THE LINEAR SAMPLING METHOD FOR ANISOTROPIC MEDIA: PART II.

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ABSTRACT. We reconsider the linear sampling method for solving the inverse scattering problem of determining the support of an anisotropic inhomogeneous medium from a knowledge of the incident and scattered time harmonic acoustic wave at fixed frequency. We extend the results of a previous paper [1] concerning with the same problem to the case where the norm of the matrix that describes the physical properties of the medium is less than one.

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1. Introduction

The *linear sampling method* has been successfully used to solve the inverse scattering problem of determining the support of the scatterer from the knowledge of the far field pattern at a fixed frequency, in the case of both acoustic and electromagnetic waves [1, 2, 3]. The scalar anisotropic case was first considered in [5] and [1] where the uniqueness results and the mathematical theory of the sampling method were developed. The techniques used in the above papers are based on an analysis of a boundary value problem called the *interior transmission problem* and only covers the case where the norm of the matrix A describing the anisotropic behaviour is greater than one. The purpose of this paper is to extend the results of [1] and [5] to the case $\|A\| < 1$.

2. The direct and inverse scattering problems for an anisotropic medium

Let $D \subset \mathbb{R}^3$ be a nonempty, open, connected and bounded set having a C^2-boundary ∂D with unit outward normal ν. Let A be a 3×3 matrix-valued function whose entries a_{jk}, $j = 1, 2, 3$, $k = 1, 2, 3$ are continuously differentiable complex-valued functions in $\overline{D}$ such that A is symmetric and satisfies $\bar{\xi} \cdot \mathcal{I}m(A)\, \xi \leq 0$ and $\bar{\xi} \cdot \mathcal{R}e(A)\, \xi \geq \gamma |\xi|^2$ for all $\xi \in \mathbb{C}^3$ and $x \in \overline{D}$ where γ is a positive constant. Note that due to the symmetry of A, $\mathcal{I}m\left(\bar{\xi} \cdot A\, \xi\right) = \bar{\xi} \cdot \mathcal{I}m(A)\, \xi$ and $\mathcal{R}e\left(\bar{\xi} \cdot A\, \xi\right) = \bar{\xi} \cdot \mathcal{R}e(A)\, \xi$. For a function $u \in C^1(\overline{D})$ we define the conormal derivative by

$$\frac{\partial u}{\partial \nu_A}(x) := \nu(x) \cdot A(x) \nabla u(x), \qquad x \in \partial D.$$

We can now formulate the direct scattering problem for an anisotropic medium. In particular, let $k > 0$ be the wave number and $n \in C(\overline{D})$ such that $\mathcal{I}m\, n \geq 0$. Then letting H^k denote the usual

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Sobolev space we want to find functions $w \in H^1(D)$, $u \in H^1(\mathbb{R}^3 \setminus \overline{D})$ such that

$$
\begin{array}{llll}
 & (i) & \operatorname{div} A\nabla w + k^2 n\, w = 0 \quad \text{in} \quad D \\
 & (ii) & \Delta u + k^2\, u = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{D} \\
(1) & (iii) & w - u = f \quad \text{on} \quad \partial D \\
 & (iv) & \dfrac{\partial w}{\partial \nu_A} - \dfrac{\partial u}{\partial \nu} = h \quad \text{on} \quad \partial D \\
 & (v) & \lim\limits_{r\to\infty} r\left(\dfrac{\partial u}{\partial r} - iku\right) = 0
\end{array}
$$

where $f := e^{ikx\cdot d}$ and $h := \frac{\partial}{\partial \nu} e^{ikx\cdot d}$, $d \in \Omega := \{x : |x| = 1\}$, $r = |x|$, the boundary conditions are assumed in the sense of the trace operator, and the radiation condition $(1v)$ holds uniformly with respect to $\hat{x} = x/|x|$.
More generally we consider (1) with $f \in H^{\frac{1}{2}}(\partial D)$ and $h \in H^{-\frac{1}{2}}(\partial D)$ arbitrary and in the sequel will refer to this more general problem as the *transmission problem* (TP). The existence of a unique solution to (TP) has been established by Hähner in [5]. Moreover, he has proved that this solution depends continuously on the boundary data in the sense that the following estimate holds

