AN EXAMPLE OF A PROPERTY Γ FACTOR WITH COUNTABLE FUNDAMENTAL GROUP

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0. Introduction.

The first examples of type II_1 factors M having countable fundamental group $\mathcal{F}(M)$ have been constructed by Connes in ([C1]), as group algebras $M = L(G)$ for G discrete ICC groups with the property T of Kazhdan. Since then, many more examples of such factors have been constructed (see e.g., [Po1,3] or [Gol]). But all such constructions are using property T groups, in a way or another. For instance, it is proved in ([Po1]) that if a type II_1 factor M contains a group von Neumann algebra $L(G)$ for some ICC property T group G then $\mathcal{F}(M)$ is countable.

All the factors M having $\mathcal{F}(M)$ countable that have been constructed so far do not have non-trivial asymptotically central sequences, i.e., they do not have the property Γ of Murray and von Neumann (equivalently, they are full factors, in the sense of [C]). In this respect, note that if a factor M has non-commuting such central sequences then by a result of McDuff ([McD]) they split-off the hyperfinite type II_1 factor, thus having fundamental group equal to $\mathbb{R}^*_+$. Thus, if it is for a property Γ factor M to have fundamental group $\neq \mathbb{R}^*_+$, then its central sequences must commute.

In this paper we construct a class of examples of factors which do have the property Γ yet have countable fundamental group. The construction does not use property T groups, but instead uses the rigidity properties of the inclusion $\mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$, also due to Kazhdan ([Kaz]). We will also use perturbation results, separability arguments and the recent striking results of Gaboriau on the cost of equivalence relations.

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The paper is organised as follows: In Section 1 we prove some perturbation result for Cartan subalgebras in type II_1 factor, in the spirit of some well known results of Eric Christensen. In Section 2 we prove the main result of the paper (Theorem 2.3), providing some classes of factors with countable fundamental group, both with and without property Γ. These factors do not contain type II_1 factors with the property T of ([CJ1]), nor can be embedded into free group factors, yet they seem to be closer to the latter class. In Section 3 we make some comments, and sketch another construction of property Γ factors with countable fundamental group, this time by using property T groups (Remark 3.1). We also construct a class of examples of property Γ factors with fundamental group $\mathbb{R}^*_+$ which however are not McDuff (Remark 3.2).

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1. Some perturbation results.

In this Section we prove some perturbation results for subalgebras in type II_1 factors.

The first such perturbation result concerns maximal abelian $*$-subalgebras (abreviated as m.a.s.a. hereafter) of type II_1 factors. Besides the technique from [Ch], the proof uses the "pull down" lemma in the basic construction ([PiPo]) and some considerations on the geometry of projections, notably a result of Kadison ([K]).

1.1. Theorem. *Let N be a type II_1 factor and $B_0, B_1 \subset N$ be m.a.s.a.'s of N such that* $\sup\{\|u - E_{B_1}(u)\|_2 \mid u \in \mathcal{U}_0\} < 1$, *for some unitary subgroup $\mathcal{U}_0 \subset B_0$ satisfying $\mathcal{U}_0'' = B_0$. Then there exists a non-zero partial isometry $v \in N$ such that $v^*v \in B_0, vv^* \in B_1$ and $vB_0v^* = B_1vv^*$.*

Proof. Consider first the basic construction for the inclusion $B_1 \subset N$: Thus, we let e be the orthogonal projection of $L^2(N, \tau)$ onto $L^2(B_1, \tau)$, and which is known to satisfy $exe = E_{B_1}(x)e, \forall x \in N$. Then we let $N_1 = \langle N, e\rangle$ be the von Neumann algebra generated inside $\mathcal{B}(L^2(N, \tau))$ by N and e. Note that $eN_1e = B_1e$. We endow N_1 with the unique normal semifinite faithful trace Tr which satisfies $Tr(xey) = \tau(xy), \forall x, y \in N$. Note that there exists a unique $N - N$ bilinear map Φ from $\mathrm{sp}NeN \subset N_1$ into N satisfying $\Phi(xey) = xy, \forall x, y \in N$. This bilinear map satisfies the "pull down" identity $eX = e\Phi(eX), \forall X \in N_1$ from ([PiPo]).

Let now $K_e = \overline{\mathrm{co}}^{\mathrm{w}}\{u_0eu_0^* \mid u_0 \in \mathcal{U}_0\}$. We clearly have $0 \leq a \leq 1$ and $Tr(a) \leq 1, \forall a \in K_e$. Moreover, K_e is contained in the Hilbert space $L^2(N_1, Tr)$, where it is still weakly closed. Let $h \in K_e$ be the unique element of minimal norm $\| \|_{2,Tr}$ in

K_e. Since $uhu^* \in K_e$ and $\|uhu^*\|_{2,Tr} = \|h\|_{2,Tr}, \forall u \in \mathcal{U}_0$, by the uniqueness of h it follows that $uhu^* = h, \forall u \in \mathcal{U}_0$. Thus $h \in \mathcal{U}_0' \cap N_1 = B_0' \cap N_1$.

On the other hand, since $Tr(eueu^*) = \|E_{B_1}(u)\|_2^2 = 1 - \|u - E_{B_1}(u)\|_2^2$, if we denote $\delta = 1 - \sup\{\|u - E_{B_1}(u)\|_2 \mid u \in \mathcal{U}_0\}$, it follows that $Tr(eueu^*) \geq \delta, \forall u \in \mathcal{U}_0$. Taking appropriate linear combinations and weak limits, it follows that $Tr(eh) \geq \delta$. Since by hypothesis we have $\delta > 0$, it follows that $h \neq 0$.

