REDUCTION THEORY, ELLIPTIC SOLITONS AND INTEGRABLE SYSTEMS

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Abstract. One goal of this paper is to connect two of the thesis topics of Sophie Kowalevski, namely reduction theory for abelian functions and integrability of a dynamical system. Part I is a digest of classical and current research on splittable Jacobians and curves with automorphisms. Part II is a collection of applications of the splittable situation to the explicit integration of Hamiltonian systems of finite and infinite dimension. Both fields are full of open questions, and the last section in each part is a collection of such.

Introduction

This article will be divided in two parts, each having to do with work of S. Kowalevski's. Kowalevski devoted part of her thesis to the study of abelian surfaces whose function theory reduces to elliptic functions. She also spent time pursuing the question of integrability of a rotating body, which had been proposed by her mentor, K. Weierstrass, but didn't obtain her striking results until later, when they earned her the Bordin prize. In the light of more recent discoveries in soliton equations and completely integrable systems, these two apparently unrelated areas of her work have come in close connection, as was remarked already in [BBME]. We give a survey of classical and recent questions of splittability for abelian varieties and attempt to connect them with related questions arising for differential equations. Indeed, it is an established feature of this area of mathematics that connections prompt unexpected progress. In the theory of the KP equation, the question of reduction becomes that of expressing KP solutions algebraically by means of elliptic functions. The question of characterizing solutions to the KP equation that are elliptic in the first variable is surprisingly connected with an important hamiltonian system. It was solved by Krichever, who defined an algebraically completely integrable hamiltonian system parametrizing them all. This system was identified as a rich algebro-geometric object by Treibich and Verdier, and it was also identified as a special case of a (generalized) Hitchin system by Donagi and Markman. We describe the system and several generalizations of it, along with its salient features, including the question of reduction, where the word has a different meaning than

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in Part I. The KP solution reduces to, e.g., a KdV or Boussinesq solution according to certain flows being stationary, and the question of non-emptiness of these loci is quite difficult.

I am deeply indebted to the organizers of the Kowalevski meeting, for an opportunity of mathematical excitement and also for the moving exposure to details of her life in Russia. It was a special event, and it is the more special for me to have been asked to write a survey of problems that stemmed from her work. I am also thankful to my collaborators, J.C. Eilbeck and V.Z. Enolskii, who asked me to present on their behalf our joint work on splittability of certain genus-3 varieties of elliptic solitons (section II-3), despite being among the participants: it was an honor and a pleasure for me.

Part I: Reduction for abelian integrals

1. Weierstrass, Kowalevski, Poincaré

Sophie Kowalevski spelled her name thus, when she published her paper on reduction of abelian integrals [Ko1]. She found a necessary and sufficient condition for a curve of genus 3 to cover an elliptic curve. This was one of the three results each of which Weierstrass "without any hesitation (...) would have accepted (...) as a doctoral dissertation." [Ke, Appendix C] Part of the depth of the work is by reason of the interplay between algebra and analysis; indeed, the condition can be stated in several different ways, and several variants of the question could be posed. Some of these variants are still actively pursued in genus 3, besides being wide open in higher genus, and the applications of reduction theory to another area of Kowalevski's work, integrable systems, are also the object of current research.

Weierstrass phrased the problem of reduction of abelian integrals as follows [Ke, Appendix C]:

Give a condition that an algebraic function $f(x)$ must satisfy if among the integrals

$$\int F(x, f(x))dx,$$

where F is a rational function of x and $f(x)$, there exist some that can be transformed into elliptic integrals. This means that the curve X whose function field is $\mathbb{C}(x, f(x))$ has a holomorphic differential which is elliptic, or equivalently Jac X contains an elliptic curve. Weierstrass gave the following answer: the required condition is the existence of a standard homology basis for X (namely, cycles $\alpha_1, \dots, \alpha_g,\ \beta_1, \dots, \beta_g$ whose intersection matrix is $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$) such that the corresponding basis of holomorphic differentials has period matrix $\tau_{ij},\ 1 \le i,\ j \le g$, with $\tau_{12} = \frac{m}{n},\ \tau_{13} = \ldots = \tau_{1g} = 0$ for some integers m, n. The algebraic side of this condition is then the determination of the moduli of the curves that satisfy it; L. Könisberger treated the case $g = 2, n = 2$ and Kowalevski the case $g = 3, n = 2$. In her case, the algebraic condition is that the canonical model of the curve, namely a plane smooth quartic, should have bitangents meeting at one point [B, §76].

In order to formulate a more general problem, we briefly recall some terminology; more details and proofs can be found in a standard text like [LB].

Definition. (1) An abelian variety A of dimension g is a complex torus $\mathbb{C}/\Lambda$, where Λ is a lattice of rank $2g$, which is also an algebraic variety. If L is a period matrix for the torus, namely a $g \times 2g$ complex matrix whose columns span Λ, then a principal polarization of A is determined by a skew-symmetric matrix P in $SL(2g, \mathbb{Z})$ such that $LP^tL = 0$ and the hermitian matrix $iLP^t\overline{L}$ is positive definite. The $g \times 2g$ complex matrices arising this way are called Riemann matrices.

