega\in Y_1$. Then, for each $\omega\in Y_1$ and $g\in G$, there exists a unique $c_{(\omega,g)}H\in H\backslash K$ such that $\omega\cdot g=c_{(\omega,g)}\cdot(\omega\circ g)$. Using a Borel cross section from $H\backslash K$ to H again, we can take $c(\omega,g)\in K$ to be a Borel map from $Y_1\times G$ to K. Let $\pi:K\longrightarrow H\backslash K$ be the quotient map. Commutativity of the K-action and G-action implies that for each fixed $g\in G$, c takes its values in $N(H)$ almost everywhere. Moreover, $\pi\cdot c$ is an $N(H)/H$-valued cocycle and

$$(\omega,\pi(k))\cdot g=(\omega\circ g,\pi(kc(\omega,g))),\quad \omega\in Y_1,\ g\in G,\ k\in K.$$

This implies that for every subset $U\subset K$, $Y_1\times\pi(U\cdot N(H))$ is G-invariant. Thus, thanks to the ergodicity of the G-action, we conclude that whenever U is an open subset of K,

$$m_K(U\cdot N(H))=m_{H\backslash K}(\pi(U\cdot N(H)))=1,$$

which implies $N(H)=K$, that is, H is a normal subgroup of K. Since the K-action is faithful, we get $H=\{e\}$.

Uniqueness of c up to equivalence is obvious. ∎

A.2. **Non-commutative extensions with relatively discrete spectrum.** In [80], Zimmer introduced the notion of ergodic extensions with relatively discrete spectrum. The following is a slight generalization of this notion to the non-commutative case.

Definition A.2. Let G be locally compact group and (X,μ) be an ergodic G-space. Let A be a von Neumann algebra with an ergodic G-action Φ such that the center of A equivariantly includes $Z = L^\infty(X,\mu)$, that is

$$\Phi_g(f)(\omega) = f(\omega\cdot g), \quad f \in Z,\ \omega \in X,\ g \in G.$$

Let u be the canonical implementation of Φ on the standard Hilbert space for A. We say that A is an *extension of Z with relatively discrete spectrum* if the von Neumann algebra $u(G)'\cap Z'$ is decomposed into a direct sum of type I factors, and for each minimal projection $p \in u(G)'\cap Z'$, $\natural(p)$ is bounded, where $\natural$ is the (non-normalized) center valued trace on Z', (in other words, the restriction of $\natural$ to $u(G)'\cap Z'$ is semifinite).

Note that the above definition coincides with Zimmer's one [80, Definition 4.2] when A is commutative (although the two definitions appear different).

Lemma A.3. *Let G be a locally compact group, K be a compact group, (X,μ) be a G-space, and $c : X\times G \longrightarrow K$ be a cocycle. If Ψ is an action of K on a von Neumann algebra B standardly acting on a Hilbert space H, then there exists an action Ψ^c of G on $L^\infty(X,\mu)\otimes B$ such that for all $g\in G$, $f\in L^\infty(X,\mu)$, $x\in B$, $\xi\in L^2((X,\mu),H)$,*

$$(\Psi^c_g(f\otimes x)\xi)(\omega) = f(\omega\cdot g)\Psi_{c(\omega,g)}(x)\xi(\omega)$$

holds for almost every $\omega\in X$.

Proof. Let v be the canonical implementation of Ψ on H. We introduce a unitary $u(g)$, $g\in G$ on $L^2((X,\mu),H)$ by

$$(u(g)\xi)(\omega) = \frac{dg\cdot\mu}{d\mu}(\omega)^{1/2} v(c(\omega,g))\xi(\omega\cdot g), \quad \xi\in L^2((X,\mu),H),\ \omega\in X.$$

Then, u is a representation of G, which is a Borel map from G to the unitary group of $B(L^2((X,\mu),H))$ with respect to the Borel structure coming from the weak (and strong) topology. Since any unitary Borel representation of a locally compact group is continuous, (or more strongly, any Borel homomorphism between Polish groups is continuous [4, Chapter I, Propositon 3.3]), so is u. Ψ^c_g is defined by the restriction of $\mathrm{Ad}(u(g))$ to $L^\infty(X,\mu)\otimes B$. $\square$

The proof of (1) of the following theorem is a straightforward generalization of Zimmer's argument in [80, Theorem 4.3] except for the proof of uniqueness, which does not exist there. (Zimmer's comment about uniqueness in [80, pp.401] does not seem to be relevant.)

