ON A CLASS OF TYPE II_1 FACTORS WITH BETTI NUMBERS INVARIANTS

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ABSTRACT. We prove that a type II_1 factor M has at most one Cartan subalgebra A satisfying a combination of rigidity and compact approximation properties. In particular, this shows that the Betti numbers of $A \subset M$, as defined in ([G2]), are isomorphism invariants for the factors M having such Cartan subalgebras. Examples of factors with this property are the group measure space algebras associated with certain actions of the free groups with finitely many generators on the probability space, for which the Betti numbers are calculated in ([G2]). As a consequence, we obtain the first examples of type II_1 factors M with trivial fundamental group and solve a problem formulated in ([K1]). Also, our results bring some insight into a recent problem of A. Connes.

0. INTRODUCTION.

We consider in this paper the class of type II_1 factors M having maximal abelian von Neumann subalgebras A with normalizer acting ergodically on it and so that $A \subset M$ satisfies both a rigidity property (in the spirit of Kazhdan and Connes-Jones, [Kaz], [CJ1]) and a compact approximation property (in the spirit of Haagerup, [H]). We call $\mathcal{HT}$ the class of all these factors.

We prove that if $M \in \mathcal{HT}$, then any two such maximal abelian subalgebras of M are conjugate by a unitary element of M. More generally, any isomorphism between factors $M \in \mathcal{HT}$ must carry the corresponding maximal abelian $*$-subalgebras A one onto the other (modulo perturbation by an inner automorphism). As a result, for factors $M \in \mathcal{HT}$, any isomorphism invariant for the inclusion $A \subset M$ becomes an isomorphism invariant for M.

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In particular, invariants for the measure-preserving countable orbit equivalence relation $\mathcal{R}_M$ induced on the standard probability space (X,μ) (for some identification $A \simeq L^\infty(X,\mu)$) by the normalizer of A in M, are invariants for $M \in \mathcal{HT}$.

In a recent paper, D. Gaboriau introduced a notion of ℓ^2 Betti numbers for arbitrary countable orbit equivalence relations $\mathcal{R}$, $\{\beta_n(\mathcal{R})\}_{n\geq 0}$, starting from ideas of Atiyah ([A]) and Connes ([C4]), which generalizes the notion of Betti numbers for measurable foliations in ([C4]). This notion also generalizes the more combinatorial ℓ^2 Betti numbers for discrete groups of Cheeger-Gromov ([ChGr]), $\beta_n(G)$, for which he shows that $\beta_n(G) = \beta_n(\mathcal{R}_G)$, for any countable equivalence relation $\mathcal{R}_G$ implemented by a free, ergodic, measure-preserving action of the group G on a standard probability space (X,μ) ([G2]).

We define the Betti numbers $\{\beta_n(M)\}_{n\geq 0}$ of a factor M in the class $\mathcal{HT}$ as the Betti ℓ^2- numbers of the corresponding orbit equivalence relation $\mathcal{R}_M$, $\{\beta_n(\mathcal{R}_M)\}_n$.

Some of the general properties of the Betti numbers for countable equivalence relations proved in ([G2]) easily entail similar properties for the Betti numbers of the factors in the class $\mathcal{HT}$. For instance, noticing that $\mathcal{HT}$ is closed under amplifications by arbitrary $t > 0$, it follows from ([G2]) that $\beta_n(M^t) = \beta_n(M)/t, \forall n$. Also, $\mathcal{HT}$ is closed under tensor products and a Künneth type formula holds for $\beta_n(M_1 \otimes M_2)$ in terms of the Betti numbers for $M_1, M_2 \in \mathcal{HT}$, as a result of the similar formula for equivalence relations in Gaboriau's paper ([G2]).

Our main example of a factor in the class $\mathcal{HT}$ is the group von Neumann algebra $L(G)$ associated with $G = \mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$, regarded as the group-measure space construction $L^\infty(\mathbb{T}^2,\mu) = A_0 \subset A_0 \rtimes_{\sigma_0} SL(2,\mathbb{Z})$. More generally, we consider cross product factors of the form $A \rtimes_\sigma SL(2,\mathbb{Z})$ with A abelian and σ a product of the action σ_0 with other ergodic actions of $SL(2,\mathbb{Z})$. By a recent result of G. Hjorth ([Hj]), based on the notion and results on tree-ability in ([G1]), these factors are in fact amplifications of group-measure space factors of the form $L^\infty(X,\mu) \rtimes \mathbb{F}_n$, where $\mathbb{F}_n$ is the free group on n generators, $n = 2, 3, \ldots$.

To prove that M belongs to the class $\mathcal{HT}$, with A its "distinguished" maximal abelian subalgebra, we use Kazhdan's rigidity of the inclusion $\mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$ and Haagerup's compact approximation property of $SL(2,\mathbb{Z})$. Note that in this case the algebra A is in fact a Cartan subalgebra of M, i.e., the normalizer of A in M generates all the von Neumann algebra M.

Since the orbit equivalence relation $\mathcal{R}_M$ implemented by $SL(2,\mathbb{Z})$ on A has non-trivial Betti numbers (cf. [G1, 2]), it follows that the factors $M = A \rtimes_\sigma SL(2,\mathbb{Z})$ have non-trivial Betti numbers.

Other examples of factors in the class $\mathcal{HT}$ are obtained by taking discrete groups Γ that can be embedded as lattices in $SU(n,1)$ or $SO(n,1)$ and suitable actions σ of such Γ on abelian von Neumann algebras $A \simeq L(\mathbb{Z}^N)$. Indeed, these groups Γ

have the Haagerup approximation property by ([dCaH], [CowH]) and their action on A can be taken to be rigid by ([Val]).

As a consequence of these considerations, we are able to answer a number of open questions in the theory of type II_1 factors. Thus, the factors $M = A \rtimes_\sigma SL(2,\mathbb{Z})$ (more generally, $A \rtimes_\sigma \Gamma$ with Γ, σ as above) provide the first class of type II_1 factors with trivial fundamental group, i.e. $\mathcal{F}(M) \stackrel{\text{def}}{=} \{t > 0 \mid M^t \simeq M\} = \{1\}$. Indeed, we mentioned that $\beta_n(M^t) = \beta_n(M)/t, \forall n$, so that if $\beta_n(M) \neq 0$ for some n then $\mathcal{F}(M) = \{1\}$.

In particular, the factors M are not isomorphic to the algebra of n by n matrices over M, for any $n \geq 2$ (see Kadison's Problem 3 in [K1]). Also, through appropriate choice of the action σ, we obtain factors of the form $M \rtimes_\sigma SL(2,\mathbb{Z})$ having the property Γ of Murray and von Neumann, yet trivial fundamental group.

The fundamental group $\mathcal{F}(M)$ of a II_1 factor M was defined by Murray and von Neumann in the early 40's, in connection with their notion of continuous dimension. They noticed that $\mathcal{F}(M) = \mathbb{R}_+^*$ when M is isomorphic to the hyperfinite type II_1 factor R, and more generally when M "splits off" R.

The first examples of type II_1 factors M with $\mathcal{F}(M) \neq \mathbb{R}_+^*$, and the first occurence of rigidity in the von Neumann algebra context, were discovered by Connes in ([C1]). He proved that if G is an infinite conjugacy class discrete group with the property T of Kazhdan then its group von Neumann algebra $M = L(G)$ is a type II_1 factor with countable fundamental group. It was then proved in ([Po1]) that this is still the case for factors M which contain some irreducible copy of $L(G)$. It was also shown that there exist type II_1 factors M with $\mathcal{F}(M)$ countable and containing any presribed countable set of numbers ([Gol], [Po3, 5]). However, the fundamental group $\mathcal{F}(M)$ could never be computed exactly, in any of these examples.

In fact, more than proving that $\mathcal{F}(M) = \{1\}$ for $M = A \rtimes_\sigma SL(2,\mathbb{Z})$, the calculation of the Betti numbers shows that $M^{t_1} \otimes M^{t_2} ... \otimes M^{t_n}$ is isomorphic to $M^{s_1} \otimes M^{s_2} ... \otimes M^{s_m}$ if and only if $n = m$ and $t_1 t_2 ... t_n = s_1 s_2 ... s_m$. In particular, all tensor powers of M, $M^{\otimes^n}, n = 2, 3, ...$, are mutually non-isomorphic. (N.B. The first examples of factors satisfying this latter property were constructed in [C4]; another class of examples was obtained in [CowH]). Moreover, since $M^t \simeq L^\infty(X,\mu) \rtimes \mathbb{F}_n$ for $t = (12(n-1))^{-1}$ (cf. [Hj]), it follows that there exists a free ergodic action σ_n of $\mathbb{F}_n$ on the standard probability space (X,μ) such that $M_n = L^\infty(X,\mu) \rtimes_{\sigma_n} \mathbb{F}_n$ satisfy $M_{k_1} \otimes ... \otimes M_{k_n} \simeq M_{l_1} \otimes ... \otimes M_{l_m}$ if and only if $n = m$ and $k_1 k_2 ... k_n = l_1 l_2 ... l_m$. Moreover, in case $n = m$, the factors are stably isomorphic for any choice of k_i, l_j. Similar results hold true in the case of lattices Γ in $SO(n,1)$ or in $SU(n,1)$. These classes of factors are non- stably equivalent, for different values of n.

