Cyclomorphy

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Abstract. A main result is that, roughly, a dense set of the infinitesimal trace-preserving deformations of a semicircular system $s_1, \dots, s_n$ arise from one-parameter groups of automorphisms of the free-group factor $L(F(n))$ generated by $s_1, \dots, s_n$. More generally the paper studies cyclic gradients in von Neumann algebras, Lie algebras of noncommutative trace-preserving vector fields and the class of cyclomorphic maps which preserve the orthogonals of spaces of cyclic gradients.

0 Introduction

The word "cyclomorphy" is meant to remind the reader of "holomorphy" and of the "cyclic derivative" of Rota-Sagan-Stein ([8]). The paper is about cyclic derivatives in von Neumann algebras.

Several questions around free probability theory lead to cyclomorphy: moments of noncommutative random variables, free entropy ([12],[14]) and its connection with large deviations for random matrices ([4],[5]), orbits of the equivalence in distribution and the free Wasserstein distance ([1]). Last but not least I was motivated by curiositiy about $\mathrm{Aut}(L(F(n)))$, the automorphism group of a free group factor.

In general, if $Y_j = Y_j^*$, $1 \le j \le m$, generate a II_1 factor M, the orbit of $Y = (Y_j)_{1 \le j \le n}$ under $\mathrm{Aut}(M)$ provides a parametrization of $\mathrm{Aut}(M)$. The derivations in the "Lie algebra of $\mathrm{Aut}(M)$" which have the Y_j's in their domain of definition, are then parametrized by the "tangent space" of the orbit at Y. This "tangent space" is a more tractable object as we shall see.

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Since automorphisms of M preserve its trace-state τ, the"tangent space" is orthogonal to the space of cyclic gradients of noncommutative polynomials in $(Y_1, \ldots, Y_n)$. The question then is which of the self-adjoint n-tuples in the orthogonal of cyclic gradients give rise to derivations which exponentiate to one-parameter groups of automorphisms? We show that exponentiation is possible if the n-tuple is polynomial, or more generally is given by noncommutative power series of sufficiently large radius of convergence evaluated at Y.

The polynomial elements in the orthogonal to cyclic gradients form a Lie algebra, which is a natural noncommutative relative of the Lie algebra of vector fields which preserve a volume form.

A natural generator for $L(F(n))$ is provided by a semicircular system $s_1, \ldots, s_n$ ([10],[11]). In this case we can give a complete description of the orthogonal to cyclic gradients. In particular we show that polynomial elements are dense in this space. Exponentiation therefore works for the corresponding derivations and we have an "infinitesimally rich" subalgebra of the "Lie algebra" of the automorphism group in this case.

From this study of automorphic orbits it becomes clear that it is important to understand the differential geometric picture arising on M^n (or on the hermitian subspace M_h^n) from the field of subspaces of the cotangent spaces arising from cyclic gradients and their orthogonals which are a field of subspaces in the tangent spaces. In particular there is a natural class of maps, those for which the differentials preserve the orthogonals to cyclic gradients. We look at basic properties of cyclomorphic maps and of a generalization of these to the case of B-von Neumann algebras (B a von Neumann subalgebra).

The paper has eight sections without the introduction. Section 1 deals with preliminaries. Section 2 is about the estimates which will prove that certain derivations can be exponentiated to automorphism groups. Section 3 combines the estimates of section 2 with the condition of orthogonality to cyclic gradients to obtain results about exponentiation of derivations. Section 4 is a brief look at endomorphic orbits and their tangent spaces. In section 5 we study real and complex cyclomorphic maps. In Theorem 5.13 and Proposition 5.14 we point out the role of independence and of free independence in this context. Section 6 is a collection of generalities about the Lie algebras of "noncommutative vector-fields" which occur in this paper. Section 7 deals with the semicircular case. We compute the orthogonal to the cyclic gradients in this case. Section 8 is about B-morphic maps, the extension to von Neumann algebras "over B" of cyclomorphic maps. We only sketch how this generalization is done and will develop the details elsewhere.

1 Preliminaries

1.1 Noncommutative polynomials

If $X_1, \ldots, X_n$ are noncommuting indeterminates, the ring of noncommutative polynomials will be denoted by $\mathbb{C}_{\langle n\rangle} = \mathbb{C}\langle X_1, \ldots, X_n\rangle$. If $a_1, \ldots, a_n$ are elements in an unital algebra over $\mathbb{C}$ and $P \in \mathbb{C}_{\langle n\rangle}$ then $P(a_1, \ldots, a_n)$ or $\varepsilon_{(a_1,\ldots,a_n)}P$ denotes the evaluation of P at $a_1, \ldots, a_n$.

We shall also use the notation $\mathbb{C}\langle a_1, \ldots, a_n\rangle$ for $\varepsilon_{(a_1,\ldots,a_n)}\mathbb{C}\langle X_1, \ldots, X_n\rangle$, i.e. for the algebra generated by $1, a_1, \ldots, a_n$. Clearly this algebra is isomorphic to the algebra of noncommutative polynomials only if $a_1, \ldots, a_n$ are algebraically free. When we will use the notation it will be clear from the context whether $a_1, \ldots, a_n$ are algebraically free or not.

1.2 Cyclic gradients and free difference quotients

The partial free difference quotients are the derivations

$$\partial_j : \mathbb{C}_{\langle n\rangle} \to \mathbb{C}_{\langle n\rangle} \otimes \mathbb{C}_{\langle n\rangle}$$

such that $\partial_j X_k = 0$ if $j \neq k$ and $\partial_j X_j = 1 \otimes 1$, where $j, k \in \{1, \ldots, n\}$. The $\mathbb{C}_{\langle n\rangle}$-bimodule structure on $\mathbb{C}_{\langle n\rangle} \otimes \mathbb{C}_{\langle n\rangle}$ is the obvious one $a_1(b \otimes c)a_2 = a_1 b \otimes c a_2$.

The partial cyclic derivatives are then $\delta_j = \tilde{\mu} \circ \partial_j : \mathbb{C}_{\langle n\rangle} \to \mathbb{C}_{\langle n\rangle}$ where $\tilde{\mu}(a \otimes b) = ba$. Thus

$$\delta_j X_{i_1} \ldots X_{i_p} = \sum_{1k|1 \leq k \leq p,\ i_k = j} X_{i_{k+1}} \ldots X_{i_p} X_{i_1} \ldots X_{i_{k-1}} .$$

The map $\delta : \mathbb{C}_{\langle n\rangle} \to (\mathbb{C}_{\langle n\rangle})^n = \mathbb{C}_{\langle n\rangle} \oplus \cdots \oplus \mathbb{C}_{\langle n\rangle}$ given by

$$\delta P = (\delta_j P)_{1 \leq j \leq n} \ = \ \delta_1 P \oplus \cdots \oplus \delta_n P$$

is the cyclic gradient.

Similarly $\partial : \mathbb{C}_{\langle n\rangle} \to (\mathbb{C}_{\langle n\rangle} \otimes \mathbb{C}_{\langle n\rangle})^n$ denotes the free difference quotient gradient.

In case $a_1, \ldots, a_n$ are algebraically free the corresponding maps for $\mathbb{C}_{\langle n\rangle}\langle a_1, \ldots, a_n\rangle$ will be denoted

$$\partial_j^{(a_1,\ldots,a_n)}, \partial^{(a_1,\ldots,a_n)}, \delta_j^{(a_1,\ldots,a_n)}, \delta^{(a_1,\ldots,a_n)}$$

or $\partial_j^\alpha, \partial^\alpha, \delta_j^\alpha, \delta^\alpha$ where $\alpha = (a_1, \ldots, a_n)$.

The notation for cyclic derivatives and gradients should not be confused with Kronecker symbols δ_{ij} which have two lower indices.

If $a, a_1, \dots, a_n$ are elements in some unital algebra A over $\mathbb{C}$ let $m_a : \mathbb{C}\langle a_1, \dots, a_n\rangle \otimes \mathbb{C}\langle a_1, \dots, a_n\rangle \to A$ be the map given by

$$m_a(P_1 \otimes P_2) \;=\; P_1 a P_2 \ .$$

In particular if $a_1, \dots, a_n$ are algebraically free, then a derivation of $\mathbb{C}\langle a_1, \dots, a_n\rangle$ into A can always be written

$$P \rightsquigarrow \sum_{1 \le j \le n} m_{b_j}\, \partial_j^\alpha P$$

where $b_1, \dots, b_n \in A$. The elements $b_1, \dots, b_n$ are uniquely determined, being the values of the derivation on $a_1, \dots, a_n$.

1.3 The exact sequence for cyclic gradients

In [13] we described the set of cyclic gradients by an exact sequence

$$0 \to \mathbb{C}1 + [\mathbb{C}_{\langle n\rangle}, \mathbb{C}_{\langle n\rangle}] \to \mathbb{C}_{\langle n\rangle} \xrightarrow{\delta} (\mathbb{C}_{\langle n\rangle})^n \xrightarrow{\theta} \mathbb{C}_{\langle n\rangle}$$

where $\theta(P_1 \oplus \cdots \oplus P_n) = \sum_j [X_j, P_j]$. Moreover

$$\mathbb{C}1 + [\mathbb{C}_{\langle n\rangle}, \mathbb{C}_{\langle n\rangle}] = \mathbb{C}1 + \sum_{1 \le k \le n} [X_k, \mathbb{C}_{\langle n\rangle}] = \mathrm{Ker}\ C$$

where $C : \mathbb{C}_{\langle n\rangle} \to \mathbb{C}_{\langle n\rangle}$ is the cyclic symmetrization map $C1 = 0$ and

$$C\, X_{i_1} \dots X_{i_p} = \sum_{1 \le j \le p} X_{i_{j+1}} \dots X_{i_p} X_{i_1} \dots X_{i_j}$$

if $p \ge 1$.

If instead of $X_1, \dots, X_n$ we have algebraically free $a_1, \dots, a_n$ we shall use the notations

$$\theta^\alpha \ , \quad C^\alpha$$

where $\alpha = (a_1, \dots, a_n)$.

Also, if $a_1, \dots, a_n$ are not algebraically free, we have the set of cyclic gradients evaluated at $\alpha = (a_1, \dots, a_n)$

$$(\varepsilon_\alpha)^n (\delta \mathbb{C}_{\langle n\rangle}) \ .$$

1.4 Semicircular systems

Recall that in a C^*-probability space (A,φ) a semicircular system is an n-tuple $(S_1,\dots,S_n)$ of selfadjoint noncommutative random variables which are freely independent and have (0,1) semicircle distributions. The von Neumann algebra of $(S_1,\dots,S_n)$ in the GNS representation associated with the restriction of φ is isomorphic to the free group factor $L(F(n))$ and the restriction of φ is the trace-state.

There is a natural realization of a semicircular system on the full Fock space

$$\mathcal{T}(\mathbb{C}^n) = \bigoplus_{k\geq 0}(\mathbb{C}^n)^{\otimes k}$$

where $(\mathbb{C}^n)^{\otimes 0} = \mathbb{C}1$ with 1 the vacuum vector. If e_j, $1\leq j\leq n$ are the standard orthonormal basis vectors in $\mathbb{C}^n$, let

$$\ell_j\xi = e_j\otimes\xi\ , \qquad r_j\xi = \xi\otimes e_j$$

be the left and right creation operators. Then $s_j = \ell_j + \ell_j^*\quad(1\leq j\leq n)$ is a semicircular system w.r.t. the vacuum expectation $\langle\cdot 1, 1\rangle$ and similarly $d_j = r_j + r_j^*\quad(1\leq j\leq n)$.

In particular $\mathcal{T}(\mathbb{C}^n)$ identifies with $L^2(W^*(s_1,\dots,s_n))$ and denoting by $P_0, P_1,\dots$ the orthonormal polynomials w.r.t. a semicircular distribution we have

$$e_{i_1}^{\otimes k_1}\otimes\ldots\otimes e_{i_p}^{\otimes k_p} = P_{k_1}(s_{i_1})\ldots P_{k_p}(s_{i_p})1 = \ell_{i_1}^{k_1}\ldots\ell_{i_p}^{k_p}1$$

where $i_j\neq i_{j+1}\quad(1\leq j\leq p-1)$ and $k_j>0\quad(1\leq j\leq p)$.

The polynomials $P_k(x),\quad k\geq 0$ are equal to $C_k^1(x/2)$ where $C_k^1(t)$ are Gegenbauer polynomials, which satisfy the generating function relation

$$(1-2rt+r^2)^{-1} = \sum_{n\geq 0} C_n^1(t)r^n$$

(see [9, ch.IX, §4, sec.12]).

2 Exponentiating noncommutative vector fields

2.1 If $\mathcal{E}$ is a $\mathbb{C}_{\langle n\rangle}$-bimodule, let $\operatorname{Vect}\mathcal{E}$ denote $\mathcal{E}^n$. If $K = (K_j)_{1\leq j\leq n}\in\operatorname{Vect}\mathcal{E}$ let $D_K:\mathbb{C}_{\langle n\rangle}\to\mathcal{E}$ be the derivation such that $D_K X_j = K_j$, i.e.,

$$D_K = \sum_{1\leq j\leq n} m_{K_j}\partial_j\ .$$

In particular if $\mathcal{E}$ is a unital Banach-algebra and the bimodule structure arises from a unital homomorphism of $\mathbb{C}_{\langle n\rangle}$ into $\mathcal{E}$ which takes X_j to T_j, then

$$\frac{d}{d\varepsilon}\, P(T_1+\varepsilon K_1,\ldots,T_n+\varepsilon K_n)|_{\varepsilon=0} = D_K P$$

where $P\in\mathbb{C}_{\langle n\rangle}$.