Theorem A.4. *Let G be a locally compact group, (X,μ) be an ergodic G-space, and Φ be an action of G on a von Neumann algebra A. We assume that A is an extension of $Z := L^\infty(X,\mu)$ with relatively discrete spectrum. Then,*

(1) *There exist a compact group K, a faithful ergodic action Ψ of K on a von Neumann algebra B, and a minimal cocycle $c : X \times G \longrightarrow K$ with $K_c = K$ such that*

$$(Z \subset A, \Phi) \cong (Z \otimes \mathbb{C} \subset Z \otimes B, \Psi^c),$$

that is, there exists an isomorphism from A onto $Z \otimes B$ that intertwines Φ and Ψ^c, and is identity on Z.

(2) *A is always an injective von Neumann algebra, and there exists a unique faithful normal conditional expectation ϵ from A onto Z satisfying $\Phi_g \cdot \epsilon = \epsilon \cdot \Phi_g$, $g \in G$ and $\epsilon(ab) = \epsilon(ba)$, $a, b \in A$.*

(3) *(K, c, Ψ, B) satisfying the condition of (1) are unique in the following sense: if (K^1, c^1, Ψ^1, B^1) satisfy the same condition, then there exist a group isomorphism ν from K onto K_1 and an isomorphism σ from B onto B^1 such that $\nu \cdot c$ and c^1 are equivalent and $\Psi^1_{\nu(k)} \cdot \sigma = \sigma \cdot \Psi_k$, $k \in K$.*

Proof. (1). By assumption, we have a direct sum decomposition

$$Z' \cap u(G)' = \bigoplus_{\lambda \in \Lambda} M_\lambda,$$

where M_λ is a type I factor. We take a system of matrix units $\{e^\lambda_{ij}\}_{i,j=1}^{n_\lambda}$ of M_λ with n_λ either a natural number or possibly infinite such that $\{e^\lambda_{ii}\}_{i=1}^{n_\lambda}$ form a partition of unity consisting of minimal projections of M_λ. Since $\mathrm{Ad}(u(g))$, $g \in G$ normalizes Z, it does Z' as well, and we set Φ'_g to be the restriction of $\mathrm{Ad}(u(g))$ to Z'. Then, we have $\Phi_g \cdot \natural = \natural \cdot \Phi'_g$, and in particular,

$$\Phi_g(\natural(e^\lambda_{11})) = \natural(e(\lambda)_{11}), \quad \forall g \in G.$$

Thus, by ergodicity $\natural(e^\lambda_{11})$ is a constant, say $d(\lambda) \in \mathbb{N}$.

In what follows, when we claim that "some statement involving $g \in G$ and $\omega \in X$ holds for almost all $\omega \in X$", we always mean that for each fixed $g \in G$, there exists a conull set $E_g \subset X$ such that the statement holds for all $\omega \in E_g$. The annoying fact that E_g varies according to $g \in G$ is usually taken care of by Mackey's argument in [54]. We consider the disintegration of H over Z

$$H = \int_X^\oplus H(\omega) d\mu(\omega).$$

Then, $\natural(e^\lambda_{11}) = d(\lambda)$ means that the dimension of $e^\lambda_{11}(\omega)H(\omega)$ is $d(\lambda)$ for almost all $\omega \in X$. We take $\xi^\lambda_a = (\xi^\lambda_a(\omega)) \in H$, $a = 1, 2, \cdots, d(\lambda)$ such that for almost all $\omega \in X$ $\{\xi^\lambda_a(\omega)\}_{a=1}^{d(\lambda)}$ is an orthonormal basis of $e^\lambda_{11}(\omega)H(\omega)$. Since $u(g)$ is a unitary satisfying $(u(g)fu(g)^*)(\omega) = f(\omega \cdot g)$ for $f \in Z$, for $\xi, \eta \in H$ and $f \in Z$ we have, on one hand,

$$\langle u(g)f\xi, u(g)\eta\rangle = \int_X f(\omega \cdot g)\langle (u(g)\xi)(\omega), (u(g)\eta)(\omega)\rangle d\mu(\omega),$$

and on the other hand,

$$\begin{aligned}\langle u(g)f\xi, u(g)\eta\rangle &= \langle f\xi,\eta\rangle = \int_X f(\omega)\langle\xi(\omega),\eta(\omega)\rangle d\mu(\omega)\\ &= \int_X f(\omega\cdot g)\frac{dg\cdot\mu}{d\mu}(\omega)\langle\xi(\omega\cdot g),\eta(\omega\cdot g)\rangle d\mu(\omega).\end{aligned}$$

