A NEW PROOF OF THE MARKOV–KREIN IDENTITY FOR DISTRIBUTIONS OF MEANS OF DIRICHLET PROCESSES

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Abstract. We present a new simple proof of the Markov–Krein identity for distributions of means of Dirichlet processes. Our approach is based on a close relation of Dirichlet processes to gamma processes and basic properties of the gamma processes.

1. Introduction. The Dirichlet processes introduced in [F] play a key role in Bayesian nonparametric statistics. The classical definition of these processes is as follows. Denote by $\Delta_n = \{x = (x_0, \dots, x_n) : x_i \geq 0, \sum x_i = 1\}$ the n-dimensional simplex. The Dirichlet distribution $\mathrm{Dir}(\tau_0, \dots, \tau_n)$ on Δ_n with parameters $\tau_0, \dots, \tau_n > 0$ is determined by the density $\frac{\Gamma(\tau_0+\ldots+\tau_n)}{\Gamma(\tau_0)\ldots\Gamma(\tau_n)} x_0^{\tau_0-1} \dots x_n^{\tau_n-1}$.

Definition 1. Let (X, ν) be a standard Borel space with a non-atomic finite positive measure ν. The Dirichlet process on the space X with parameter measure ν is a random probability distribution P on X such that for every finite measurable partition $X = A_0 \cup \ldots \cup A_n$, the vector $(P(A_0), \dots, P(A_n))$ has Dirichlet distribution $\mathrm{Dir}(\nu(A_0), \dots, \nu(A_n))$ on Δ_n.

Let $\theta = \nu(X)$ be the total charge of the measure ν, and denote by $\bar{\nu} = \nu/\theta$ the normalized measure ν. An explicit construction of the Dirichlet process is given by the following formula:

$$P = \sum_{i=1}^{\infty} Q_i \delta_{Y_i}, \tag{1}$$

where $Y = (Y_1, Y_2, \dots)$ is a sequence of of i.i.d. variables with common distribution $\bar{\nu}$, and $Q = (Q_1, Q_2, \dots)$ is a random point of the infinite-dimensional simplex $\Delta = \{x = (x_1, x_2, \dots) : x_1 \geq x_2 \geq \ldots \geq 0, \sum x_i = 1\}$ which is independent of Y and has the *Poisson–Dirichlet distribution* $PD(\theta)$ with parameter θ.

Many papers are devoted to a particular problem of studying the distributions of random means of Dirichlet processes. Let $a : X \to \mathbb{R}$ be a measurable function.

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It defines a linear functional $f_a(\xi) = \int_X a(x)d\xi(x)$ on the space of Borel measures on X. The problem is to describe the distribution μ_a of this functional with respect to the Dirichlet process P. The answer is as follows. Let ν_a be the distribution of the function a with respect to the normalized measure $\bar{\nu}$. Then the measures μ_a and ν_a are related by the following integral identity:

$$(*) \qquad \int_{\mathbb{R}} \frac{1}{(1+zu)^\theta} d\mu_a(u) = \exp\left(- \int_X \log(1+zu)^\theta d\nu_a(u) \right).$$

This formula was first obtained in [CR] by hard analytic arguments (see also simpler proofs in [DK, KT, RGN, HO]). In case of $\theta = 1$, this identity means that the distribution μ_a is the *Markov–Krein transform* of the measure ν_a. This transform arises in many contexts, such as the Markov moment problem, continued fractions theory, exponential representations of functions of negative imaginary type, the spectral shift function of a self-adjoint operator, the Plancherel growth of Young diagrams, etc. (see [Ke] for a detailed survey). It appeared first in A. A. Markov's paper [M] and was intensively studied by M. G. Krein and his school. In particular, this identity gives a link between the so-called Markov moment problem and the well-studied Hausdorff moment problem, see [KN].