2.2 On $\mathbb{C}_{\langle n\rangle}$ we define seminorms

$$\left|\sum_{p\geq 0}\ \sum_{1\leq j_1,\ldots,j_p\leq n} c_{j_1,\ldots,j_p} X_{j_1}\ldots X_{j_p}\right|_{R,k} = \sum_{p\geq k}\ \sum_{1\leq j_1,\ldots,j_p\leq n} |c_{j_1,\ldots,j_p}| R^{p-k} p!((p-k)!)^{-1}\ .$$

The completion of $\mathbb{C}_{\langle n\rangle}$ w.r.t. $|\cdot|_{R,j}$, $0\leq j\leq k$ will be denoted $\mathbb{C}_{\langle n\rangle,R,k}$ or $\mathbb{C}\langle X_1,\ldots,X_n\rangle_{R,k}$ and identifies with a subalgebra of the algebra of noncommutative formal power series denoted $\mathbb{C}_{\langle\langle n\rangle\rangle}$ or $\mathbb{C}\langle\langle X_1,\ldots,X_n\rangle\rangle$.

On $\mathrm{Vect}\,\mathbb{C}\langle X_1,\ldots,X_n\rangle$ we have corresponding seminorms

$$|(K_1,\ldots,K_n)|_{R,k} = \max_{1\leq s\leq n} |K_s|_{R,k}\ .$$

The completions coincide with $\mathrm{Vect}\,\mathbb{C}_{\langle n\rangle,R,k}$.

If $K\in\mathrm{Vect}\,\mathbb{C}_{\langle n\rangle}$ and $P\in\mathbb{C}_{\langle n\rangle}$, then

$$|D_K P|_{R,0} \leq |K|_{R,0}\,|P|_{R,1}\ .$$

In particular $D_K P$ can be defined for $K\in\mathrm{Vect}\,\mathbb{C}_{\langle n\rangle,R,0}$ and $P\in\mathbb{C}_{\langle n\rangle,R,1}$ as an element in $\mathbb{C}_{\langle n\rangle,R,0}$. Also if $0<R<R'$ then $|P|_{R,1}\leq C|P|_{R',0'}$ for some constant independent of P, so that $D_K P\in\mathbb{C}_{\langle n\rangle,R,0}$ if $P\in\mathbb{C}_{\langle n\rangle,R',0}$ and $K\in\mathrm{Vect}\,\mathbb{C}_{\langle n\rangle,R,0}$. Iterating, we have $D_K^m P\in\mathbb{C}_{\langle n\rangle,R,0}$ if $P\in\mathbb{C}_{\langle n\rangle,R',0}$ and $K\in\mathrm{Vect}\,\mathbb{C}_{\langle n\rangle,R',0}$ for some $R'>R$.

2.3 The map $\Phi_n:\mathbb{C}_{\langle n\rangle}\to\mathbb{C}_{\langle 1\rangle}$ is defined by

$$\Phi_n\left(\sum_{p\geq 0}\ \sum_{1\leq j_1,\ldots,j_p\leq n} c_{j_1,\ldots,j_p} X_{j_1}\ldots X_{j_p}\right) = \sum_{p\geq 0}\ \sum_{1\leq j_1,\ldots,j_p\leq n} |c_{j_1,\ldots,j_p}| X_1^p\ .$$

Clearly $|\Phi_n(P)|_{R,k} = |P|_{R,k} = (D_{X_1}^k\Phi_n(P))(R)$ where $R>0,\ k\geq 0$.

Obviously Φ_n extends to a map of $\mathbb{C}_{\langle n\rangle,R,k}$ to $\mathbb{C}_{\langle 1\rangle,R,k}$.

2.4 If $P\in\mathbb{C}_{\langle n\rangle}$ let $|P|$ be the noncommutative polynomial with coefficients the absolute values of the coefficients of P. Also, if $P,Q\in\mathbb{C}_{\langle n\rangle}$ we shall write $|P|\leq|Q|$ if the inequality holds among the coefficients of $|P|$ and $|Q|$ which are nonnegative numbers).

We extend this definition to $\operatorname{Vect}\mathbb{C}_{\langle n\rangle}$ by putting $|(K_j)_{1\le j\le n}| = (|K_j|)_{1\le j\le n}$ and $|K| \le |K'|$ if $|K_j| \le |K'_j|$ $(1 \le j \le n)$ where $K = (K_j)_{1\le j\le n}$, $K' = (K'_j)_{1\le j\le n}$.

We shall also write $|P| \vee |Q|$ for the coefficientwise maximum.

2.5 The analogue of Φ_n on $\operatorname{Vect}\mathbb{C}_{\langle n\rangle}$ is the map $\Psi_n : \operatorname{Vect}\mathbb{C}_{\langle n\rangle} \to \operatorname{Vect}\mathbb{C}_{\langle 1\rangle}$ defined by

$$\Psi_n((K_j)_{1\le j\le n}) = \Phi_n(K_1) \vee \cdots \vee \Phi_n(K_n) \ .$$

We have

$$|K|_{R,k} \le (D^k_{X_1}\,\Psi_n(K))(R)$$

with equality if $k = 0$.

2.6 Lemma. *Let $K, K' \in \operatorname{Vect}\mathbb{C}_{\langle n\rangle}$, and $P, P' \in \mathbb{C}_{\langle n\rangle}$. Then $|D_K P| \le D_{|K|}|P|$. If $|K| \le |K'|$ and $|P| \le |P'|$, then*

$$D_{|K|}|P| \le D_{|K'|}|P'| \quad \text{and} \Phi_n(P) \le \Phi_n(P') \ , \quad \Psi_n(K) \le \Psi_n(K') \ .$$

Moreover we have

$$\Phi_n(D_{|K|}|P|) \le D_{\Psi_n(K)}\Phi_n(P)$$

The proof reduces to the most obvious majorizations and is left to the reader.

2.7 Theorem. *Let $0 < R < R'$ and let $K \in \operatorname{Vect}\mathbb{C}_{\langle n\rangle, R',0}$ and $P \in \mathbb{C}_{\langle n\rangle, R',0}$. Then*

$$|D^m_K P|_{R,0} \le 1\cdot 3\cdot 5\cdot \ldots (2m-1)(R')^{m+1}(R'-R)^{-2m-1}|K|^m_{R',0}|P|_{R',0} \ .$$

In particular, $D^m_K P \in \mathbb{C}_{\langle n\rangle, R,0}$ and if $0 < r < |K|^{-1}_{R',0}(R'-R)^2(2R')^{-1}$, then

$$\sum_{m\ge 0} |D^m_K P|_{R,0}\frac{r^m}{m!} \ < \ \infty$$

i.e., P is an analytic vector for D_K in $\mathbb{C}_{\langle n\rangle, R,0}$.

Proof. Clearly all assertions are an easy consequence of the estimate for $|D^m_K P|_{R,0}$ and it suffices to prove it when $K \in \operatorname{Vect}\mathbb{C}_{\langle n\rangle}$ and $P \in \mathbb{C}_{\langle n\rangle}$.

If $|P|_{R',0} = M$ then $(\Phi_n(P))(R') = M$ and the coefficient of X^k_1 of $\Phi_n(P)$ is then majorized by $M(R')^{-k}$ so that

$$\Phi_n(P) \le M(1 - X_1/R')^{-1} \ .$$

Similarly, if $|K|_{R',0} = N$ we have

$$\Psi_n(K) \leq N(1 - X_1/R')^{-1} .$$

Using Lemma 2.6 we get

$$D^m_{\Psi_n(K)} \Phi_n(P) \leq |K|^m_{R',0}\, |P|_{R',0}\, D^m_\Lambda \Lambda$$

where $\Lambda = (1 - X_1/R')^{-1}$. By induction we easily get

$$D^m_\Lambda \Lambda = 1 \cdot 3 \cdot 5 \cdot \ldots (2m-1)(R')^{-m} \Lambda^{2m+1} .$$

It follows that

$$\begin{aligned} |D^m_K P|_{R,0} &\leq |K|^m_{R',0}\, |P|_{R',0}\, D^m_\Lambda \Lambda(R) \\ &= (2m)!(m!)^{-1}\, 2^{-m} (R')^{m+1} (R' - R)^{-2m-1} |K|^m_{R',0}\, |P|_{R',0} \end{aligned}$$

□

3 Cyclic gradients and exponentiation

3.1 The basic property of cyclic gradients

Let (M, τ) be a von Neumann algebra with a normal faithful trace state, i.e., a tracial W^*-probability space, and let $T_j \in M$, $K_j \in M$, $1 \leq j \leq n$. If $P \in \mathbb{C}_{\langle n \rangle}$, then

$$\frac{d}{d\varepsilon} \tau(P(T_1 + \varepsilon K_1, \ldots, T_n + \varepsilon K_n))|_{\varepsilon=0} = \sum_{1 \leq j \leq n} \tau((\delta_j P)(T_1, \ldots, T_n) K_j) .$$

Thus the cyclic gradient $(\delta P)(T_1, \ldots, T_n)$ is precisely the gradient at $(T_1, \ldots, T_n)$ of the function

$$M^n \ni (T_1, \ldots, T_n) \to \tau(P(T_1, \ldots, T_n)) \in \mathbb{C}$$

when we use the bilinear scalar product given by $\sum_{1 \leq j \leq n} \tau(a_j b_j)$ on $M^n = \mathrm{Vect}\, M$ (w.r.t. the bimodule structure induced by $\varepsilon_{(T_1, \ldots, T_n)}$). Under the sesquilinear scalar product $\sum_{1 \leq j \leq n} \tau(a_j b_j^*)$ the gradient would be $(\delta P(T_1, \ldots, T_n))^* = ((\delta_j P(T_1, \ldots, T_n))^*)_{1 \leq j \leq n}$.

3.2 On $\mathbb{C}_{\langle n \rangle}$ we consider the involution $P \to P^*$ such that $X_j = X_j^*$, i.e., $(cX_{i_1} \ldots X_{i_p})^* = \bar{c} X_{i_p} \ldots X_{i_1}$. If $T_j \in M$ then

$$(P(T_1, \ldots, T_n))^* = (P^*(T_1^*, \ldots, T_n^*)$$

and if $T_j = T_j^*$, then

$$(P(T_1, \ldots, T_n))^* = (P^*(T_1, \ldots, T_n) \ .$$

It is also easily seen that $\delta_j P^* = (\delta_j P)^*$. Similarly $\partial_j P^* = \widetilde{(\partial_j P)}^*$ where on $\mathbb{C}_{\langle n\rangle} \otimes \mathbb{C}_{\langle n\rangle}$ we define $(\xi \otimes \eta)^* = \xi^* \otimes \eta^*$ and $\widetilde{\xi \otimes \eta} = \eta \otimes \xi$.

3.3 Let $Y_j = Y_j^* \in M$, $1 \leq j \leq n$ be algebraically free so that $\mathbb{C}_{\langle n\rangle}$ and $\mathbb{C}\langle Y_1, \ldots, Y_n\rangle$ are isomorphic, via $\varepsilon_{(Y_1,\ldots,Y_n)}$. This turns $L^2(M,\tau)$ into a $\mathbb{C}_{\langle n\rangle}$ bimodule. Assume also $\{Y_1, \ldots, Y_n\}$ generates M. If $K \in \operatorname{Vect} L^2(M,\tau)$ we shall consider $D_K^\varepsilon = D_K \circ \varepsilon^{-1}_{(Y_1,\ldots,Y_n)}$ which is a derivation of $\mathbb{C}\langle Y_1, \ldots, Y_n\rangle$ into $L^2(M,\tau)$. We shall view D_K^ε as an unbounded densely defined operator $L^2(M)$ with domain of definition $\mathbb{C}\langle Y_1, \ldots, Y_n\rangle$.

3.4 Proposition. *The following are equivalent conditions on K.*

(i) $\sum_{1\leq j\leq n} \tau((\delta_j P)(Y_1, \ldots, Y_n)K_j) = 0$ *for all* $P \in \mathbb{C}_{\langle n\rangle}$.

(ii) $\tau(D_K^\varepsilon Q) = 0$ *for all* $Q \in \mathbb{C}\langle Y_1, \ldots, Y_n\rangle$

(iii) *If* $P_1, P_2 \in \mathbb{C}\langle Y_1, \ldots, Y_n\rangle$ *then in* $L^2(M,\tau)$, $\langle D_K^\varepsilon P_1, P_2\rangle = -\langle P_1, D_{K^*}^\varepsilon P_2\rangle$ *where* $K^* = (K_1^*, \ldots, K_n^*)$.

Proof. (i) $\Leftrightarrow$ (ii) follows from 2.1 and 3.1. We have

$$\begin{aligned}
\langle D_K^\varepsilon P_1, P_2\rangle + \langle P_1, D_{K^*}^\varepsilon P_2\rangle &= \tau((D_K^\varepsilon P_1)P_2^* + P_1(D_{K^*}^\varepsilon P_2)^*) \\
&= \tau((D_K^\varepsilon P_1)P_2^* + P_1(D_K^\varepsilon P_2^*)) \\
&= \tau(D_K^\varepsilon(P_1 P_2^*)) \ .
\end{aligned}$$

Clearly $\tau(D_K(P_1 P_2^*)) = 0$ for all $P_1, P_2 \in \mathbb{C}\langle Y_1, \ldots, Y_n\rangle$ is equivalent to (ii) so that (ii) $\Rightarrow$ (iii). $\square$

3.4 Corollary. *Assume K satisfies the equivalent conditions of Proposition* 3.3. *Then:*

a) D_K^ε *is closable*
b) *If* $K = K^*$, *the densely defined operator* D_K^ε *is antisymmetric.*

3.5 *If K satisfies the equivalent conditions of Proposition* 3.3, *we shall say K is a trace-preserving or τ-preserving (if we want to specify the trace) noncommutative vector field.*

Note also that the equivalence in Proposition 3.3 actually holds more generally for $K \in \operatorname{Vect} L^1(M,\tau)$ while (iii) can be adapted to this case in terms of the duality of $L^1(M,\tau)$ and M.