This implies,

$$\langle (u(g)\xi)(\omega), (u(g)\eta)(\omega)\rangle = \frac{dg\cdot\mu}{d\mu}(\omega)\langle\xi(\omega\cdot g),\eta(\omega\cdot g)\rangle,$$

for almost all ω. Thus,

$$\{\frac{dg\cdot\mu}{d\mu}(\omega)^{-1/2}(u(g)\xi_a^\lambda)(\omega)\}_{i=1}^{d(\lambda)}$$

is an orthonormal basis of $e_{11}^\lambda(\omega)H(\omega)$ for almost all ω. We set

$$c^\lambda(\omega,g)_{ab} = \frac{dg\cdot\mu}{d\mu}(\omega)^{-1/2}\langle(u(g)\xi_a^\lambda)(\omega),\xi_b^\lambda(\omega\cdot g)\rangle.$$

Then, $c^\lambda(\omega,g) = (c^\lambda(\omega,g)_{ab})$ is an element of $Z^1(G,U(d(\lambda)))$. We define a compact group $\mathcal{K}$ and a cocycle c by

$$\mathcal{K} = \prod_{\lambda\in\Lambda} U(d(\lambda)),$$

$$c = (c^\lambda)_{\lambda\in\Lambda} \in Z^1(G,\mathcal{K}).$$

Changing the system $\{\xi_a^\lambda\}_{a=1}^{d(\lambda)}$ if necessary, we may and do assume that c is minimal. We denote by K the minimal subgroup of c. Note that c^λ is minimal for all $\lambda\in\Lambda$ and that if $\lambda_1\neq\lambda_2$, c_{λ_1} and c_{λ_2} are not equivalent (otherwise, $Z'\cap u(G)'$ would be bigger).

Let H_λ^1 and H_λ^2 be Hilbert spaces with basis $\{f_a^\lambda\}_{a=1}^{d(\lambda)}$ and $\{e_i^\lambda\}_{i=1}^{n_\lambda}$ respectively. We set

$$H_0 = \bigoplus_{\lambda\in\Lambda} H_\lambda^1\otimes H_\lambda^2.$$

Since $\{(e_{i1}^\lambda\xi_a^\lambda)(\omega)\}_{\lambda,i,a}$ is an orthonormal basis of $H(\omega)$ for almost all ω, we identify $(e_{i1}^\lambda\xi_a^\lambda)(\omega)$ with $f_a^\lambda\otimes e_i^\lambda$, and H with $L^2(X,\mu)\otimes H_0$. There exists a natural embedding of $\mathcal{K} = \prod_{\lambda\in\Lambda} U(H_\lambda^1)$ in the group $U(H_0)$ of all unitaries H_0 as those unitaries acting "only on H_λ^1, $\lambda\in\Lambda$". Through this embedding, we also regard K as a subgroup of $U(H_0)$. Then, we have

$$(u(g)\xi)(\omega) = (\frac{dg\cdot\mu}{d\mu}(\omega))^{1/2}c(\omega,g)\xi(\omega\cdot g),\quad \xi\in L^2((x,\mu),H_0), \tag{A.1}$$

for almost all $\omega\in X$. Let

$$A = \int_X^\oplus A(\omega)d\mu(\omega)$$

be the disintegration of A over $Z = L^\infty(X,\mu)\otimes\mathbb{C}1_{H_0}$. Then, the map $X \ni \omega \mapsto A(\omega)$ is measurable with respect to the Effros Borel structure of the set of von Neumann algebras on H_0 [59, Chapter 4], [75, Chapter 4, Section 8]. For $x \in A$, (A.1) implies

$$\alpha_g(x)(\omega) = c(\omega,g)x(\omega\cdot g)c(\omega,g)^{-1}, \tag{A.2}$$

for almost all ω. Since A has countable generators (we assume that Hilbert spaces are separable throughout this paper), this implies

$$c(\omega,g)A(\omega\cdot g)c(\omega,g)^{-1} = A(\omega), \tag{A.3}$$

for almost all ω.

Using the Effros Borel structure in the same way as in the proof of [80, Theorem 4.3] and passing to an equivalent cocycle if necessary (and changing $A(\omega)$ and $c(\omega,g)$ on a null set), we may and do assume that there exists a von Neumann algebra $B \subset B(H_0)$ such that $A(\omega) = B$ and $c(\omega,g)$ normalizes B for all $\omega \in X$. Since c is a minimal cocycle with $K_c = K$, K normalizes B, and so we define Ψ to be the restriction of $\mathrm{Ad}(k)$, $k \in K$ to B. Then, (A.2) implies $\Phi = \Psi^c$. Since Φ is ergodic, so is Ψ.