We also prove that the factors $M = A \rtimes_\sigma SL(2,\mathbb{Z})$, as well as any tensor product of amplifications of M, have only subfactors of integer Jones index, if finite. In

particular, they have no subfactors with values in the Jones series $4\cos^2\pi/n$, except for $n = 3, 4, 6$.

As for the "size" of $\mathcal{HT}$, we mention that we do not know of other examples than the ones mentioned above, namely the factors M of the form $A \rtimes G$, with $G = SL(2,\mathbb{Z})$ or G a lattice in $SU(n,1), SO(n,1)$, their amplifications, tensor products and certain cross-products. The class $\mathcal{HT}$ does not contain the hyperfinite type II_1 factor. The factors $M \in \mathcal{HT}$ cannot contain property T factors and cannot be embedded into free group factors (due to arguments similar to [CJ1]).

Besides these concrete applications, our results give a partial answer to a challenging problem recently raised by Alain Connes, on defining a notion of Betti numbers for type II_1 factors, from similar conceptual grounds as in the case of orbit equivalence relations (simplicial structure, L^2-cohomology, etc). In this respect, our definition is only for the special class of factors $\mathcal{HT}$ and falls short of being conceptual in the above sense. Yet it already produces significant applications and emphasizes phenomena that could be used as a test case for the general problem.

The paper is organized as follows: In Section 1 we prove a key perturbation result for maximal abelian von Neumann subalgebras (hereafter abbreviated as m.a.s.a.) in type II_1 factors. The proof uses Christensen's set-up and techniques ([Chr]), and a careful analysis of the geometry of projections. In Section 2 we consider a relative version of Haagerup's compact approximation property. In Section 3 we use the Connes-Jones idea of translating rigidity in terms of properties of completely positive maps ([CJ1]), to define the notion of rigid inclusion (or property T pair) of algebras, by analogy with the case of groups.

In Section 4 we define the class $\mathcal{HT}$ of factors M having m.a.s.a.'s A with the property that $A \subset M$ is rigid and M has the Haagerup property relative to A. We prove the main technical result of the paper, showing that such m.a.s.a.'s are unique. We also prove the stability of $\mathcal{HT}$ with respect to various operations (amplification, tensor product, finite index induction/restriction, etc). In Section 5 we construct examples of such factors. In Section 6 we define the Betti numbers $\{\beta_n(M)\}_n$ for $M \in \mathcal{HT}$ and use ([G2]) to deduce various properties of this invariant. We end with applications, as well as some remarks and open questions.

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1. A perturbation result.

In this Section we prove a perturbation result concerning abelian von Neumann

subalgebras of type II_1 factors. This result will play a key role in the proof of Theorem 4.1, which is the main technical result in this paper.

The proof uses the perturbation techniques from [Chr], the "pull down" identity ([PiPo]) and a careful handling of the geometry of projections, including a result from ([K2]).

Since we will also use it in the next Section, let us recall from ([Chr], [J1], [Po2]) some well known facts about the *basic construction* for an inclusion of a von Neumann algebra B into a type II_1 factor N. Thus, we denote by $\langle N, B\rangle$ the von Neumann algebra generated in $\mathcal{B}(L^2(N,\tau))$ by N (regarded as the algebra of left multiplication operators by elements in N) and by the orthogonal projection e_B of $L^2(M,\tau)$ onto $L^2(B,\tau)$.

Since $e_B x e_B = E_B(x)e_B, \forall x \in N$, where E_B is the unique τ-preserving conditional expectation of N onto B, it follows that $\mathrm{sp}Ne_BN$ is a *-algebra with support equal to 1 in $\mathcal{B}(L^2(N,\tau))$. Thus, $\langle N, B\rangle = \overline{\mathrm{sp}}^{\mathrm{w}}\{xe_By \mid x, y \in N\}$ and $e_B\langle N, B, \rangle e_B = Be_B$.

One can also readily see that if J denotes the canonical conjugation on the Hilbert space $L^2(N,\tau)$, then $\langle N, B\rangle = JBJ'$. This shows in particular that $\langle N, B\rangle$ is semifinite.

We endow $\langle N, B\rangle$ with the unique normal semifinite faithful trace Tr satisfying $Tr(xe_By) = \tau(xy), \forall x, y \in N$. Note that there exists a unique N bilinear map Φ from $\mathrm{sp}Ne_BN \subset \langle N, B\rangle$ into N satisfying $\Phi(xey) = xy, \forall x, y \in N$, and $\tau \circ \Phi = Tr$. The map Φ extends uniquely to a N-bilinear map from $L^1(\langle N, B\rangle, Tr)$ onto $L^1(N,\tau)$, still denoted Φ. This bilinear map satisfies the "pull down" identity $eX = e\Phi(eX), \forall X \in \langle N, B\rangle$ (see [PiPo], or [Po7]). Note that $\Phi(eX)$ actually belongs to $L^2(N,\tau) \subset L^1(N,\tau)$, for $X \in \langle N, B\rangle$.

Let now $\mathcal{U}_0 \subset N$ be a subgroup of unitary elements and $B_0 = \mathcal{U}_0''$ the von Neumann algebra it generates. If b is an element in $\langle N, B\rangle$ with $Tr(b^*b) < \infty$ then we denote by $K_{\mathcal{U}_0}(b) = \overline{\mathrm{co}}^{\mathrm{w}}\{u_0bu_0^* \mid u_0 \in \mathcal{U}_0\}$. Note that $K_{\mathcal{U}_0}(b)$ is also contained in the Hilbert space $L^2(\langle N, B\rangle, Tr)$, where it is still weakly closed.

Let $h = h_{\mathcal{U}_0}(b) \in K_{\mathcal{U}_0}(b)$ be the unique element of minimal norm $\| \|_{2,Tr}$ in $K_{\mathcal{U}_0}(b)$. Since $uK_{\mathcal{U}_0}(b)u^* = K_{\mathcal{U}_0}(b)$ and $\|uhu^*\|_{2,Tr} = \|h\|_{2,Tr}, \forall u \in \mathcal{U}_0$, by the uniqueness of h it follows that $uhu^* = h, \forall u \in \mathcal{U}_0$. Thus $h \in \mathcal{U}_0' \cap \langle N, B\rangle = B_0' \cap \langle N, B\rangle$. Moreover, by the definitions, we see that if $0 \leq b \leq 1$ then $0 \leq k \leq 1$ for all $k \in K_{\mathcal{U}_0}(b)$.

1.1. Theorem. *Assume B is a maximal abelian $*$-subalgebra in N and B_0 is an abelian von Neumann subalgebra such that $B_{01} \stackrel{\mathrm{def}}{=} B_0' \cap N$ is abelian (thus maximal abelian). Then the following conditions are equivalent:*

1°. The element $h_{\mathcal{U}_0}(e_B)$, of minimal norm $\| \|_{2,Tr}$ in $K_{\mathcal{U}_0}(e_B)$, is non-zero and

belongs to $B_0' \cap \langle N, B\rangle$.

2°. *There exists a non-zero projection* $e_0 \in B_0' \cap \langle N, B\rangle$ *with* $Tr(e_0) < \infty$.

3°. *There exist non-zero projections* $q_0 \in B_0' \cap N$, $q \in B$ *and a partial isometry* $v \in N$ *such that* $v^*v = q_0, vv^* = q$ *and* $vB_0v^* \subset Bq$.

Proof. 3° $\Longrightarrow$ 1°. If v satisfies condition 3° then B_0q_0 is contained in v^*Bv. Thus, $e_0 = v^*e_Bv$ belongs to $B_0' \cap \langle N, B\rangle$ and has finite trace.

1° $\Longrightarrow$ 2°. Let e_0 be the spectral projection of $h = h_{\mathcal{U}_0}(e_B)$ corresponding to the interval $(\|h\|/2, \infty)$. Then $e_0 \neq 0$ and $h \geq \frac{1}{2}e_0$. Thus,

$$Tr(e_0) \leq 2Tr(h) \leq 2Tr(e_B) < \infty.$$

Thus, e_0 is a finite projection in $N_1 = \langle N, B\rangle$ and e_0 commutes with B_0 (since h does).

2° $\Longrightarrow$ 3°. Since B_0e_0 is abelian, it is contained in a maximal abelian subalgebra B_1 of $e_0N_1e_0$.

By a result of Kadison ([K2]), B_1 contains a non-zero abelian projection e_1 of N_1 (i.e., $e_1N_1e_1$ is abelian). Since e_B has central valued (semifinite) trace equal to 1 (it is also a maximal abelian projection in N_1), it follows that e majorizes e_1.