3.6 If $K = K^* \in \operatorname{Vect} M$ then $D^\varepsilon_K P \in M$ if $P \in \mathbb{C}\langle Y_1,\ldots,Y_n\rangle$ and $D^\varepsilon_K P^* = (D^\varepsilon_K P)^*$ so that D^ε_K is a symmetric derivation (see 3.2.21 in [3]) of $C^*(Y_1,\ldots,Y_n)$.

If additionally K is in $\mathbb{C}\langle Y_1,\ldots,Y_n\rangle$ and is trace-preserving then by Corollary 3.4, $H = -iD^\varepsilon_K$ is a symmetric unbounded operator defined on $\mathbb{C}\langle Y_1,\ldots,Y_n\rangle$ which is a dense subspace in $L^2(M,\tau)$. Since D^ε_K is a derivation, we have

$$D^\varepsilon_K P \ = \ i[H,P]$$

if $P \in \mathbb{C}\langle Y_1,\ldots,Y_n\rangle$. This makes D^ε_K a *spatial derivation of* $C^*(Y_1,\ldots,Y_n)$ *implemented by* H (see Definition 3.2.54 in [3]). Summarizing, we have proved

Corollary. *Assume* $K = K^* \in \operatorname{Vect} \mathbb{C}\langle Y_1,\ldots,Y_n\rangle$ *is trace-preserving, then* D^ε_K *is a spatial derivation of* $C^*(Y_1,\ldots,Y_n)$ *implemented on* $L^2(M,\tau)$.

3.7 Combining the preceding corollary with results on exponentiation of derivations to automorphisms, one gets results about when D^ε_K can be exponentiated. For instance, using Proposition 3.2.58 in [3] we get the following.

Corollary. *Assume* $K = K^* \in \operatorname{Vect} \mathbb{C}\langle Y_1,\ldots,Y_n\rangle$ *is trace-preserving. If* $\mathbb{C}\langle Y_1,\ldots,Y_n\rangle \subset L^2(M,\tau)$ *consists of analytic vectors for* D^ε_K, *viewed as an unbounded operator in* $L^2(M,\tau)$, *then* D^ε_K *exponentiates to a one-parameter automorphism group* $\exp(tD^\varepsilon_K)$ *of* M *which is trace-preserving* $\tau \circ \exp(tD_K) = \tau$.

3.8 Let $R \geq \|Y_j\|$, $1 \leq j \leq n$. Then $|P(Y_1,\ldots,Y_n)|_2 \leq \|P(Y_1,\ldots,Y_n)\| \leq |P(X_1,\ldots,X_n)|_{R,0}$. Then Theorem 2.7 implies that for some $r > 0$ we have

$$\sum_{m \geq 0} |(D^\varepsilon_K)^m P|_2 \frac{r^m}{m!} \ < \ \infty$$

if $P \in \mathbb{C}\langle Y_1,\ldots,Y_n\rangle$, i.e., P is an analytic vector in $L^2(M)$. Thus we have actually proved:

Theorem. *Assume* $Y_j = Y_j^* \in M$, $1 \leq j \leq n$, *and* $K = K^* \in \operatorname{Vect} \mathbb{C}\langle Y_1,\ldots,Y_n\rangle$. *Assume* $\{Y_1,\ldots,Y_n\}$ *generates* M *and the* Y_j *'s are algebraically free. If* K *is trace-preserving, then* D^ε_K *exponentiates to a one-parameter group of trace-preserving automorphism of* M.

3.9 Let $R \geq \|Y_j\|$, $1 \leq j \leq n$. We could have replaced during all steps the ring $\mathbb{C}\langle Y_1, \dots, Y_n\rangle$ with the ring

$$\varepsilon_{(Y_1,\dots,Y_n)}\mathbb{C}_{\langle n\rangle, > R}$$

of power-series of radius of convergence $> R$. The only additional assumption is that algebraic freeness has to be replaced with the stronger requirement that

$$\varepsilon_{(Y_1,\dots,Y_n)}\mathbb{C}_{\langle n\rangle, > R} \ \to \ M$$

is one-to-one. With these amendments trace-preserving self-adjoint $K \in \mathrm{Vect}\,\varepsilon_{(Y_1,\dots,Y_n)}\mathbb{C}_{\langle n\rangle, > R}$ also "exponentiate" to one-parameter groups of automorphisms of M.

Of course exponentiate both here and in 3.7, 3.8 means either that we take a weak closure of the exponential or use the automorphism group implemented by the unitary group $\exp(it\overline{H})$ where $\overline{H}$ is the closure of H.

4 Endomorphic orbits

4.1 Let $Y_j = Y_j^* \in (M, \tau)$, $1 \leq j \leq n$. We shall denote by $M_h = \{a \in M \mid a = a^*\}$ the hermitian elements of M and by $L^p(M_h, \tau)$ the corresponding L^p-space. The scalar product $\sum_{1 \leq j \leq n} \tau(a_j b_j^*)$ on $(L^2(M,\tau))^n$ restricts to a real scalar product on $(L^2(M_h,\tau))^n$. Note also that the set of cyclic gradients is self-adjoint when evaluated at $Y_1, \dots, Y_n$. The *orthogonal to cyclic gradients at* $Y_1, \dots, Y_n$ is:

$$(\delta\mathbb{C}_{\langle n\rangle}^{\perp})(Y_1, \dots, Y_n) = \{Z \in (L^2(M_h,\tau))^n \mid \langle Z, \delta P(Y_1, \dots, Y_n)\rangle = 0 \ \text{ if } \ P \in \mathbb{C}_{\langle n\rangle}\} \ .$$

Clearly in the previous formula we might consider only $P = P^*$. Also, we have corresponding p-spaces replacing L^2 by L^p.

4.2 By $\mathrm{Aut}(M,\tau)$ (resp. $\mathrm{End}(M,\tau)$) we denote automorphisms (resp. unital endomorphisms) α of M such that $\tau \circ \alpha = \tau$. If M is a II_1-factor then preservation of the trace is automatic and we can write $\mathrm{Aut}(M)$ (resp. $\mathrm{End}(M)$). *If* $Y \in M_h^n$ *let* $AO(Y) = \mathrm{Aut}(M,\tau) \cdot Y$ *(resp.* $EO(Y) = \mathrm{End}(M,\tau) \cdot Y$*) be the automorphic (resp. endomorphic) orbit of* Y.

We define the tangent set to $AO(Y)$ (resp. $EO(Y)$) at Y to be

$$\begin{aligned} TAO(Y) \ &= \ \{\xi \in (L^2(M_h,\tau))^n \mid \exists\ \eta_k \in AO(Y), \\ &\qquad \exists\ \mu_k \in \mathbb{R}\backslash\{0\},\ \mu_k \to 0,\ \lim_{k\to\infty} |\mu_k^{-1}(\eta_k - Y) - \xi|_2 = 0\} \end{aligned}$$

(resp. $TEO(Y)$ defined similarly using EO instead of AO).

There are several variants of these sets $TAO_p(Y)$, $wTAO_p(Y)$ (resp. $TEO_p(Y)$, $wTEO_p(Y)_n$) where we take $x \in (L^p(M_h, \tau))^n$ and require the limit to hold in p-norm ($1 \le p \le \infty$) or for $wTAO_p(Y)$ (resp. $TEO_p(Y)$) to hold w.r.t. the weak convergence in the duality with $(L^q(M_h, \tau))^n$ ($p^{-1} + q^{-1} = 1$). If $p = 2$ we simply write $wTAO(Y)$ (resp. $wTEO(Y)$). Remark also that $wTEO(Y) \supset TEO(Y) \cup TAO(Y) \cup wTAO(Y)$.

4.3 Proposition. $wTEO(Y) \subset (\delta\mathbb{C}^{\perp}_{\langle n\rangle})(Y)$.

Proof. Let $\eta_k \in EO(Y)$ so that $w - \lim_{k\to\infty}(\mu_k^{-1}(\eta_k - Y) - \xi) = 0$ in $(L^2(M_h, \tau))^n$. Since the weak convergence is for a sequence, we have uniform boundedness in L^2 and hence $\mu_k^{-1} = O(|\eta_k - Y|_2^{-1})$. On the other hand, since η_k, Y are in $(M_h)^n$, we have

$$\tau(P(\eta_k) - P(Y) - \sum_j (\eta_k - Y)_j(\delta_j P)(Y)) = O(|\eta_k - Y|_2^2)$$

where $P \in \mathbb{C}_{\langle n\rangle}$. Since

$$\begin{aligned} \mu_k^{-1} O(|\eta_k - Y|_2^2) &= O(|\eta_k - Y|_2^{-1} |\eta_k - Y|_2^2) \\ &= O(|\eta_k - Y|_2) = o(1) \end{aligned}$$

and $\tau(P(\eta_k)) = \tau(P(Y))$ we have

$$\lim_{k\to\infty} \tau\left(\sum_j \mu_k^{-1}(\eta_k - Y)_j(\delta_j P)(Y)\right) = 0 \ .$$

We infer $\sum_j \tau(\xi_j(\delta_j P)(Y)) = 0$, i.e., $\xi \in (\delta\mathbb{C}^{\perp}_{\langle n\rangle})(Y)$. □

4.4 If $Y \in (M_h)^n$ is a generator of M, then $EO(Y)$ parametrizes $\mathrm{End}(M, \tau)$.

It is also easily seen that $EO(Y)$ is a closed set in the topology of the L^2-metric.

On the other hand, if $Y = (Y_1, \ldots, Y_n)$ *is a generating semicircular system of* $L(F(n))$, *then* $AO(Y)$ *being* L^2*-closed would imply the non-isomorphism of free group factors.*

Indeed $L(F(\infty))$ has a sequence of automorphisms α_k so that $\alpha_k(\lambda(g_p)) = \lambda(g_q)$ if $q - p \equiv 1 \pmod{k}$ when $1 \le p \le k$ and $p = q$ if $p > k$, which converge pointwise on L^2 to a non-invertible endomorphism. Thus the isomorphism of $L(F(n))$ and $L(F(\infty))$ would imply the automorphic orbit of Y is not closed.

4.5 Remark. In connection with free probability, there is a third kind of orbits which occur: the orbits of the relation of equivalence in distribution. The *distribution orbit of* Y denoted

$DO(Y)$ is the set $\{Y' \in M_h^n \mid Y'$ and Y equivalent in distribution$\}$ where equivalence in distribution means $\tau(Y_{i_1} \dots Y_{i_p}) = \tau(Y'_{i_1} \dots Y'_p)$ for all $p \in \mathbb{N}$ and $i_j \in \{1, \dots, n\}$. Equivalently, $Y' \in DO(Y)$ if there is a unital trace-preserving isomorphism $\alpha : W^*(I, Y_1, \dots, Y_n) \to W^*(I, Y'_1, \dots, Y'_n)$ so that $\alpha(Y_j) = Y'_j$. Clearly $DO(Y) \supset EO(Y)$. Then the tangent sets $TDO(Y)$, $T_pDO(Y)$, $wTDO(Y)$, $wTDO_p(Y)$ are defined similarly to the endomorphic and automorphic cases and we have the analogue of Proposition 4.3, i.e.:

$$wTDO(Y) \subset (\delta\mathbb{C}_{\langle n\rangle}^{\perp})(Y) \ .$$

5 Real and complex cyclomorphic maps

5.1 Throughout section 5, by (M, τ), (N, ν), (A, α), (B, β), (C, γ) we shall denote von Neumann algebras with normal faithful trace-states.

Definition. A differentiable map $f : \Omega \to \mathbb{R}$, where $\Omega \subset M_h^n$ is an open set in the norm-topology, is $\mathbb{R}$-cyclomorphic if for every $Y \in \Omega$ there is $T \in (L^1(M_h, \tau))^n$ such that for $K \in M_h^n$ we have a

$$df[Y](K_1, \dots, K_n) = \sum_{1 \le j \le n} \tau(T_j K_j)$$

and T is in the L^1-norm closure of $\delta\mathbb{C}_{\langle n\rangle}(Y) \cap M_h^n$. We shall abbreviate saying f is $\mathbb{R}cm$.

5.2 Definition. A differentiable map $f : \Omega \to N_h^p$, where $\Omega \subset M_h^n$ is open in the norm-topology, is $\mathbb{R}$-cyclomorphic if for every $\mathbb{R}$-cyclomorphic map $f : \omega \to \mathbb{R}$, where $\omega \subset N_h^p$ is open in the norm-topology, we have that $f \circ F : \Omega \cap F^{-1}(\omega) \to \mathbb{R}$ is $\mathbb{R}$-cyclomorphic. Also in this case this will be abbreviated by F is $\mathbb{R}cm$.

5.3 Since $\mathbb{R} = \mathbb{C}_h$, a $\mathbb{R}cm$ map $f : \Omega \to \mathbb{R}$ is also a map $f : \Omega \to \mathbb{C}_h$. It is easily seen that definition 5.1 for $f : \Omega \to \mathbb{R}$ and definition 5.2 for $f : \Omega \to \mathbb{C}_h$ are equivalent. For this it suffices to remark that the $\mathbb{R}cm$ maps $g : \omega \to \mathbb{R}$, $\omega \subset \mathbb{C}_h$ are just the differentiable maps and that if $g : \omega \to \mathbb{R}$ is differentiable, $f(Y) \in \omega$, $Y \in \Omega$, then $d(g \circ f)[Y] = g'(f(Y))dg[Y]$.

5.4 Examples. a) If $P = P^* \in \mathbb{C}_{\langle n\rangle}$ then the map $f_P : M_h^n \to \mathbb{R}$, defined by $f_P(Y) = \tau(P(Y))$ is $\mathbb{R}cm$. Indeed $df_P[Y](K) = \tau(\sum_j (\delta_j P)(Y) K_j)$.

b) If $f_1, f_2 : \Omega \to \mathbb{R}$, where $\Omega \subset M_h^n$, are $\mathbb{R}cm$, then also $f_1 + f_2$ and $f_1 f_2$ are $\mathbb{R}cm$. The $\mathbb{R}cm$-maps from Ω to $\mathbb{R}$ form an algebra over $\mathbb{R}$ with unit.