Let e_0 be a non-zero spectral projection of h. Thus, e_0 is a finite projection in N_1 and e_0 commutes with B_0 (since h does). Since B_0e_0 is abelian, it is contained in a maximal abelian subalgebra B of $e_0N_1e_0$. (Note that any element in B commutes with B_0.) By a result of Kadison ([K]), B contains a non-zero abelian projection e_1 of N_1 (i.e., $e_1N_1e_1$ is abelian). Since e has central valued (semifinite) trace equal to 1, it follows that e majorizes e_1.

Let $V \in N_1$ be a partial isometry such that $V^*V = e_1 \leq e_0$ and $VV^* \leq e$. Moreover VBe_1V^* is a subalgebra of $eN_1e = B_1e$. Since e_1 commutes with B_0, it follows that if we denote by f' the maximal projection in B_0 such that $f'e_1 = 0$ and let $f_0 = 1 - f'$, then there exists a unique isomorphism φ from B_0f_0 into B_1 such that $\varphi(b)e = VbV^*, \forall b \in B_0f$. Let $f_1 = \varphi(f_0) \in B_1$.

It follows that $\varphi(b)eV = eVb, \forall b \in B_0f_0$. By applying Φ to both sides and denoting $a = \Phi(eV) \in N$, it follows that $\varphi(b)a = ab, \forall b \in B_0$. Since $ea = eV = V$, it follows that $a \neq 0$.

By the usual trick, if we denote by $v_0 \in N$ the unique partial isometry in the polar decomposition of a such that the right supports of a and v_0 coincide, then $p_0 = v_0^*v_0 \in B_0' \cap N = B_0$, $p_1 = v_0v_0^* \in \varphi(B_0)f_1' \cap f_1Nf_1$ and $\varphi(b)v_0 = v_0b, \forall b \in B_0f_0$.

But B_0f_0 maximal abelian in f_0Nf_0 implies that $vB_0v^* = \varphi(B_0)p_1$ is maximal abelian in $v_0v_0^*Nv_0v_0^*$. Since any element in $p_1B_1p_1$ commutes with $\varphi(B_0)p_1$, which is maximal abelian, it follows that $p_1B_1p_1 = \varphi(B_0)p_1$. Thus, if P_1 denotes the von Neumann algebra generated by p_1 and B_1f_1 inside f_1Nf_1, then P_1 is like a basic construction for the inclusion $\varphi(B_0)f_1 \subset B_1f_1$, with p_1 playing the role of the "Jones projection". In particualar, $p_1P_1p_1 = \varphi(B_0)p_1$ is abelian.

Thus, p_1 is an abelian projection in P_1. Since P_1 is a finite von Neumann algebra, there exists a central projection z_1 of P_1, under the central support of p_1 in P_1, such that $p_1z_1 \neq 0$ and such that P_1z_1 is homogeneous of type n, for some $n \geq 1$. Note that the center of P_1 is included in B_1f_1 the latter being maximal abelian in f_1Nf_1, thus in $P_1 \subset f_1Nf_1$. Thus, $z_1 \in B_1f_1$.

Now, since p_1z_1 has central trace equal to $1/n$ in P_1z_1 and B_1z_1 is maximal abelian in P_1z_1, it follows that there exists a projection $f_{11} \in B_1f_1$ such that f_{11} is equivalent to p_1z_1 in P_1z_1 (see [K]). Let $v_1 \in P_1z_1$ be such that $v_1v_1^* = f_{11}, v_1^*v_1 = p_1z_1$. Since p_1z_1 is abelian in P_1, f_{11} is also abelian, thus $f_{11}P_1f_{11} = B_1f_{11}$. This implies that $v_1(\varphi(B_0)p_1z_1)v_1^* = B_1f_{11}$.

Finally, since $B_0f_0 \ni b \mapsto \varphi(b) \in \varphi(B_0)f_1$ and p_1z_1 belongs to $\varphi(B_0)p_1$, it follows that there exists a projection f_{00} in B_0f_0 such that $\varphi(f_{00})p_1 = z_1p_1$. In particular, $v_1^*v_1 = v_0f_{00}v_0^*$.

Altogether, this shows that if we denote $v = v_1v_0f_{00}$, then v is a partial isometry satsifying $v^*v = f_{00} \in B_0, vv^* = f_{11} \in B_1$ and $vB_0v^* = B_1vv^*$. Q.E.D.

In Section 2 we will in fact need a consequence of Theorem 1.1. To state it, recall from ([D1]) that a m.a.s.a. A of a von Neumann factor N is called *semiregular* if the set of unitaries of N that normalize A generate a factor. Also, A is called *regular* if this normalizer generetes all the ambient factor N. Such regular m.a.s.a.'s were later called *Cartan subalgebras* in ([FM]), a terminology that seems to have prevailed and which we will therefore adopt.