(2) The algebraic counterpart of (1) is the following: a principal polarization is an ample line bundle $\mathcal{L}$ over A, such that the homomorphism

$$\phi_{\mathcal{L}} : A \to Pic^0 A \qquad a \mapsto T_a^* \mathcal{L} \otimes \mathcal{L}^{-1}$$

is an isomorphism.

The link between (1) and (2) is given by Riemann's theta function ϑ: if A has a principal polarization, then L can be normalized to $[I\ \Omega]$ with Ω a $g \times g$ symmetric matrix and Im Ω positive definite, so that for $z \in \mathbb{C}^g$

$$\vartheta(z, \Omega) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i^t n \Omega n + 2\pi i^t z n)$$

is an analytic function. $\mathcal{L}$ corresponds to Riemann's theta divisor $\Theta = \{z | \vartheta(z, \Omega) = 0\}$.

(3) Two complex tori A and A' with period matrices L and L' are said to be isomorphic if and only if there are matrices $C \in GL(n, \mathbb{C})$ and $T \in GL(2n, \mathbb{Z})$ such that $CL = L'T$. Notice that T can be taken to be in $SL(2n, \mathbb{Z})$ by changing $C \mapsto C \cdot \det T$. Then, if $\phi : A' \to A$ is the isomorphism, the pull-back of a principal polarization P gives a principal polarization $P' = TP^tT$ for A'.

Remark. By the previous definition then, an abelian variety of dimension one (also called an elliptic curve) has a unique principal polarization, whereas for $g > 1$ it may have none – infinitely many examples of 2-dimensional abelian varieties that do not are given in [E3] – or several, as the following theorem illustrates.

Theorem. [L2, Th. 1.5] *If the abelian variety A has a principal polarization, then the set of isomorphism classes of principal polarizations is in one-to-one correspondence with the set of automorphisms of A, which are symmetric w.r.t. any fixed principal polarization and have positive eigenvalues, modulo the natural action of $AutA$.*

In particular, the set of non-isomorphic principal polarizations is finite, taking care of a question posed by Martens [Mar1].

Before returning to Weierstrass' question let's recall two classical facts:

Theorem. *A Jacobian $A = \mathrm{Jac}(X)$ has a canonical principal polarization given by the injective homomorphism $H_1(X,\mathbb{Z}) \cong \mathbb{Z}^{2g} \hookrightarrow H^0(X,\Omega^1)^*$ (integration over a cycle). The corresponding theta divisor is only determined up to translation, giving isomorphic principally polarized abelian varieties, and is irreducible, i.e., the Jacobian with its canonical principal polarization is indecomposable.*

Theorem. (Poincaré's Complete Reducibility) *If X is an abelian variety and Y an abelian subvariety of X, there exists an abelian subvariety Z of X such that $Y \cap Z$ is finite and $Y + Z = X$.*

Remark. (1) The statement of this theorem is equivalent to saying that X is isogenous to the product $Y \times Z$, where an isogeny is a surjective group homomorphism with finite kernel.

(2) This statement is the algebraic version. Analytically, it could be stated as follows:

Theorem. (Weierstrass-Poincaré, [Mar2]). *Let $[E\ Z]$ be a $g \times 2g$ Riemann matrix admitting reduction: $H \cdot [E\ Z] = \Pi \cdot M$, where H is a maximal-rank $p \times g$ complex matrix, Π is a $p \times 2p$ complex matrix and M is a maximal-rank $2p \times 2g$ integral matrix, with $1 \le p \le g$. Then there exist matrices $A \in GL(g,\mathbb{C})$, $T \in \mathrm{Sp}(2g,\mathbb{C})$ such that*

$$[E\ Z] \cdot T = A \cdot \begin{bmatrix} E_1 & 0 & Z_1 & Q \\ 0 & E_2 & {}^tQ & Z_2 \end{bmatrix}$$

where Z_1, Z_2 are $p \times p$ and $(g-p) \times (g-p)$ Riemann matrices, and Q is a $p \times (g-p)$ rational matrix, whose only possible non-zero entries occur at the beginning of the main diagonal.

(3) The analytic version of the result has the practical consequence that Riemann's theta function reduces to theta functions of lower dimension. Indeed (cf. [Mar2]), for a period matrix $[I\ Z]$ with

$$Z = \begin{bmatrix} Z_1 & Q \\ {}^tQ & Z_2 \end{bmatrix}$$

as above,

$$\vartheta(z;Z) = \sum_{m_2} \vartheta(z_1 + Qm_2; Z_1)\vartheta \begin{bmatrix} D^{-1}\ m_2 \\ 0 \end{bmatrix} (Dz_2; DZ_2D)$$

where $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$, $D = \mathrm{diag}(d^1, \dots, d^{g-p})$ is an integral matrix such that QD is integral, we rewrite any integral vector of length $g-p$ as $n_2 = m_2 + Dk_2$ with $0 \le m_2^j < d^j$, and $\vartheta \begin{bmatrix} D^{-1}m_2 \\ 0 \end{bmatrix} (Dz_2; DZ_2D)$ is the theta function with characteristics, defined more generally for real g-vectors r and s, in dimension g,

$$\vartheta \begin{bmatrix} r \\ s \end{bmatrix} (z;Z) = \sum_m \exp(\pi i^t(m+r)(Z(m+r) + 2(z+s))).$$

It is now apparent that we can ask several important types of reduction questions, with practical applications to solving integrable systems (of finite and infinite dimension!); that there are advantages in formulating both the algebraic and the analytic versions of the questions; and that, even if we decide to restrict our attention to Jacobians, more general abelian varieties may come into play, as we will see next.