$$
(2) \qquad \|w\|_{H^1(D)} + \|u\|_{H^1(B\setminus\overline{D})} \le C\left(\|f\|_{H^{\frac{1}{2}}(\partial D)} + \|h\|_{H^{-\frac{1}{2}}(\partial D)}\right)
$$

where B is a ball containing D and $C = C(B)$ is a positive constant.
Since u satisfies the radiation condition $(1v)$ we can conclude (see [3]) that u has the asymptotic behavior

$$
u(x) = \frac{e^{ikr}}{r} u_\infty(\hat{x}, d) + O\left(\frac{1}{r^2}\right)
$$

where $u_\infty(\hat{x}, d)$ is the *far field pattern* of the scattered field u. We recall the definition of a *Herglotz wave function*, which is a solution v_g to the Helmholtz equation in $\mathbb{R}^3$ of the form

$$
v_g(x) = \int_\Omega e^{ikx\cdot d} g(d)\, ds(d), \qquad x \in \mathbb{R}^3.
$$

The inverse scattering problem we are concerned with is to determine D from a knowledge of $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in \Omega$. Our approach for solving this inverse scattering problem is the *linear sampling method* as described in [1]. In particular, we pick a parameter $y \in \mathbb{R}^3$ and then look for a solution $g \in L^2(\Omega)$ of the *far field equation*

$$
(3) \qquad (Fg)(\hat{x}) := \int_\Omega u_\infty(\hat{x}, d) g(d)\, ds(d) = \Phi_\infty(\hat{x}, y)
$$

where $\Phi_\infty(\hat{x}, y) := e^{-ik\hat{x}\cdot y}$ is the far field pattern of the fundamental solution to the Helmholtz equation $\Phi(x, y) := \frac{e^{ik|x-y|}}{|x-y|}$ and $F : L^2(\Omega) \longrightarrow L^2(\Omega)$ is called the *far field operator*. The linear sampling method is based on an examination of the *interior transmission problem* associated to (TP) (see [1]), which in the sequel will be referred to as (ITP). The problem (ITP) is: given $f \in H^{\frac{1}{2}}(\partial D)$

and $h \in H^{-\frac{1}{2}}(\partial D)$ find two functions $w \in H^1(D)$ and $v \in H^1(D)$ satisfying

$$\begin{aligned} &(i) && \operatorname{div} A\nabla w + k^2 n\, w = 0 \quad \text{in} \quad D \\ &(ii) && \Delta u + k^2\, u = 0 \quad \text{in} \quad D \\ &(iii) && w - u = f \quad \text{on} \quad \partial D \\ &(iv) && \frac{\partial w}{\partial \nu_A} - \frac{\partial u}{\partial \nu} = h \quad \text{on} \quad \partial D. \end{aligned} \tag{4}$$

The questions of the existence and uniqueness of (ITP) as well as a-priori estimate for the solution are already considered in [1] where partial results are obtained under an additional assumption on the matrix A. For readers convenience we formulate these results in the following theorem.

Theorem 2.1. *Assume that either $\mathcal{I}m\, n > 0$ or $\mathcal{I}m\left(\bar{\xi} \cdot A\,\xi\right) < 0$ in a neighborhood B_{x_0} of a point $x_0 \in D$ and that there exists a constant $\gamma > 1$ such that for almost every $x \in D$,*

$$\mathcal{R}e\left(\bar{\xi} \cdot A(x)\,\xi\right) \geq \gamma|\xi|^2 \quad \forall\, \xi \in \mathbb{C}^3. \tag{5}$$

Then (ITP) *has a unique solution $(w, v) \in H^1(D) \times H^1(D)$. This solution satisfies the a priori estimate*

$$\|w\|_{H^1(D)} + \|v\|_{H^1(D)} \leq C\left(\|f\|_{H^{\frac{1}{2}}(\partial D)} + \|h\|_{H^{-\frac{1}{2}}(\partial D)}\right), \tag{6}$$

where the constant C is independent of f and h.