We will also use the following notations from ([Ch]):

1.2. Notation. Let $\mathcal{B}_0, \mathcal{B}_1$ be von Neumann subalgebras of a type II_1 factor N. If $\sup\{\|x_0 - E_{\mathcal{B}_1}(x_0)\|_2 \mid x_0 \in \mathcal{B}_0, \|x_0\| \leq 1\} \leq \varepsilon$ then we write $\mathcal{B}_0 \subset_\varepsilon \mathcal{B}_1$. Also, we denote by $d(\mathcal{B}_0, \mathcal{B}_1)$ the maximum between $\sup\{\|x_0 - E_{\mathcal{B}_1}(x_0)\|_2 \mid x_0 \in \mathcal{B}_0, \|x_0\| \leq 1\}$ and $\sup\{\|x_0 - E_{\mathcal{B}_1}(x_0)\|_2 \mid x_0 \in \mathcal{B}_0, \|x_0\| \leq 1\})$.

1.3. Corollary. *Let N be an arbitrary type II_1 factor and $A_0, A_1 \subset N$ be semiregular m.a.s.a.'s of N. If $A_0 \subset_{1-\delta} A_1$ for some $\delta > 0$ then there exists a unitary element $u \in N$ such that $uA_0u^* = A_1$. In particular, this is the case if A_0, A_1 are Cartan subalgebras of N.*

Proof. The condition implies that $\sup\{\|u - E_{A_1}(u)\|_2 \mid u \in \mathcal{U}(A_0)\} < 1$. Thus, by Theorem 1.1 there exists a non-zero partial isometry $v \in N$ such that $v^*v \in A_0, vv^* \in A_1, vA_0v^* = A_1vv^*$. Moreover, by cutting v from the right with a smaller projection in A_0, we may clearly assume $\tau(vv^*) = 1/n$ for some integer n.

Since A_0, A_1 are semiregular, there exist partial isometries $v_1, v_2, ..., v_n$, respectively $w_1, w_2, ..., w_n$ in the normalizing groupoids of A_0 respectively A_1 such that $\Sigma_i v_iv_i^* = 1, \Sigma_j w_jw_j^* = 1$ and $v_i^*v_i = v^*v, w_j^*w_j = vv^*, \forall i, j$. But then $u = \Sigma_i w_ivv_i^*$ is a unitary element and $vA_0v^* = A_1$. Q.E.D.

Let us also mention another application of Theorem 1.1 which, although not needed later in this paper, has some independent interest. Thus, recall from ([D1]) that a m.a.s.a. A of a von Neumann algebra N is *singular* if the only unitaries in N that normalize A are the unitaries of A. When N is a type II_1 factor, in ([Po3]) a numerical invariant $\delta(A)$ was associated to m.a.s.a.'s A of N, as a "measure of singularity", as follows:

For each non-zero partial isometry $v \in N$ with vv^*, v^*v mutually orthogonal projections in A, denote $\delta(vAv^*, A) = \sup\{\|x - E_A(x)\|_2 \mid x \in vAv^*, \|x\| \leq 1\}$.

Then define $\delta(A)$ to be the infimum of $\delta(vAv^*, A)/\|vv^*\|_2$, as v runs over the set of all such partial isometries. In ([Po3]) it was noted that $\delta(A) > 0$ implies A is singular and examples of m.a.s.a.'s with $\delta(A) > 0$ were constructed in any type II_1 factor with separable predual. But it was left as an open question to calculate the possible values the constants $\delta(A)$ can take. This problem was recently revived in ([SiSm]). From 1.1 we get:

1.4. Corollary. *Let N be an arbitrary type II_1 factor and $A \subset N$ a singular m.a.s.a. Then $\delta(A) = 1$.*

Proof. If for some partial isometry $v \in N$ with $vv^*, v^*v \in A$, $v \neq 0, v^2 = 0$ we have $\sup\{\|x - E_{Avv^*}(x)\|_2/\|vv^*\|_2 \mid x \in vAv^*, \|x\| \leq 1\} < 1$, then it follows that the m.a.s.a.'s $B_0 = vAv^*, B_1 = Avv^*$ in the factor vv^*Nvv^* (with its normalized trace) verify the condition in Theorem 1.1. Thus, there exists a non-zero partial isometry $v_0 \in vv^*Nvv^*$ such that $v_0^*v_0 \in vAv^*, v_0v_0^* \in Avv^*$ and $v_0vAv^*v_0^* \subset A$. But this contradicts the singularity of A. Q.E.D.

We end this section by mentioning two well known perturbation results from ([Ch]), needed in the sequel. We have included a proof for the sake of completeness.

1.5. Lemma. [Ch]. *Let N be a type II_1 factor and $B_0, B_1 \subset N$ be von Neumann subalgebras of N. Assume there exists a subgroup $\mathcal{U}_0$ of the unitary group of B_0 such that $\mathcal{U}_0'' = B_0$ and $\|u_0 - E_{B_1}(u_0)\|_2 \leq \varepsilon, \forall u_0 \in \mathcal{U}_0$. Then $B_1' \cap N \subset_{2\varepsilon} B_0' \cap N$.*

Proof. Let $x \in B_1' \cap N, \|x\| \leq 1$. Since $E_{B_0'\cap N}(x)$ is the element of minimal norm-2 in the weakly compact convex set $\overline{\mathrm{co}}^{\mathrm{w}}\{u_0xu_0^* \mid u_0 \in \mathcal{U}_0\}$, it follows that $\forall\delta > 0, \exists u_1, u_2, ..., u_n \in \mathcal{U}_0$ such that $\|1/n\Sigma_i u_ixu_i^* - E_{B_0'\cap N}(x)\|_2 < \delta$.