The purpose of this paper is to present a new proof of the Markov–Krein identity for means of Dirichlet processes based on a relation of Dirichlet processes to gamma processes. Let $\gamma(x)$ be the gamma process on (X, ν). Then *the Dirichlet process on X with parameter measure ν is just the normalized gamma process $\bar{\gamma}(x) = \gamma(x)/\gamma(X)$*. Thus the above problem about the distributions of means of Dirichlet processes may be formulated in terms of gamma processes as follows: *given a linear functional $f_a(\xi) = \int_X a(x)d\xi(x)$ defined by a function $a : X \to \mathbb{R}_+$, describe the distribution μ_a of the functional f_a with respect to the normalized gamma process $\bar{\gamma}$ in terms of the distribution ν_a of a.* The key independence property of the gamma processes (the normalized gamma process $\bar{\gamma}$ is independent on the total charge $\gamma(X)$) which leads to many distinguished properties of the gamma processes, e.g. the so-called multiplicative quasi-invariance, see [TV,TVY2], also allows one to obtain easily the Markov–Krein formula $(*)$ relating these two distributions. This formula may also be interpreted as a relation between the Laplace transform of the distribution of the functional f_a with respect to the gamma process and the Cauchy transform of the distribution of the same functional with respect to the normalized gamma process.

Our interpretation of the Markov–Krein identity leads to the following general problem: given an arbitrary Lévy process η on (X, ν) and a function $a : X \to \mathbb{R}_+$, describe the relation between the distribution of the functional f_a with respect to η and the distribution of the same functional with respect to the normalized Lévy process $\bar{\eta}$.

2. General Lévy processes. It is natural to consider Dirichlet processes in the context of general Lévy processes. We present this framework following [TV, TVY2].

Let (X, ν) be a standard Borel space with a non-atomic finite non-negative measure ν, and let $\nu(X) = \theta$ be the total charge of ν. Denote by

$$D = \left\{ \sum z_i \delta_{x_i},\ x_i \in X,\ z_i \in \mathbb{R}, \sum |z_i| < \infty \right\}$$

the real linear space of all finite real discrete measures on X, and by $D^+ = \{\sum z_i \delta_{x_i} \in D :\ z_i > 0\} \subset D$ the cone in D consisting of all positive measures.

Let Λ be a measure on $\mathbb{R}_+$ satisfying the following conditions:

$$\begin{aligned} &\Lambda(0, \infty) = \infty, && \Lambda(1, \infty) < \infty, \\ &\int_0^1 s d\Lambda(s) < \infty, && \Lambda(\{0\}) = 0. \end{aligned} \tag{2}$$

Denote by ψ_Λ the Laplace transform of the infinitely divisible distribution F_Λ with Lévy measure Λ:

$$\psi_\Lambda(t) = \exp\left(- \int_0^\infty (1 - e^{-ts}) d\Lambda(s) \right).$$

Each bounded Borel function $a : X \to \mathbb{R}$ defines a linear functional f_a on D, where $f_a(\eta) = \int_X a(x) d\eta(x)$ for $\eta \in D$.

Definition 2. A Lévy process on the space (X, ν) with Lévy measure Λ satisfying (2) is a generalized process on D whose law P_Λ has Laplace transform

$$\mathbb{E}\left[\exp\left(- \int_X a(x) d\eta(x) \right) \right] = \exp\left(\int_X \log \psi_\Lambda(a(x)) d\nu(x) \right), \tag{3}$$

where a is an arbitrary non-negative bounded Borel function on X.

Consider the cone $C = \{z = (z_1, z_2, \dots) :\ z_1 \geq z_2 \geq \dots \geq 0, \sum z_i < \infty\} \subset l^1$, and define a map $T : D^+ \to C \times X^\infty$ by

$$T\eta = \big((Q_1, Q_2, \dots),\ (X_1, X_2, \dots) \big), \quad \text{if} \quad \eta = \sum Q_i \delta_{X_i}.$$

Definition 3. Let P be a distribution on the space D^+, and let η be a random process obeying P. The random sequence of charges $(Q_1, Q_2, \dots)$ is called the *conic part* of the process η, and its distribution on the cone C is called the *conic part* of the law P.