Let $V \in N_1$ be a partial isometry such that $V^*V = e_1 \leq e_0$ and $VV^* \leq e_B$. Moreover $VB_1e_1V^*$ is a subalgebra of $e_BN_1e_B = Be$. Since e_1 commutes with B_0, it follows that if we denote by f' the maximal projection in B_0 such that $f'e_1 = 0$ and let $f_0 = 1 - f'$, then there exists a unique isomorphism φ from B_0f_0 into B such that $\varphi(b)e_B = VbV^*, \forall b \in B_0f$. Let $f = \varphi(f_0) \in B$.

It follows that $\varphi(b)e_BV = e_BVb, \forall b \in B_0f_0$. By applying Φ to both sides and denoting a the square integrable operator $a = \Phi(eV) \in L^2(N, \tau)$, it follows that $\varphi(b)a = ab, \forall b \in B_0$. Since $ea = eV = V$, it follows that $a \neq 0$.

By the usual trick, if we denote by $v_0 \in N$ the unique partial isometry in the polar decomposition of a such that the right supports of a and v_0 coincide, then $p_0 = v_0^*v_0$ belongs to the algebra $B_0' \cap N = B_{01}$, which is abelian by hypothesis, $p = v_0v_0^*$ belongs to $(\varphi(B_0)f)' \cap fNf$ and $\varphi(b)v_0 = v_0b, \forall b \in B_0f_0$.

But $B_{01} = B_0' \cap N$ maximal abelian in N implies $B_{01}f_0$ maximal abelian in f_0Nf_0. Noreover, since $v_0B_0v_0^* = \varphi(B_0)p$, if we denote $B_{11} = v_0B_{01}v_0^*$, then by spatiality we have

$$B_{11} = v_0B_{01}v_0^* = v_0(B_0' \cap N)v_0^*$$
$$= v_0B_0v_0^{*\prime} \cap pNp = \varphi(B_0)p' \cap pNp.$$

Thus, the commutant of $\varphi(B_0)p$ is abelian. Since any element in pBp commutes with $\varphi(B_0)p$, it follows that $pBp \subset (\varphi(B_0)p)' \cap pNp = B_{11} \subset (\varphi(B_0)f)' \cap fNf$. Moreover, $(p(\varphi(B_0)f)' \cap fNf)p = pBp \subset (\varphi(B_0)p)' \cap pNp = B_{11}$.

It follows that if P_1 denotes the von Neumann algebra generated by p and $(\varphi(B_0)f)' \cap fNf \supset Bf$ inside fNf, then $pP_1p = (p(\varphi(B_0)f)' \cap fNf)p = B_{11}$ is abelian.

Thus, p is an abelian projection in P_1. Since P_1 is a finite von Neumann algebra, there exists a central projection z_1 of P_1, under the central support of p in P_1, such that $pz_1 \neq 0$ and such that P_1z_1 is homogeneous of type n, for some $n \geq 1$. Note that the center of P_1 is included in Bf the latter being maximal abelian in fNf, thus in $P_1 \subset fNf$. Thus, $z_1 \in Bf$.

Now, since pz_1 has central trace equal to $1/n$ in P_1z_1 and Bz_1 is maximal abelian in P_1z_1, it follows that there exists a projection $f_{11} \in Bf$ such that f_{11} is equivalent to p_1z_1 in P_1z_1 (see [K2]). Let $v_1 \in P_1z_1$ be such that $v_1v_1^* = f_{11}, v_1^*v_1 = p_1z_1$. Since p_1z_1 is abelian in P_1, f_{11} is also abelian, thus $f_{11}P_1f_{11} = Bf_{11}$. This implies that $v_1pz_1(\varphi(B_0)' \cap fNf))pz_1v_1^* = Bf_{11}$.

Finally, since $pz_1 \leq p$, $\varphi(B_0)p = v_0B_0v_0^*$ and pz_1 belongs to $\varphi(B_0)'$, it follows that $v_0^*pz_1v_0 \in B_0'$. This shows that if we denote $v = v_1v_0$, then v is a partial isometry satsifying $v^*v \in B_0', vv^* = f_{11} \in B$ and $vB_0v^* \subset Bvv^*$. Q.E.D.

In Section 4 we will in fact need a consequence of Theorem 1.1. To state it, recall from ([D1]) that a maximal abelian *-subalgebra (abreviated m.a.s.a.) B of a von Neumann factor N is called *semiregular* if the set of unitaries of N that normalize B, i.e., $\mathcal{N}(B) = \{u \in \mathcal{U}(N) \mid uBu^* = B\}$, generate a factor, equivalently, $\mathcal{N}(B)' \cap N = \mathbb{C}$. Also, B is called *regular* if this normalizer generetes all the ambient factor N. Such regular MASA's were later called *Cartan subalgebras* in ([FM]), a terminology that seems to have prevailed and which we will therefore adopt.

1.2. Corollary. *Let N be an arbitrary type II_1 factor. Let B be a semiregular m.a.s.a. of N and $B_0 \subset N$ an abelian von Neumann subalgebra such that $B_0' \cap N$ is a semiregular m.a.s.a. of N. If B, B_0 satisfy the equivalent conditions in* Theorem 1.1 *then there exists a unitary element $u \in N$ such that $u(B_0' \cap N)u^* = B$. In particular, this is the case if $B, B_0' \cap N$ are Cartan subalgebras of N.*

Proof. By Theorem 1.1 there exists a non-zero partial isometry $v \in N$ such that $v^*v \in B_0' \cap N, vv^* \in B, vB_0v^* = Bvv^*$. Thus, $v(B_0' \cap N)v^* = vB_0v^{*\prime} \cap vv^*Nvv^*$ contains $Bvv^{*\prime} \cap vv^*Nvv^* = Bvv^*$. Since both are maximal abelian, we get $v(B_0' \cap N)v^* = Bvv^*$.

Moreover, by cutting v from the right with a smaller projection in $B_0' \cap N$, we may clearly assume $\tau(vv^*) = 1/n$ for some integer n.

Since $B, B_0' \cap N$ are semiregular, there exist partial isometries $v_1, v_2, ..., v_n$, respectively $w_1, w_2, ..., w_n$ in the normalizing groupoids of B respectively $B_0' \cap N$ such that $\Sigma_i v_iv_i^* = 1, \Sigma_j w_jw_j^* = 1$ and $v_i^*v_i = v^*v, w_j^*w_j = vv^*, \forall i, j$. But then $u = \Sigma_i w_ivv_i^*$ is a unitary element and $u(B_0' \cap N)u^* = B$. Q.E.D.

2. Relative Haagerup property.

Recall that a type II_1 factor N has Haagerup's compact approximation property (*property* H hereafter) if there exists a sequence of compact completely positive, unital, trace preserving maps $\Phi_n : N \to N$ such that

$$\lim_{n\to\infty} \|\Phi_n(x) - x\|_2 = 0, \forall x \in N.$$

In this paper we need to introduce a relative version of this property. To this end, let us first note the following:

2.1. Lemma. *Let N be a type II_1 factor and $B \subset N$ a von Neumann subalgebra. Let $\Phi : N \to N$ be a unital, trace preserving, completely positive map satisfying*

$$\Phi(b_1 x b_2) = b_1 \Phi(x) b_2, \forall x \in N, b_1, b_2 \in B.$$

Then there exists a unique bounded operator T_Φ on the Hilbert space $L^2(N,\tau)$ such that $T_\Phi(x) = \Phi(x)$, for $x \in N$. Moreover this operator lies in $B' \cap \langle N, B\rangle$.

Proof. By Kadison's inequality, for $x \in M$ we have

$$\langle T_\Phi(\hat{x}), T_\Phi(\hat{x})\rangle = \tau(\Phi(x)^*\Phi(x)) \leq \tau(\Phi(x^*x)) = \tau(x^*x), \forall x \in N.$$

Thus, $\|T_\Phi\| \leq 1$.

If $b \in B$ is regarded as operator of left multiplication by b on $L^2(N,\tau)$, then we have

$$bT_\Phi(\hat{x}) = b\Phi(x) = \Phi(bx) = T_\Phi(b\hat{x}).$$

Thus, $T_\Phi \in B'$.

Similarily,

$$JbJ(T_\Phi(\hat{x})) = \Phi(x)b = \Phi(xb) = T_\Phi(JbJ(\hat{x}))$$

showing that $T_\Phi \in JBJ' = \langle N, B\rangle$. Q.E.D.

Note that if we take Φ to be the τ-preserving conditional expectation E_B, of N onto B, then T_Φ is the orthogonal projection e_B of $L^2(N,\tau)$. The next Lemma, which is an immediate extension of a well known result of Haagerup, gives examples of B-bimodular completely positive maps in the case of cross product inclusions.