But by hypothesis, for all $i = 1, 2, .., n$ we have the estimates:

$$\|x - u_ixu_i^*\|_2 \leq \|x - E_{B_1}(u_i)xu_i^*\|_2 + \|u_i - E_{B_1}(u_i)\|_2$$
$$= \|x - xE_{B_1}(u_i)u_i^*\|_2 + \|u_i - E_{B_1}(u_i)\|_2 \leq 2\|u_i - E_{B_1}(u_i)\|_2 \leq 2\varepsilon.$$

Altogether, this implies that $\|x - E_{B_0'\cap N}(x)\|_2 < 2\varepsilon + \delta$, with δ arbitrary, showing that $x \in_{2\varepsilon} B_0' \cap N$. Since x was taken arbitrary in the unit ball of $B_1' \cap N$, we have shown that $B_1' \cap N \subset_{2\varepsilon} B_0' \cap N$. Q.E.D.

1.6. Lemma. [Ch]. *Let N be a type II_1. Assume there exists a subgroup $\mathcal{U}_0$ of the unitary group of N and a group morphism $\rho : \mathcal{U}_0 \to \mathcal{U}(N)$ such that $\|\rho(u_0) - u_0\|_2 \leq \varepsilon, \forall u_0 \in \mathcal{U}_0$. Then there exists a partial isometry $v \in N$ such that $v^*v \in \mathcal{U}_0' \cap N$, $vv^* \in \rho(\mathcal{U}_0)' \cap N$, $\|1 - v\|_2 \leq 2\varepsilon$ and $\mathrm{Ad}(v)(u_0) = \rho(u_0)vv^*, \forall u_0 \in \mathcal{U}_0$.*

Proof. Let $K_\rho = \overline{\mathrm{co}}^{\mathrm{w}}\{\rho(u_0)u_0^* \mid u_0 \in \mathcal{U}_0\}$. Let $k \in K_\rho$ be the unique element of minimal norm-2. Since $\rho(u_0)K_\rho u_0^* \subset K_\rho$ and $\|\rho(u_0)k'u_0^*\|_2 = \|k'\|_2$, forall $u_0 \in$

$\mathcal{U}_0, k' \in K_\rho$, it follows that $\rho(u_0)ku_0^* = k, \forall u_0 \in \mathcal{U}_0$. Thus, $\rho(u_0)k = ku_0, \forall u_0 \in \mathcal{U}_0$. By the usual trick, it follows that if v denotes the partial isometry in the polar decomposition of k with right support equal to the right support of k then v is an intertwiner between ρ and id on $\mathcal{U}_0$. Also, since any element k' in K_ρ satisfies $\|k' - 1\|_2 \leq \varepsilon$, by standard estimates (see e.g., [C2] or [Ch]) one gets $\|v - 1\|_2 \leq 2\varepsilon$. Q.E.D.

2. Examples of factors M with countable $\mathcal{F}(M)$.

In this section we construct a class of examples of type II_1 factors with countable fundamental group. Some of them have the property Γ of Murray and von Neumann while others don't. The construction relies on the rigidity properties of the embedding of the group $G_0 = \mathbb{Z}^2$ inside the group $G = \mathbb{Z}^2 \rtimes SL(2, \mathbb{Z}^2)$, discovered by Kazhdan in the late 60's ([Kaz]). However, as we will prove in the next section, the factors that we construct in this section do not contain any subfactor with the property T, other than the finite dimensional ones.

2.1. The construction. Let (X_0, μ_0) be the 2-dimensional thorus, regarded as the dual group of $\mathbb{Z}^2$, endowed with the Haar measure μ_0. Note that μ_0 is the same as the Lebesgue measure on $X_0 = \mathbb{T}^2$. Let σ_0 be the action of $SL(2, \mathbb{Z})$ on X_0 implemented by the action of $SL(2, \mathbb{Z})$ on $\mathbb{Z}^2$. Let (X_1, μ_1) be a probability space with a measure preserving ergodic transformation σ_1 of $SL(2, \mathbb{Z})$ on it. Let $\sigma = \sigma_0 \times \sigma_1$ be the product action on the probability space $(X, \mu) = (X_0 \times X_1, \mu_0 \times \mu_1)$. Denote $A = L^\infty(X, \mu)$ and $M = A \rtimes_\sigma SL(2, \mathbb{Z})$. Also, denote $A_0 = L^\infty(X_0, \mu_0)$ and $M_0 = A_0 \rtimes_{\sigma_0} SL(2, \mathbb{Z})$.

Note that we can regard A_0 as a subalgebra A, in which case the canonical unitaries $u_g, g \in SL(2, \mathbb{Z}) \subset M$ implementing the action σ on A also implement the action σ_0 on A_0. Thus, M_0 can be viewed as a subfactor of M. Moreover, we can view A_0 as $L(\mathbb{Z}^2)$, in which case M_0 is identified with $L(\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z}))$.