In the case when the assertion of Theorem 2.1 holds one can prove by using the technique of Hähner in [5] that for all incident directions d and all observation directions $\hat{x}$ $u_\infty(\hat{x}, d)$ uniquely determines the support D (we remind the reader that A is *not* uniquely determined by u_∞ [4]). Moreover in this case the regularized solution to the far field equation (3) leads to a simple reconstruction algorithm for determining the support D of the inhomogeneity from the far field pattern. In particular in [1] it is proved that

1. if $y \in D$ then for every $\epsilon > 0$ there exists a solution $g^\epsilon(\cdot, y) \in L^2(\Omega)$ of the inequality

$$\|Fg_\epsilon(\cdot, y) - \Phi_\infty(\cdot, y)\|_{L^2(\Omega)} < \epsilon$$

   such that

$$\lim_{y \to \partial D} \|g_\epsilon(\cdot, y)\|_{L^2(\Omega)} = \infty \quad \text{and} \quad \lim_{y \to \partial D} \|v_{g_\epsilon}(\cdot, y)\|_{H^1(D)} = \infty,$$

   where v_{g_ϵ} is the Herglotz wave function with kernel g_ϵ, and

2. if $y \in \mathbb{R}^2 \setminus \overline{D}$ then for every $\epsilon > 0$ and $\delta > 0$ there exists a solution $g_{\epsilon,\delta}(\cdot, y) \in L^2(\Omega)$ of the inequality

$$\|Fg_{\epsilon,\delta}(\cdot, y) - \Phi_\infty(\cdot, y)\|_{L^2(\Omega)} < \epsilon + \delta$$

   such that

$$\lim_{\delta \to 0} \|g_{\epsilon,\delta}(\cdot, y)\|_{L^2(\Omega)} = \infty \quad \text{and} \quad \lim_{\delta \to 0} \|v_{g_{\epsilon,\delta}}(\cdot, y)\|_{H^1(D)} = \infty,$$

   where $v_{g_{\epsilon,\delta}}$ is the Herglotz wave function with kernel $g_{\epsilon,\delta}$.

One can now use regularization methods to solve the far field equation $Fg = \Phi_\infty(\cdot, y)$ for y on an appropriate grid containing D. In this way an approximation to $g_{(\cdot,y)}$ can be obtained and hence ∂D can be determined by those points where $\|g(\cdot, y)\|_{L^2(\Omega)}$ becomes unbounded (c.f. [2]).

3. The interior transmission problem

Theorem 2.1 addresses the case $\|A\| > 1$ and consequently the mathematical theory of the sampling method holds for such matrices A. Our aim here is to complete this result by studying the solvability of (ITP) in the case of $\|A\| < 1$. To this end we formulate the *modified interior transmission problem* (MITP) which later will be seen as a compact perturbation of our original interior transmission problem (ITP): given D and A as above, a positive constant m, functions $\ell_1 \in L^2(D)$, $\ell_2 \in L^2(D)$, $f \in H^{\frac{1}{2}}(\partial D)$, and $h \in H^{-\frac{1}{2}}(\partial D)$, find $w \in H^1(D)$ and $v \in H^1(D)$ satisfying

$$(7) \qquad \left\{ \begin{array}{lll} (i) & \operatorname{div} A\nabla w - m\, w = \ell_1 & \text{in } D \\ (ii) & \Delta v - v = \ell_2 & \text{in } D \\ (iii) & w - v = f & \text{on } \partial D \\ (iv) & \partial_{\nu_A} w - \partial_\nu v = h & \text{on } \partial D. \end{array} \right.$$

Such a pair (w, v) will be referred to as a strong solution to (7). Note that in the following for simplicity we use the notations $\partial_{\nu_A} := \frac{\partial}{\partial \nu_A}$ and $\partial_\nu := \frac{\partial}{\partial \nu}$.

We will now reformulate (7) as a variational problem. Let us introduce the Hilbert space

$$(8) \qquad W(D) = \left\{ \mathbf{w} \in L^2(D)^3 \ : \ \operatorname{div} \mathbf{w} \in L^2(D) \text{ and } \operatorname{curl} A^{-1}\mathbf{w} = 0 \right\}$$

equipped with the natural norm $\|\mathbf{w}\|_W = \left(\|\mathbf{w}\|^2_{L^2} + \|\operatorname{div} \mathbf{w}\|^2_{L^2} \right)^{\frac{1}{2}}$ and denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $H^{\frac{1}{2}}(\partial D)$ and $H^{-\frac{1}{2}}(\partial D)$. We recall the duality identity

$$(9) \qquad \langle \varphi, \boldsymbol{\psi} \cdot \nu \rangle = \int_D \varphi \operatorname{div} \boldsymbol{\psi} \, dx + \int_D \nabla \varphi \cdot \boldsymbol{\psi} \, dx$$