5.5 Lemma. *A differentiable map $F : \Omega \to N_h^p$, where $\Omega \subset M_h^n$ is $\mathbb{R}cm$ iff for every $P = P^* \in \mathbb{C}_{\langle p\rangle}$ the map $f_P \circ F$ is $\mathbb{R}cm$, where $f_P : N_h^p \to \mathbb{R}$, $f_P(T) = \nu(P(T))$.*

Proof. The "only if" is obvious by 5.4a). To prove the "if"-part, remark that Definition 5.1 for a differentiable map $f : \omega \to \mathbb{R}$, $\omega \subset N^p$ open, is equivalent to saying that f is $\mathbb{R}cm$ iff for every $T \in N_h^p$ there is a sequence of polynomials $P_s = P_s^* \in \mathbb{C}_{\langle p\rangle}$ such that $\|df_{P_s}[T] - df[T]\| \to 0$ as $s \to \infty$ in $\mathcal{L}(N_h^p, \mathbb{R})$. If $Y \in \Omega$, $F(Y) \in \omega$, then $d(f \circ F)[Y] = df[F(Y)] \circ dF[Y]$, so that

$$\|d(f \circ F)[Y] - d(f_{P_s} \circ F)[Y]\| \leq \|df[F(Y)] - df_{P_s}[F(Y)]\| \, \|dF[Y]\| \, .$$

Hence $d(f \circ F)[Y]$ is in the closure of differentials $d(f_{P_s} \circ F)[Y]$, i.e. in the closure of linear maps given by cyclic gradients. Hence $f \circ F$ is $\mathbb{R}cm$. □

5.6 Examples. a) If $P_j = P_j^* \in \mathbb{C}_{\langle n\rangle}$, $1 \leq j \leq p$, then the map $F_p : M_h^n \to M_h^p$ given by $F_P(Y) = (P_j(Y))_{1 \leq j \leq p}$ is $\mathbb{R}cm$. Indeed if $Q = Q^* \in \mathbb{C}_{\langle p\rangle}$, then

$$(f_Q \circ F_P)(Y) = \tau(Q(P(Y)) = f_{Q \circ P}$$

and the assertion follows from Lemma 5.5 and Example 5.4a).

b) Let $\Omega_1 \subset A_h^n$, $\Omega_2 \subset B_h^p$, be open sets in the norm-topology, and let $F : \Omega_1 \to B_h^p$, $G : \Omega_2 \to C_h^q$ be $\mathbb{R}cm$ maps, so that $F(\Omega_1) \subset \Omega_2$. Then $F \circ G$ is also an $\mathbb{R}cm$ map.

c) If $f : \Omega \to \mathbb{R}$ and $F : \Omega \to N_h^p$, where $\Omega \subset M_h^n$, are $\mathbb{R}cm$, then $H : \Omega \to N_h^p$ defined as $H(Y) = f(Y)F(Y)$ is $\mathbb{R}cm$. Indeed by Lemma 5.5 it suffices to show that if $P = P^* \in \mathbb{C}_{\langle p\rangle}$ we have $f_P \circ H$ is $\mathbb{R}cm$. Clearly there is no loss of generality to assume P is homogeneous of degree q. Then $(f_P \circ H)(Y) = (f(Y))^q (f_P \circ H)(Y)$ is $\mathbb{R}cm$ by 5.4b).

5.7 Proposition. *Let $f : \Omega \to \mathbb{R}$, $\Omega \subset M_h^n$ be a $\mathbb{R}cm$ map and let $(\alpha_t)_{t \geq 0} \subset \mathrm{End}(M, \tau)$ be a semigroup, i.e., $\alpha_0 = \mathrm{id}_M$, $\alpha_{s+t} = \alpha_s \circ \alpha_t$. Let further $Y \in \omega$ be such that for $t \in [0,1]$, $\alpha_t(Y) \in \Omega$, $t \to \alpha_t(Y)$ is norm-continuous and the right derivative*

$$\lim_{t \downarrow 0} t^{-1}(\alpha_t(Y) - Y) = K \in M_n^h$$

exists in norm-convergence. Then

$$f(\alpha_t(Y)) = f(Y)$$

for all $t \in [0,1]$.

Proof. By Proposition 4.3 we have that $K \in \delta\mathbb{C}^{\perp}_{\langle n\rangle}(Y)$ which implies $df[Y](K) = 0$, i.e. $\lim_{t\downarrow 0} t^{-1}(f(\alpha_t(Y)) - f(Y)) = 0$. By the semigroup property we have $\lim_{t\downarrow 0} t^{-1}(\alpha_{s+t}(Y) - \alpha_s(Y)) = \alpha_s(K)$ and the same argument applied to $\alpha_s(Y)$, $0 \le s < 1$ gives

$$\lim_{t\downarrow 0} t^{-1}(f(\alpha_{t+s}(Y)) - f(\alpha_s(Y))) = 0 \ .$$

Therefore $f(\alpha_t(Y))$ as a function of $t \in [0,1]$ is continuous and has zero right derivatives at each point of [0,1). We infer $f(\alpha_t(Y))$ is constant on [0,1]. $\square$

5.8 Corollary. *If $f : M_h^n \to \mathbb{R}$ is $\mathbb{R}cm$ then f is unitary-invariant, i.e., if $U \in M$ is unitary and $Y = (Y_j)_{1\le j\le n} \in M_h^n$ then $f((UY_jU^*)_{1\le j\le n}) = f(Y)$.*

5.9 Proposition. *Let $\Omega \subset M_h$ be open in the norm-topology. A differentiable map $f : \Omega \to \mathbb{R}$ is $\mathbb{R}cm$ iff $df[Y](m) = \tau(\xi m)$ for some $\xi = \xi^* \in L^1(W^*(Y), \tau)$ for each $Y \in \Omega$.*

Proof. This follows from the fact that $\delta\mathbb{C}_{\langle 1\rangle}(Y) \cap M_h = M_h \cap \mathbb{C}_{\langle 1\rangle}(Y)$ is a dense subset of $L^1(W^*(Y)_h, \tau)$, which is clearly a closed subset of $\mathcal{L}(M_h, \mathbb{R})$ via the identification $\xi \rightsquigarrow \tau(\xi\cdot)$. $\square$

5.10 Corollary. *Let M_h^{inv} be the elements with bounded inverse in M_h. The map*

$$M_h^{\mathrm{inv}} \ni Y \ \to \ Y^{-1} \in M_h$$

is $\mathbb{R}cm$.

Proof. By 5.5 it suffices to check that

$$M_h^{\mathrm{inv}} \ni Y \ \to \ \tau(Y^{-k}) \in \mathbb{R}$$

is $\mathbb{R}cm$ for all $k > 0$. Since the differential is

$$M_h \ni K \ \to \ -k\tau(Y^{-k-1}K)$$

the assertion follows from Proposition 5.9.

5.11 Proposition. *Let $F : \Omega \to N_h^p$ be a differential map, where $\Omega \subset M_h^n$ is open in the norm-topology. If for each Y there are $\mathbb{R}cm$ maps $F_k : \Omega \to N_h^p$ such that for $k \to \infty$,*

$$\|F_k(Y) - F(Y)\| \ \to \ 0$$
$$\|dF_k[Y] - dF[Y]\| \ \to \ 0$$

then F is $\mathbb{R}cm$.

Proof. We shall use Lemma 5.5. Let $P \in \mathbb{C}_{\langle p\rangle}$ and $f_P : N^p \to \mathbb{R},\ f_P(Y) = \nu(P(Y))$. We must prove that $f_P \circ F$ is $\mathbb{R}cm$. Remark that for $k \to \infty$ we have

$$\begin{aligned} \|(f_P \circ F)(Y) - (f_P \circ F_k)(Y)\| &\to 0 \\ \|d(f_P \circ F)[Y] - d(f_P \circ F_k)[Y]\| &\to 0 \end{aligned}$$

where for the second equality we used the fact that

$$\|df_P[F(Y)] - df_P[F(Y_k)]\| \to 0 .$$

Since $d(f_P \circ F_k)[Y]$ converges to $d(f_P \circ F)[Y]$ we infer $f_P \circ F$ is $\mathbb{R}cm$. □

5.12 Corollary. *Let $\Omega = \{Y \in M_h^n \mid \|Y_j\| < R,\ 1 \le j \le n\}$ and let $G_j = G_j^* \in \mathbb{C}_{\langle n\rangle,R,0}$ be a noncommutative power series. Then the map F_G defined by $\Omega \ni Y \to (G_j(Y))_{1\le j\le p} \in M_h^p$ is $\mathbb{R}cm$.*

5.13 Theorem. *Let $A \subset B$ be a unital inclusion of von Neumann algebras and let $Z = (Z_1, \ldots, Z_m) \subset B_h^m$. Let $\Phi_Z : A_h^n \to B_h^{n+m}$ be the map defined by $\Phi_Z(Y_1, \ldots, Y_n) = (Y_1, \ldots, Y_n, Z_1, \ldots, Z_m) \in B_h^{n+m}$. If $\{Z_1, \ldots, Z_m\}$ and A are independent or freely independent in (B, β), then Φ_Z is $\mathbb{R}cm$.*

Proof. Using Lemma 5.5 it suffices to prove under the given assumptions that $f_P \circ \Phi_Z$ is $\mathbb{R}cm$ where $P = P^* \in \mathbb{C}_{\langle n+m\rangle}$. In both cases $\tau(P(Y_1, \ldots, Y_n, Z_1, \ldots, Z_m)$ is a polynomial in the moments of $(Y_1, \ldots, Y_n)$ and the moments of $(Z_1, \ldots, Z_m)$. The moments of $(Z_1, \ldots, Z_m)$ are constants, so $f_P \circ \Phi_Z$ is a polynomial in the moments of $(Y_1, \ldots, Y_m)$, or being a real polynomial in the real and imaginary parts of such moments it is $\mathbb{R}cm$ by 5.4b). □

5.14 Proposition. *Let $Y_j \in M_h$, $Z_j \in M_h$, $1 \le j \le n$ be such that $\{Y_1, \ldots, Y_n\}$ and $\{Z_1, \ldots, Z_n\}$ are independent or freely independent. Let W_{Y+Z} and W_Z be the von Neumann subalgebras of M generated by $\{I, Y_1 + Z_1, \ldots Y_n + Z_n\}$ and respectively $\{I, Z_1, \ldots, Z_n\}$ and let E be the conditional expectation of M onto W_{Y+Z}. Then, if $\xi \in ((W_Z, \tau))_h^n$ is such that $\tau(\sum_{1\le j\le n} \xi_j \delta_j P(Z)) = 0$ for all $P \in \mathbb{C}_{\langle n\rangle}$, then $\tau(\sum_{1\le j\le n} \xi_j \delta_j P(Y + Z)) = 0$ for all $P \in \mathbb{C}_{\langle n\rangle}$. In particular $(E\xi_j)_{1\le j\le n}$ is in the orthogonal to $\delta\mathbb{C}_{\langle n\rangle}(Y + Z)$* in $(W_{Y+Z,h})^n$.

Proof. The map $S : M_h^{2n} \to M_h^n$, $S((R_j)_{1\le j\le 2n} = (R_j + R_{n+j})_{1\le j\le n}$ is $\mathbb{R}cm$ and hence $S \circ \Phi_Z = \Psi_Z$ is $\mathbb{R}cm$, where Φ_Z is the map in Theorem 5.13. We have

$$\Psi_Z : M_h^n \to M_h^n , \quad \Psi_Z(Y_1, \ldots, Y_n) = (Y_j + Z_j)_{1\le j\le n} .$$

If $P = P^* \in \mathbb{C}_{\langle n\rangle}$ then $d(f_P \circ \Psi_Z)[Y]$ is of the form $\sum_j \tau(T_j \cdot)$ with $(T_j)_{1\le j\le n}$ in the L^1 closure of $\delta\mathbb{C}_{\langle n\rangle}(Y+Z)$ (the L^1 closure is actually superfluous here). Since $\xi \in \operatorname{Ker} d(f_P \circ \Psi_Z)[Y]$ we infer $d\Psi_Z[Y](\xi) \in \operatorname{Ker} df_P[Y+Z]$ for all $P = P^* \in \mathbb{C}_{\langle n\rangle}$. Since $d\Psi_Z[Y](\xi) = \xi$ this yields the assertion of the proposition. $\square$

5.15 On M_h^n we define the $\mathbb{R}cm$-equivalence relation, by $Y \sim Y'$, if $f(Y) = f(Y')$ for all $\mathbb{R}cm$ maps $f : M_h^n \to \mathbb{R}$. In particular taking $f = f_P$ where $P = P^* \in \mathbb{C}_{\langle n\rangle}$ we see that $Y \sim Y'$ implies that $(Y_1, \dots, Y_n)$ and $(Y_1', \dots, Y_n')$ are equivalent in distribution. This means that $Y \sim Y'$ implies there is an isomorphism $\rho : W^*(I, Y_1, \dots, Y_n) \to W^*(I, Y_1', \dots, Y_n')$ such that $\rho(Y_j) = Y_j'$ and which is trace-preserving. On the other hand, 5.7 and 5.8 give some partial converses.

5.16 A $k+1$-times differentiable map $F : \Omega \to N_h^p$, where $\Omega \subset M_h^n$ is open in the norm-topology, is $k - \mathbb{R}cm$ $(k \in \mathbb{N} \cup \infty)$ if for all $i \le j \le k$ the j-th order differential $d^j F : \Omega \times M_h^j \to N_h^p$ is $\mathbb{R}cm$.

5.17 Examples. The maps F_P defined in 5.6a) are $\infty - \mathbb{R}cm$. Indeed $d^j F_P : M_h^{j+1} \to M_h^p$ are also maps of the type defined in 5.6a).