It is not difficult to show that the conic part of the Lévy process with Lévy measure Λ is the ordered sequence of points of the Poisson process on $\mathbb{R}_+$ with mean measure $|\nu|\Lambda$. Thus the conic part depends only on Λ and on the full charge of the parameter measure ν. In fact, the following theorem shows that studying a Lévy process may be essentially reduced to studying its conic part, since the construction of the process involves the parameter measure in a trivial way. This fundamental property of Lévy processes is a particular case of the representation theorem first proved in [FK]. A simpler proof of this fact is presented in [TVY1].

Theorem 1 ([FK, TVY1]). *Let $\eta = \sum Q_i \delta_{X_i}$ be a Lévy process on the space (X, ν) with Lévy measure Λ. Then $TP_\Lambda = \varkappa_{|\nu|\Lambda} \times \bar{\nu}^\infty$, i.e. $X_1, X_2, \dots$ is a sequence of i.i.d. random variables with common distribution $\bar{\nu} = \nu/|\nu|$, and this sequence is independent of the conic part $(Q_1, Q_2, \dots)$.*

Denote by $D_1^+ \subset D^+$ the simplex of all normalized atomic measures. Then $D^+ = D_1^+ \times [0, \infty)$, i.e. each $\eta \in D^+$ can be represented as

$$\eta = (\eta/\eta(X), \eta(X)).$$

The second coordinate in this decomposition is the total charge of the measure η. It follows from the definition of the Lévy process that $\eta(X)$ obeys the infinite divisible law F_Λ corresponding to the Lévy measure Λ. The first coordinate is called the *normalization* of the measure η. In general, the law of a Lévy process is not a product measure in this decomposition. It will be a product measure only for gamma processes, see below.

Using this decomposition, consider a map $T' : D^+ \to \mathbb{R}_+ \times \Sigma \times X^\infty$, where $\Sigma = \{y = (y_1, y_2, \dots) : \ y_1 \ge y_2 \ge \dots \ge 0, \ y_1 + y_2 + \dots = 1\}$ is the infinite-dimensional simplex, and

$$T'\eta = \big(\eta(X), \ (Q_1/\eta(X), Q_2/\eta(X), \dots), \ (X_1, X_2, \dots)\big), \quad \text{if} \quad \eta = \sum Q_i \delta_{X_i}.$$

Definition 4. The normalized sequence of charges $(Q_1/\eta(X), Q_2/\eta(X), \dots)$ is called the *simplicial part* of the process and its distribution on Σ is called the *simplicial part* of the law P_Λ.

See [TVY1] for a detailed treatment of general Lévy prosesses, characterization of conic parts, etc.

3. The gamma process. In this section we summarize basic properties of the gamma processes which we need to prove the Markov–Krein identity $(*)$.

Definition 5. The standard gamma process on the space (X, ν) is a Lévy process γ on (X, ν) with Lévy measure $d\Lambda(z) = z^{-1} dz$, $z > 0$. Thus the law $\mathcal{G}$ of the gamma process has Laplace transform

$$\mathbb{E}_{\mathcal{G}} \left[\exp \left(- \int_X a(x) d\gamma(x) \right) \right] = \exp \left(- \int_X \log\left(1 + a(x)\right) d\nu(x) \right). \tag{4}$$

It is easy to show that formula (4) holds for any measurable function $a \in \mathcal{M} = \{a : X \to \mathbb{R}_+ : \int_X \log(a(x) + 1) d\nu(x) < \infty\}$.

Lemma 1 presents the key independence property of the gamma process. It follows immediately from the corresponding property of gamma variables: if Y and Z are independent gamma variables with the same scale parameter, then the variables $Y + Z$ and $\frac{Y}{Y+Z}$ are independent.