(2). Thanks to [30], B is injective and so is A.

Let ϵ be a normal conditional expectation from A onto Z. Then, since H is the standard Hilbert space of A, there exists $\xi = (\xi(\omega)) \in L^2((X,\mu),H_0)$ whose vector state is $\mu\cdot\epsilon$, where μ means the state of Z corresponding to the measure μ. Thus, for all $x \in A$ and $f \in Z$ we have

$$\int_X f(\omega)\epsilon(x)(\omega)d\mu(\omega) = \langle \epsilon(fx)\xi,\xi\rangle. \tag{A.4}$$

We introduce a measurable field of normal states $\{\varphi_\omega\}$ of B by

$$\varphi_\omega(b) = \langle b\xi(\omega),\xi(\omega)\rangle, \quad b \in B.$$

Then, (A.4) implies $\epsilon(x)(\omega) = \varphi_\omega(x(\omega))$ for almost all ω. Moreover, if ϵ commutes with Φ_g, we get

$$\varphi_{\omega\cdot g} = \varphi_\omega\cdot\Psi_{c(\omega,g)},$$

for almost all ω. Let $\mathfrak{S}(B)$ be the set of all normal states of B, which is a Polish space in the norm topology. Since Ψ is continuous in u-topology, K acts on $\mathfrak{S}(B)$ as a continuous transformation group. Applying the same argument as above to $\mathfrak{S}(B)$ and passing to an equivalent cocycle (and changing φ_ω and $c(\omega,g)$ on a null set), we may assume that there exists $\varphi \in \mathfrak{S}(B)$ such that $\varphi_\omega = \varphi$ for all ω and $\varphi\cdot\Psi_{c(\omega,g)} = \varphi$ holds. Thus, φ is a Ψ-invariant state, which is unique because of ergodicity of Ψ. Thus, $\epsilon = \mathrm{id}\otimes\varphi$ is uniquely determined. ϵ has the trace property as φ is a trace thanks to [30].

(3). We assume that (A,Φ) are also given by $A = Z\otimes B^1$, $\Phi = (\Psi^1)^{c^1}$. Let τ be the unique Ψ^1-invariant trace, $L^2(B^1)$ be the GNS Hilbert space of τ, and $v(k)$ be the canonical implementation of Ψ^1_k, $k \in K^1$. Then, we have $H = L^2(X,\mu)\otimes L^2(B^1)$ and u is given by

$$u(g)\xi(\omega) = \frac{dg\cdot\mu}{d\mu}(\omega)^{1/2}v(c^1(\omega,g))\xi(\omega\cdot g), \quad \xi \in L^2((X,\mu),L^2(B^1)).$$

Since K^1 is a compact group, we have the irreducible decomposition

$$(L^2(B^1), v(k)) = (\bigoplus_{(\pi,H_\pi)\in\widehat{K}^1} H_\pi \otimes L_\pi, \bigoplus_{(\pi,H_\pi)\in\widehat{K}^1} \pi(k)\otimes 1_{L_\pi}),$$

where L_π is the multiplicity space, which may be zero. Since the G-action on (X,μ) is ergodic and c^1 is a minimal cocycle with $K_{c^1} = K$ we have

$$Z' \cap u(G)' = \bigoplus_{(\pi,H_\pi)\in\widehat{K}^1} \mathbb{C}\otimes\mathbb{C}\otimes B(L_\pi).$$

Thus, we identify Λ with the subset of $\widehat{K}^1$ consisting of the irreducibles appearing in $(v, L^2(B^1))$, and identify H^2_λ with L_π when $\lambda\in\Lambda$ is identified with $(\pi,H_\pi)\in\widehat{K}^1$. In what follows, we abuse notation and use only Λ, instead of $\widehat{K}^1$, if there is no possibility of confusion. We set

$$\mathcal{K}_1 = \prod_{\lambda\in\Lambda} U(H_\lambda),$$

and regard it as a subgroup of the unitary group $U(H)$ as before. Since Ψ^1 is faithful, we identify K_1 with $v(K_1)\subset\mathcal{K}_1$.