5.18 We shall now take a look at some weakenings of the requirement of cyclomorphy.

Definition. Let $\Omega \subset M_h^n$ be open. A map $F : \Omega \to N_h^p$ is weakly $\mathbb{R}cm$ (abbreviated $w\mathbb{R}cm$) at a given point $Y \in \Omega$ if F is differentiable at Y and

$$(dF[Y])(\delta\mathbb{C}_{\langle n\rangle}^{\perp}(Y) \cap M_h^n) \subset \delta\mathbb{C}_{\langle n\rangle}^{\perp}(F(Y)) \cap N_h^p .$$

In particular a map $f : \Omega \to \mathbb{R}$ is $w\mathbb{R}cm$ if f is differentiable at Y and

$$\delta\mathbb{C}_{\langle n\rangle}^{\perp}(Y) \cap M_h^n \subset \operatorname{Ker} df .$$

A map $f : \Omega \to \mathbb{R}$ is pseudo $\mathbb{R}cm$ (abbreviated $\psi\mathbb{R}cm$) at $Y \in \Omega$ if f is continuous at Y and $f(Y+T) = f(Y) + o(\|T\|)$ for $T \in (Y + \delta\mathbb{C}_{\langle n\rangle}^{\perp}(Y) \cap M_h^n) \cap \Omega$.

5.19 The difference between global $w\mathbb{R}cm$ and $\mathbb{R}cm$ has to do with certain ultraweak continuity requirements for the differential. We record these facts as the next proposition, the easy proof of which is left to the reader.

Proposition. *Let $\Omega \subset M_h^n$ be open and let $F : \Omega \to N_h^p$ and $f : \Omega \to \mathbb{R}$ be differentiable maps. Then f is $\mathbb{R}cm$ iff f is $w\mathbb{R}cm$ at Y and $df[Y]$ is ultraweakly continuous for each*

$Y \in \Omega$. If F is $\mathbb{R}cm$ then F is $w\mathbb{R}cm$ at each point $y \in \Omega$. If F is $w\mathbb{R}cm$ at Y and $dF[Y]$ is ultraweakly continuous at each point $Y \in \Omega$, then F is $\mathbb{R}cm$.

5.20 We pass now to our brief discussion of complex cyclomorphy. We shall need to consider spaces of cyclic gradients evaluated at non-selfadjoint n-tuples $Z = (Z_j)_{1 \le j \le n} \in M^n$.

Definition. Let $\Theta \subset M^n$ be an open set. A complex-analytic map $h : \Theta \to \mathbb{C}$ is complex cyclomorphic (abbreviated $\mathbb{C}cm$) if for each $Z \in \Omega$ there is $S \in (L^1(M,\tau))^n$ such that for $W \in M^n$ we have $dh[Z](W_1, \ldots, W_n) = \sum_{1 \le j \le n} \tau(S_j W_j)$ and S is in the L^1-norm-closure of $\delta\mathbb{C}_{\langle n\rangle}(Z) \cap M^n$. A complex-analytic map $H : \Theta \to N^p$ is $\mathbb{C}cm$ if for every $h : \theta \to \mathbb{C}$, where $\theta \subset N^p$ is open, we have that $h \circ H : \Theta \cap h^{-1}(\theta) \to \mathbb{C}$ is $\mathbb{C}cm$.

5.21 Many of the results for $\mathbb{R}$-cyclomorphic maps have correspondents for $\mathbb{C}$-cyclomorphic maps and the proofs are usually analogous. We shall briefly state, without proofs, a few of these.

Proposition. *A complex analytic map $H : \Theta \to N^p$, where $\Theta \subset M^n$ is open, is $\mathbb{C}cm$ iff for every $P \in \mathbb{C}_{\langle p\rangle}$ the map $h_p \circ F$ is $\mathbb{C}cm$, where $h_P : N^p \to \mathbb{C}$, $h_P(T) = \nu(P(T))$.*

5.22 Examples. a) If $P \in \mathbb{C}_{\langle n\rangle}$, then $h_P : M^n \to \mathbb{C}$ defined by $h_P(Z = \tau(P(Z))$ is $\mathbb{C}cm$.

b) If $h \in \Omega \to \mathbb{C}$ is $\mathbb{C}cm$, $H : \Omega \to N^p$ is $\mathbb{C}cm$, where Ω is open, then also $Hh : \Omega \to N^p$ is $\mathbb{C}cm$. In particular the $\mathbb{C}cm$ maps from Ω to $\mathbb{C}$ form an algebra over $\mathbb{C}$ with unit.

c) If $P_j \in \mathbb{C}_{\langle n\rangle}$, $1 \le j \le p$, then the map $H_P : M^n \to M^p$, given by $H_P(Z) = (P_j(Z))_{1 \le j \le p}$ is $\mathbb{C}cm$.

d) Let $\Theta_1 \subset A^n$, $\Theta \subset B^p$ be open sets, and let $H : \Theta_1 \to B^p$, $K : \Theta_2 \to C^q$ be $\mathbb{C}cm$ maps such that $H(\Theta_1) \subset \Theta_2$. Then $K \circ H$ is $\mathbb{C}cm$.

5.23 Proposition. *Let $\Theta \subset M$ be open. A complex analytic map $h : \Theta \to \mathbb{C}$ is $\mathbb{C}cm$ iff $dh[Z](T) = \tau(\xi T)$ for some $\xi \in L^1$-closure of $\mathbb{C}_{\langle 1\rangle}(Z)$.*

5.24 Proposition. *Let M^{inv} be the elements with bounded inverse in M. The map $M^{\text{inv}} \ni Z \to Z^{-1} \in M$ is $\mathbb{C}cm$.*

5.25 Proposition. *Let $H : \Theta \to N^p$ be a complex-analytic map, where $\Theta \subset M^n$ is open. If for each $Z \in M^n$ there are $\mathbb{C}cm$ maps $H_k : \Theta \to N^p$ such that for $k \to \infty$*

$$\|H_k(Z) - H(Z)\| \to 0$$
$$\|dH_k[Z] - dH[Z]\| \to 0$$

then H is $\mathbb{C}cm$.

5.26 Proposition. *Let $\Theta = \{Z \in M^n \mid \|Z_j\| < R,\ 1 \leq j \leq n\}$ and let $G_j \in \mathbb{C}_{\langle n\rangle,R,0}$ be noncommutative power series. Then the map H_G defined by*

$$\Theta \ni Z \to (G_j)(Z))_{1 \leq j \leq p} \in M^p$$

is $\mathbb{C}cm$.

5.27 Theorem. *Let $A \subset B$ be a unital inclusion of von Neumann algebras and let $W = (W_1, \ldots, W_m) \in B^m$. Let $\Xi_W : A^n \to B^{n+m}$ be the map defined by $\Xi_W(Z_1, \ldots, Z_n) = (Z_1, \ldots, Z_n, W_1, \ldots, W_m) \in B^{n+m}$. If $\{W_1, \ldots, W_m\}$ and A are independent or freely independent in (B, β) then Ξ_W is $\mathbb{C}cm$.*

5.28 Definition. A complex-analytic map $H : \Theta \to N^p$, where $\Theta \subset M^n$ is open, is $k{-}\mathbb{C}cm$ $(k \in \mathbb{N} \cup \infty)$ if for all $1 \leq j \leq k$, the j-th order complex differential $d^j H : \Omega \times M^j \to N^p$ is $\mathbb{C}cm$.

5.29 Example. If $P_i \in \mathbb{C}_{\langle n\rangle}$, $1 \leq i \leq p$, then the map $H_P : M^n \to M^p$ given by $H_P(Z) = (P_i(Z))_{1 \leq i \leq p}$ is $\infty - \mathbb{C}cm$.

5.30 To define weak $\mathbb{C}cm$ maps we first must adapt some notation. For non-selfadjoint $Z \in M^n$ we shall denote by $\delta\mathbb{C}^{\perp}_{\langle n\rangle}(Z)$ the set

$$\{T \in (L^1(M, \tau))^n \mid \sum_{1 \leq j \leq n} \tau(T_j(\delta_j P)(Z)) = 0 \text{ for all } P \in \mathbb{C}_{\langle n\rangle}\}\ .$$

In case there may be some confusion with the notation in the selfadjoint case, we may emphasize the non-selfadjointness by writing $(\delta\mathbb{C}^{\perp}_{\langle n\rangle})_{nh}(Z)$.

Definition. Let $\Theta \subset M^n$ be open. A map $H : \Theta \to N^p$ is weakly $\mathbb{C}cm$ (abbreviated $w\mathbb{C}cm$) at a point $Z \in \Theta$ if H is $\mathbb{C}$-differentiable at Z and its $\mathbb{C}$-differential satisfies

$$(dH[Z])(\delta\mathbb{C}^{\perp}_{\langle n\rangle}(Z) \cap M^n) \subset \delta\mathbb{C}^{\perp}_{\langle p\rangle}(H(Z)) \cap N^p\ .$$

In particular a map $h : \Theta \to \mathbb{C}$ is $w\mathbb{C}cm$ at Z if h is $\mathbb{C}$-differentiable at Z and $\delta\mathbb{C}^{\perp}_{\langle n\rangle}(Z) \cap M^n \subset \operatorname{Ker} df$.

5.31 Proposition. *Let $\Theta \subset M^n$ be open and let $H : \Theta \to N^p$ and $h : \Theta \to \mathbb{C}$ be maps. Then h is $\mathbb{C}cm$ iff h is $w\mathbb{C}cm$ at Z and $dh[Z]$ is ultraweakly continuous for each $Z \in \Theta$.*

If H is $\mathbb{C}cm$ then H is $w\mathbb{C}cm$ at each point $Z \in \Theta$. If H is $w\mathbb{C}cm$ at Z and $dH[Z]$ is ultraweakly continuous at each point $Z \in \Theta$, then H is $\mathbb{C}cm$.

5.32 $\mathbb{R}cm$ and $\mathbb{C}cm$ manifolds. If Ω_1 and Ω_2 are open in M_h^n (resp. in M^n) then a $\mathbb{R}cm$-isomorphism (resp. $\mathbb{C}cm$ isomorphism) is a $\mathbb{R}cm$ (resp. $\mathbb{C}cm$) map $F : \Omega_1 \to \Omega_2$ which is a bijection and the inverse of which $F^{-1} : \Omega_2 \to \Omega_1$ is also $\mathbb{R}cm$ (resp. $\mathbb{C}cm$).

A $\mathbb{R}cm$ (resp. $\mathbb{C}cm$) n-manifold over M_h (resp. M) is then a manifold modeled on M_h^n (resp. M^n) with an equivalence class of atlases (U_1, φ_1) such that the maps $\varphi_j\varphi_i^{-1}$: $\varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ are $\mathbb{R}cm$-isomorphisms (resp. $\mathbb{C}cm$ isomorphisms) (See for instance [7] for the definitions in the differentiable case.)

5.33 Remarks. a) It is easy to produce examples of $\mathbb{R}cm$ or $\mathbb{C}cm$ manifolds by gluing open sets in M_h^n, or respectively M^n by $\mathbb{R}cm$, or respectively $\mathbb{C}cm$ isomorphisms.

b) It seems that some natural candidates for $\mathbb{R}cm$ manifolds don't satisfy the definitions. For instance the unitary group $U(M) = \{u \in M \mid u^*u = uu^* = I\}$ is such a candidate, but there is a problem with the charts which one would expect to be connected with the Cayley transform $u \to i(u - e^{i\theta}I)(u - e^{-i\theta}I)^{-1}$ or to logarithms. Unfortunately the natural domain for the Cayley transform are the unitaries such that $\mathrm{Ker}(u - e^{-i\theta}I) = \{0\}$, which is not an open set while the range would consist of unbounded selfadjoint operators affiliated with M. Since these charts are the natural ones, the solution could be to find relaxations of the definitions so that also these charts be admissible.

5.34 Cyclicgrad submanifolds. To conclude the discussion around cyclomorphy here is another interesting class of geometric objects that arise in this context: the real (respectively, complex) cyclicgrad submanifolds in M_h^n (respectively, in M^n). A differentiable (respectively, complex-analytic) submanifold $V \subset M_h^n$ (respectively, $W \subset M^n$) is real-cyclicgrad (respectively, complex-cyclicgrad) if for every $Y \in V$ (respectively, $Z \in W$), $T_Y V \subset L^2(\delta\mathbb{C}_{\langle n\rangle}[Y])$ (respectively, for the complex tangent space $T_Z W \subset L^2(\delta\mathbb{C}_{\langle n\rangle}[Z])$. Note that these manifolds are perpendicular to automorphic orbits since the latter have tangent spaces contained in the orthogonal to cyclic gradients.

6 The Lie algebras

6.1 Vect $\mathbb{C}_{\langle n\rangle}$ is a Lie algebra under the bracket

$$[P, Q] = (D_P Q_j - D_Q P_j)_{1 \le j \le n}$$

where $P = (P_j)_{1\leq j\leq n}$, $Q = (Q_j)_{1\leq j\leq n}$. Possible confusion with the commutator on $\mathbb{C}_{\langle n\rangle}$ which is denoted the same way can be avoided by checking the context.