for $(\varphi, \boldsymbol{\psi}) \in H^1(D) \times W(D)$ that will be of particular interest in the sequel.
We now introduce the sesquilinear form $\mathcal{A}$ defined on $\{H^1(D) \times W(D)\}^2$ by

$$(10) \qquad \begin{array}{rcl} \mathcal{A}(U, V) & = & \displaystyle\int_D \nabla v \cdot \nabla \bar{\varphi} \, dx + \int_D v \, \bar{\varphi} \, dx + \int_D \operatorname{div} \mathbf{w} \operatorname{div} \bar{\boldsymbol{\psi}} \, dx \, dx \\ & + & m \displaystyle\int_D A^{-1}\mathbf{w} \cdot \bar{\boldsymbol{\psi}} - m \, \langle v, \bar{\boldsymbol{\psi}} \cdot \nu \rangle - \langle \bar{\varphi}, \mathbf{w} \cdot \nu \rangle \end{array}$$

where $U = (v, \mathbf{w})$ and $V = (\varphi, \boldsymbol{\psi})$ are in $H^1(D) \times W(D)$. We also introduce for $V = (\varphi, \boldsymbol{\psi}) \in H^1(D) \times W(D)$ the antilinear form

$$(11) \qquad L(V) = \int_D (\ell_1 \, \bar{\varphi} + \ell_2 \operatorname{div} \bar{\boldsymbol{\psi}}) \, dx - \langle \bar{\varphi}, h \rangle + m \, \langle f, \bar{\boldsymbol{\psi}} \cdot \nu \rangle \, .$$

The variational formulation of the problem (7) is

$$(12) \qquad \left\{ \begin{array}{l} \text{Seek } U \in H^1(D) \times W(D) \text{ such that} \\ \mathcal{A}(U, V) = L(V), \quad \forall \, V \in H^1(D) \times W(D), \end{array} \right.$$

and the following theorem proves the equivalence between the two formulations.

Theorem 3.1. *Problem (7) has a unique strong solution (w, v) if and only if problem (12) has a unique solution $U \in H^1(D) \times W(D)$. Moreover if (w, v) is the unique strong solution to (7) then $U = (v, A\nabla w)$ is the unique strong solution to (12). Conversely, if U is the unique solution to (12) then the unique strong solution (w, v) to (7) is such that $U = (v, A\nabla w)$.*

Proof. Let us prove first the equivalence between the existence of a strong solution (w, v) to (7) and the existence of a solution U to (12).

a) $\exists \, (w, v) \Rightarrow \exists \, U$: Let (w, v) be a strong solution to (7). We set $\mathbf{w} = A\nabla w$. From $(7i)$ we see that

$\operatorname{div}\mathbf{w} = w + \ell_1 \in L^2(D)$ then $\mathbf{w} \in W(D)$. Taking the L^2 scalar product of $(7i)$ with $\operatorname{div}\boldsymbol{\psi}$ for some $\boldsymbol{\psi} \in W(D)$ and using (9) shows that

$$\int_D \operatorname{div}\mathbf{w} \operatorname{div}\bar{\boldsymbol{\psi}}\, dx + m\int_D A^{-1}\mathbf{w}\cdot\bar{\boldsymbol{\psi}}\, dx - m\left\langle w,\, \bar{\boldsymbol{\psi}}\cdot\nu\right\rangle = \int_D \ell_1 \operatorname{div}\bar{\boldsymbol{\psi}}\, dx.$$

Hence by $(7iii)$

$$\begin{aligned}\int_D \operatorname{div}\mathbf{w} \operatorname{div}\bar{\boldsymbol{\psi}}\, dx \quad &+ \quad m\int_D A^{-1}\mathbf{w}\cdot\bar{\boldsymbol{\psi}}\, dx - m\left\langle v,\, \bar{\boldsymbol{\psi}}\cdot\nu\right\rangle \\ &= \quad m\left\langle f,\, \bar{\boldsymbol{\psi}}\cdot\nu\right\rangle + \int_D \ell_1 \operatorname{div}\bar{\boldsymbol{\psi}}\, dx.\end{aligned} \tag{13}$$