2. Automorphisms

By Poincaré's complete reducibility, if the abelian variety A contains an abelian subvariety B, it is isogenous to a product $B \times C$ (with C isogenous to A/B). One question is, when is A actually isomorphic to a product of lower-dimensional abelian varieties? A second question is whether A is isomorphic to a product of lower-dimensional abelian varieties that admit a principal polarization; and if so, whether the product polarization induces a given principal polarization on A. These three questions are actually inequivalent. Let's follow Ries [R] and call a principally polarized abelian variety "splittable" if it is isogenous or isomorphic to a product of lower-dimensional abelian varieties. One way, though not the only one, to find splittable Jacobians is to consider curves with automorphisms.

Hurwitz [Hu2] proved that a curve of genus g can have at most $84(g-1)$ automorphisms. Klein's quartic, in homogeneous coordinates $X_0X_1^3 + X_1X_2^3 + X_0^3X_2 = 0$, is the only curve of genus 3 for which the Hurwitz bound is attained. As a consequence, it is actually possible to calculate the period matrix, a rare situation for a curve given algebraically ([BT], [LB §11.7], [RL]), because it is preserved by many symmetries:

$$Z = \begin{bmatrix} -\frac{1}{8} + \frac{3\sqrt{7}i}{8} & -\frac{1}{4} - \frac{\sqrt{7}i}{4} & -\frac{3}{8} + \frac{\sqrt{7}i}{8} \\ -\frac{1}{4} - \frac{\sqrt{7}i}{4} & \frac{1}{2} + \frac{\sqrt{7}i}{2} & -\frac{1}{4} - \frac{\sqrt{7}i}{4} \\ -\frac{3}{8} + \frac{\sqrt{7}i}{8} & -\frac{1}{4} - \frac{\sqrt{7}i}{4} & \frac{7}{8} + \frac{3\sqrt{7}i}{8} \end{bmatrix}.$$

As observed in [RL], all entries lie in the field generated (over the field k of definition of the curve, $k = \mathbb{Q}$, e.g.) by the character of the representation induced on the differentials of the first kind by the automorphism group of the curve. But another interesting phenomenon occurs: $\mathrm{Jac}(X) = \mathbb{C}^3/\Lambda$, where Λ is the lattice corresponding to $[I\ Z]$, is actually isomorphic to the product of 3 elliptic curves.

Baker [B, §76] shows that

$$\begin{bmatrix} 1\ 0\ 0 & \frac{1+i\sqrt{7}}{4} & 0 & 0 \\ 0\ 1\ 0 & 0 & 2\ \frac{1+i\sqrt{7}}{4} & 0 \\ 0\ 0\ 1 & 0 & 0 & 2\frac{1+i\sqrt{7}}{4} \end{bmatrix}$$

is a matrix of abelian integrals that gives an isomorphic abelian variety. Since Jacobians are indecomposable, however, this choice of homology basis cannot be canonical, or the matrices would determine isomorphic polarizations, when a polarization is viewed as a bilinear form: $H = H((0, x_2), (0, y_2)) = {}^t x_2 (\mathrm{Im} Z)^{-1} y_2$ on the complex structure $x = \Omega x_1 + x_2$ on $\mathbb{R}^{2n} = \{(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n\}$, according to the dictionary given in [Mu, Ch.II §4].

The point of this example is to exhibit a Jacobian that is not only isogenous to a product, as predicted by Poincaré's Reducibility, but actually isomorphic to one. Earle [E1] constructs 1-dimensional families of Jacobians $\mathrm{Jac}(X)$ of any even genus $2k$ which are both isogenous and isomorphic to $\mathrm{Jac}(Y_1) \times \mathrm{Jac}(Y_2)$, with Y_i quotients of X by an automorphism; X is the hyperelliptic curve given by

$$y^2 = (x^n - 1)(x^n - t), \; n = 2k+1, \; t \in \mathbb{C}\backslash\{0, 1\}.$$

It is then a natural question whether, for a Jacobian isomorphic to a product of lower-dimensional tori, these tori "will necessarily be Jacobians themselves, or at least will admit principal polarization" [E3]. Earle replies "emphatically no" by constructing perturbations $\mathrm{Jac}(X)$ of the Klein Jacobian and showing that $\mathrm{Jac}(X)$ admits to isomorphisms onto the product of two complex tori of dimensions 1 and 2, and for any such isomorphism the 2-dimensional torus admits no principal polarization.