Let $w(\omega) : H_0 \longrightarrow L^2(B^1)$ be the measurable field of unitaries describing the identity map on H with respect to the two distinct splitting $H = L^2(X,\mu)\otimes H_0$ and $H = L^2(X,\mu)\otimes L^2(B^1)$. $w(\omega)$ is decomposed as

$$w(\omega) = \bigoplus_{\lambda\in\Lambda} w_\lambda(\omega)\otimes 1_{H^2_\lambda},$$

where $w_\lambda(\omega) : H^1_\lambda \longrightarrow H_\lambda$ is a measurable field of unitaries. Then, we have

$$B^1 = w(\omega)Bw(\omega)^*,$$
$$c^1(\omega,g) = w(\omega)c(\omega,g)w(\omega\cdot g)^*,$$

for almost all $\omega\in X$. Thanks to uniqueness of the minimal subgroup up to conjugacy, there exists a unitary w^0_λ from H^1_λ to H_λ such that if we set

$$\nu_0 = (\prod_{\lambda\in\Lambda}\mathrm{Ad}(w^0_\lambda)) : \mathcal{K}\longrightarrow\mathcal{K}_1,$$

then $\nu_0(K) = K_1$. We denote by ν_1 the restriction of ν_0 to K. Let $w^1(\omega) = w_\lambda(\omega)w^{0*}_\lambda$, and

$$w^0 = \bigoplus_{\lambda\in\Lambda} w^0_\lambda\otimes 1_{H^2_\lambda},$$
$$w^1(\omega) = \bigoplus_{\lambda\in\Lambda} w^1_\lambda(\omega)\otimes 1_{H^2_\lambda},$$

Then, we have

$$B^1 = w^1(\omega)w^0Bw^{0*}w^1(\omega)^*,$$
$$c^1(\omega,g) = w^1(\omega)\nu^1\cdot c(\omega,g)w^1(\omega\cdot g)^*,$$

for almost all $\omega\in X$. Since c^1 and $\nu_1\cdot c$ are mutually equivalent minimal $\mathcal{K}_1$-valued cocycles with $(\mathcal{K}_1)_{c^1} = (\mathcal{K}_1)_{\nu\cdot c} = K_1$, [80, Theorem 6.1] implies that there exist

$\kappa \in \mathcal{K}_1$ normalizing K_1 and a Borel map $a : X \longrightarrow K_1$ such that $w^1(\omega) = a(\omega)\kappa$ for almost all $\omega \in X$. Let $\nu(k) = \kappa\nu_1(k)\kappa^{-1}$, then we get

$$B^1 = \kappa w^0 B w^{0*} \kappa^*,$$

$$c^1(\omega, g) = a(\omega)\nu \cdot c(\omega, g) a(\omega \cdot g)^*,$$

for almost all ω. Let σ be the restriction of $\mathrm{Ad}(\kappa w^0)$ to B. Then ν and σ have the desired property. □

A.3. **Existence of minimal cocycles on ergodic flows.** The following is an easy generalization of [81, Theroem 2] to the non-singular case, which asserts that there are plenty of non-trivial modular endomorphisms for any type III_0 factors.

Proposition A.5. *Let (X, μ) be an ergodic non-transitive $\mathbb{R}$-space. Then, for every compact group K, there exists a minimal cocycle $c : X \times \mathbb{R} \longrightarrow K$ with $K_c = K$.*

Proof. As in [81, Theorem 2], it suffices to show the statement for a $\mathbb{Z}$-action instead of the $\mathbb{R}$-action thanks to Ambrose-Kakutani's theorem [1]. Let S be a given non-singular and non-transitive ergodic transformation on (X, μ). We show that there exists a minimal cocycle $c : X \times \mathbb{Z} \longrightarrow K$ with $K_c = K$. We take a non-transitive ergodic transformation T on (Y, ν) such that ν is T-invariant. Thanks to [81, Theorem 1], there exists a minimal cocycle $c' : Y \times \mathbb{Z} \longrightarrow K$ such that $K_{c'} = K$. Let R be the $\mathbb{Z}^2$-action on $X \times Y$ given by $R_{(n_1,n_2)} = S^{n_1} \times T^{n_2}$. We define a cocycle c'' on $(X \times Y) \times \mathbb{Z}^2$ by

$$c''((\omega_1, \omega_2), (n_1, n_2)) = c'(\omega_2, n_2),$$

which is a minimal cocycle with $K_{c''} = K$. Since (X, μ, S) is orbit equivalent to $(X \times Y, \mu \times \nu, R)$ [40], we get the result. □

References

[1] Ambrose, W; Kakutani, S. *Structure and continuity of measurable flows.* Duke Math. J. 9, (1942). 25–42.