2.2. Lemma. *Let G be a discrete group and σ an action of G on the finite von Neumann algebra (B,τ) by τ-preserving automorphisms. Let $N = B \rtimes_\sigma G$. Let $\varphi : G \to \mathbb{C}$ be a positive definite function on G with $\varphi(e) = 1$. Define M_φ on N*

by $M_\varphi(\Sigma_g b_g u_g) = \Sigma_g \varphi(g) b_g u_g$. *Then* M_φ *is a unital, trace preseving, completely positive map and it is* B*-bilinear.*

Proof. The proof of the case $B = \mathbb{C}$ of (1.1 in [H]) works exactly the same in this case. Q.E.D.

We can now state the cross product version of Haagerup's result. We will use the following:

2.3. Definition. 1°. A discrete group G has the *property* H if there exist positive definite functions ϕ_n on G such that $\phi_n(e) = 1$, $\lim_{g\to\infty} \phi_n(g) = 0, \forall n$, and

$$\lim_{n\to\infty} \phi_n(g) = 1, \forall g \in G.$$

It should be noted that Gromov calls such groups G a-T-*menable*.

2°. Let N be a type II_1 factor and $B \subset N$ a von Neumann subalgebra. We say that N has the *property* H *relative to* B, or that $B \subset N$ has the *property* H, if there exists a sequence of unital, completely positive, trace preserving maps Φ_n on N, with $\Phi(b_1 x b_1) = b_1 \Phi(x) b_2, \forall x \in N, b_1, b_2 \in B$, such that

$$\lim_{n\to\infty} \|\Phi_n(x) - x\|_2 = 0, \forall x \in M$$

and such that all the corresponding operators T_{Φ_n} defined in 2.1 belong to the ideal of compact operators of the semifinite von Neumann algebra $\langle N, B\rangle$.

Note that, following the above remark, one could also use the terminology: The factor N is a-T-*menable relative to* B.

2.4. Proposition. *Let* B, G, σ, N *be like in* Lemma 2.2. *Then* N *has the property* H *relative to* B *if and only if* G *has the property* H.

Proof. Straighforward. Q.E.D.

Let us recall some well known results from ([H], [dCaH], [CowH]), that we need later:

2.5. Proposition. *Let* B, G, σ, N *be like in* Lemma 2.2. *Assume* G *is one of the following:*

1°. $G = SL(2, \mathbb{Z})$ *or* $G = \mathbb{F}_n$, *for some* $2 \leq n \leq \infty$,

2°. G *is a discrete subgroup of* $SO(n, 1)$, *for some* $n \geq 2$.

3°. G *is a discrete subgroup of* $SU(n, 1)$, *for some* $n \geq 2$.

4°. $G = G_1 *_H G_2$ *for some discrete groups* G_1, G_2 *with the property* H, *with* $H \subset G_1, H \subset G_2$ *a common subgroup.*

Then $B \subset N$ *has the relative property* H.

Proof. Part 1° is a result from ([H]), then 2° is in ([dCaH]) and 3° in ([CowH]). Part 4° is also well known. Q.E.D.

We will also need the following simple observations:

2.6. Proposition. 1°. *If $B \subset N$ has the property* H *and $p \in B$ is a projection, then $pBp \subset pNp$ has the property* H.

2°. *If $B_0 \subset N_0$ and $B_1 \subset N_1$ have the property* H *then $B_0 \otimes B_1 \subset N_0 \otimes N_1$ has the property* H.

3°. *If $B \subset N$ has the property* H *and $B \subset N_0 \subset N$, then $B \subset N_0$ has the property* H.

4°. *If $B \subset B_0 \subset N$ and $B \subset B_0$ has a finite orthonormal basis then $B_0 \subset N$ has* H *implies $B \subset N$ has* H.

Proof. Immediate by the definitions. Q.E.D.

3. Rigid embeddings.

In this section we consider a notion of rigid m.a.s.a., and more generally of a rigid embedding of a von Neumann algebra into a type II_1 factor, inspired by Kazhdan's example of the embedding $\mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$. Thus, our definition will be the operator algebraic version of the de la Harpe-Valette notion of property T for pairs of groups ([dlHVa]), in the same spirit as the Connes-Jones definition of property T for von Neumann algebras ([CJ1]). To this end, we use Connes' correspondences as a set-up, like in ([CJ1]).

First recall the definition for inclusions of groups:

3.1 Definition. An embedding of discrete groups $H \subset G$ is called *rigid* if there exist finitely many elements $g_1, g_2, ..., g_n \in G$ and $\varepsilon > 0$, such that if $\pi : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation of the group G on the Hilbert space $\mathcal{H}$ with a unit vector $\xi \in \mathcal{H}$ such that $\|\pi(g_i)\xi - \xi\| < \varepsilon, \forall i$, then there exists a non-zero vector $\xi_0 \in \mathcal{H}$ such that $\pi(h)\xi_0 = \xi_0, \forall h \in H$. Note that in ([dHVa]), the terminology used is: *the pair (G,H) has the property* T.

The following equivalent characterisation of the condition 3.1 is well known and thus we omit the proof:

3.2. Lemma. *$H \subset G$ satisfies condition* 3.1 *if and only if it satisfies the following: There exist $K \geq 1$, $g_1, g_2, ..., g_n \in G$ and $\varepsilon_0 > 0$, such that if $\pi : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation with a unit vector $\xi \in \mathcal{H}$ satisfying $\|\pi(g_i)\xi - \xi\| < \delta, \forall i$, for some $\delta \leq \varepsilon_0$, then $\|\pi(h)\xi - \xi\| \leq K\delta, \forall h \in H$.*

3.3. Definition. Let N be a finite von Neumann algebra and $B \subset N$ a von Neumann subalgebra. We say that the embedding $B \subset N$ is *rigid* (or that *the pair (N,B) has the property* T) if there exists a finite subset $x_1, x_2, ..., x_n \in N$ and $\varepsilon > 0$ such that if $\mathcal{H}$ is a N bimodule with $\xi \in \mathcal{H}$ a unit vector such that $\|x_i\xi - \xi x_i\| \leq \varepsilon, \forall i$, then there exist a non-zero vector $\xi_0 \in \mathcal{H}$ such that $b\xi_0 = \xi_0 b, \forall b \in B$.

Note that when $B = N$ this notion coincides with the notion of property T for von Neumann algebras of Connes and Jones in ([CJ1]).

3.4. Remark. In ([Po1]) we defined a notion of relative property T for inclusions $B \subset N$ (or property T of N relative to B) which is complementary to the one above. This notion was also considered independently by Anantharam-Delaroche in ([A-De]). In the case of groups, the notion corresponds to the situation in which $H \subset G$ is so that G is the semi-direct product of the group H by a property T group G_0. It should be noted that both the notion 3.3 considered here and the notion considered in ([Po1]) are in some sense "relative property T" for an inclusion $B \subset N$, but while the notion in ([Po1]) means "N has the property T relative to B", the notion considered in this paper is a "property T of B relative to its embedding into N".

3.5. Lemma. *$H \subset G$ is rigid if and only if $L(H) \subset L(G)$ is rigid.*

Proof. The proof is identical to the proof of the case $H = G$ in ([CJ1]) and of the more general such statements in ([Po1]). Q.E.D.

3.6. Lemma. *The following conditions are equivalent:*

1°. *$B_0 \subset N$ is rigid.*

2°. *$\exists x_1, x_2, ..., x_n \in N, \varepsilon > 0, K \geq 1$, such that if $\delta > 0$ and $(\mathcal{H}, \xi)$ is a Hilbert space with a unit vector satisfying $\|x_i\xi - \xi x_i\| \leq \delta, \forall i$ then there exists a vector $\xi_0 \in \mathcal{H}$ such that $\|\xi_0 - \xi\| \leq K\delta$ and $b\xi_0 = \xi_0 b, \forall b \in B_0$.*

3°. *$\exists x_1, x_2, ..., x_n \in N, \varepsilon > 0, K \geq 1$, such that if $\delta > 0$ and $\Phi N \to N$ is a normal, completely positive map with $\tau(\Phi(1)) = 1$ and $\|\Phi(x_i) - x_i\|_2 \leq \delta, \forall i$, then $\|\Phi(b) - b\|_2 \leq K\delta, \forall b \in B_0$, $\|b\| \leq 1$.*

Proof. $1^\circ \implies 2^\circ$. The proof is identical to the proof of 4.1.5 in ([Po1]).

$2^\circ \implies 3^\circ$. By using the equivalence between the bimodules and completely positive maps (Section 1.2 in [Po1]), this becomes a tautology (see also 4.2.1 in [Po1]).

$3^\circ \implies 1^\circ$. This is immediate from (1.2 in [Po1]). Q.E.D.