Since the map

$$\mathrm{Vect}\,\mathbb{C}_{\langle n\rangle} \ni P \to D_P \in \mathrm{Der}\,\mathbb{C}_{\langle n\rangle}$$

into the derivations of $\mathbb{C}_{\langle n\rangle}$ is injective, the equality $D_{[P,Q]} = [D_P, D_Q]$ implies that $[P,Q]$ is indeed a Lie algebra bracket. The last equality in turn is a straightforward computation

$$\begin{aligned} D_P D_Q X_{i_1}\dots X_{i_p} &= D_P \sum_{1\leq j\leq p} X_{i_1}\dots X_{i_{j-1}} Q_{i_j} X_{i_{j+1}}\dots X_{i_p} \\ &= \sum_{\substack{1\leq j,k\leq p\\ j\neq k}} X_{i_1}\dots X_{i_j} P_{ij} X_{i_{j+1}}\dots X_{i_{k-1}} Q_{i_k} X_{i_k}\dots X_{i_p} \\ &\qquad + \sum_{1\leq j\leq p} X_{i_1}\dots X_{i_{j-1}} (D_P Q_{i_j}) X_{i_{j+1}}\dots X_{i_p} \ . \end{aligned}$$

Combining with the analogues formula for $D_Q D_P$ we get

$$\begin{aligned} [D_P D_Q] X_{i_1}\dots X_{i_p} &= \sum_{1\leq j\leq p} X_{i_1}\dots X_{i_{j-1}} (D_P Q_{i_j} - D_Q P_{i_j}) X_{i_{j+1}}\dots X_{i_p} \\ &= D_{[P,Q]} X_{i_1}\dots X_{i_p} \ . \end{aligned}$$

6.2 We have $|[P,Q]|_{R,0} \leq |P|_{R,0}|Q|_{R,1} + |P|_{R,1}|Q|_{R,0}$. In particular if $R' > R$ the definition of the bracket extends to $P, Q \in \mathrm{Vect}\,\mathbb{C}_{\langle n\rangle,R',0}$ with $[P,Q] \in \mathrm{Vect}\,\mathbb{C}_{\langle n\rangle,R,0}$. This turns

$$\mathrm{Vect}\,\mathbb{C}_{\langle n\rangle,>R} = \bigcup_{R'>R} \mathrm{Vect}\,\mathbb{C}_{\langle n\rangle,R,0}$$

into a Lie algebra.

6.3 Lemma. *Let* $H, K \in \mathrm{Vect}\,\mathbb{C}_{\langle n\rangle}$ *and assume*

$$\begin{aligned} \Psi_n(H) &\leq C_1(1-\alpha X_1)^{-e} \\ \Psi_n(K) &\leq C_2(1-\alpha X_1)^{-f} \end{aligned}$$

Then

$$\Psi_n([H,K]) \leq C_1 C_2 (e+f)(1-\alpha X_1)^{-e-f-1} \ .$$

Proof. This is a straightforward computation based on

$$
\begin{aligned}
\Psi_n([H,K]) &\le \Psi_n((D_{|H|}|K_1|),\ldots,(D_{|H|}|K_n|)) + \Psi_n((D_{|K|}|H_1|),\ldots,(D_{|K|}|H_n|)) \\
&\le D_{\Psi_n(H)}\Psi_n(K) + D_{\Psi_n(K)}\Psi_n(H) \\
&\le C_1C_2(D_{(1-\alpha X_1)^{-e}}(1-\alpha X_1)^{-f} + D_{(1-\alpha X_1)^{-f}}(1-\alpha X_1)^{-e})
\end{aligned}
$$

□

6.4 Proposition. *Let* $K^{(j)} \in \mathrm{Vect}\,\mathbb{C}_{\langle n\rangle}$, $0 \le j \le m$, *and let* $M \ge |K^{(j)}|_{R',0}$, $0 \le j \le m$. *If* $R' > R > 0$, *then*

$$|\mathrm{ad}\,K^{(n)}\mathrm{ad}\,K^{(n-1)}\mathrm{ad}\,K^{(1)}K^{(0)}|_{R,0} \;\le\; M^{m+1}2^m m!(1-R/R')^{-2-1}$$

Proof. The assumption $M \ge |K^{(j)}|_{R',0}$ gives $\Psi_n(K^{(j)}) \le M(1-X_1/R')^{-1}$. We then prove by induction that

$$\Psi_n(\mathrm{ad}\,K^{(n)}\mathrm{ad}\,K^{(n-1)}\ldots\mathrm{ad}\,K^{(1)}\mathrm{ad}\,K^{(0)}) \le M^{m+1}2^m m!(1-X_1/R')^{-2m-1}$$

using Lemma 2.9. Setting $X_1 = R$ then yields the desired result. □

6.5 It is easily seen that the Lie bracket we considered extends to Vect $\mathbb{C}\langle\langle X_1,\ldots,X_n\rangle\rangle$ (see for instance the discussion fo the grading in 6.9). It is immediate from Lemma 6.3 that the following proposition holds.

Proposition. *The set* $\{H \in \mathrm{Vect}\,\mathbb{C}\langle\langle X_1,\ldots,X_n\rangle\rangle \mid \psi_n(H) \le C(1-\alpha X_1)^{-p}$ for some $C > 0$ and $p \in \mathbb{N}\}$ *is a Lie subalgebra.*

6.6 Remark that the Lie algebra in Proposition 6.5 is contained in Vect $\mathbb{C}_{\langle n\rangle,R',0}$, where $\alpha R' < 1$, and contain Vect $\mathbb{C}_{\langle n\rangle,R'',0}$, when $\alpha R'' > 1$. Combining this with Proposition 6.5 gives the following result.

Corollary. Vect $\mathbb{C}_{\langle n\rangle,>R}$ *is a Lie subalgebra of* Vect $C\langle\langle X_1,\ldots,X_n\rangle\rangle$.

6.7 Remark. Unless in some unexpected way the estimates in Proposition 6.4 can be vastly improved, it seems unlikely that convergence of the Campbell-Hausdorff series (see [2]) can be obtained in the operator algebra context.

6.8 We set aside the completions now to point out some simple features of Vect $\mathbb{C}_{\langle n\rangle}$ and we leave the reader to carry over the obvious considerations for the completions.

6.9 Vect $\mathbb{C}_{\langle n\rangle}$ has a grading $\bigoplus_{k\geq -1} V_k$ where V_k consists of n-tuples $P=(P_1,\ldots,P_n)$ which are homogeneous of degree $k+1$. It is easily seen that $[V_k, V_\ell] \subset V_{k+\ell}$.

6.10 It is easily seen that V_{-1} is isomorphic to the commutative Lie algebra $\mathbb{C}^n$. Also $V_{\geq 0} = \bigoplus_{k\geq 0} V_k$ is a Lie subalgebra and $V_{\geq p} = \bigoplus_{k\geq p} V_k$ where $p \in \mathbb{N}$ are ideals in $V_{\geq 0}$. V_0 is also a Lie algebra. Let $E_{ik} = (\delta_{ij} X_k)_{1\leq j\leq n}$ (here δ_{ij} is the Kronecker symbol). Then $[E_{ab}, E_{cd}] = \delta_{bc} E_{ad} - \delta_{da} E_{cb}$, i.e., V_0 is isomorphic to $\mathfrak{gl}(n,\mathbb{C})$ with E_{ab} corresponding to the matrix with (i,j)-entry equal to $\delta_{ai}\delta_{bj}$. In particular the center of V_0 is $\mathbb{C}(X_j)_{1\leq j\leq n}$.

It is also easily seen that $V_{\geq 0}$ is the semidirect product of $V_{\geq 1}$ and V_0 (V_0 acting on $V_{\geq 1}$).

6.11 The involution on Vect $\mathbb{C}_{\langle n\rangle}$ being defined component-wise, we have $D_{P^*} Q_j^* = (D_P Q_j)^*$ so that $P \rightsquigarrow P^*$ is a conjugate-linear automorphism of Vect $\mathbb{C}_{\langle n\rangle}$. Then the selfadjoint part Vect $\mathbb{C}_{\langle n\rangle}^{sa}$ is a real Lie-algebra. Remark also that V_0^{sa} is isomorphic to $\mathfrak{gl}(n,\mathbb{R})$.

6.12 If $\tau : \mathbb{C}_{\langle n\rangle} \to \mathbb{C}$ is a trace, we define

$$\operatorname{Vect} \mathbb{C}_{\langle n|\tau\rangle} = \{P \in \operatorname{Vect} \mathbb{C}_{\langle n\rangle} \mid \sum_{1\leq j\leq n} \tau(P_j(\delta_j R)) = 0 \text{ for all } R \in \mathbb{C}_{\langle n\rangle}\} .$$

Then Vect $\mathbb{C}_{\langle n|\tau\rangle}$ is a Lie subalgebra of Vect $\mathbb{C}_{\langle n\rangle}$. It can be viewed as a noncommutative analogue of the Lie algebras of volume-preserving vector fields. The defining relations can be also written $\tau(D_P R) = 0$. If $P, Q \in \mathbb{C}_{\langle n|\tau\rangle}$ then $\tau(D_{[P,Q]}R) = 0$ since

$$\tau(D_{[P,Q]}R) = \tau(D_P D_Q R - D_Q D_P R) = 0 .$$

6.13 Let $P \in \mathbb{C}_{\langle n\rangle}$ and consider $([P, X_j])_{1\leq j\leq n} \in \operatorname{Vect} \mathbb{C}_{\langle n\rangle}$ (these brackets are commutators). Then $D_{([P,X_j])_{1\leq j\leq n}} Q = [P, Q]$, i.e. the derivation defined by this element of Vect $\mathbb{C}_{\langle n\rangle}$ is just the commutator with P. We have

$$\begin{aligned}
[([P,X_1],\ldots,[P,X_n]),(P_1,\ldots,P_n)] &= (D_{([P,X_j])_{1\leq j\leq n}} P_k)_{1\leq k\leq n} \\
&\quad -([D_{(P_k)_{1\leq k\leq n}} P, X_j])_{1\leq j\leq n} - ([P,P_j])_{1\leq j\leq n} \\
&= -([D_{(P_k)_{1\leq k\leq n}} P, X_j])_{1\leq j\leq n} .
\end{aligned}$$

So $\{([P, X_j])_{1\leq j\leq n} \in \operatorname{Vect} \mathbb{C}_{\langle n\rangle} \mid P \in \mathbb{C}_{\langle n\rangle}\}$ is an ideal in Vect $\mathbb{C}_{\langle n\rangle}$. This ideal corresponds to the inner derivations. In particular since a trace applied to a commutator gives zero, we infer that

$$\{([P, X_j])])_{1\leq j\leq n} \in \operatorname{Vect} \mathbb{C}_{\langle n\rangle} \mid P \in \mathbb{C}_{\langle n\rangle}\} \subset \operatorname{Vect} \mathbb{C}_{\langle n|\tau\rangle} .$$

6.14 Clearly

$$\text{Vect } \mathbb{C}^{sa}_{\langle n|\tau\rangle} = \text{Vect } \mathbb{C}^{sa}_{\langle n\rangle} \cap \text{Vect } \mathbb{C}_{\langle n|\tau\rangle}$$

is a real Lie subalgebra of Vect $\mathbb{C}_{\langle n\rangle}$. Note also that

$$\{(i[P, X_j])_{1\le j\le n} \in \text{Vect } \mathbb{C}_{\langle n\rangle} \mid P \in \mathbb{C}^{sa}_{\langle n\rangle}\} \subset \text{Vect } \mathbb{C}^{sa}_{\langle n|\tau\rangle} \ .$$

7 The Lie algebraVect $\mathbb{C}_{\langle n|\tau\rangle}$ in the semicircular case

7.1 We return to the context of 1.4 and we will compute Vect $\mathbb{C}_{\langle n|\tau\rangle}$ in the semicircular case. An important consequence will be that the assumptions for the exponentiation result, Theorem 3.8, are satisfied in the semicircular case.

$\mathcal{T}(\mathbb{C}^n)$ identifies on one hand with the L^2-space of $W^*(s_1, \ldots, s_n)$, but also with the L^2-space of the non-selfadjoint algebra generated by $\ell_1, \ldots, \ell_n$, w.r.t. the scalar product which corresponds to the vacuum on the C^*-algebra generated by $\ell_1, \ldots, \ell_n$. Thus we will be able to look at cyclic gradients w.r.t. $s_1, \ldots, s_n$ and $\ell_1, \ldots, \ell_n$ within the same Hilbert space. We shall denote by

$$\delta^\ell = (\delta^\ell_1, \ldots, \delta^\ell_n)\mathbb{C}\langle \ell_1, \ldots, \ell_n\rangle \to \text{Vect}\,\mathbb{C}\langle \ell_1, \ldots, \ell_n\rangle$$

and

$$\delta^s = (\delta^s_1, \ldots, \delta^s_n) : \mathbb{C}\langle s_1, \ldots, s_n\rangle \to \text{Vect}\,\mathbb{C}\langle s_1, \ldots, s_n\rangle$$

the corresponding cyclic gradients. Similarly ∂^ℓ and ∂^s will denote the two free difference quotient gradients.