We now take the L^2 scalar product of $(7ii)$ with φ in $H^1(D)$ and integrate by parts. Using the boundary condition $(7iv)$, this shows that

$$\int_D \nabla v\cdot\nabla\bar{\varphi}\, dx + \int_D v\,\bar{\varphi}\, dx - \left\langle\bar{\varphi},\, \mathbf{w}\cdot\nu\right\rangle = -\left\langle\bar{\varphi},\, h\right\rangle + \int_D \ell_2\,\bar{\varphi}\, dx. \tag{14}$$

Adding (13) and (14) together shows that $U = (v, \mathbf{w})$ is a solution to (12).

b) $\exists\ U \Rightarrow \exists\ (w,v)$: We set $U = (v,\mathbf{w}) \in H^1 \times W(D)$. Since $\operatorname{curl} A^{-1}\mathbf{w} = 0$ and D is simply connected, we deduce the existence of a function $w \in H^1(D)$ such that $\mathbf{w} = A\nabla w$ where w is determined up to an additive constant. As we shall see later, this constant can be adjusted such that (w,v) is a strong solution to (7). Obviously, if U satisfies (12) then $(v,\mathbf{w})$ satisfies (13) and (14) for all $(\varphi,\boldsymbol{\psi}) \in H^1(D) \times W(D)$. One easily see from (14) that the couple (w,v) satisfies

$$\begin{cases} \Delta v - v = \ell_2 & \text{in } D \\ \partial_{\nu_A} w - \partial_\nu v = h & \text{on } \partial D. \end{cases} \tag{15}$$

On the other hand, substituting for $\mathbf{w}$ in (13) and using the duality identity (9) in the second integral shows that

$$\begin{aligned}\textstyle\int_D (\operatorname{div} A\nabla w - mw)\operatorname{div}\bar{\boldsymbol{\psi}}\, dx + m\left\langle w - v,\, \bar{\boldsymbol{\psi}}\cdot\nu\right\rangle \\ = m\left\langle f,\, \bar{\boldsymbol{\psi}}\cdot\nu\right\rangle + \textstyle\int_D \ell_2 \operatorname{div}\bar{\boldsymbol{\psi}}\, dx\end{aligned} \tag{16}$$

for all $\boldsymbol{\psi}$ in $W(D)$. Now consider a function $\phi \in L^2_0(D) = \left\{\phi \in L^2(D) \,:\, \int_D \phi\, dx = 0\right\}$ and let $\chi \in H^1(D)$ be a solution to

$$\Delta\chi = \bar{\phi} \text{ in } D; \quad \partial_\nu\chi = 0 \text{ on } \partial D. \tag{17}$$

Taking $\boldsymbol{\psi} = \nabla\chi$ in (16) ($\Rightarrow \operatorname{div}\bar{\boldsymbol{\psi}} = \phi$ in D and $\bar{\boldsymbol{\psi}}\cdot\nu = 0$ on ∂D) shows that

$$\int_D (\operatorname{div} A\nabla w \,-\, mw - \ell_2)\,\phi\, dx = 0 \quad \forall\, \phi \in L^2_0(D)$$

which implies the existence of a constant c_1 such that

$$\operatorname{div} A\nabla w - mw - \ell_2 = c_1 \quad \text{in } D. \tag{18}$$

We now take $\phi \in L^2_0(\partial D)$ and let $\chi \in H^1(D)$ be the solution to

$$\Delta\chi = 0 \text{ in } D; \quad \partial_\nu\chi = \bar{\phi} \text{ on } \partial D. \tag{19}$$

Taking $\boldsymbol{\psi} = \nabla\chi$ in (13) ($\Rightarrow \operatorname{div}\bar{\boldsymbol{\psi}} = 0$ in D and $\bar{\boldsymbol{\psi}}\cdot\nu = \phi$ on ∂D) shows that

$$m\int_{\partial D} (w \,-\, v \,-\, f)\,\phi\, d\gamma = 0 \quad \forall\, \phi \in L^2_0(\partial D)$$

which implies the existence of a constant c_2 such that

$$w \,-\, v \,-\, f = c_2 \quad \text{on } \partial D. \tag{20}$$

Substituting (18) and (20) into (16) and using (9) now shows that

$$(c_1 - c_2)\int_D \operatorname{div}\bar{\boldsymbol{\psi}}\, dx = 0 \quad \forall\, \boldsymbol{\psi} \in W(D)$$

which implies $c_1 = c_2 = c$ (take $\psi = \nabla\chi$ where $\chi \in H_0^1(D)$ and $\Delta\chi = 1$ in D). Equations (15), (18) and (20) show that $(w - c, v)$ is a strong solution to (7).