Since $X \hookrightarrow \mathrm{Jac}(X)$ projects to each elliptic curve E_i, there exist finite maps $X \to E_i$; in fact each of them is obtained by moding out by an automorphism of order 2, as illustrated in [E2]. First, we connect the actual period matrix of $\mathrm{Jac}(X)$ with Baker's matrix:

$$C[I \; Z] = [I \; Z']N$$

where $Z' = \begin{bmatrix} \tau & 0 & 0 \\ 0 & 2\tau & 0 \\ 0 & 0 & 2\tau \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & -1 \\ 2\tau - 1 & -1 & 0 \\ -2\tau & 0 & 1 \end{bmatrix}$ and the unimodular matrix

$$N = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & -1 \\ -1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is not symplectic, since the polarization of $\mathrm{Jac}(X)$ is indecomposable. But then the three resulting equations

$$[1 \; 0 \; -1][I \; Z] = [1 \; \tau] \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

$$[2\tau - 1 \; -1 \; 0][I \; Z] = [1 \; 2\tau] \begin{bmatrix} -1 & -1 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$$[-2\tau \; 0 \; 1][I \; Z] = [1 \; 2\tau] \begin{bmatrix} 0 & 0 & 1 & 1 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

each describe a map of Jac(X) to the corresponding elliptic curve E_i; the corresponding maps $X \to E_i$ have degree $a \cdot d - b \cdot c$ $(= 2$, in our case) where $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the matrix of the induced map $H_1(X) \to H_1(E_i)$ with respect to a canonical homology basis. Since all elements of order 2 in Aut$X$ are conjugate, the three elliptic curves must be isomorphic, as seen by the equation

$$\tau[1\ 2\tau] = [1\ \tau] \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

We note that the elliptic curve $E = E_i$ does not have extra automorphisms. However, it does have complex multiplication, in fact Ries [R] proves that E is the only elliptic curve with complex multiplication by τ and that End $E = \mathbb{Z}[\tau]$.

Again the interplay of analysis and algebra has a role in some of the many, still wide-open questions that have been asked about the automorphism groups of Riemann surfaces. Hurwitz besides finding the bound $84(g-1)$ proved that it is achieved if and only if the group can be generated by two elements, t, u such that $t^2 = u^3 = (tu)^7 = 1$ [Hu2]. Later it was determined that the maximum possible order of an automorphism is $2(2g+1)$ (and this is optimal). Since I have not read the original papers I refer instead to the survey [M] where they are cited. Call $N(g)$ the maximum order of Aut$X$, X a curve of genus g. Accola [A1] proved that $8(g+1) \leq N(g)$ and that equality obtains infinitely often, and on hyperelliptic curves, however Maclachlan independently verified the bound on a disjoint family of curves [Ma]. What seems to be more elusive is an understanding of the subvariety of moduli space where on particular group acts (or even its dimension!), except for the fact that the bound $84(g-1)$ can only be achieved on isolated curves (but we know now, not a unique one as is the case for the first two genera, $g = 3$ and 7 [M §7]) despite the importance of the locus of curves with automorphisms, in view of the fact that it is the branch locus of the covering map $\mathcal{T}_g \to \mathcal{M}_g$ from Teichmüller space (under the action of the mapping class group). As noted in [R], we can view the question of characterizing the family of principally polarized abelian varieties with the action of a given group G of polarization-preserving automorphisms as a Schottky-type problem. For example, Ries proves that the Klein curve belongs to a 1-parameter family of principally polarized abelian varieties which split completely and whose automorphism group contains $SL(2, 2^3)$; he uses the automorphism group and the attendant group ring in End$A$ to produce a splitting of A.

Finally Accola [A2], starting with an analytic property of theta functions, constructs a plane model of the curve whose properties allow him to prescribe certain automorphisms of the curve and to calculate the dimension of the locus of such curves in Teichmüller space.

The lessons we learned in this section are that it is rare for a curve to have automorphisms, and that even if it has many, the question of splitting its Jacobian is by no means settled, in particular as regards polarization.

Before leaving this section we mention another classical problem, having to do with ordinary differential equations, where the Klein curve stars again. We use it as motivation a sketch of differential Galois theory which we take from [SU].

The third-order equation

$$H(y) = x^2(x-1)^2y''' + (7x-4)x(x-1)y''+$$

$$\left(\frac{72}{7}(x^2-x) - \frac{20}{9}(x-1) + \frac{3}{4}x\right)y' + \left(\frac{792}{7^3}(x-1) + \frac{5}{8} + \frac{2}{63}\right)y$$

has regular-singular points at $x = 0, 1, \infty$. Hurwitz [Hu1] proved that its solutions are algebraic functions defined on Klein's curve. But more is true. The field generated by a basis of solutions for a linear differential equation $L(y) = 0$ is a differential field E, called a Picard-Vessiot extension. The differential Galois group $\mathcal{G}(L)$ of $E/\mathbb{C}(z)$ consists of all automorphisms of E that fix $\mathbb{C}(x)$ elementwise and commute with differentiation. The monodromy group is always a subgroup of the Galois group. In fact, let $c_1, \ldots, c_n$ be the singular points of $L(y) = 0$ (including infinity if it is a singular point) and let c_0 be an ordinary point of the equation. Consider these points as lying on the Riemann sphere S^2. Let $\{y_1, \ldots, y_n\}$ be a fundamental set of solutions of $L(y) = 0$ analytic at c_0 and let γ be a closed path in $S^2 - \{c_1, \ldots, c_n\}$ that begins and ends at c_0. One can analytically continue $\{y_1, \ldots, y_n\}$ along γ and get new fundamental solutions $\{\overline{y}_1, \ldots, \overline{y}_n\}$ analytic at c_0. These two sets must be related via $(\overline{y}_1, \ldots, \overline{y}_n)^T = M_\gamma(y_1, \ldots, y_n)^T$ where $M_\gamma \in GL(n, \mathbb{C})$. One can show that M_γ depends only on the homotopy class of γ and that the map $\gamma \mapsto M_\gamma$ defines a group homomorphism from $\pi_1(S^2 - \{c_1, \ldots, c_n\})$ to $\mathcal{G}(L)$. The image of this map depends on the choice of c_0 and $\{y_1, \ldots, y_n\}$ but is unique up to conjugacy and is called the monodromy group of $L(y)$. In general the image of this group will be a proper subgroup of $\mathcal{G}(L)$ but when $L(y)$ is fuchsian, the Zariski closure of this group will be the full Galois group $\mathcal{G}(L)$. Thus, the Galois group of $H(y)$ coincides with the automorphism group of the Klein curve and Singer and Ulmer [SU] check this computationally.