7.2 Proposition. *Assume $i_j \ne i_{j+1}$ $(1 \le j < p)$ and assume $k_1 > 0, \ldots, k_p > 0$. Then:*

$$(\partial^s_j P_{k_1}(s_{i_1}) \ldots P_{k_p}(s_{i_p}))(1 \otimes 1) = (\partial^\ell_j \ell^{k_1}_{i_1} \ldots \ell^{k_p}_{i_p})(1 \otimes 1)$$

If moreover $i_1 \ne i_p$ if $p > 1$, then

$$(\delta^s_j P_{k_1}(s_{i_1}) \ldots P_{k_p}(s_{i_p}))1 = (\delta^\ell_s \ell^{k_1}_{i_1} \ldots \ell^{k_p}_{i_p})1 \ .$$

Proof. By the recurrence relation for Gegenbauer polynomials which we did recall in 1.4 we have

$$(1 - rt + r^2)^{-1} = \sum_{n\ge 0} P_n(t) r^n \ .$$

In particular computing the difference quotient for the left- and right-hand sides, we get

$$\begin{aligned}\sum_{n\geq 0}\frac{P_n(t_1)-P_n(t_2)}{t_1-t_2}\,r^n &= r(1-rt_1+r^2)^{-1}(1-rt_2+r^2)^{-1}\\ &= \sum_{n\geq 1}\left(\sum_{0\leq k\leq n-1}P_k(t_1)P_{n-1-k}(t_2)\right)r^n\ .\end{aligned}$$

It follows that

$$\frac{P_n(t_1)-P_n(t_2)}{t_1-t_2}=\sum_{0\leq k\leq n-1}P_k(t_1)P_{n-1-k}(t_2)$$

i.e.,

$$\partial P_n=\sum_{0\leq k\leq n-1}P_k\otimes P_{n-1-k}\ .$$

This in turn gives

$$\begin{aligned}&(\partial_j^s P_{k_1}(s_{i_1})\dots P_{k_p}(s_{i_p}))(1\otimes 1)\\ &\sum_{\{m|i_m=j\}}\ \sum_{0\leq h<k_m}(P_{k_1}(s_{i_1})\dots P_h(s_{i_m})1)\otimes(P_{k_m-1-h}(s_{i_m})P_{k_{m+1}}(s_{i_{m+1}})\dots P_{k_p}(s_{i_p})1)\\ &=\sum_{\{m|i_m=j\}}\ \sum_{0\leq h<k_m}\big(e_{i_1}^{\otimes k_1}\otimes\dots\otimes e_{i_m}^{\otimes h}\big)\otimes\big(e_{i_m}^{\otimes k_m-1-h}\otimes\dots\otimes e_{i_p}^{\otimes k_p}\big)\\ &=\sum_{\{m|i_m=j\}}\ \sum_{0\leq h<k_m}\big(\ell_{i_1}^{k_1}\dots\ell_{i_m}^{h}1\big)\otimes\big(\ell_{i_m}^{k_m-1-h}\dots\ell_{i_p}^{k_p}1\big)\\ &=\big(\partial_j^\ell\,\ell_{i_1}^{k_1}\dots\ell_{i_p}^{k_p}\big)1\otimes 1\ .\end{aligned}$$

Similarly under the additional assumption that $i_p\neq i_1$ in case $p>1$, we have

$$\begin{aligned}&\delta_j^s P_{k_1}(s_{i_1})\dots P_{k_p}(s_{i_p})1\\ &=\sum_{\{m|i_m=j\}}\ \sum_{0\leq h<k_m}P_{k_m-1-h}(s_{i_m})P_{k_m+1}(s_{i_{m+1}})\dots P_{k_p}(s_{i_p})P_{k_1}(s_{i_1})\dots P_{k_{m-1}}(s_{i_{m-1}})P_h(s_{i_m})1\\ &=\sum_{\{m|i_m=j\}}\ \sum_{0\leq h<k_m}e_{i_m}^{\otimes k_m-1-h}\otimes e_{i_{m+1}}^{\otimes k_{m+1}}\otimes\dots\otimes e_{i_p}^{\otimes k_p}\otimes e_{i_1}^{\otimes k_1}\otimes\dots\otimes e_{i_{m-1}}^{\otimes k_{m-1}}\otimes e_{i_m}^{\otimes h}\\ &=\sum_{\{m|i_m=j\}}\ell_{i_m}^{k_m-1-h}\ell_{i_{m+1}}^{k_{m+1}}\dots\ell_{i_p}^{k_p}\ell_{i_1}^{k_1}\dots\ell_{i_{m-1}}^{k_{m-1}}\ell_{i_m}^{h}1\\ &=\big(\delta_j^\ell\,\ell_{i_1}^{k_1}\dots\ell_{i_p}^{k_p}\big)1\ .\end{aligned}$$

$\square$

7.3 Lemma. *Let* $\mathcal{F}_k \subset \mathbb{C}\langle s_1, \ldots, s_n\rangle$ *be the linear span of* $P_{k_1}(s_{i_1}) \ldots P_{k_p}(s_{i_p})$ *with* $k_j > 0$ $(1 \le j \le p)$, $k_1 + \cdots + k_p = k$, $i_j \ne i_{j+1}$ $(1 \le j < p)$ *and* $i_1 \ne i_p$ *if* $p > 1$, *and let* $\mathcal{F} = \sum_{k\ge 0} \mathcal{F}_k$. *Then* $\mathcal{F}+ \operatorname{Ker}\ \delta^s = \mathbb{C}\langle s_1, \ldots, s_n\rangle$.

Proof. Recall from 1.3 that Ker $\delta^s = \operatorname{Ker} C^s$ where C^s is the cyclic symmetrization. Moreover Ker $\delta^s = \sum_{k\ge 0} \operatorname{Ker} \delta^{s,k}$ is a direct sum decomposition where $\delta^{s,k}$ is the restriction of δ^s to the homogeneous noncommutative polynomials of degree k and similarly

$$\operatorname{Ker} C^s = \sum_{k\ge 0} \operatorname{Ker} C^{s,k}$$

so that Ker $\delta^{s,k} = \operatorname{Ker} C^{s,k}$.

Let $\mathcal{G}_k$ be the linear span of $s_{i_1}^{k_1} \ldots s_{i_p}^{k_p}$ with $k_j > 0$ $(1 \le j \le p)$, $k_1 + \cdots + k_p = k$, $i_j \ne i_{j+1}$ $(1 \le j < p)$ and $i_1 \ne i_p$ if $p \ge 2$, and put $\mathcal{G} = \sum_{k\ge 0} \mathcal{G}_k$. It is easily seen that

$$\mathcal{G}_k + \operatorname{Ker} C^{s,k} = (\mathbb{C}\langle s_1, \ldots, s_n\rangle)_k \ .$$

It will suffice to prove that

$$\sum_{0\le k\le m} (\mathcal{F}_k + \operatorname{Ker} C^{s,k}) \ = \ \sum_{0\le k\le m} (\mathcal{G}_k + \operatorname{Ker} C^{s,k}) \ .$$

This is shown by induction on m, using

$$\mathcal{F}_m + \sum_{0\le k<m} (\mathbb{C}\langle s_1, \ldots, s_n\rangle)_k \ = \ \mathcal{G}_m + \sum_{0\le k<m} (\mathbb{C}\langle s_1, \ldots, s_n\rangle)_k$$

which follows from $P_k(s) = cs^k \pmod{\sum_{0\le h<k} (\mathbb{C}\langle s\rangle)_h}$ for some $c \ne 0$. $\square$

7.4 Theorem. $(\delta^s\mathbb{C}\langle s_1, \ldots, s_n\rangle)(1 \oplus \cdots \oplus 1) = (\delta^\ell\mathbb{C}\langle \ell_1, \ldots, \ell_m\rangle)(1 \oplus \cdots \oplus 1)$.

Proof. Let $\mathcal{L}_k$ be the linear span of $\ell_{i_1}^{k_1} \ldots \ell_{i_p}^{k_p}$, $k = k_1 + \cdots + k_p$, $i_j \ne i_{j+1}$, $(1 \le j \le p)$, $i_1 \ne i_p$ if $p \ge 2$, and $\mathcal{L} = \sum_{k\ge 0} \mathcal{L}_k$. Then $\mathcal{L} + \operatorname{Ker} C^\ell = \mathbb{C}\langle \ell_1, \ldots, \ell_n\rangle$ and hence $\mathcal{L} + \operatorname{Ker} \delta^\ell = \mathbb{C}\langle \ell_1, \ldots, \ell_n\rangle$.

It follows that

$$\begin{aligned}
&(\delta^\ell\mathbb{C}\langle \ell_1, \ldots, \ell_n\rangle)(1 \oplus \cdots \oplus 1)\\
&= (\delta^\ell\mathcal{L})(1 \oplus \cdots \oplus 1)\\
&= (\delta^s\mathcal{F})(1 \oplus \cdots \oplus 1)\\
&= (\delta^s\mathbb{C}\langle s_1, \ldots, s_n\rangle)(1 \oplus \cdots \oplus 1)
\end{aligned}$$

where we used 7.2 and 7.3 in the last two steps. □

7.5 Theorem. *Identifying* $\mathcal{T}(\mathbb{C}^n)$ *and* $L^2(W(s_1,\dots,s_n),\tau)$ *we have:*

$$(\mathcal{T}(\mathbb{C}^n))^n \ominus \overline{\delta^s\mathbb{C}\langle s_1,\dots,s_n\rangle} \;=\; \{((\ell_j^* - r_j^*)\xi)_{1\le j\le n} \mid \xi \in \mathcal{T}(\mathbb{C}^n)\}\ .$$

In particular

$$\mathrm{Vect}\,\mathbb{C}\langle s_1,\dots,s_n \mid \tau\rangle = \sum_{k\ge 0} \mathcal{X}_k$$

where $\mathcal{X}_k \subset (\mathcal{T}_k(\mathbb{C}^n))^n$, *where* $\mathcal{X}_0 = 0$ *and* $\mathcal{X}_k$ *for* $k \ge 1$ *is spanned by*

$$\left(\delta_{j,i_0} e_{i_1} \otimes \ldots \otimes e_{i_k} - \delta_{j,i_k} e_{i_0} \otimes \ldots \otimes e_{i_{k-1}}\right)_{1\le j\le n}$$

where $(i_0,\dots,i_k) \in \{1,\dots,n\}^{k+1}$.

Proof. $\delta^s\mathbb{C}\langle s_1,\dots,s_n\rangle$ identifies with

$$\delta^s\mathbb{C}\langle s_1,\dots,s_n\rangle)(1\oplus\cdots\oplus 1) = (\delta^\ell\mathbb{C}\langle \ell_1,\dots,\ell_n\rangle)(1\oplus\cdots\oplus 1)\ .$$

To compute $(\mathcal{T}(\mathbb{C}^n))^n \ominus \overline{\delta^\ell\mathbb{C}\langle \ell_1,\dots,\ell_n\rangle(1\oplus\cdots\oplus 1)}$ we use the exact sequence 1.3. The maps in the exact sequence are homogeneous, so we get exact sequences

$$(\mathbb{C}_{\langle n\rangle})_k \;\xrightarrow{\delta}\; V_{k-2} \;\xrightarrow{\theta}\; (\mathbb{C}_{\langle n\rangle})_k$$

(recall that V_k are the noncommutative vector fields with components homogeneous of degree $k+1$). Endowing $(\mathbb{C}_{\langle n\rangle})_k$ with the scalar product in which the monomials $X_{i_1}\dots X_{i_k}$ form an orthonormal basis we have isometries

$$\begin{aligned}(\mathbb{C}_{\langle n\rangle})_k &\longrightarrow \mathcal{T}_k(\mathbb{C}^n)\\ V_{k-2} &\longrightarrow (\mathcal{T}_{k-1}(\mathbb{C}^n))^n\end{aligned}$$

which map $X_{i_1}\dots X_{i_k}$ to $e_{i_1}\otimes\ldots\otimes e_{i_k}$. In this correspondence δ identifies with δ^ℓ acting in the L^2-space and θ with the map

$$\theta^\ell((\xi_j)_{1\le j\le n}) = \sum_{1\le j\le n} (\ell_j - r_j)\xi_j\ .$$

Clearly, then

$$(\mathcal{T}_k(\mathbb{C}^n))^n \ominus \delta^\ell(\mathbb{C}\langle \ell_1,\dots,\ell_n\rangle)_{k+1} = (\theta^\ell)^*\xi = ((\ell_j^* - r_j^*)\xi)_{1\le j\le n}\ .$$

Taking $\xi = e_{i_0} \otimes e_{i_1} \otimes \ldots \otimes e_{i_k}$ we get the last part of the statement. $\square$

7.6. Transcribing the last part of the preceding theorem in terms of the s_j's instead of the e_j's, we get the following result.

Corollary *The elements*

$$F_I = \big(\delta_{i_0,j} P_{k_0-1}(s_{i_0}) P_{k_1}(s_{i_1}) \ldots P_{k_p}(s_{i_p}) - \delta_{i_p,j} P_{k_0}(s_{i_0}) P_{k_1}(s_{i_1}) \ldots P_{k_p-1}(s_{i_p})\big)_{1 \le j \le n}$$

where $I = (\underbrace{i_0, \ldots, i_0}_{k_0\text{-times}}, \ldots, \underbrace{i_p, \ldots, i_p}_{k_p\text{-times}})$, $k_r > 0$ $(0 \le r \le p)$, *span* $\mathrm{Vect}\, \mathbb{C}\langle s_1, \ldots, s_n \mid \tau\rangle$.

7.7. We will now clarify the relations among the elements F_I. On $\mathcal{T}_k(\mathbb{C}^n)$ let R_k be the cyclic permutation $R_k e_{i_1} \otimes \ldots \otimes e_{i_k} = e_{i_k} \otimes e_{i_1} \otimes \ldots \otimes e_{i_{k-1}}$ (in particular $R_0 1 = 0$, $R_1 e_j = e_j$ and $R_2 e_i \otimes e_j = e_j \otimes e_i$). Let $R = \bigoplus_{k \ge 1} R_k$ be the operator on $\mathcal{T}(\mathbb{C}^n)$. Let also P be the projection onto $\mathbb{C}1$.