Given any arbitrary ergodic action σ_1 as in 2.1, the action σ and the algebras defined above have the following properties:

2.2. Lemma. 1°. *The action σ is a free, ergodic action of $SL(2, \mathbb{Z})$ on the probability space (X, μ).*

2°. *The action σ_0 is strongly ergodic. The action $\sigma = \sigma_0 \times \sigma_1$ is strongly ergodic if and only if σ_1 is strongly ergodic.*

3°. *There exist ergodic actions σ_1 of $SL(2, \mathbb{Z})$ that are not strongly ergodic.*

4°. *M_0 is a non-Γ type II_1 factor and $A_0 \subset M_0$ is a Cartan subalgebra of M_0. More generally, M is a type II_1 factor, $A \subset M$ is a Cartan subalgebra in M and M is non-Γ if and only if σ_1 is strongly ergodic.*

5°. *When regarded as a subalgebra in M, A_0 satisfies $A_0' \cap M = A$.*

Proof. 1°. It is well known that a non-inner automorphism of a group G implements a properly outer automorphism on $L(G)$ that preserves the canonical trace of $L(G)$. Thus, since each non-trivial element g in the group $SL(2,\mathbb{Z})$ implements a non-inner automorphism of $\mathbb{Z}^2$ it follows that $\sigma_0(g)$ is properly outer $\forall g \in SL(2,\mathbb{Z}), g \neq e$.

Since the tensor product of any properly outer automorphism with an arbitrary automorphism is properly outer, it follows that $\sigma(g)$ is properly outer $\forall g \in SL(2,\mathbb{Z}), g \neq e$, as well.

Furthermore, the action σ_0 is well known to be mixing. More precisely, it is easy to see that given any finite set $F \subset SL(2,\mathbb{Z}), e \notin F$, there exists $g \in SL(2,\mathbb{Z})$ such that $gF \cap F = \emptyset$. But this implies not only that σ_0 is ergodic, but also that its tensor product with any ergodic action is still ergodic.

2°. The first part is a well known result of Klaus Schmidt ([S1]). The second part is an immediate consequence of the proof of this result in ([S1]).

3°. This is a consequence of a theorem of Connes and Weiss ([CW]), showing that any discrete group G_1 which doesn't have the property T has a free, ergodic but not strongly ergodic action σ_1 on a probability space. Thus, one simply applies this result to $G_1 = SL(2,\mathbb{Z})$, which doesn't have the property T.

One can in fact avoid using the general result in ([CW]), by noticing that since $SL(2,\mathbb{Z})$ has an infinite amenable group H as a quotient (see e.g., [dHV]), any ergodic action of H on a non-atomic probability space (e.g., a Bernoulli shift action of H) composed with the quotient map $G_1 \to H$ gives an ergodic but not strongly ergodic action of $SL(2,\mathbb{Z})$ (note that the resulting action of $SL(2,\mathbb{Z})$ is not free though, in fact not even faithful, but freeness is not necessary in the construction 1.1).

4°. Since $SL(2,\mathbb{Z})$ is close to be a free group (see e.g., [dHV]), it is easy to see that any central sequences in a type II_1 factor of the form $B \rtimes_\sigma SL(2,\mathbb{Z})$ obtained as the cross product of a finite von Neumann algebra (B,τ) by a free, τ-preserving action σ of $SL(2,\mathbb{Z})$ on it must be supported on B.

5°. Note that if $b = \Sigma_g a_g u_g \in M = A \rtimes_\sigma SL(2,\mathbb{Z})$ commutes with all $a \in A_0$ and $a_g \neq 0$ for some $g \neq e$ then $a a_g u_g = a_g u_g a, \forall a \in A_0$, implying that $a_g a = a_g \sigma_g(a), \forall a \in A_0$. This in turn contradicts the fact that σ_g is properly outer on A_0. Thus, all a_g with $g \neq 0$ must be equal to 0, implying that b lies in A. Q.E.D.

2.3. Theorem. *Given any ergodic action σ_1 of $SL(2,\mathbb{Z}^2)$ on a probability space (X_1,μ_1), the type II_1 factor M constructed in 2.1 has countable fundamental group.*

To prove the Theorem, we'll use first a "separability argument", then the result of Kazhdan on the rigidity of the embedding of the group $\mathbb{Z}^2$ inside $\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$ ([Kaz]), then the perturbation results from the previous section, and finally the recent rigidity result of the free ergodic actions of the free groups on probability spaces of Gaboriau ([G]).

We split the corresponding arguments in a series of lemmas.

2.4. Lemma. *Let N be a separable type II_1 factor. If $\mathcal{F}(N)$ is uncountable then there exist projections $p_n \in \mathcal{P}(N)$, with $p_n \neq 1, p_n \nearrow 1$, and isomorphisms $\theta_n : N \simeq p_n N p_n$, such that $\lim_{n\to\infty} \|\theta_n(x) - x\|_2 = 0, \forall x \in N$. Moreover, the projections $\{p_n\}_n$ can be taken to lie in any given diffuse abelian von Neumann subalgebra A_0 of N.*

Proof. Let $\{q_t \mid 0 \leq t \leq 1\} \subset N$ be a totally ordered set of projections with $\tau(q_t) = t, \forall 0 \leq t \leq 1$. For each $t \in S = (1/2, 1) \cap \mathcal{F}(N)$ choose an isomorphism $\theta'_t : N \simeq q_t N q_t$ and denote $\mathcal{I} = \{\theta'_t \mid t \in S\}$. Note that, since $\mathcal{F}(N)$ is uncountable, S is uncountable.

Let $\{x_n\}_{n\geq 1}$ be a sequence of elements in the unit ball N_1 of N, such that as a set it is dense in N_1 in the norm $\| \,\|_2$ and such that each element appears infinitely many times in the sequence.