3. Poncelet

As we saw in the previous sections, the reduction problem can be approached in two different ways: by cover, or by symmetries. The question of whether a Jacobian is splittable is still more general, but these are sufficient conditions:

- if a curve Γ of genus h covers a curve X of genus g then JacΓ is splittable, more precisely it is isogenus to $B \times C$ with $B = \mathrm{Jac}(X)$,
- if σ is an automorphism of a curve Γ then Γ covers $X = \Gamma/\sigma$.

However, the question of whether JacΓ contains an elliptic curve is equivalent to the fact that Γ covers an elliptic curve. For $h(\Gamma) = 2$, several results are known and work is in progress. Lange [L1] proved that the set $\mathcal{M}_2(d)$ of curves of genus 2 that admit a morphism of degree n to an elliptic curve is a subvariety of the moduli space $\mathcal{M}_2$ that has dimension 2 for each n divisible by 2, 3 or 5, and dimension ≤ 2 for any $n \leq 2$. He obtains algebraic equations for this subvariety by defining the cover explicitly and using the fact that Γ covers an elliptic curve E if and only if the surface $\Gamma \times E$ has a nontrivial section. However, in [Kuh] Kuhn produces a more combinatorial construction which shows (by Riemann's existence theorem) that the dimension of $\mathcal{M}_2(d)$ is 2 for every d and from which one could presumably calculate the number of its connected components (as [L1] notices, $\mathcal{M}_2(2)$ is connected). But a difficult question, which has arithmetic significance, remains: what is an algebraic equation for a complementary elliptic curve $J(\Gamma)/E$? Even ignoring the fact that a complementary curve is not canonical (cf. [Kuh]) and eliminating all the interesting arithmetic features that occur in finite characteristic, this question remains challenging. Kuwata [Kuw] had the idea of interpreting it in the framework of Poncelet's theorem and we sketch his construction in view of the significance of this theorem in dynamics, where it can be generalized to hyperelliptic curves of any genus $g > 1$ in place of E [P].

First we describe Kuhn's combinatorial parametrization.

Theorem. [Kuh] *Let Γ be a hyperelliptic curve of genus $g = 2$ covering a curve E of genus 1 by a map $f : \Gamma \to E$, ι the hyperelliptic involution on $\Gamma, C_1, \dots, C_6$ the Weierstrass points of X and $D_1, \dots, D_4$ the ramification points of π_E in the following commutative diagram (where superscript ι means modulo ι):*

$$\begin{array}{ccc} \Gamma & \xrightarrow{f} & E \\ \pi_\Gamma \downarrow & & \downarrow \pi_E \\ \Gamma^\iota & \xrightarrow{f^\iota} & E^\iota \end{array}$$

Let c_j and d_i be the images of C_j and D_i in Γ^ι and E^ι; then for all i, $f^{\iota *}(d_i)$ contains an odd number of c_j's (counted with multiplicity) and any other point of Γ with even multiplicity; every such ramification configuration $f^\iota : \mathbb{P}^1 \to \mathbb{P}^1$ lifts to a morphism $f : \Gamma \to E$. For a given hyperelliptic curve Γ, the existence of a lifting of $f^\iota : \Gamma^\iota \to \mathbb{P}^1$ to a map from Γ to an elliptic curve is an algebraic condition in terms of the points c_j, and it falls into two different cases:

(a) (generic) If f is unramified over the D_i, the ramification of f^ι over the d_i is as follows (with one more point of Γ^ι where f^ι is doubly ramified), with the symbol – representing an unramified point of Γ^ι over one of the d_i, which is therefore one of the c_j, and $*$ representing a doubly ramified point:

case 1 case 2 case 3

even d:

$$\begin{array}{cccc} * & * & * & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & * & * \\ * & = & = & = \\ d_1 & d_2 & d_3 & d_4 \end{array} \qquad \begin{array}{cccc} * & * & * & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & * & * \\ * & = & * & * \\ = & = & * & * \\ d_1 & d_2 & d_3 & d_4 \end{array} \qquad \begin{array}{cccc} * & * & * & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & * & * \\ = & * & * & * \\ = & * & * & * \\ = & * & * & * \\ d_1 & d_2 & d_3 & d_4 \end{array}$$

$$\text{odd } d\text{:}\qquad \begin{array}{cccc} * & * & * & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & * & * \\ = & * & * & * \\ - & - & - & - \\ d_1 & d_2 & d_3 & d_4 \end{array}$$

(b) (special) If f is ramified over some D_i, then the ramification of f^ι over the d_i is as shown in (a) except that either one of the c_j has ramification of degree 3, or there is a unique point, not one of the c_j, with ramification of degree 4.