We now consider the uniqueness equivalence.

c) Uniqueness of (w, v) $\Rightarrow$ Uniqueness of U: Assume that problem (7) has a unique strong solution and consider two solutions $U_1 = (v_1, \mathbf{w}_1)$ and $U_2 = (v_2, \mathbf{w}_2)$ to (12). From step **b)** we deduce the existence of w_1 and w_2 in $H^1(D)$ such that $\mathbf{w}_1 = A\nabla w_1$ and $\mathbf{w}_2 = A\nabla w_2$ and (w_1, v_1) and (w_2, v_2) are strong solutions to (7). Hence $(w_1, v_1) = (w_2, v_2)$ and $(v_1, \mathbf{w}_1) = (v_2, \mathbf{w}_2)$.

d) Uniqueness of U $\Rightarrow$ Uniqueness of (w, v): Assume that problem (12) has a unique solution and consider two strong solutions (w_1, v_1) and (w_2, v_2) to (7). We deduce from step **a)** that $(v_1, A\nabla w_1)$ and $(v_2, A\nabla w_2)$ are two solutions to (12). Hence $v_1 = v_2$ and $w = w_1 - w_2$ is a function in $H^1(D)$ that satisfies

$$\begin{cases} \operatorname{div} A\nabla w - mw = 0 & \text{in } D \\ w = \partial_{\nu_A} w = 0 & \text{on } \partial D \end{cases}$$

which implies $w = 0$ (since A is coercive). $\square$

Theorem 3.2. *Assume that there exists a constant $\gamma > 1$ such that for $x \in D$,*

$$\mathcal{R}e\left(\bar{\xi} \cdot A^{-1}(x)\,\xi\right) \geq \gamma|\xi|^2 \quad \forall\, \xi \in \mathbb{C}^3 \quad \text{and} \quad \gamma^{-1} \leq m < 1. \tag{21}$$

Then problem (12) has a unique solution $U = (v, \mathbf{w}) \in H^1(D) \times W(D)$. This solution satisfies the a priori estimate

$$\begin{aligned} \|v\|_{H^1(D)} + \|\mathbf{w}\|_W \leq \tfrac{4c}{1-m} \Big(&\|\ell_1\|_{L^2(D)} + \|\ell_2\|_{L^2(D)} \\ &+ \|f\|_{H^{\frac{1}{2}}(\partial D)} + \|h\|_{H^{-\frac{1}{2}}(\partial D)} \Big) \end{aligned} \tag{22}$$

where the constant c is independent of ℓ_1, ℓ_2, f, h and γ.

Proof. Classical trace theorems and Schwarz's inequality insure the continuity of the antilinear form L on $H^1(D) \times W(D)$ and the existence of a constant c independent of ℓ_1, ℓ_2, f and h such that

$$\|L\| \leq c\left(\|\ell_1\|_{L^2} + \|\ell_2\|_{L^2} + \|f\|_{H^{\frac{1}{2}}} + \|h\|_{H^{-\frac{1}{2}}}\right) \tag{23}$$

On the other hand, if $U = (v, \mathbf{w}) \in H^1(D) \times W(D)$ then, by assumption (21),

$$|\mathcal{A}(U, U)| \geq \|v\|_{H^1}^2 + \|\mathbf{w}\|_W^2 + (m\gamma - 1)\,\|\mathbf{w}\|_{L^2}^2 - (1 + m)\mathcal{R}e\,\langle \bar{v}, \mathbf{w}\rangle\,. \tag{24}$$

According to the duality identity (9), one has by Schwarz's inequality that

$$|\langle \bar{v}, \mathbf{w}\rangle| \leq \|v\|_{H^1}\,\|\mathbf{w}\|_W$$

and on the other hand $m\gamma - 1 \geq 0$, therefore

$$|\mathcal{A}(U, U)| \geq \|v\|_{H^1}^2 + \|\mathbf{w}\|_W^2 - (m + 1)\,\|w\|_{H^1}\,\|\mathbf{w}\|_W$$