[2] Araki, H.; Haag, R.; Kastler, D.; Takesaki, D. *Extension of KMS states and chemical potential.* Commun. Math. Phys. **53** (1977), 97–134.

[3] Arveson, W. *An Invitation to C^*-algebras.* Graduate Texts in Mathematics, No. 39. Springer-Verlag, New York-Heidelberg.

[4] Auslander, L.; Moore, C. C. *Unitary representations of solvable Lie groups.* Mem. Amer. Math. Soc. **62** 1966.

[5] Connes, A. *Une classification des facteurs de type* III. Ann. Sci. École Norm. Sup. (4) **6** (1973), 133–252.

[6] Connes, A. *On hyperfinite factors of type* III_0 *and Krieger's factors.* J. Funct. Anal. **18** (1975), 318–327.

[7] Connes, A. *Outer conjugacy classes of automorphisms of factors.* Ann. Sci. École Norm. Sup. (4) **8** (1975), 383–419.

[8] Connes, A. *Classification of injective factors. Cases* II_1, II_∞, III_λ, $\lambda \neq 1$. Ann. of Math. (2) **104** (1976), 73–115.

[9] Connes, A. *Periodic automorphisms of the hyperfinite factor of type* II_1. Acta Sci. Math. (Szeged) **39** (1977), 39–66.

[10] Combes, F.; Delaroche, C. *Groupe modulaire d'une espérance conditionnele dans une algèbre de von Neumann.* Bull. Soc. Math. France *103* (1975), 385–426.

[11] Connes, A.; Takesaki, M. *The flow of weights on factors of type* III. Tohoku Math. J. **29** (1977), 473–575.
[12] Digernes, T.; *Duality for weights on covariant systems and its applications* thesis, UCLA, 1975.
[13] Doplicher, S.; Roberts, J. E. *Endomorphisms of C^*-algebras, crossed products and duality for compact groups.* Ann. Math. **130** (1989), 75–119.
[14] Doplicher, S.; Roberts, J. E. *A new duality theory for compact groups.* Invent. Math. **98** (1989), 157–218.
[15] Fabec, R. C. *Normal ergodic actions and extensions.* Israel J. Math. **40** (1981), 175–186.
[16] Fell, J. M. *A Hausdorff topology for the closed subsets of a locally compact non-Hausdorff space.* Proc. Amer. Math. Soc. **13** (1962), 472–476.
[17] Fidaleo, F.; Isola, T. *On the conjugate endomorphism in the infinite index case.* Math. Scand. **77** (1995), 289–300.
[18] Haagerup, U. *On the dual weights for crossed products of von Neumann algebras. I. Removing separability conditions.* Math. Scand. **43** (1978/79), 99–118.
[19] Haagerup, U. *Operator-valued weights in von Neumann algebras. II.* J. Funct. Anal. **33** (1979), 339–361.
[20] Haagerup, U. *Connes' bicentralizer problem and uniqueness of the injective factor of type* III_1. Acta Math. **158** (1987), 95–148.
[21] Haagerup, U.; Størmer, E. *Equivalence of normal states on von Neumann algebras and the flow of weights.* Adv. Math. **83** (1990), 180–262.
[22] Haagerup, U.; Størmer, E. *Pointwise inner automorphisms of injective factors.* J. Funct. Anal. **122** (1994), 307–314.
[23] Hamachi, T. *Suborbits and group extensions of flows.* Israel J. Math. **100** (1997), 249–283. Correction **109** (1999), 379–380.
[24] Hamachi, T. *Canonical subrelations of ergodic equivalence relations-subrelations.* J. Operator Theory **43** (2000), 3–34.
[25] Hamachi, T.; Kosaki, H. *Index and flow of weights of factors of type* III. Proc. Japan Acad. Ser. A Math. Sci. **64** (1988), 11–13.
[26] Hamachi, T.; Kosaki, H. *Inclusion of type* III *factors constructed from ergodic flows.* Proc. Japan Acad. Ser. A Math. Sci. **64** (1988), 195–197.
[27] Hamachi, T.; Kosaki, H. *Orbital factor map.* Ergodic Theory Dynam. Systems **13** (1993), 515–532.
[28] Hewitt, E.; Ross, K. A. *Abstract Harmonic Analysis. Vol. II: Structure and Analysis for Compact Groups. Analysis on Locally Compact Abelian groups.* Die Grundlehren der mathematischen Wissenschaften, Band 152 Springer-Verlag, New York-Berlin 1970.
[29] Hiai, F. *Minimizing indices of conditional expectations onto a subfactor.* Publ. Res. Inst. Math. Sci. **24** (1988), 673–678.
[30] Høegh-Krohn, R.; Landstad, M. B.; Størmer, E. *Compact ergodic groups of automorphisms.* Ann. of Math. (2) **114** (1981), 75–86.
[31] Izumi, M. *On type* II *and type* III *principal graphs of subfactors*, Math. Scand., **73** (1993), 307-319.
[32] Izumi, M. *Canonical extension of endomorphisms on factors.* "Subfactors" (Kyuzeso, 1993), 129–138, World Sci. Publishing, River Edge, 1994.
[33] Izumi, M. *Goldman's type theorems in index theory.* "Operator Algebras and Quantum Field Theory" (Rome, 1996), 249–269, Internat. Press, Cambridge, MA, 1997.
[34] Izumi, M. *Subalgebras of infinite C^*-algebras with finite Watatani indices. II. Cuntz-Krieger algebras.* Duke Math. J. **91** (1998), 409–461.
[35] Izumi, M.; Longo, R.; Popa, S. *A Galois correspondence for compact groups of automorphisms of von Neumann algebras with a generalization to Kac algebras.* J. Funct. Anal. **155** (1998), 25–63.
[36] Izumi, M; Kosaki, H. in preparation.