We assume from now on that $d = 3$. There are two possible cases for the map $f : \Gamma \to E$, after normalizing the choice of origin in E so that the induced involution ι is ± 1:

case 1. f ramifies at two distinct points R and $\iota(R)$;
case 2. f ramifies at a Weierstrass point C_1.

After relabeling, we have the following situation:

$$f^{\iota *}(d_i) \ni c_i, e_i \ (i = 1, 2, 3); \qquad f^{\iota *}(d_4) = \{c_4, c_5, c_6\}$$

and the ramification points of f^ι are

case 1. $r = \pi_\Gamma(R)$ and the e_i, $i = 1, 2, 3$;
case 2. e_1, e_2 and $c_3 = e_3$ with multiplicity 2.

Now we recall Poncelet's result in the briefest possible terms, only for the case of triangles and for a generic situation (the conics in question are assumed to meet at four distinct points, e.g.)

Theorem. *If there is a triangle inscribed in a conic Q_1 and circumscribed to a conic Q_2, then there is a 1-parameter family of such triangles, obtained by moving the divisor $D = P_1 + P_2 + P_3$ (where P_i are the tangency points of the sides of the triangle to Q_2) in a linear equivalence class. This family of triangles corresponds to a curve of genus 1, $I = \{(P, \ell) | P \in Q_1,\ \ell \in Q_2^*,\ P \in \ell\}$ whose Jacobian $Pic^0(I)$ is therefore an elliptic curve; and to the point of order three $\mathcal{D} =$ the divisor class $[\tau(P, \ell) - (P, \ell)]$ where τ is the fixed-point free involution that sends a pair (P, ℓ) to the pair (Q, m), (P, Q) being the side of a triangle.*

We now return to the map $f : \Gamma \to E$, case 1. We choose a conic $Q_2 \cong \mathbb{P}^1 \cong \Gamma^\iota$. This is then Kuwata's observation [Kuw, Lemma 3.4]

Proposition. *Let $\mathcal{D}$ be the divisor class determined by $f^{\iota *}$. Then the corresponding elliptic curve $E' = \mathrm{Pic}^0(Q_1, Q_2)$ can be viewed as a double covering of Q_2 branched over c_1, c_2, c_3 and r', where rr' is the side of a triangle, while r is the image of $\mathcal{D}$ under the projection $E' \to Q_2$. If $\overline{E}'$ is the quotient of E' by the subgroup $\langle \mathcal{D} \rangle$ of* *order 3, then* $\overline{E}'$ *can be viewed as a double covering of* E^ι *ramified at* $f^\iota(r)$, q_1, q_2, q_3, *as in the following commutative diagram:*

$$\begin{array}{ccc} \Gamma & \stackrel{f}{\rightarrow} & E \\ \pi_\Gamma \downarrow & & \downarrow \pi_E \\ \Gamma^\iota & \stackrel{f^\iota}{\rightarrow} & E/\{\pm 1\} \\ \uparrow & & \uparrow \\ E' & \stackrel{\phi}{\rightarrow} & \overline{E}' \end{array}$$

With this motivation, Kuwata constructs explicitly a family of curves Γ, parametrized by an open set U of $\mathbb{P}^1$, that admit a morphism of degree 3 to a fixed elliptic curve E. Let E be given in Weierstrass form: $y^2 = x^3+ax+b$, $\delta = 4a^3+27b^2$ and for $u \in \mathbb{C}$ let

$$\begin{array}{rcc} d(u) & = & 27bu^3 - 18a^2u^2 - 27abu - 2a^3 - 27b^2 \\ c(u) & = & 3au^4 + 18bu^3 - 6a^2u^2 - 6abu - a^3 - 9b^2 \\ f_u(t) & = & 4c(u)t^3 - 3\delta(3u^2+a)t - 3\delta(2au+3b) \\ g_u(t) & = & (u^3+au+b)t^3 - (3au^2+9bu-a^2)t^2 + \delta \end{array}$$

Proposition. *The set of all curves* Γ *of genus 2 that admit a morphism* $\Gamma \to E$ *of degree 3 can be parameterized by the open set* $U = \{u \in \mathbb{P}^1 : \ c(u)d(u) \neq 0\}$, *with the curve* Γ_u *given by equation:* $-\delta d(u)s^2 = f_u(t)g_u(t)$. *The morphism* $f : \Gamma \to E$ *is given by the formula:*

$$f(t,s) = \Bigg(\frac{\delta d(u)(3(u^3+au+b)t + 3au^2 + 9bu - a^2)}{c(u)f_u(t)}$$

$$+ \frac{9bu^4 - 8a^2u^3 - 18abu^2 - 18b^2u + a^2b}{c(u)}, \frac{\delta d(u) g_u(-2t)}{f_u(t)^2} s \Bigg)$$

and is in case 2 if and only if $u^3 + au + b = 0$.