The next result lists the main properties of rigid embeddings:

3.7. Proposition. 1°. *If $B_0 \subset N$ is a rigid embedding and $N \subset M$ then $B_0 \subset M$ is a rigid embedding. Conversely, if $[M : N] < \infty$ and $B_0 \subset M$ is a rigid embedding then $B_0 \subset N$ follows a rigid embedding.*

2°. *If $B_0 \subset N$ is a rigid embedding and $B \subset B_0$ then $B \subset N$ is a rigid embedding. Conversely, if B has a finite orthonormal basis with respect to B_0 and $B \subset N$ is a rigid embedding, then $B_0 \subset N$ is a rigid embedding.*

3°. *Let $B \subset N$ and $p \in \mathcal{P}(B)$ a projection. Assume that there exist finitely many partial isometries $v_1, v_2, ..., v_n \in N$ such that $\Sigma_i v_i v_i^* = 1$ and $v_i^* v_i \in pBp$. Then $B \subset N$ is a rigid embedding if and only if $pBp \subset pNp$ is a rigid embedding.*

4°. *If* $(B_i \subset N_i)$ *are rigid embeddings for* $i = 1, 2$ *then* $(B_1 \otimes B_2 \subset N_1 \otimes N_2)$ *is a rigid embedding.*

Proof. 1°. The first part is trivial. The second part has the same proof as (4.1.8 in [Po1]).

2°. The first part is trivial. The second part has same proof as (4.1.8 in [Po1]).

3°. If $pBp \subset pNp$ is a rigid embedding then clearly $B \subset N$ forllows rigid, by the definitions, and by taking p to be in the critical set for $B \subset N$, with ε sufficiently small. The converse has the same proof as (4.1.9 in [Po1]).

4°. This has the same proof as analogue statements in ([Po1]). Q.E.D.

Let us end this Section with some examples of rigid inclusions, following from results of Kazhdan ([Kaz]) and respectively Valette ([Va]).

3.8. Proposition. 1°. *The inclusion* $\mathbb{Z}^2 \subset Z^2 \rtimes SL(2, \mathbb{Z})$ *is rigid. Thus,* $L(\mathbb{Z}^2) \subset L(\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z}))$ *is a rigid embedding of algebras.*

2°. *For each* $n \geq 2$ *there exist free ergodic measure preserving actions* σ *of* $\mathbb{F}_n$ *on* $A \simeq L^\infty(X, \mu)$ *such that* $A \subset A \rtimes_\sigma \mathbb{F}_n$ *is rigid.*

3°. *For each* $n \geq 2$ *there exist lattices* Γ *in* $SO(n, 1)$ *(respectively in* $SU(n, 1)$*) which admit free ergodic measure preserving actions* σ *on* $A \simeq L^\infty(X, \mu)$ *such that* $A \subset A \rtimes_\sigma \Gamma$ *is rigid.*

Proof. Part 1° is a well known result in ([Kaz]). Part 2° then follows trivially from 1° and by the result of Hjorth, showing that if $p \in L(\mathbb{Z}^2)$ is a projection of trace $(12(n-1))^{-1}$ then $pL(\mathbb{Z}^2)p \subset pL(\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z}^2))p$ is isomorphic to an inclusion of the form $A \subset A \rtimes_\sigma \mathbb{F}_n$, for some appropriate free, ergodic, measure preserving action σ of $\mathbb{F}_n$ on $A \simeq L^\infty(X, \mu)$. The rigidity of this latter inclusion follows by property 3.7.3°.

Part 3° is a recent result of Alain Valette ([Va]). Q.E.D.

4. HT FACTORS.

In this Section we prove the main technical result of this paper, showing that if a type II_1 factor has Cartan subalgebras which have at the same time the property H (as considered in Sec. 2) and the relative property T (considered in Sec. 3), then such Cartan subalgebras are unique modulo conjugacy by unitary elements. More generally, we have:

4.1. Theorem. *Let* N *be a type* II_1 *factor with two abelian von Neumann subalgebras* B, B_0. *Assume that* $B_0' \cap N$ *is abelian and that both* B *and* $B_0' \cap N$ *are semiregular m.a.s.a.'s. Assume further that* N *has property H relative to* B *and that* $B_0 \subset N$ *is a rigid inclusion. Then there exists a unitary element* u *in* N *such that* $uB_0u^* \subset B$, *and thus* $u(B_0' \cap N)u^* = B$.

Proof. By Corollary 1.2, if we assume by contradiction that $B_0' \cap N$ and B are not unitary conjugate, then by Theorem 1.1, $0 \in K_{\mathcal{U}_0}(e_B) \subset N_1 = \langle M, B \rangle$. This implies that given any finite projection $f \in N_1$ and any $\varepsilon > 0$, there exists a unitary element $u \in \mathcal{U}_0$ such that $Tr(fue_Bu^*) < \varepsilon$. Indeed, because if for some $\varepsilon_0 > 0$ we would have $Tr(fue_Bu^*) \geq \varepsilon_0$ then by taking appropriate convex combinations and weak limits, we would get that $0 = Tr(f0) \geq \varepsilon_0$, a contradiction.

Let $\Phi_n : N \to N$ be completely positive, unital trace preserving, B-bimodular maps tending strongly to the identity.

By the rigidity of the embedding $B_0 \subset N$, there exists n large enough such that $\Phi = \Phi_n$ satisfies

$$\|\Phi(v) - v\|_2 \leq 1/4, \forall v \in \mathcal{U}(B_0) \tag{1}$$

On the other hand, since T_Φ is in the compact ideal space of N_1, it follows that there exists a finite projection $f \in N_1$ such that

$$\|T_\Phi(1-f)\|_2 < 1/4 \tag{2}$$

Let then $u \in \mathcal{U}_0$ satisfy the condition

$$Tr(fue_Bu^*) < 1/4 \tag{3}$$

Let $\{m_j\}_j \subset L^2(N,\tau)$ be such that $m_j e_B m_j^* = f$. Equivalently, $\oplus_j L^2(m_jB) = fL^2(N,\tau)$. Thus, if $x \in N \subset L^2(N,\tau)$ then $f(x) = \Sigma_j m_j E_B(m_j^* x)$ and $\|f(x)\|_2^2 = \Sigma_j \|m_j E_B(m_j^* x)\|_2^2$.

It follows that we have:

$$Tr(fue_Bu^*) = Tr((\Sigma_j m_j e_B m_j^*)ue_Bu^*) = \Sigma_j \tau(m_j E_B(m_j^* u)u^*)$$

$$= \Sigma_j \|m_j E_B(m_j^* x)\|_2^2 = \|f(u)\|_2^2$$

Taking this into account, by (2) and (3) we have:

$$\|T_\Phi(u)\|_2 \leq \|T_\Phi((1-f)(u))\|_2 + \|f(u)\|_2$$

$$< 3/4 \tag{4}$$

But by (1), this implies:

$$\|u\|_2 \leq \|T_\Phi(u)\|_2 + \|\Phi(u) - u\|_2$$

$$< 3/4 + 1/4 = 1.$$

Thus $1 = \tau(uu^*) < 1$, a contradiction.

We conclude that the semiregular m.a.s.a's $B, B_0' \cap N$ must be conjugate by a unitary element in N. Q.E.D.

4.2. Definition. Let M be a type II_1 factor. An HT *m.a.s.a.* of M is a maximal abelian $*$-subalgebra $A \subset M$ satisfying the conditions:

1°. The normalizer of A in M acts ergodically on A, i.e., A is a semiregular m.a.s.a. in M.

2°. M has the property H relative to A.

3°. A has a von Neumann subalgebra $A_0 \subset A$ such that $A_0' \cap M = A$ and such that $A_0 \subset M$ is a rigid embedding (equivalently, the pair (M, A_0) has the property T).

If a type II_1 factor M has a HT m.a.s.a then we say that M is a HT *factor*. We denote by $\mathcal{HT}$ the class of HT factors. If $M \in \mathcal{HT}$ and $A \subset M$ is a HT m.a.s.a. of M then we say that $A \subset M$ is a HT *inclusion*, or that (M, A) is a HT *pair*.

We will often consider the case when A is a Cartan subalgebra of N (i.e., when the normalizer of A in N generates all N, $\mathcal{N}(A)'' = N$), in which case we call A a HT *Cartan subalgebra*.

The next 2 Corollaries are immediate consequences of the Theorem 4.1:

4.3. Corollary. 1°. *Any two* HT *m.a.s.a.'s of a type* II_1 *factor are conjugate by a unitary element of that factor.*

2°. *If* $(A_1 \subset M_1), (A_2 \subset M_2)$ *are* HT *inclusions and* θ *is an isomorphism from* M_1 *onto* M_2 *then there exists a unitary element* $u \in M_2$ *such that* $u\theta(A_1)u^* = A_2$.

4.4. Corollary. *Let* $A \subset M$ *be a* HT *inclusion. Any automorphism of* M *can be perturbed by an inner automorphism to an automorphism that leaves* A *invariant, i.e.,*

$$\mathrm{Aut}M/\mathrm{Int}M = \mathrm{Aut}(M, A)/\mathrm{Int}(M, A).$$

The next two results give some basic properties of the HT inclusions, and thus of the $\mathcal{HT}$ class of factors.