Lemma 1. *The total charge $\gamma(X)$ of the gamma process and the normalized gamma process $\bar\gamma = \gamma/\gamma(X)$ are independent. The distribution of the total charge is the gamma distribution with shape parameter $\theta = |\nu|$.*

As follows from a remarkable result of Lukacs [L] for gamma variables, this property is characteristic of the gamma processes. That is, if η is a Lévy process such that $\bar\eta$ and $\eta(X)$ are independent, then η is a gamma process (maybe with some scale parameter).

Lemma 2 ([Ki]). *The simplicial part of the gamma process with $|\nu| = \theta$ is the Poisson–Dirichlet measure $PD(\theta)$.*

It is easy to see from (3) that for every measurable subset $A \subset X$, the random variable $\gamma(A)$ is governed by the gamma distribution with shape parameter $\nu(A)$, and for every measurable partition $A_0, \dots, A_n$ of X, the variables $\gamma(A_0), \dots \gamma(A_n)$ are independent. Then it follows from the known property of the Dirichlet distributions that the vector $(\gamma(A_0)/\gamma(X), \dots, \gamma(A_n)/\gamma(X))$ has Dirichlet distribution on the simplex with parameters $(\nu(A_0), \dots, \nu(A_n))$. Thus *the Dirichlet process on the space X with parameter measure ν is just the normalized gamma process on $D(X, \nu)$*. One can also establish this relation in terms of the explicit construction (1). Indeed, in view of Theorem 1 and Lemmas 1 and 2, the right-hand side of (1) presents the normalized gamma process on (X, ν).

4. The Markov–Krein identity for means of Dirichlet processes. In this section we prove the Markov–Krein identity for means of Dirichlet processes which may be interpreted as a formula relating the distribution of a linear functional with respect to the gamma process and the distribution of the same functional with respect to the normalized gamma process. This interpretation allows one to prove it immediately, using only a formula for the Laplace transform of the gamma process and the independence property of the gamma processes.

Given a function $a \in \mathcal{M}$, denote by μ_a the distribution of the linear functional $\eta \mapsto f_a(\eta) = \int_X a(x)d\eta(x)$ on D with respect to the law $\mathcal{D}$ of the Dirichlet process on (X, ν) (i.e., the law of the normalized gamma process), and let ν_a be the distribution of the function a with respect to the normalized parameter measure $\bar\nu$.

Theorem 2. *The measures μ_a and ν_a are related by the following integral identity,*

$$\int_{\mathbb{R}} \frac{1}{(1+zu)^\theta} d\mu_a(u) = \exp\left(- \int_X \log(1+zu)^\theta d\nu_a(u) \right). \tag{5}$$

Note that the left-hand side of (5) is the generalized Cauchy–Stieltjes transform of the distribution μ_a. It is natural to call the right-hand side the multiplicative version of the generalized Cauchy–Stieltjes transform of the distribution ν_a. In view of (4), it is equal to the Laplace transform of the gamma process calculated on the function a. Hence one may regard formula (5) as relating an integral transform (Cauchy–Stieltjes) of the distribution μ_a of the functional f_a with respect to the normalized gamma process and an integral transform (Laplace) of its distribution with respect to the non-normalized gamma process.

Proof. Using (4), Lemma 1 and the Fubini theorem we obtain that the right-hand side of (5) equals

$$
\begin{aligned}
&\exp\left(-\int_X \log(1+za(x))d\nu(x)\right) = \mathbb{E}_{\mathcal{G}}\left[\exp\left(-z\int_X a(x)d\gamma(x)\right)\right] \\
=&\mathbb{E}_{\mathcal{G}}\left[\exp\left(-z\gamma(X)\int_X a(x)d\bar{\gamma}(x)\right)\right] \\
=&\mathbb{E}_{\mathcal{D}}\left[\frac{1}{\Gamma(\theta)}\int_0^\infty t^{\theta-1}\exp\left(-t-zt\int_X a(x)d\bar{\gamma}(x)\right)\right] \\
=&\mathbb{E}_{\mathcal{D}}\left[\frac{1}{(1+z\int_X a(x)d\bar{\gamma}(x))^\theta}\right],
\end{aligned}
$$

and Theorem follows.