The reason for such polynomial expressions in the parameter is for the result to hold over any field of definition k (except the formulas need to be modified in characteristic 3). So, the point of order 3 $\left(x(u) = \frac{-2au+3b}{3u^2+a},\ y(u)\right)$ on the curve $\overline{E}'$ will be defined over k, with $\overline{E}'$:

$$\begin{array}{cl} d(u)y^2 & = (x - e(u))(x^3 + ax + b) \\ e(u) & = \frac{9bu^4 - 8a^2u^3 - 18abu^2 - 18b^2u + a^2b}{c(u)} \end{array}$$

as well as an explicit isogeny of degree 3: $E' \to \overline{E}'$. Γ_u is then $E'/\{\pm 1\} \times_{\overline{E}'/\{\pm 1\}} E$.

Poncelet's theorem now gives a way of writing an algebraic equation for the second elliptic curve $F = \mathrm{Jac}(\Gamma_u)/E$, which in this case is canonical.

Theorem. *$F_u = \mathrm{Jac}(\Gamma_u)/E$ can be given by the equation:*

$$3\delta y^2 = x^3 - 3\delta(3u^4 + 6au^2 + 12bu - a^2)x - 6\delta(9bu^6 - 12a^2u^5 - 45abu^4$$

$$-90b^2u^3 + 15a^2bu^2 - (4a^4 + 18ab^2)u - 3a^2b - 18b^3).$$

The morphism $\Gamma \to F$ and the j-invariant of F are also computed explicitly in [Kuw]. The fact that allows F to be identified is the following:

Proposition. [Kuw, 5.2] *Let Q_2 be a conic in $\mathbb{P}^2$ and let $w_1, \dots, w_6$ be mutually distinct points in Q_2. We denote by ℓ_i the tangent line to Q_2 at w_i and we set $p_{ij} = \ell_i \cap \ell_j$. Then the following two conditions are equivalent.*

(1) There exists a conic Q' passing through p_{12}, p_{23} and p_{31}, and tangent to ℓ_4, ℓ_5 and ℓ_6.

(2) There exists a conic Q'' passing through p_{45}, p_{56} and p_{64}, and tangent to ℓ_1, ℓ_2 and ℓ_3.

Let Γ be a curve of genus 2 that is a double covering of Q_2 ramifying at $w_1, \dots, w_6$. If one of the above conditions is satisfied, then the Jacobian $\mathrm{Jac}(\Gamma)$ of Γ is isogenous to a product of two elliptic curves E and F.

The key to this "duality" is the following lemma due to Kuhn, which generalizes Jacobi's description of all the degree-2 morphisms from a curve of genus 2 to an elliptic curve: given an involution α of $\mathbb{P}^1$ and three points a_1, a_2, a_3 not fixed by the involution, the curve of genus 2 ramified over $a_1, a_2, a_3,\ \alpha(a_1), \alpha(a_2), \alpha(a_3)$ maps to the two elliptic curves ramified over $\mathbb{P}^1/\alpha$ at the images of the a_i and one of the two fixed points of the involution.

Lemma. [Kuh, p.47] *If $f : \Gamma \to E$ and $g : \Gamma \to F = \mathrm{Jac}(\Gamma)/E$ are a given morphism of degree 3 and the canonically associated one, there is a labeling of the branchpoints of Γ such that $f^{*\iota}(d_i) \ni c_i,\ i = 1,2,3,\ f^{\iota *}(d_4) = \{c_4, c_5, c_6\}$ and $g^{*\iota}(b_i) \ni c_{i+3},\ i = 1,2,3,\ g^{*\iota}(b_4) = \{c_1, c_2, c_3\}$ where $b_1, \dots, b_4$ correspond to the points of period 2 of F, and d_4, b_4 correspond to the origins of E, F:*

$$\begin{array}{ccccc} F & \overset{g}{\leftarrow} & \Gamma & \overset{f}{\rightarrow} & E \\ \downarrow & & \downarrow & & \downarrow \\ F^\iota & \leftarrow & \Gamma^\iota & \rightarrow & E^\iota \end{array}$$

Remark. There are exactly two conics (up to projective equivalence) that are both 3-inscribed in and 3-circumscribed to a given conic Q', according to Barth and Michel [BM]. They correspond to "control points": $[1, \omega, \omega^2]$ and $[1, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}] \in \mathbb{P}^2$, resp. Since Kuwata points out that Q', Q'' in the previous proposition are mutually 3-circumscribed and the corresponding Poncelet elliptic curve has an automorphism of order 3, it seems reasonable that it must correspond to the solution $[1, \omega, \omega^2]$ (checking this would be a straightforward calculation).

4. Problems

In this part we have seen that many beautiful classical questions having to do with special abelian varieties are still quite open, and in fact despite the progress of algebraic geometry no particularly powerful techniques have been introduced, as opposed to a classical area such as enumerative geometry, which has received so much impetus lately from the theory of Gromov-Witten invariants. While it is often the case that in group theory the methods are *ad hoc* (cf. [A1]), still I would pose some questions that seem both tractable and useful, because they have arisen in several contexts, one of which will be the theme of the next part.

4.1 Genus 2, splittable case. We refer to the notation of section 3. Use the invariant theory of 6 points on a line, and the topology of finite covers of this moduli space, to investigate the number of connected components of $\mathcal{M}_2(d)$.