Since $\{\theta'_t(x_1)\}_{t\in S}$ is a subset of N_1, by covering N_1 with countably many open balls of $\| \,\|_2$-radius 1/2 and using the separability of $(N_1, \| \,\|_2)$ it follows that there exists an uncountable subset $S_1 \subset S$ such that $\|\theta_t(x_1) - \theta'_{t'}(x_1)\|_2 < 1, \forall t, t' \in S_1$. Similarily, by repeatedly using the separability of N_1 to cover it by countably many balls of radius $1/2n$, we construct recursively a decreasing sequence of uncountable sets $S_n \subset S, n \geq 1$, such that

$$\|\theta'_t(x_n) - \theta'_{t'}(x_n)\|_2 < 1/n, \forall t, t' \in S_n.$$

For each $n \geq 1$ choose now two distinct $t_n, t'_n \in S_n$, say with $t_n > t'_n$. Since $\theta'_{t_n}(1) \geq \theta'_{t'_n}(1)$,

$$\theta_n(x) = {\theta'}^{-1}_{t_n}(\theta'_{t'_n}(x)), x \in N,$$

gives a well defined isomorphism of N onto $p_n N p_n$, where

$$p_n = {\theta'}^{-1}_{t_n}(\theta'_{t'_n}(1)) = \theta^{-1}_{t_n}(q_{t'_n}) \in \mathcal{P}(N).$$

Moreover, since $t_n, t'_n \in S_j, \forall j \leq n$, we have for each $n \geq 1$ and $1 \leq j \leq n$ the estimates:

$$\|\theta_n(x_j) - x_j\|_2^2 = \tau(q_{t_n})^{-1} \|\theta'_{t'_n}(x_j) - \theta'_{t_n}(x_j)\|_2^2 \leq 2/j.$$

Since each x_j appears infinitely many times, it follows that $\lim_{n\to\infty} \|\theta_n(x_j) - x_j\|_2 = 0, \forall j \geq 1$. By the density of $\{x_n\}_n$ in N_1, it follows that $\lim_{n\to\infty} \|\theta_n(x) - x\|_2 = 0, \forall x \in N_1$, and thus for all $x \in N$.

Finally, in case we want the projections $\{p_n\}_n$ to lie in a given diffuse abelian von Neumann subalgebra A_0, then we first choose projections $\{p'_n\}_n \subset A_0$ with $\tau(p'_n) = \tau(p_n), \forall n$, and note that $\|p'_n - p_n\|_2 \to 0$ (because both sequences tend to 1). Thus, there exist partial isometries $v_n \in N$ such that $v_n v_n^* = p_n, v_n^* v_n = p'_n, \forall n$ and $\|v_n - 1\|_2 \to 0$ (see e.g., [C2] or [Ch]). But then, by replacing n p'_n for p_n and $\mathrm{Ad}(v_n)\theta_n$ for θ_n, $\forall n \geq 1$, the last condition will also be satisfied. Q.E.D.

We are now going to use the rigidity of the embedding of the group $\mathbb{Z}^2$ inside $\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$ (cf. [Kaz]). We recall that if G_0 is a subgroup of a discrete group G then we say that G_0 *has the property* T *inside* G if there exist $g_1, g_2, ..., g_n \in G$ and $\varepsilon > 0$ such that if π is a unitary representation of G on a Hilbert space $\mathcal{H}$ such that for some unit vector $\xi \in \mathcal{H}$ one has $\|\pi(g_i)\xi - \xi\| \leq \varepsilon, \forall i$ then there exists a non-zero vector $\xi_0 \in \mathcal{H}$ such that $\pi(h)\xi_0 = \xi_0, \forall h \in G_0$. Note that if this is the case, then by a well known argument it follows that $\exists K \geq 1$ such that $\forall \delta < \varepsilon$, if $\|\pi(g_i)\xi - \xi\| \leq \delta, \forall i$ then $\|\pi(h)\xi - \xi\| \leq K\delta, \forall h \in G_0$ (see e.g., [De-Ki]).

2.5. Lemma. *Let G be a discrete group and $G_0 \subset G$ a property T embedding of a group G_0 into G (e.g., $G_0 = \mathbb{Z}^2, G = \mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}^2)$). Let $(A_0 \subset M_0) = (L(G_0) \subset L(G))$. Denote by $u_g, g \in G$, the canonical unitaries in M_0. Let N be a type II_1 factor that contains M_0. Assume $\{\theta_n\}_n$ is a sequence of non-necessarily unital endomorphisms of N such that $\lim_{t\to 0}\|\theta_n(u_g) - u_g\|_2 = 0, \forall g \in G$. Then for any $\varepsilon > 0$ there exists n_ε such that if $n \geq n_\varepsilon$ then $\|\theta_n(u_h) - u_h\|_2 \leq \varepsilon$, $\forall h \in G_0$.*

Proof. Let $\pi_n : \mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}^2) \to \mathcal{U}(\mathcal{H}_n)$ be the unitary representation on the Hilbert space $\mathcal{H}_n = \theta_n(1)L^2(M,\tau))$ defined by $\pi_n(g)(\xi) = \theta_n(u_g)\xi u_g^*, g \in \mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}^2)$. If we let $\xi_n \in \mathcal{H}_n$ be the projection $p_n \in M$ regarded as a vector in $\mathcal{H}_n$ then

$$\|\pi_n(g)\xi_n - \xi_n\| = \|\theta_n(u_g)u_g^* - p_n\|_2$$
$$= \|\theta_n(u_g) - p_n u_g\|_2 \leq \|\theta_n(u_g) - u_g\|_2 + \|1 - p_n\|_2, \forall n \geq 1, g \in G.$$