According to a personal communication of P. Diaconis to S. Kerov, the idea of proving formula (6) using the Laplace transform formula for the gamma process was used by F. Huffer in case of *discrete* parameter measure ν (in this case the gamma process is just a sum of independent gamma variables, and the normalized gamma process is a random point of a finite-dimensional simplex obeying a Dirichlet distribution). But the fact that this argument works as well for continuous parameter measures seems to have been overlooked. But after the preliminary version [TVY1] of this paper had been published, other independent close proofs of the identity (5) appeared, see [RGN, HO].

A multivariate version of the Markov–Krein identity for the common distributions of several linear functionals of the Dirichlet process was first obtained in [KT]. It is easy to extend our proof of Theorem 2 to obtain this result.

Theorem 3. *Let $a_1, \dots, a_n \in \mathcal{M}$. Denote by μ_a the common distribution of the linear functionals $(f_{a_1}, \dots, f_{a_n})$ on D with respect to the law $\mathcal{D}$ of the Dirichlet process. Let ν_a be the common distribution of the functions $a_1, \dots, a_n$ with respect to the normalized parameter measure $\bar{\nu}$. Then the measures μ_a and ν_a are related by the multivariate Markov–Krein identity*

$$
\begin{aligned}
(6)\quad &\int_{\mathbb{R}^n} \frac{1}{(1+z_1u_1+\ldots+z_nu_n)^\theta} d\mu_a(u) \\
&\qquad = \exp\left(-\int_{\mathbb{R}^n} \log(1+z_1u_1+\ldots+z_nu_n)^\theta d\nu_a(u)\right).
\end{aligned}
$$

Proof. Goes exactly as the proof of Theorem 2 with function $za(x)$ replaced by $z_1a_1(x)+\ldots+z_na_n(x)$.

5. Two-parameter generalization of the Markov–Krein identity. S. Kerov [Ke] and J. Pitman [P] suggested independently the same class of generalizations of the Dirichlet processes. These generalizations are indexed by an arbitrary distribution σ on the simplex Δ, and they are defined by (1), where the sequence Q obeys

σ instead of $PD(\theta)$. An important particular case of these generalized Dirichlet processes is obtained when σ is the so-called two-parameter Poisson–Dirichlet distribution $PD(\alpha, \theta)$ [PY]. The range of admissible parameters is the union of the sets $\{(\alpha, \theta) : \alpha \in (0, 1), \theta > -\alpha\}$ and $\{(\alpha, -m\alpha) : \alpha < 0, m \in \mathbb{N}\}$. When $\alpha = 0$, the measure $PD(0, \theta)$ coincides with the ordinary Poisson–Dirichlet distribution $PD(\theta)$. Denote by $\mathcal{D}(\alpha, \theta)$ the law of the generalized Dirichlet process associated with the two-parameter Poisson–Dirichlet distribution $PD(\alpha, \theta)$.

An analogue of the Markov–Krein identity for the distribution of a linear functional with respect to $\mathcal{D}(\alpha, \theta)$ is obtained in [T]. We present here a new proof of this identity based on relation of the two-parameter Poisson–Dirichlet distributions to the stable processes.

Definition 6. Let $\alpha \in (0, 1)$. The standard α-stable process on the space (X, ν) is a Lévy process with Lévy measure $d\Lambda_\alpha = \frac{\alpha}{\Gamma(1-\alpha)} s^{-\alpha-1} ds$, $s > 0$. Thus the law P_α of the α-stable process has Laplace transform

$$\mathbb{E}_{P_\alpha} \left[\exp \left(- \int_X a(x) d\eta(x) \right) \right] = \exp \left(- \int_X a(x)^\alpha d\nu(x) \right), \tag{7}$$

where $a : X \to \mathbb{R}_+$ is an arbitrary measurable function with $\int_X a(x)^\alpha d\nu(x) < \infty$.