4.5. Theorem. 1°. *If* $M \in \mathcal{HT}$ *and* $t > 0$ *then* $M^t \in \mathcal{HT}$.

2°. *If* $M_1, M_2 \in \mathcal{HT}$ *then* $M_1 \otimes M_2 \in \mathcal{HT}$. *More precisely, if* $(A_i \subset M_i), i = 1, 2,$ *are* HT *inclusions then* $(A_1 \otimes A_2 \subset M_1 \otimes M_2)$ *is a* HT *inclusion.*

Proof. 1° By 2.6 (respectively 3.7) it follows that if $A \subset M$ has the relative property H (resp. is a rigid embedding) and D denotes the diagonal of $M_0 = M_{n\times n}(\mathbb{C})$ then $A \otimes D \subset M_n(M)$ has the property H (resp., is rigid). Then by 2.6 (resp. 3.7) again, if we cut this latter inclusion by a projection in $A \otimes D$ of trace $\tau(p) = t/n$

then we get that M^t has both property H and T with respect to an appropriate "amplification" of A by t. Thus, M^t is a HT factor.

2°. This follows by applying 2.6 and 3.7 again. Q.E.D.

4.6. Theorem. 1°. *Let $N \subset M$ be an inclusion of type* II_1 *factors of finite Jones index. Then N is* HT *if and only if M is* HT.

2°. *Let $N \subset M$ be an irreducible inclusion of finite index. Assume both N, M are* HT *factors having* HT *Cartan subalgebras. Then* $[M : N] \in \mathbb{N}$.

Proof. 1°. By taking into account that the basic construction $N \subset M \subset M_1$ has the property that M_1 is an amplification of N, it follows that it is sufficient to prove that if $N \in \mathcal{HT}$ then $M \in \mathcal{HT}$. Let $A \subset N$ be a HT m.a.s.a. and $A_0 \subset A$ be a von Neumann subalgebra such that $A_0 \subset N$ is a rigid embedding and $A_0' \cap N = A$. Then the inclusion $A \subset A' \cap M$ has index $\leq [M : N]$ by commuting squares and it also has a finite orthonormal basis (see e.g., [Po7]). Thus, by Sections 2 and 3 it follows that $A' \cap M \subset M$ has the relative property H and is rigid. Moreover, by (Section 1 in [Po7]), there exists a projection $p \in A' \cap M$ such that $Ap = p(A' \cap M)p$. Thus, Ap follows a m.a.s.a. in pMp. Let us show that it is semiregular in pMp.

Let $f \in A$ be the support of p in A. Note that Af is semiregular in fNf. For each $u \in \mathcal{N}(Af)$ let σ_u be the automorphism of Af implemented by Adu. It follows that $sigma_u(a)u = ua, \forall a \in Af$, so that $\sigma_u(a)pup = pupa, \forall a \in Af$. It follows that the $(pu^*pup)^{1/2} \in Ap$ (because Ap is maximal abelian) and that the partial isometry u_p in the polar decomposition of pup is in the normalizing groupoid $\mathcal{G}_p$ of Ap in pMp. Thus, $(Ap \cup \mathcal{G}_p)''$ contains the linear $p(\mathrm{sp}\mathcal{N})p$. Thus, if there exists a projection $q \in (\mathcal{N}_{pMp}(Ap))' \cap pMp$ then q commutes with $p(\mathrm{sp}\mathcal{N})p$.

Let N_0 denote the von Neumann algebra generated the normalizer of A in N and N_1 denote the von Neumann algebra generated by p and N_0 in M. Since the $*$-algebra $\mathrm{sp}\mathcal{N}(A)$ is weakly dense in N_0 it follows from the above that the von Neumann algebra generated by Ap and $\mathcal{G}_p$ contains pN_0p and the von Neumann algebra it generates, thus it contains pN_1p, thus it is equal to pN_1p.

It follows that q commutes with pN_1p. By (1.2 in [Po]) it follows that there exists a projection q_1 in ${N_1}' \cap M$ such that $q = q_1p$. But ${N_1}' \cap M$ is contained in ${N_0}' \cap M$, which is a finite dimensional von Neumann algebra (because any projection in ${N_0}' \cap M$ expects on N as a scalar). Let q_0 be a minimal projection in ${N_1}' \cap M$ under q_1 and denote $e = q_0p$. Then $e \in Ap$ (the latter being maximal abelian in pMp) and the normalizing groupoid of Ae in eMe generates a von Neumann algebra equal to eN_1e, which since q_0 was minimal in ${N_1}' \cap M$ follows a factor.

Altogether, this shows that Ae is a semiregular m.a.s.a. in eMe. Also, by (Section 1 in [Po7]) it follows that there exists a finite dimensional subalgebra A_1 is Ae such that if we denote $A_{01} = A_0e \vee A_1$ then $A_{01}' \cap eMe \subset Ae$. But by Section

3, A_0e is rigid in eMe, and by applying again Section 3, it follows that A_{01} is rigid in eMe as well.

2°. Let $A \subset N$ be a HT Cartan subalgebra of N. Since the normalizer $\mathcal{N}_N(A)$ leaves $A' \cap M$ invariant and $\mathcal{N}_N(A)' \cap M = \mathbb{C}$, it follows by (Section 1 in [Po7]) that the center of $A' \cap M$ has uniform number of points above each point of A and that $A' \cap M$ is homogeneous of type I_n. Thus, there exists a finite orthonormal basis $\{m_j\}_j$ of $A' \cap M$ over $A' \cap N = A$ such that $E_A(m_j^* m_j) = 1, \forall j$. By commuting squares, $E_N(m_j^* m_j) = 1$ as well. Thus, N has integer index in the factor $N_0 = N \vee A' \cap M$.

Moreover, like in the proof of part 1°, there exists a projection $e \in A' \cap M$ such that Ae is maximal abelian in eMe and Ae is a semiregular HT m.a.s.a. in eMe. By Theorem 4.1, it follows that Ae is conjugate to the HT Cartan subalgebra of eMe (which exists by hypothesis). Thus, eN_oe must have integer index in eMe because it contains a Cartan subalgebra of eMe (cf. [Po8]). Q.E.D.

4.7. Corollary. *If $M \in \mathcal{HT}$ has* HT *Cartan subalgebras and $N \subset M$ is a subfactor with finite Jones index then:*

1°. $[M : N] \leq 4$ *implies* $[M : N] \in \{1, 2, 3, 4\}$.

2°. *For each irreducible M-bimodule $\mathcal{H}$ appearing in some $L^2(M_n), n \geq 0$, we have* $\dim({}_M\mathcal{H}_M) \in \mathbb{N}$.

5. Exemples of HT factors.

In this section we provide several examples of HT factors. All these factors have HT Cartan subalgebras, being obtained via the group measure space construction using cross products by certain special groups. We begin with the simplest example:

5.1. Exemple. Let (X_0, μ_0) be the 2-dimensional thorus, regarded as the dual group of $\mathbb{Z}^2$, endowed with the Haar measure μ_0. Note that μ_0 is the same as the Lebesgue measure on $X_0 = \mathbb{T}^2$. Let σ_0 be the action of $SL(2, \mathbb{Z})$ on X_0 implemented by the action of $SL(2, \mathbb{Z})$ on $\mathbb{Z}^2$. Let (X_1, μ_1) be a probability space with a measure preserving ergodic transformation σ_1 of $SL(2, \mathbb{Z})$ on it. Let $\sigma = \sigma_0 \times \sigma_1$ be the product action on the probability space $(X, \mu) = (X_0 \times X_1, \mu_0 \times \mu_1)$. Denote $A = L^\infty(X, \mu)$ and $M = A \rtimes_\sigma SL(2, \mathbb{Z})$. Also, denote $A_0 = L^\infty(X_0, \mu_0)$ and $M_0 = A_0 \rtimes_{\sigma_0} SL(2, \mathbb{Z})$.

Note that we can regard A_0 as a subalgebra A, in which case the canonical unitaries $u_g, g \in SL(2, \mathbb{Z}) \subset M$ implementing the action σ on A also implement the action σ_0 on A_0. Thus, M_0 can be viewed as a subfactor of M. Moreover, we can view A_0 as $L(\mathbb{Z}^2)$, in which case M_0 is identified with $L(\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z}))$.

Given any arbitrary ergodic action σ_1 as in 5.1, the action σ and the algebras defined above have the following properties:

5.2. Proposition. 1°. *The action σ is a free, ergodic action of $SL(2,\mathbb{Z})$ on the probability space (X,μ).*

2°. *The action σ_0 is strongly ergodic. The action $\sigma = \sigma_0 \times \sigma_1$ is strongly ergodic if and only if σ_1 is strongly ergodic.*

3°. *There exist ergodic actions σ_1 of $SL(2,\mathbb{Z})$ that are not strongly ergodic.*

4°. *M_0 is a non-Γ type II_1 factor and $A_0 \subset M_0$ is a Cartan subalgebra of M_0. More generally, M is a type II_1 factor, $A \subset M$ is a Cartan subalgebra in M and M is non-Γ if and only if σ_1 is strongly ergodic.*

5°. *When regarded as a subalgebra in M, A_0 satisfies $A_0' \cap M = A$.*

6°. *A_0 is rigid in M and M has the property* H *relative to A. Thus, M is a* HT *factor and $A \subset M$ is a* HT *Cartan subalgebra in M.*

Proof. 1°. It is well known that a non-inner automorphism of a group G implements a properly outer automorphism on $L(G)$ that preserves the canonical trace of $L(G)$. Thus, since each non-trivial element g in the group $SL(2,\mathbb{Z})$ implements a non-inner automorphism of $\mathbb{Z}^2$ it follows that $\sigma_0(g)$ is properly outer $\forall g \in SL(2,\mathbb{Z}), g \neq e$.