[37] Jones, V. F. R. *Actions of finite groups on the hyperfinite type* II_1 *factor.* Mem. Amer. Math. Soc. **28** (1980), no. 237.
[38] Jones, V. F. R. *Index for subfactors.* Invent. Math. **72** (1983), 1-25.
[39] Kawakami, S.; Watatani, Y. *The multiplicativity of the minimal index of simple C^*-algebras.* Proc. Amer. Math. Soc. **123** (1995), 2809–2813.
[40] Krieger, W. *On ergodic flows and the isomorphism of factors.* Math. Ann. **223** (1976), 19–70.
[41] Kosaki, H. *Extension of Jones' theory on index to arbitrary factors.* J. Funct. Anal. **66** (1986), 123–140.
[42] Kosaki, H. *AFD factor of type* III_0 *with many isomorphic index 3 subfactors.* J. Operator Theory **32** (1994), 17–29.
[43] Kosaki, H. *Sector theory and automorphisms for factor-subfactor pairs.* J. Math. Soc. Japan **48** (1996), 427–454.
[44] Kosaki, H.; Longo, R. *A remark on the minimal index of subfactors.* J. Funct. Anal. **107** (1992), 458–470.
[45] Kosaki, H.; Sano, T. *Non-splitting inclusions of factors of type* III_0. Pacific J. Math. **178** (1997), 95–125.
[46] Kodiyalam, V.; Sunder, V. S. *The subgroup-subfactor.* Math. Scand. **86** (2000), 45–74.
[47] Loi, Ph. *On the theory of index for type* III *factors.* J. Operator Theory **28** (1992), 251–265.
[48] Longo, R. *Simple injective subfactors.* Adv. in Math. **63** (1987), 152–171.
[49] Longo, R. *Index of subfactors and statistics of quantum fields I.* Commun. Math. Phys. **126** (1989), 217–247.
[50] Longo, R. *Index of subfactors and statistics of quantum fields II.* Commun. Math. Phys. **130** (1990), 285–309.
[51] Longo, R. *Minimal index and braided subfactors.* J. Funct. Anal. **109** (1992), 98–112.
[52] Longo, R. *An analogue of the Kac-Wakimoto formula and black hole conditional entropy.* Comm. Math. Phys. **186** (1997), 451–479.
[53] Longo, R.; Roberts, J. E. *A theory of dimension.* K-Theory **11** (1997), 103–159.
[54] Mackey, G. W. *Point realizations of transformation groups.* Illinois J. Math. **6** (1962), 327–335.
[55] Mackey, G. W. *Ergodic transformation groups with a pure point spectrum.* Illinois J. Math. **8** 1964, 593–600.
[56] Mackey, G. W. *Ergodic theory and virtual groups.* Math. Ann. **166** (1966), 187–207.
[57] Masuda, Toshihiko. *On non-strongly free automorphisms of subfactors of type* III_0. in preparation.
[58] Nakagami, Y. Takesaki, M. *Duality for crossed products of von Neumann algebras.* Lecture Notes in Mathematics, 731. Springer, Berlin, 1979.
[59] Nielsen, O. A. *Direct Integral Theory.* Lecture Notes in Pure and Applied Mathematics, 61. Marcel Dekker, Inc., New York, 1980.
[60] Pimsner, M.; Popa, S. *Entropy and index for subfactors.* Ann. Sci. École Norm. Sup. (4) **19** (1986), 57–106.
[61] Popa, S. *Classification of amenable subfactors of type* II. Acta Math. **172** (1994), 163–255.
[62] Popa, S.; Wassermann, A. *Actions of compact Lie groups on von Neumann algebras.* C. R. Acad. Sci. Paris Sér. I Math. **315** (1992), 421–426.
[63] Roberts, J. E. *Crossed product of von Neumann algebras by group duals.* Symposia Math., Vol.XX (1976).
[64] Sauvageot, J.-L. *Produits tensoriels de $\mathcal{Z}$-modules et applications.* "Operator Algebras and Their Connections with Topology and Ergodic Theory" (Buşteni, 1983), 468–485, Lecture Notes in Math., 1132, Springer, Berlin, 1985.
[65] Sekine,Y. *Flows of weights of crossed products of type* III *factors by discrete groups.* Publ. Res. Inst. Math. Sci. **26** (1990), 655–666.
[66] Schaflitzel R. II_1*-subfactors associated with the C^*-tensor category of a finite group.* Pacific J. Math. **184** (1998), 333–348.