Using the identity $x^2 + y^2 - (m+1)xy = \frac{m+1}{2}\,(x - y)^2 + \frac{1-m}{2}(x^2 + y^2)$, we conclude that

$$|\mathcal{A}(U, U)| \geq \frac{1 - m}{2}\left(\|\mathbf{w}\|_W^2 + \|v\|_{H^1}^2\right)$$

and thus $\mathcal{A}$ is coercive. The continuity of $\mathcal{A}$ follows easily from Schwarz's inequality and classical trace theorems. Theorem 3.2 is therefore a direct consequence of Lax-Milgram theorem applied to (12). $\square$

Theorem 3.3. *Under the assumptions of Theorem 3.2, problem (7) has a unique strong solution (w, v) that satisfies*

$$\begin{aligned} \|w\|_{H^1(D)} + \|v\|_{H^1(D)} \leq C \Big(&\|\ell_1\|_{L^2(D)} + \|\ell_2\|_{L^2(D)} \\ &+ \|f\|_{H^{\frac{1}{2}}(\partial D)} + \|h\|_{H^{-\frac{1}{2}}(\partial D)} \Big) \end{aligned} \tag{25}$$

where the constant C in independent of ℓ_1, ℓ_2, f and h.

Proof. The existence and uniqueness of a strong solution follows from Theorem 3.1 and Theorem 3.2. The a priori estimate (25) can be obtained directly from (7) but can be also deduced from (22) as follows. Theorem 3.1 tells us that $(v, A\nabla w)$ is the unique solution to (12). Hence, according to (22)

$$\|v\|_{H^1} + \|A\nabla w\|_{L^2} \leq c_1 \left(\|\ell_1\|_{L^2} + \|\ell_2\|_{L^2} + \|f\|_{H^{\frac{1}{2}}} + \|h\|_{H^{-\frac{1}{2}}} \right).$$

But from Poincaré inequality,

$$\|\nabla w\|_{H^1(D)} \leq c_2 \left(\|\nabla w\|_{L^2(D)} + \|w\|_{L^2(\partial D)} \right),$$

and using the boundary condition ($7iii$) and the trace theorem one deduces that

$$\|w\|_{H^1(D)} \leq c_2 \left(\|\nabla w\|_{L^2(D)} + \|v\|_{H^1(D)} + \|f\|_{L^2(\partial D)} \right)$$

for some positive constant c_2. Estimate (25) follows straightforwardly. □

Now we can state the main result of this paper:

Theorem 3.4. *Assume that either $\mathcal{I}m\, n > 0$ or $\mathcal{I}m \left(\bar{\xi} \cdot A\, \xi \right) < 0$ in a neighborhood B_{x_0} of a point $x_0 \in D$ and that there exists a constant $\gamma > 1$ such that for almost every $x \in D$,*

$$\mathcal{R}e \left(\bar{\xi} \cdot A(x)^{-1} \xi \right) \geq \gamma |\xi|^2 \quad \forall\, \xi \in \mathbb{C}^3. \tag{26}$$

Then (ITP) *has a unique solution $(w, v) \in H^1(D) \times H^1(D)$. This solution satisfies the a priori estimate*

$$\|w\|_{H^1(D)} + \|v\|_{H^1(D)} \leq C \left(\|f\|_{H^{\frac{1}{2}}(\partial D)} + \|h\|_{H^{-\frac{1}{2}}(\partial D)} \right), \tag{27}$$

where the constant C is independent of f and h.

Proof. Let us set

$$\mathcal{X}(D) = \left\{ (w, v) \in H^1(D) \times H^1(D) \;:\; \operatorname{div} A\nabla w \in L^2(D) \text{ and } \Delta v \in L^2(D) \right\}$$

and consider the operator $\mathcal{G}$ from $\mathcal{X}(D)$ into $L^2(D) \times L^2(D) \times H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$ defined by

$$\mathcal{G}(w, v) = (\operatorname{div} A\nabla w - mw, \Delta v - v, (w - v)|_{\partial D}, (\partial_{\nu_A} w - \partial_\nu v)|_{\partial D}) \tag{28}$$