Since the tensor product of any properly outer automorphism with an arbitrary automorphism is properly outer, it follows that $\sigma(g)$ is properly outer $\forall g \in SL(2,\mathbb{Z}), g \neq e$, as well.

Furthermore, the action σ_0 is well known to be mixing. More precisely, it is easy to see that given any finite set $F \subset SL(2,\mathbb{Z}), e \notin F$, there exists $g \in SL(2,\mathbb{Z})$ such that $gF \cap F = \emptyset$. But this implies not only that σ_0 is ergodic, but also that its tensor product with any ergodic action is still ergodic.

2°. The first part is a well known result of Klaus Schmidt ([S1]). The second part is an immediate consequence of the proof of this result in ([S1]).

3°. This is a consequence of a theorem of Connes and Weiss ([CW]), showing that any discrete group G_1 which doesn't have the property T has a free, ergodic but not strongly ergodic action σ_1 on a probability space. Thus, one simply applies this result to $G_1 = SL(2,\mathbb{Z})$, which doesn't have the property T.

One can in fact avoid using the general result in ([CW]), by noticing that since $SL(2,\mathbb{Z})$ has an infinite amenable group H as a quotient (see e.g., [dHV]), any ergodic action of H on a non-atomic probability space (e.g., a Bernoulli shift action of H) composed with the quotient map $G_1 \to H$ gives an ergodic but not strongly ergodic action of $SL(2,\mathbb{Z})$ (note that the resulting action of $SL(2,\mathbb{Z})$ is not free though, in fact not even faithful, but freeness is not necessary in the construction 1.1).

4°. Since $SL(2,\mathbb{Z})$ is close to be a free group (see e.g., [dHV]), it is easy to see that any central sequences in a type II_1 factor of the form $B \rtimes_\sigma SL(2,\mathbb{Z})$ obtained

as the cross product of a finite von Neumann algebra (B,τ) by a free, τ-preserving action σ of $SL(2,\mathbb{Z})$ on it must be supported on B.

5°. Note that if $b=\Sigma_g a_g u_g \in M = A\rtimes_\sigma SL(2,\mathbb{Z})$ commutes with all $a\in A_0$ and $a_g\neq 0$ for some $g\neq e$ then $aa_gu_g = a_gu_ga, \forall a\in A_0$, implying that $a_ga = a_g\sigma_g(a), \forall a\in A_0$. This in turn contradicts the fact that σ_g is properly outer on A_0. Thus, all a_g with $g\neq 0$ must be equal to 0, implying that b lies in A.

6°. By ([Kaz]), $\mathbb{Z}^2\subset\mathbb{Z}^2\rtimes SL(2,\mathbb{Z})$ is a rigid inclusion of groups. Equivalently, $A_0\subset M_0$ is rigid, so even more so $A_0\subset M$ is rigid. Moreover, since $A\subset M$ is a cross-product inclusion by an action of $SL(2,\mathbb{Z})$, which has Haagerup compact approximation property, it follows that M has the property H relative to A. Q.E.D.

5.3. Corollary. *Let M be any of the factors constructed in 5.1, with its corresponding* HT *Cartan subalgebra $A\subset M$. For each $n\geq 2$ let $p_n\in A$ be a projection of trace $\tau(p_n)=1/12(n-1)$. Then $(Ap_n\subset p_nMp_n)$ is isomorphic to $(L^\infty([0,1])\subset L^\infty([0,1])\rtimes\mathbb{F}_n)$, for some free ergodic action of $\mathbb{F}_n$ on $L^\infty([0,1])$, and $Ap_n\subset p_nMp_n$ are* HT *inclusions. The corresponding* HT *factors $M_n = L^\infty([0,1])\rtimes\mathbb{F}_n, n\geq 2$, are stably isomorphic, namely, if $n>m$ and $q_{nm}\in M_m$ satisfies $\tau(q_{nm})=(m-1)/(n-1)$ then $q_{nm}M_mq_{nm}\simeq M_n$.*

Proof. The first part is a consequence of a recent result of G. Hjorth ([Hj]). The second part is then an immediate consequence of 2.5 and 3.8. Q.E.D.

The second class of examples is based on the result of A. Valette ([Va]), mentioned in 3.8, and on ([dCaH], [CowH]), mentioned in 2.5:

5.4. Proposition. *For any $n\geq 2$ there exists a lattice Γ in $SO(n,1)$ (respectively in $SU(n,1)$) for which there exist free, ergodic, measure-preserving actions σ on the standard probability space (X,μ) such that $A=L^\infty(X,\mu)$ is a* HT *Cartan subalgebra in $M=A\rtimes_\sigma\Gamma$.*

Our final set of examples shows that for some special classes of HT factors (the ones obtained by group measure space construction), taking cross products by certain actions of property H groups, keeps us in the $\mathcal{HT}$ class.

5.5. Proposition. *Let $A_0\subset A_0\rtimes_{\sigma_0}G_0$ be a* HT *inclusion and G_1 a discrete group with the property* H. *Let*

$$A=\overline{\bigotimes_{h\in G_1}}(A_0,\tau_{|A_0})_h$$

and still denote by σ_0 the action of G_0 on this infinite tensor product given by $\sigma_0(g)(\otimes_h a_h)=\otimes_h\sigma_0(g)(a_h)$, $\forall g\in G_0$. Denote by σ_1 the Bernoulli shift action of G_1 on A Noting that σ_0 commutes with σ_1, let σ denote the product action $\sigma_0\times\sigma_1$ of $G=G_0\times G_1$ on A. Then $A\subset A\rtimes_\sigma G$ is a HT *inclusion.*

Proof. Since G_0, G_1 have the property H, it follows that $G = G_0 \times G_1$ has the property H. Also, if $A_{00} \subset A_0$ is so that $A'_{00} \cap M_0 = A_0$ and $A_0 \subset M_0$ is rigid, where $M_0 = A_0 \rtimes_{\sigma_0} G_0$, then by the construction of A and σ_1, we clearly have $A'_{00} \cap M = A$, where $M = A \rtimes_{\sigma} G$. Q.E.D.

6. Betti numbers for HT factors and applications.

6.1. Definition. Let M be a HT factor and $A \subset M$ a HT m.a.s.a. of M. Let $\mathcal{R}_M$ be the countable measurable orbit equivalence relation (called standard equivalence relation in [G1,2]) implemented on A by $\mathcal{N}(A)$ (cf. [FM]). Let $\{\beta_n(\mathcal{R}_M)\}_{n\geq 0}$ be the ℓ^2 Betti numbers of $\mathcal{R}_M$, as defined by D. Gaboriau in ([G2]). We denote $\beta_n(M) \stackrel{\text{def}}{=} \beta_n(\mathcal{R}_M), n = 0, 1, 2, ...$ and call them the *Betti numbers of* M. Note right away that by ([G2]), $\beta_0(M)$ is always equal to zero.

From the results in Section 4 and the properties proved by Gaboriau for the Betti numbers of the standard equivalence relations, one immediately gets:

6.2. Corollary. . 1°. *If $A \subset A \rtimes_{\sigma} G$ is a* HT *inclusion, for some countable discrete group G acting freely and ergodically on $A \simeq L^{\infty}(X, \mu)$, then $\beta_n(M)$ is equal to the n'th ℓ^2 Betti number of G, as defined in* [ChG]).

2°. *If $M \in \mathcal{HT}$ and $t > 0$ then $\beta(M^t) = \beta(M)/t, \forall n$.*

3°. *If $M_1, M_2 \in \mathcal{HT}$ then for each $n \geq 0$ we have:*

$$\beta_n(M_1 \otimes M_2) = \sum_{i+j=n} \beta_i(M_1)\beta_j(M_2)$$

Proof. 1°. By 6.1, we have $\beta_n(M) = \beta_n(\mathcal{R}_M)$. But $\mathcal{R}_M = \mathcal{R}_G$, and by Gaboraiu's theorem the latter has Betti numbers $\beta(\mathcal{R}_G)$ equal to the Cheeger-Gromov ℓ^2 Betti numbers $\beta_n(G)$ of the group G.

2°. By Section 4 we know that the class $\mathcal{HT}$ is closed to amplifications and tensor products. Moreover, it is immedaiate to see that the "amplification" of a semiregular m.a.s.a. $A \subset M$ has a normalizer that gives rise to the standard equivalence relation $(\mathcal{R}_M)^t$, as defined in [G2]. Then formula 2° is a consequence of Gaboriau's similar result for standard equivalence relations.