Lemma *Let* $T_j = \ell_j - r_j$. *Then* $\sum_{1 \le j \le n} T_j T_j^* = 2I - 2P - (R + R^*)$. *In particular*

$$\mathrm{Ker}\,(\theta^\ell)^* = \mathrm{Ker} \sum_{1 \le j \le n} T_j T_j^* = \mathrm{Ker}(I - P - R)\ .$$

Proof.

$$\begin{aligned} \sum_{1 \le j \le n} T_j T_j^* &= \sum_{1 \le j \le n} \big(\ell_j \ell_j^* + r_j r_j^* - \ell_j r_j^* - r_j \ell_j^*\big) \\ &= (I - P) + (I - P) - R - R^*\ . \end{aligned}$$

Since $R\,|_{\mathcal{T}_+}$ is a unitary operator, we have
$\mathrm{Ker}\,((2I - R - R^*)|_{\mathcal{T}_+} = \mathrm{Ker}(I - R)|_{\mathcal{T}_+}$. $\square$

7.8. Remark. Let N be the number operator, i.e. $N1 = 0$, $N e_{i_1} \otimes \ldots \otimes e_{i_k} = k e_{i_1} \otimes \ldots \otimes e_{i_k}$. Then

$$\mathrm{Ker}((I - R)|_{\mathcal{T}_+} = \mathrm{Ker}(N - C)|_{\mathcal{T}_+}$$

7.9. Remark. To get a basis of $\mathrm{Vect}\, \mathbb{C}\langle s_1, \ldots, s_n | \tau\rangle$ we shall use a basis of $(I - R)(\mathcal{T}_+(\mathbb{C}^n))$. Consider on $\{1, \ldots, n\}^{k+1}$ the lexicographic order $\prec$ and let

$\Omega(k+1) = \{(i_0, \ldots, i_k) \in \{i, \ldots, n\}^{k+1} \,|\, (i_0, \ldots, i_k) \prec (i_k, i_0, \ldots, i_{k-1})$ and $(i_0, \ldots, i_k) \ne (i_k, i_0, \ldots, i_{k-1})\}$

Then

$$\Psi = \bigcup_{k\geq 1} \{F_{i_0\dots i_k} - F_{i_k i_0 \dots i_{k-1}} \mid (i_0 \dots i_k) \in \Omega(k+1)\}$$

is a basis of $\mathrm{Vect}\,\mathbb{C}\langle s_1, \dots, s_n|\tau\rangle$ over $\mathbb{C}$. Indeed the set

$$\Phi = \bigcup_{k\geq 1} \{e_{i_0} \otimes \dots \otimes e_{i_k} - e_{i_k} \otimes e_{i_0} \dots e_{i_{k-1}} \mid (i_0 \dots i_k) \in \Omega(k+1)\}$$

is a homogeneous basis of $(\bigoplus_{k\geq 2} \mathcal{T}_k(\mathbb{C}^n) \ominus \mathrm{Ker}(I - P - R)$ which in view of Lemma 7.7 is mapped by $(\theta^\ell)^*$ to $\mathrm{Vect}\,\mathbb{C}\langle s_1, \dots, s_n|\tau\rangle$. We have used here that $(\theta^\ell)^*\xi = (T_j^*\xi)_{1\leq j\leq n}$, so that $\mathrm{Ker}\,(\theta^\ell)^* = \mathrm{Ker}\,\sum_{1\leq j\leq n} T_j T_j^*$. Also $F_{i_0\dots i_k} = (\theta^\ell)^* e_{i_0} \otimes \dots \otimes e_{i_k}$.

7.10. Remark. The same kind of considerations as in 7.9 yield also another natural basis. If $I = (i_0, \dots, i_k) \in \{1, \dots, n\}^{k+1}$, let $\mathrm{per}(I)$ be the least non-zero cyclic period of I, i.e., the least $m \in \{1, \dots, k\}$ such that $i_s = i_t$ whenever $s \equiv t \pmod m$. If $\zeta^m = 1$, $\zeta \neq 1$ and $m = \mathrm{per}(I)$ let $F(I, \zeta) = \sum_{0\leq j<m} \zeta^j F_{(i_j, i_{j+1}, \dots, i_k, i_0, \dots, i_{j-1})}$. Let also $\omega(k+1) = \{(i_0 \dots i_k) \in \{1, \dots, h\}^{k+1} \mid (i_0, \dots i_k) \prec (i_j, i_{j+1}, \dots, i_k, i_0, \dots, i_{j-1}), j = 1, \dots, k\}$ and $\rho(m) = \{\zeta \in \mathbb{C} \mid \zeta \neq 1,\ \zeta^m = 1\}$. Then

$$\bigcup_{k\geq 1} \{F(I, \zeta) \mid I \in \omega(k+1),\ \zeta \in \rho(\mathrm{per}(I))\}$$

is a basis of $\mathrm{Vect}\,\mathbb{C}\langle s_1, \dots, s_n|\tau\rangle$.

7.11. To construct a basis over $\mathbb{R}$ of the real Lie algebra $\mathrm{Vect}\,\mathbb{C}^{sa}\langle s_1, \dots, s_n|\tau\rangle$ we can take the hermitian and antihermitian parts of the elements of the previous bases. In doing so one must pay attention to the fact that if $\mathrm{per}(I) = 2$ then $F_{i_0\dots i_k} - F_{i_k i_0\dots i_{k-1}}$ is antihermitian. For instance the basis in 7.9 then gives the basis:

$$\bigcup_{k\geq 1} \big\{F_{i_0\dots i_k} + F_{i_k\dots i_0} - F_{i_k i_0\dots i_{k-1}} - F_{i_{k-1}\dots i_0 i_{k-1}} \mid (i_0, \dots, i_k) \in \Omega(k+1), \mathrm{per}(i_0, \dots, i_k) > 2\big\}$$
$$\cup \bigcup_{k\geq 1} \big\{\sqrt{-1}(F_{i_0\dots i_k} - F_{i_k\dots i_0} - F_{i_k i_0\dots i_{k-1}} - F_{i_{k-1}\dots i_0 i_{k-1}} \mid (i_0, \dots, i_k) \in \Omega(k+1)\big\}$$

7.12. Putting together the results of this section and the results about exponentiation and automorphic orbits we see that we have proved the following result.

Theorem *If $s_1, \dots, s_n$ is a semicircular system and $M = W^*(s_1, \dots, s_n)$ then $TAO_\infty(s_1, \dots, s_n) \cap \mathrm{Vect}\,\mathbb{C}^{sa}\langle s_1, \dots, s_n\rangle = \mathrm{Vect}\,\mathbb{C}^{sa}\langle s_1, \dots, s_n \mid \tau\rangle$ is dense in $(\delta\mathbb{C}^{\perp}_{\langle n\rangle})(s_1, \dots, s_n)$.*

In particular for K in this dense subset of $(\delta\mathbb{C}^{\perp}_{\langle n\rangle})(s_1,\ldots,s_n)$, D^{ε}_K exponentiates to a one-parameter automorphism group of $W^*(s_1,\ldots,s_n)$. Moreover $C^*(s_1,\ldots,s_n)$ is invariant under these $\exp(tD^{\varepsilon}_K)$.

7.13. Remark It is easily seen that $V_0 \cap \mathrm{Vect}\ \mathbb{C}^{sa}\langle s_1,\ldots,s_n \mid \tau\rangle = 0$ and $V_1 \cap \mathrm{Vect}\ \mathbb{C}^{sa}\langle s_1,\ldots,s_n \mid \tau\rangle$ is isomorphic to the Lie algebra of the orthogonal group $\mathfrak{o}(n,\mathbb{R})$.

8 B-morphic maps

8.1 Cyclomorphic maps have a natural generalization to the context of B-von Neumann algebras, i.e. the complex field is replaced by B. More precisely we deal with (M,τ) von Neumann algebras with specified normal faithful trace-state and there is a specified unital inclusion $B \hookrightarrow M$ so that B is a von Neumann subalgebra. For lack of better names the analogues of cyclomorphic maps will be called B-morphic. If $(Y_j)_{1\le j\le n} \in M_h^n$ we shall denote by $\Delta_h(Y_1,\ldots,Y_m : B_h)$ the norm-closed linear span in $\mathcal{L}(M_h^n, B_h)$ of the maps

$$\begin{aligned} M_h^n \ni m &= (m_j)_{1\le j\le n} \rightsquigarrow T(m) + (T(m))^* \\ &\text{and} \\ M_h^n \ni m &= (m_j)_{1\le j\le n} \rightsquigarrow \sqrt{-1}\ (T(m) - (T(m))^* \end{aligned}$$

where $T(m)$ is of the form

$$T(m) = \sum_{1\le j\le p} E_B(b_0 Y_{i_1} b_1 \ldots Y_{i_{j-1}} b_{j-1} m_{i_j} b_j Y_{i_{j+1}} b_{j+1} \ldots Y_{i_p} b_p)\ .$$

Similarly when $(Y_j)_{1\le j\le n} \in M^n$ (no selfadjointness requirement) then $\Delta(Y_1,\ldots,Y_n : B)$ denotes the norm-closed linear span in $\mathcal{L}(M^n, B)$ of the maps $m \rightsquigarrow T(m)$. Note that $T(\cdot)$ is the differential at $(Y_1,\ldots,Y_n)$ of the map $M^n \ni (X_j)_{1\le j\le n} \rightsquigarrow E_B(b_0 X_{i_1} b_1 \ldots X_{i_n} b_n)$. Remark that $\Delta(Y_1,\ldots,Y_n : B)$ is a $B-B$-bimodule, while $\Delta_h(Y_1,\ldots,Y_n : B_h)$ is stable under $L \rightsquigarrow bL + Lb^*$, $L \rightsquigarrow (bL - Lb^*)\sqrt{-1}$. This last feature is why we will prefer to work with $\Delta_h(Y_1,\ldots,Y_n : B)$ in the selfadjoint case, which is the norm closure of the linear span of the $T(\cdot)$ in $\mathcal{L}(M^n, B)$.

By $\Delta_h^{\perp}(Y_1,\ldots,Y_n : B)$ we denote the set of $K = (K_1,\ldots,K_n) \in M_h^n$ so that $T(K) = 0$ for all $T \in \Delta_h(Y_1,\ldots,Y_n : B)$. Similarly, $\Delta^{\perp}(Y_1,\ldots,Y_n : B)$ is the set of $K \in M^n$ so that $T(K) = 0$ for all $T \in \Delta(Y_1,\ldots,Y_n : B)$.

Our discussion of B-morphic maps will be rather sketchy in this paper, we just want to explain the main idea of the generalization.

8.2 Definition. Let $\Omega \subset M_h^n$ be an open set. A differentiable map $f : \Omega \to B$ is hermitian B-morphic (abbreviated hBm) if for every $Y \in \Omega$, $df[Y] \in \Delta_h(Y_1, \ldots, Y_n : B)$. A differentiable map $f : \Omega \to B$ is weakly hermitian B-morphic (abbreviated $whBm$) if $(df[Y])(\Delta_h^\perp(Y_1, \ldots, Y_n : B)) = 0$.

8.3 Definition. Let $\Omega \subset M_h^n$ be an open set. A differentiable map $F : \Omega \to N_h^p$ where N is also a B-von Neumann algebra is hBm if for every hBm map $f : \omega \to B$, $\omega \subset N_h^p$ open, the map $f \circ (F |_{F^{-1}(\omega)\cap\Omega})$ is hBm. The differentiable map $F : \Omega \to N_h^p$ is $whBm$ if

$$(dF[Y])(\Delta_h^\perp(Y : B)) \subset \Delta_h^\perp(F(Y) : B) \ .$$

8.4 Definition. Let $\Omega \subset M^n$ be an open set. A complex analytic map $f : \Omega \to B$ is B-morphic (abbreviated Bm) if for each $Y \in \Omega$ the complex differential is such that $dF[Y] \in \Delta(Y_1, \ldots, Y_n : B)$. A complex analytic map $f : \Omega \to B$ is weakly B-morphic (abbreviated wBm) if for each $Y \in \Omega$ the complex differential is such that $dF[Y])(\Delta_h^\perp(Y_1, \ldots, Y_n : B)) = 0$.

8.5 Definition. Let $\Omega \subset M^n$ be open and let $F : \Omega \to N^p$ be a complex analytic map. F is Bm if for every Bm map $f : \omega \to B$, $\omega \subset N^p$ open, the map $f \circ F |_{F^{-1}(\omega)\cap\Omega}$ is Bm. The map F is wBm if

$$dF[Y](\Delta^\perp(Y : B)) \subset \Delta^\perp(F(Y) : B) \ .$$

8.6 If $X_1, \ldots, X_n$ are indeterminates the noncommutative polynomials with coefficients in B, i.e. $B *_{\mathbb{C}} \mathbb{C}\langle X_1, \ldots, X_n\rangle$ will be denoted by $B\langle X_1, \ldots, X_n\rangle$ or $B_{\langle n\rangle}$. $B_{\langle n\rangle}$ is spanned by noncommutative monomials $b_0 X_{i_1} b_1 \ldots X_{i_p} b_p$. The involution is defined so that $(b_0 X_{i_1} b_1 \ldots X_{i_p} b_p)^* = b_p^* X_{i_p} \ldots b_1^* X_{i_1} b_0^*$.

8.7 Examples. a) If $P \in B_{\langle n\rangle}$ then $f_P : M_h^n \to B$ defined by $f_P(Y) = E_B(P(Y))$ is hBm.

b) The hBm maps of Ω to B form an algebra.

c) If $P \in B_{\langle n\rangle}$ then $g_P : M^n \to B$ defined by $g_P(Z) = E_B(P(Z))$ is a Bm map.

d) The Bm maps $\Omega \to B$ form an algebra.

e) If $P = (P_j)_{1\le j\le m} \in (B_{\langle n\rangle})^m$, then $G_P : M^n \to M^m$ defined by $G_P(Z) = (P_j(Z))_{1\le j\le m}$ is a Bm map.

f) If $P = (P_j)_{1\le j\le m} \in (B_{\langle n\rangle})^n$ and $P_j = P_j^*$, $1 \le j \le m$, then $F_P : M_h^n \to M_h^m$ defined by $F_P(Y) = (P_j(Y))_{1\le j\le m}$ is a hBm map.

8.8 Theorem. *Assume we have* $I \in B \subset N \subset M$ *and let* $Y_1, \ldots, Y_m \in M_h$ *be such that* $\{Y_1, \ldots, Y_m\}$ *and* N *are freely independent over* B *in* (M, E_B). *Then the map* $\Phi : N_h^n \to M_h^{n+m}$ *defined by* $\Phi(T_1, \ldots, T_n) = (T_1, \ldots, T_n, Y_1, \ldots, Y_m)$ *is a* hBm *map.*

8.9 Theorem. *Assume we have* $I \in B \subset N \subset M$ *and let* $Z_1, \ldots, Z_m \in M$ *be such that* $\{Z_1, \ldots, Z_m\}$ *and* N *are freely independent over* B *in* (M, E_B). *Then the map* $\Psi : N^n \to M^{n+m}$ *defined by* $\Psi(T_1, \ldots, T_n) = (T_1, \ldots, T_n, Z_1, \ldots, Z_m)$ *is a* Bm *map.*

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