Lemma 3 [PY]. *The simplicial part of the α-stable process is the Poisson–Dirichlet distribution $PD(\alpha, 0)$.*

The Poisson–Dirichlet distribution $PD(\alpha, \theta)$ with $\alpha, \theta \neq 0$ cannot be represented as the simplicial part of any Lévy process. However, one may obtain it as the simplicial part of the process that has density with respect to a stable process. Namely, let $\theta > -\alpha$ and consider the law $P_{\alpha,\theta}$ on D which has density

$$\frac{dP_{\alpha,\theta}}{dP_\alpha}(\eta) = \frac{c_{\alpha,\theta}}{\eta(X)^\theta} \tag{8}$$

with respect to the α-stable law P_α. Here $c_{\alpha,\theta} = \frac{\Gamma(\theta+1)}{\Gamma(\theta/\alpha+1)}$ is a normalizing constant.

Lemma 4 [PY]. *The simplicial part of the measure $P_{\alpha,\theta}$ is the Poisson–Dirichlet distribution $PD(\alpha, \theta)$.*

It follows from Lemmas 3 and 4 that the generalized Dirichlet process with parameters $(\alpha, 0)$ is just the normalized α-stable process, and the generalized Dirichlet process with parameters (α, θ) is the normalized processed governed by the law $P_{\alpha,\theta}$.

Given an arbitrary measurable function $a : X \to \mathbb{R}_+$ with $\int_X a(x)^\alpha d\nu(x) < \infty$, let μ_a be the distribution of the functional f_a with respect to $\mathcal{D}(\alpha, \theta)$, and let ν_a be the distribution of a with respect to the normalized parameter measure $\bar{\nu}$.

Theorem 4. *The measures μ_a and ν_a are related by the following integral identity:*
1) *if $\theta \neq 0$,*

$$\left(\int_{\mathbb{R}} (1 + zu)^{-\theta} d\mu_a(u) \right)^{-\frac{1}{\theta}} = \left(\int_{\mathbb{R}} (1 + zu)^\alpha d\nu_a(u) \right)^{\frac{1}{\alpha}}; \tag{9}$$

2) *if* $\theta = 0$,

$$\exp\left(\int_{\mathbb{R}} \log(1+zu)^\alpha d\mu_a(u)\right) = \int_{\mathbb{R}} (1+zu)^\alpha d\nu_a(u). \tag{10}$$

Proof. 1) Denote the left-hand side of the desired identity by $A^{-1/\theta}$ and the right-hand side by $B^{1/\alpha}$. Using the identity

$$\frac{1}{r^\theta} = \frac{1}{\Gamma(\theta)} \int_0^\infty t^{\theta-1} e^{-rt} dt,$$

we obtain

$$\begin{aligned}
A =& c_{\alpha,\theta} \mathbb{E}^\alpha \left[\left(\eta(X) + z \int_X a(x) d\eta(x) \right)^{-\theta} \right] \\
=& \frac{c_{\alpha,\theta}}{\Gamma(\theta)} \mathbb{E}^\alpha \left[\int_0^\infty t^{\theta-1} \exp\left(-t \left(\eta(X) + z \int_X a(x) d\eta(x) \right) \right) dt \right] \\
=& \frac{c_{\alpha,\theta}}{\Gamma(\theta)} \int_0^\infty t^{\theta-1} \mathbb{E}^\alpha \left[\exp\left(- \int_X t(1 + za(x)) d\eta(x) \right) \right] dt.
\end{aligned}$$

By the Laplace transform formula (7), the expectation equals precisely $e^{-t^\alpha B}$, thus

$$A = \frac{c_{\alpha,\theta}}{\Gamma(\theta)} \int_0^\infty t^{\theta-1} e^{-t^\alpha B} dt,$$

and (9) follows by changing variables.

2) Follows from (9) by letting $\theta \to 0$.

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