where m is a constant satisfying $\gamma^{-1} \leq m < 1$. Theorem 3.3 shows that the inverse of $\mathcal{G}$ exists and is continuous. Since $\mathcal{G}$ is continuous, we deduce that $\mathcal{G}$ is a bijective operator. Now consider the operator $\mathcal{T}$ from $\mathcal{X}(D)$ into $L^2(D) \times L^2(D) \times H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$ defined by

$$\mathcal{T}(w, v) = \left((k^2 n + m) w,\ (k^2 + 1) v,\ 0,\ 0 \right)$$

By the compact embedding of $H^1(D)$ into $L^2(D)$, the operator $\mathcal{T}$ is compact. Hence $\mathcal{G} + \mathcal{T}$ is a Fredholm operator of index one. From Theorem 3.1 of [1] we have that $\mathcal{G} + \mathcal{T}$ is injective and therefore we deduce the existence and the continuity of $(\mathcal{G} + \mathcal{T})^{-1}$, which means in particular the existence of a unique solution to (ITP) satisfying the a priori estimate (27). □

Due to a lack of uniqueness in the case where n and A are real (see Theorem 3.1 of [1]) we cannot conclude the solvability of (ITP). This leads to the problem of transmission eigenvalues, i.e. the values of k for which the corresponding homogeneous (ITP) ($f = h = 0$) has a non trivial solution. For these eigenvalues the linear sampling method fails, and therefore it is important to characterize the set of these critical values. It is shown in [1] that this set is discrete under the assumption $\left(\bar{\xi} \cdot A(x)\, \xi \right) \geq \gamma |\xi|^2$ where $\gamma > 1$. We prove here an analogous result for the case $\left(\bar{\xi} \cdot A(x)^{-1} \xi \right) \geq \gamma |\xi|^2$.

Theorem 3.5. *Assume that $\mathcal{I}m\, n = 0$ and $\mathcal{I}m\, A = 0$ in D and that there exists a constant $\gamma > 1$ such that for almost every $x \in D$,*

$$\left(\bar{\xi} \cdot A(x)^{-1} \xi \right) \geq \gamma |\xi|^2 \quad \forall\, \xi \in \mathbb{R}^3. \tag{29}$$

Then the set of the values of $k \in \mathbb{C}$ for which (ITP) *does not have a unique solution is discrete.*

Proof. Consider the operator $\mathcal{G}$ defined by (28) where $\gamma^{-1} \leq m < 1$ and the operator $\mathcal{T}_k$ from $\mathcal{X}(D)$ into $L^2(D) \times L^2(D) \times H^{\frac{1}{2}}(\partial D) \times H^{-\frac{1}{2}}(\partial D)$ defined by

$$\mathcal{T}_k(w,v) = \big((k^2 n + m)w, (k^2+1)v,\ 0,\ 0\big)\,.$$

We want to prove that the operator $\mathcal{G} + \mathcal{T}_k$ is invertible for all $k \in \mathbb{C} \setminus S$ where S is a discrete subset of $\mathbb{C}$. Since $\mathcal{G}$ is bijective (Theorem 3.3), this is equivalent to showing that $(I + \mathcal{G}^{-1}\,\mathcal{T}_k)^{-1}$ exists, where I is the identity operator from $\mathcal{X}(D)$ into $\mathcal{X}(D)$. Let us defined the operator valued function $A : \mathbb{C} \to \mathcal{L}(\mathcal{X}(D))$ by $A(k) = -\mathcal{G}^{-1}\,\mathcal{T}_k$. Since $\mathcal{T}_k$ is compact and $\mathcal{G}^{-1}$ is bounded, $A(k)$ is then compact for all $k \in \mathbb{C}$. Moreover, since $\mathcal{T}_k$ depends polynomially on k and the function n is bounded, it is easy to see that A is analytic on $\mathbb{C}$.

We also notice that $\mathcal{G} + \mathcal{T}_i$ is injective (this corresponds to the uniqueness of the solution to problem (4) with $k = i$ (similar arguments as in the proof of theorem 3.1 in [1])). Hence the operator $I - A(i) = \mathcal{G}^{-1}(\mathcal{G} + \mathcal{T}_i)$ is injective and therefore invertible. Now applying the analytic Fredholm theory (Theorem 8.26 in [3]) shows that $(I - A(k))^{-1}$ exists for all $k \in \mathbb{C} \setminus S$ where S is a discrete subset of $\mathbb{C}$. □

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