Similarily,

$$\|\pi_n(g)\xi_n - \xi_n\| \geq \|\theta_n(u_g) - u_g\|_2 - \|1 - p_n\|_2, \forall n \geq 1, g \in G.$$

By the hypothesis and the first set of estimates it follows that $\lim_{n\to\infty}\|\pi_n(g)\xi_n - \xi_n\| = 0, \forall g \in \mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}^2)$. Since $\|\xi_n\|^2 = \tau(p_n) \to 1$ as $n \to \infty$, by the property T of $\mathbb{Z}^2$ inside $\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}^2)$ and the second set of estimates, it follows that

$$\lim_{n\to\infty} \sup_{h\in\mathbb{Z}^2} \|\theta_n(u_h) - u_h\|_2 = 0.$$

Q.E.D.

2.6. Lemma. *With the same notations as in the statement of* Lemma 2.5*, if in addition we assume the projections* $p_n = \theta_n(1)$ *lie in* $A_0, \forall n$*, then for any* $n \geq n_\varepsilon$ *we have* $d(\theta_n(A_0)' \cap N, A_0 p_n' \cap N) \leq 2\varepsilon(1-\varepsilon)^{-1}$.

Proof. By using the fact that $p_n \in A_0 = \{u_h\}''_{h\in\mathbb{Z}^2}$ and applying Lemma 1.5, first to $\mathcal{U}_0 = \theta_n(\{u_h\}_{h\in\mathbb{Z}^2})$ then to $\mathcal{U}_0 = \{u_h p_n\}_{h\in\mathbb{Z}^2}$, regarded as unitary subgroups in the type II_1 factor $p_n N p_n$, endowed with the normalized trace $\tau(p_n)^{-1}\tau$, it follows that

$$d(A_0 p_n' \cap p_n N p_n, \theta_n(A_0)' \cap p_n N p_n) \leq 2\varepsilon\tau(p_n)^{-1/2} \leq 2\varepsilon(1-\varepsilon)^{-1}.$$

Q.E.D.

Proof of Theorem 2.3. If $\mathcal{F}(M)$ is uncountable then by Lemma 2.4 there exists a sequence of projections $p_n \in A_0, p_n \neq 1$, and isomorphisms θ_n of M onto $p_n M p_n$ such that $\lim_{n\to\infty} \|\theta_n(x) - x\|_2 = 0, \forall x \in M$. In particular, aplying this to $x = u_g \in M_0 \subset M, g \in \mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$, we have

$$\lim_{n\to\infty} \|\theta_n(u_g) - u_g\|_2 = 0, \forall g \in \mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}).$$

By 2.5 and 2.6, it follows that for n large enough, if we denote $p = p_n$ and $\theta = \theta_n$ then we have $d(A_0 p' \cap pMp, \theta(A_0)p' \cap pMp) < 1$. But by Lemma 2.1, $A_0 p' \cap pMp = Ap$ is a Cartan subalgebra of pMp. Similarily,

$$\theta(A_0)p' \cap pMp = \theta(A_0' \cap M) = \theta(A)$$

is a Cartan subalgebra of pMp as well. By Corollary 1.4 it follows that the Cartan subalgebras Ap and $\theta(A)$ of pMp are conjugate by a unitary element u of pMp. Thus, $\theta' = \mathrm{Ad}(u) \circ \theta$ is an isomorphism of M onto pMp carying A onto Ap. But by the results of Gaboriau the cost $c(A \subset M)$ of the Cartan subalgebra A of M is equal to $13/12$ while the cost $c(Ap \subset pMp)$ of its restriction to p is equal to $1/12\tau(p) + 1$, which is larger than $13/12$. Since the cost is invariant to the isomorphism θ', we get a contradiction. Thus, $\mathcal{F}(M)$ is uncountable. Q.E.D.

2.7. Corollary. 1°. *The type* II_1 *factor* $M_0 = L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}))$ *is non-*Γ *and has countable fundamental group* $\mathcal{F}(M_0)$.

2°. *If* σ_1 *is chosen to be ergodic but not strongly ergodic* (*cf* $2.2.3^\circ$)*, then the type* II_1 *factor* $M = A \rtimes_\sigma SL(2,\mathbb{Z})$*, where* A *and* σ *are defined as in 2.1, then* M *has the property* Γ *and countable fundamental group* $\mathcal{F}(M)$.

Proof. This is now an immediate consequence of Lemma 2.2 and Theorem 2.3. Q.E.D.

3. Further remarks.

3.1. More examples of factors M with countable $\mathcal{F}(M)$. Recall that the first examples of factors M with countable fundamental group were constructed by Connes in ([C1]), as group von Neumann factors associated to dicrete ICC groups G having the property T of Kazhdan. More examples of factors with countable fundamental group were constructed in ([Po1,3,5], [Gol]). All these examples had relied in a way or another on constructions involving property T groups and they were non-Γ factors. The examples in the previous section are still using some rigidity phenomena, but the milder one involving the inclusion $\mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$. Although these examples seem "closer" to free group factors, they cannot be embedded in any free group factor, by the same argument as the one in [CJ1] (see [Po]). But they cannot contain property T groups either, because of the "relative Haagerup property" that the inclusion $L(\mathbb{Z}^2) \subset L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}))$ has (see [Po]). .