[67] Strătilă, Ş. *Modular Theory in Operator Algebras.* Editura Academiei Republicii Socialiste Romania, Bucharest; Abacus Press, Tunbridge Wells, 1981.
[68] Strătilă, Ş.; Zsidó, L. *Lectures on von Neumann algebras.* Translated from the Romanian by Silviu Teleman. Editura Academiei, Bucharest; Abacus Press, Tunbridge Wells, 1979.
[69] Strătilă, Ş.; Zsidó, L. *The commutation theorem for tensor products over von Neumann algebras.* J. Funct. Anal. 165 (1999), 293–346.
[70] Sutherland, C. E. *Cohomology and extensions of von Neumann algebras. I, II.* Publ. Res. Inst. Math. Sci. **16** (1980), 105–133.
[71] Sutherland, C. E. *Subfactors and ergodic theory.* "Current Topics in Operator Algebras" (Nara, 1990), 38–42, World Sci. Publishing, 1991.
[72] Sutherland, C. E.; Takesaki, M. *Actions of discrete amenable groups on injective factors of type* $\mathrm{III}_\lambda,\ \lambda \neq 1$. Pacific J. Math. **137** (1989), 405–444.
[73] Sutherland, C. E.; Takesaki, M. *Right inverse of the module of approximately finite-dimensional factors of type III and approximately finite ergodic principal measured groupoids.* "Operator algebras and their applications, II" (Waterloo, ON, 1994/1995), 149–159, Fields Inst. Commun., 20, Amer. Math. Soc., Providence, RI, 1998.
[74] Takesaki, M. *Conditional expectations in von Neumann algebras.* J. Funct. Anal. **9** (1972), 306–321.
[75] Takesaki, M. *Theory of operator algebras. I.* Springer-Verlag, New York-Heidelberg, 1979.
[76] Wassermann, A. *Ergodic actions of compact groups on operator algebras. I. General theory.* Ann. of Math. (2) **130** (1989), 273–319.
[77] Wassermann, A. *Coactions and Yang-Baxter equations for ergodic actions and subfactors.* "Operator Algebras and Applications, Vol. 2", 203–236, London Math. Soc. Lecture Note Ser., 136, Cambridge Univ. Press, Cambridge, 1988.
[78] Watatani, Y. *Index for* C^**-subalgebras.* Mem. Amer. Math. Soc. **83** (1990), no. 424.
[79] Winsløw, C. *Classification of strongly amenable subfactors of type* III_0. J. Funct. Anal. **148** (1997), 296–313.
[80] Zimmer, R. J. *Extensions of ergodic group actions.* Illinois J. Math. **20** (1976), 373–409.
[81] Zimmer, R. J. *Random walks on compact groups and the existence of cocycles.* Israel J. Math. **26** (1977), 84–90.

Department of Mathematics, Graduate School of Science, Kyoto University, Sakyo-ku, Kyoto 606-8502, Japan
E-mail address: izumi@kusm.kyoto-u.ac.jp