Part 3° follows similarily, by taking into account that if $A_1 \subset M_1, A_2 \subset M_2$ are semiregular m.a.s.a.'s then $\mathcal{N}(A_1 \otimes A_2)'' = (\mathcal{N}(A_1) \otimes \mathcal{N}(A_2))''$ (this is a trivial consequence of 1.2 in [JPo], for instance). Q.E.D.

6.3. Corollary. 1°. *If $M \in \mathcal{HT}$ has at least one non-zero Betti number, then it has trivial fundamental group, $\mathcal{F}(M) = \{1\}$.*

2°. *If $M \in \mathcal{HT}$ has at least one non-zero Betti number then $M^{t_1} \otimes ... \otimes M^{t_n}$ is isomorphic to $M^{s_1} \otimes ... \otimes M^{s_m}$ if and only if $n = m$ and $t_1...t_n = s_1...s_m$.*

Proof. Part 1° is trivial by the formula $\beta_n(M^t) = \beta(M)/t$. Part 2° follows from the fact that if $\beta_{n_i}(M_i)$ is the first non-zero Betti number for $M_i, i = 1, 2$, (which thus necessarily satisfies $n_i \geq 1$), then $\beta_n(M_1 \otimes M_2) = 0$ if $n < n_1 + n_2$. Q.E.D.

6.4. Corollary. *The factors M constructed in 5.1 have the following properties:*

1°. *For each $n \geq 2$ let $p_n \in A$ be a projection of trace $\tau(p_n) = 1/12(n-1)$. Then $(Ap_n \subset p_nMp_n) \simeq (L^\infty([0,1]) \subset L^\infty([0,1]) \rtimes \mathbb{F}_n)$, for some free ergodic action of $\mathbb{F}_n$ on $L^\infty([0,1])$. The corresponding factors $M_n = L^\infty([0,1]) \rtimes \mathbb{F}_n, n \geq 2$, are mutually non-isomorphic, but stably isomorphic, namely, if $n > m$ and $q_{nm} \in M_m$ satisfies $\tau(q_{nm}) = (m-1)/(n-1)$ then $q_{nm}M_mq_{nm} \simeq M_n$.*

2°. *Two tensor products $M^{t_1} \otimes ... \otimes M^{t_n}$ and $M^{s_1} \otimes ... \otimes M^{s_m}$ are isomorphic if and only if $n = m$ and $t_1t_2...t_n = s_1s_2...s_m$. In particular, $\mathcal{F}(M) = \{1\}$.*

3°. *All subfactors of finite index of M, as well as of the tensor products in 2° (in particular, of the factors M_n in 1°), are extremal and have integer index. Moreover, given any $n \in \mathbb{N}$ there exist irreducible subfactors $N \subset M$, $N_0 \subset M$ such that $[M : N] = [M : N_0] = m$ and such that $\beta_n(N) = m\beta_n(M)$ while $\beta_n(N_0) = \beta_n(M)$, for all $n \geq 0$.*

4°. *The factor M cannot be embedded into a free group factor. Also, M does not contain subfactors $L(G)$ with G an infinite property T group.*

5°. *The action σ_1 in the construction of M can be taken either strongly ergodic (i.e., without asymptotically invariant sequences), in which case M follows non-Γ, or non - strongly ergodic, in which case M has the property Γ. Thus, there exist property Γ factors with trivial fundamental group.*

Proof. 1°, 2°, the first part of 3° and 5° have already been proved.

For the last part of 3°, to construct N just take a subgroup of index m in $\mathbb{F}_{fin}$, which exists for any $m \in \mathbb{N}$. To obtain N_0 take the action σ_1 on (X_1, μ_1) so that to leave invariant a quotient (Y_1, λ_1) of X that has exactly m points in each fiber.

The first part of 4° has the same proof as the similar result in ([CJ1]), as for instance done in ([Po1]). The second part in 4° has a similar proof to the proof of 4.1 (except easier, in terms of proving a lemma playing the role of 1.1). Q.E.D.

The examples 5.4, 5.5 give additional, distinct families of type II_1 factors with trivial fundamental group:

6.5. Corollary. 1°. *Let A, Γ, σ be like in* Proposition 5.4, *with Γ a lattice in $SO(2n,1)$ or $SU(n,1)$. Let $M_n = A \rtimes_\sigma \Gamma$. Then $\mathcal{F}(M_n) = \{1\}$, and in fact each M_n satisfies the same property as in* 6.4.2°, *$\forall n$. Moreover, the factors M_n are mutually non-stably isomorphic, for different n's.*

2°. *Given any group G_0 which is either tree-able in the sense of ([Ga1]) (e.g. $G_0 = SL(2,\mathbb{Z})$ or $G_0 = \mathbb{F}_n$ for some $n \geq 2$), or is equal to some lattice Γ like*

in 5.4, *and any group* G_1 *with the property* H, *there exists a* HT *factor* M *with* $\beta_n(M) = \beta_n(G_0 \times G_1), \forall n \geq 0$.

Proof. By 6.2 and ([G2], [ChG]), each M_n has exactly one non-zero beta number $\beta_{k_n}(M_n) \neq 0$ and $k_n \neq k_m$ if $n \neq m$. This proves 1°.

Part 2° is a consequence of 5.5. Q.E.D.

6.6. Remarks. 1°. Note that the above Corollaries 6.4 and 6.5 solve Problem 3 from Kadison's Baton Rouge list, providing lots of examples of factors M with the property that the algebra of n by n matrices over M is not isomorphic to M, for any $n \geq 2$.

2°. In a recent conference at MSRI ([C6]). Connes addressed the question of contsructiong Betti number invariants for typer II_1 factors, building on the same conceptual grounds as in the case of measurable foliations ([C4]), dicrete groups ([ChGr]), and ultimately equivalence relations ([G2, 3]). Although our results do not quite address the question this way (see remarks $3^\circ - 6^\circ$ below), they bring some light to the problem, pointing out a significant phenomenology which we hope could serve both to obtain further results and as a test case for better definitions of Betti numbers.

3°. It should be noted that, although the notion Betti numbers introduced here is for global factors M, it is in some sense a "relative" notion, in that it only applies for factors having certain special m.a.s.a.'s $A \subset M$ and it reduces to the notion of Betti numbers for $A \subset M$, modulo proving a uniqueness result for such m.a.s.a.'s. We avoided to complicate the notations and opted for "$\beta_n(M)$", thus not indicating the "relative" aspect of this notion, mostly for convenience of notations.

Thus, if it was to indicate "r" for "relative", the fact that we take semiregular m.a.s.a. as the subalgebra "relative to which" we built the theory should also have been indicated (thus making the notation quite complicated).

This is because in exactly the same spirit as the notion introduced here based on HT Cartan subalgebras, one can consider the class of factors of the form $M = R \rtimes_\sigma G_0$, with R the hyperfinite type II_1 factor, G_0 a group with the property H and the actions σ chosen so that $R \subset M$ be a rigid inclusion, as defined in Section 3. Such actions exist for $G = SL(2, \mathbb{Z})$, by ([Cho]), so the class is non-empty. A result similar to Theorem 1.1 can then be proved for the "HT hyperfinite subfactors" R and the same proof as for 4.1 shows that a factor M contains a unique such subfactor R, modulo unitary conjugacy.

4°. The same strategy can of course be played with other invariants than the Betti numbers considered here, and possibly with respect to other type of subalgebras than m.a.s.a.'s (see also the Remark 3° above). In a forthcoming paper, we will develop an invariant for type II_1 factors M obtained from cross products of the

hyperfinite type II_1 factor R by Connes-Stormer Bernoulli shift actions of property T groups G, and use the invariants introduced in ([Po9]) for such inclusions $R \subset M$.

5°. It would be very important to extend the class of factors in the "good class" for which a certain uniqueness result can be proved for some special type of Cartan subalgebras, beyond the HT factors we considered here. Such generalizations can go in many directions: by either extending the class of groups G_0 for which $A \subset A \rtimes_\sigma G_0$ works, for certain σ, or by showing that for the groups G_0 we already considered here (e.g., the free groups) any action σ works.

6°. There is of course an analogy between factors of the form $A \rtimes \mathbb{F}_n$ and the free group factors $L(\mathbb{F}_n)$ that gives room to speculations about the possibility that, like their cross-product counter-parts, the free group factors are non-isomorphic, for different n's. In this respect, there are already many indications from results in Voiculescu's free probability that this would indeed be the case (cf. [V1,2], [Ra], [Dy], [Ge], [Sh], etc.). Note however that such free group factors have no Cartan subalgebras ([V2]), and that if some formula for Betti numbers is to hold true for group von Neumann algebras, as in Connes' problem ([C6]), then according to Voiculescu's formula ([V1]) for the number of generators of the amplifications/compressions of the free group factors (cf. also [Ra], [Dy]), the Betti number β_1 (= number of generators -1) should satisfy a formula of the type $\beta(M^t) = \beta(M)/t^2$!

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