One can actually construct another class of examples of factors having the property Γ but countable fundamental group, by using property T groups and a strategy reminiscent of ([C1], [CJ1]) and ([Po1,3]), as follows:

Let G be an ICC group with the property T of Kazhdan and satisfying the following condition:

(3.1.1). Given any finite set $F \subset G - \{e\}$ there exists $g \in G$ such that $gFg^{-1}F \cap FgFg^{-1} = \emptyset$.

Note that this implies in particular $gFg^{-1} \cap F = \emptyset$. (We are greatful to George Skandalis for confirming to us that the groups $SL(n,\mathbb{Z})$ with n odd, $n \geq 3$ do satisfy this condition, besides having the property T by Kazhdan's classical result).

Then we let $(\mathbb{T}_g, \mu_g), g \in G$, be copies of the thorus $\mathbb{T}$ with its Lebesgue measure and (X,μ) be the product of these spaces, with the corresponding product measure. We let σ be the action of G on (X,μ) by Bernoulli shifts. We let M_0 be the corresponding type II_1 factor obtained by the group-measure space construction, i.e., $M_0 = L^\infty(X,\mu) \rtimes_\sigma G$. Finally, we consider the action α of $\mathbb{Z}$ on M_0, implemented by the "transversal" action of $\mathbb{Z}$ on (X,μ) given by rotation by the same irrational number on each copy of the thorus $(\mathbb{T}_g, \mu_g), \forall g \in G$. We denote $M = M_0 \rtimes_{\sigma_0} \mathbb{Z}$.

It is easy to see that M has the property Γ. Using Lemmas 2.4 and 2.5 (the latter for $G_0 = G$) and adding some more work, one can show that if $\mathcal{F}(M)$ is uncountable then there exists a projection $p \in A_0 = \{u_h\}''_{h\in\mathbb{Z}} \subset M$, with $p \neq 1$, and an isomorphism $\theta : M \to pMp$, with $\theta(u_g) = u_g p, \forall g \in G$. This implies $\theta(A_0) = A_0 p$ and then the condition (3.1.1) easily implies that $\theta(L^\infty(X,\mu) \rtimes \mathbb{Z}) = p(L^\infty(X,\mu) \rtimes \mathbb{Z})p$. This forces each Haar unitary v_h generating $L^\infty(\mathbb{T}_h)$ to be so that $\theta(v_h)$ has a certain specific form. But then, by using again (3.1.1), it follows that for appropriate $g \in G$ the unitaries $\theta(v_h)$ and $\sigma_g(\theta(v_h))$ are "supported" on almost disjoint finite segments of the product $\otimes_g L^\infty(\mathbb{T}_g)$ cross product with $\mathbb{Z}$. But then it is easy to see that $\theta(v_h)$ and $\sigma_g(\theta(v_h))$ cannot commute, a contradiction.

3.2. Examples of factors M with $\mathcal{F}(M) = \mathbb{R}_+^*$. Recall that in ([V], [Ra]) the free group factor $L(\mathbb{F}_\infty)$ was proved to have fundamental group equal to $\mathbb{R}_+^*$. This is the most striking example, so far, of a non-Γ type II_1 factor with fundamental group same as the hyperfinite factor.

The following construction provides a class of examples of type II_1 factors which have the property Γ, are not McDuff, and yet have fundamental group $\mathbb{R}_+^*$:

Let Q be an arbitrary non-atomic finite von Neumann algebra. Let $A \subset R$ be the hyperfinite type II_1 factor with its Cartan subalgebra. Let $M = Q\overline{\otimes}A *_A R$ be the amalgamated free product with respect to the trace preserving conditional expectations, as in ([Po3]). By the proof of (4.1 and 7.1 in [Po3]), we see that any central sequence of M must lie in A. On the other hand, any central sequence of R that lies in A is also a central sequence of M. Thus, M has the property Γ, without being McDuff.

Note that M can be alternatively described as follows: Denote by σ the irrational rotation on the thorus, viewed as an automorphism of $L(\mathbb{Z})$. We still denote by σ the action $id * \sigma$ of $\mathbb{Z}$ on $Q * L(\mathbb{Z})$. Then $M \simeq Q * L(\mathbb{Z}) \rtimes_\sigma \mathbb{Z}$, with Q (respectively $A \subset R$) in the first construction corresponding to Q (resp. $\lambda(\mathbb{Z})'' \subset L(\mathbb{Z}) \rtimes_\sigma \mathbb{Z}$) in this construction. The fact that M has Γ but is not McDuff can then also be deduced from ([Po4]).

Using the first representation of M, we see that if p is a non-zero projection in A then $p(Q\overline{\otimes}A *_A R)p$ is trivially isomorphic to $Q\overline{\otimes}Ap *_{Ap} pRp$. But by Dye's theorem $(Ap \subset pRp)$ is isomorphic to $(A \subset R)$, while $Q\overline{\otimes}Ap$ is trivially isomorphic to $Q\overline{\otimes}A$. Thus, $pMp \simeq M$ for any non-zero projection in M. Thus $\mathcal{F}(M) = \mathbb{R}_+^*$.

Due to the resemblence of the above construction of M with the amalgamated free product construction in ([Po3]), it is likely that in the case $Q = L(\mathbb{F}_\infty)$ the factor M has the same universality property ([PoSh]) as the free group factor $L(\mathbb{F}_\infty)$, namely that any standard lattice $\mathcal{G}$ can "act" on it.

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