4.2 Genus 2, splittable case. Pursue Kuwata's idea and apply Poncelet's theorem to describe $\mathcal{M}_2(d)$, especially the complementary elliptic curve, which is canonical for an "optimal" cover [Kuh], namely one that doesn't factor non-trivially through another elliptic curve. In view of Kuhn's statement, to the effect that if $\Gamma \to E$ is optimal of degree d, there exists a unique E' and an optimal morphism $\Gamma \to E'$ also of degree d, with $\mathrm{Jac}(\Gamma) \sim E \times E'$, these pairs (E, E') should again give examples of the inscribed and circumscribed pairs that Barth and Michel suggest are worth understanding [BM, §6]

4.3 Higher genus hyperelliptic Γ and Poncelet's theorem. There is a higher-dimensional generalization of Poncelet's theorem, together with its interpretation as one of the aci's of II.2 below [P], and here closure holds only in the rare situation that the curve has a point P with $\mathcal{O}(P - \iota P)$ of finite order. But this might very well be related to the fact that Γ admits a morphism to an elliptic curve, with some restrictions on the inverse image of the origin. This is only speculative at this stage, but concrete calculations in genus 2 and 3 should provide a good indication.

4.4 Genus 3. Kowalevski's curve:

$$(z^2 - f_2)^2 = 4xy(ax + by)(cx + dy)$$

in projective coordinates, with f_2 a form of degree 2 in x and y, has genus 3 and admits a 2:1 morphism to an elliptic curve, yet is not hyperelliptic. She proved that a canonical quartic has splittable Jacobian if and only if it has two concurrent bitangents, as recalled above. This point of view seems to me worth pursuing in two directions. One question is the dimension of the subvariety $\mathcal{M}_g(d)$ of curves of genus g that admits a degree d morphism to an elliptic curve, then its topology and defining equations. Even the dimension question was nontrivial in $g = 2$ [L1,Kuh] and I am advocating finding geometric criteria on the canonical curve equivalent to the property of being in $\mathcal{M}_g(d)$. The technique goes back to Riemann and has been very fruitful, namely translating analytic properties of the theta function (with characteristics) into properties of linear series on the curve; as an example, again recalled above, Accola infers from the vanishing of ϑ with 1/2-period characteristics the existence of an automorphism of order 2 on the Riemann surface; then on the geometric side he uses the same vanishing to produce a plane model of the curve with given properties; by putting the two together, he derives the dimension of moduli of curves with an automorphism of order 2. In the second direction, geometric properties of (nonhyperelliptic) canonical curves reflect of course on the moduli space of rank 2-bundles with trivial determinant $\mathcal{M}_\Gamma(2,\mathcal{O}) \hookrightarrow |2\Theta_\Gamma|^* = \mathbb{P}^{2^g-1}$; for genus 3, Coble's analysis [C] highlights important features of this variety, with many questions remaining open. The special case corresponding to curves with splittable Jacobians seems fertile ground to plow. Lastly, it seems to be unknown whether the Jacobian of Kowalevski's curve is actually isomorphic to a product of three elliptic curves; number-theoretic applications of such a feature would include current techniques of coding theory and cryptography.

4.5 Differential Galois Theory. L. Greenberg [Gr] proved that for any finite group G and non-negative integer g there is a compact Riemann surface X with Aut $X \cong G$ and X/G of genus g (cf. [TT]; as those authors note, the genus of X is not controlled by this statement). Given the recent advances in differential Galois theory, particularly in computation, it would seem worth pursuing the question: when X/G is $\mathbb{P}^1$, G can be viewed as the monodromy group of a differential equation with coefficients in $\mathbb{C}(z)$; find an equation explicitly and use the coefficients to gain insight on the moduli space of curves with $\mathrm{Aut}(X) = G$.

I won't mention here any other of the many fascinating problems on automorphisms because I can't offer any suggestion of a method (that's why I included some at the end of Section 2); it should be said that automorphisms of the curve X have also been used in the construction and investigation of Prym varieties with applications to integrable systems (cf. [K], [D], and the recent [LR] for progress on the geometric side). In concluding, I will stress again the 'interdisciplinary' nature of the reduction problem; to wit, (4.1-4.3) are geometric, 4.4 links analysis and algebra (group theory), 4.5 is differential calculus; and since I am unable to touch at all on number theory I will cite a question in [TT] as an example: are the entries of the period matrix for a curve with $84(g-1)$ automorphisms algebraic numbers?

Part II: Elliptic solitons and integrable systems

1. Calogero-Moser-Krichever: Aspects

Ever since Dubrovin and Novikov [DN] obtained the superposition of 'periodic solitons',

$$2\wp(x - x_1(t)) + 2\wp(x - x_2(t)) + 2\wp(x - x_3(t))$$

the question of elliptic solutions of the KdV or KP equation has been with us. There are two (inequivalent) properties that one may seek in this regard; consider to begin with an 'algebro-geometric solution of the KP hierarchy' by which what is meant is a (complex-valued) function of g (complex) variables $\underline{t} = (t_1 = x, t_2 = y, t_3 = t, \dots, t_g)$ that is produced from algebro-geometric data (roughly speaking, a curve X of genus g with a choice of standard homology basis, a (smooth) point $\infty \in X$ and a divisor $D = P_1 + \ldots + P_g$ on X with $h^0(D - \infty) = 0$)

$$u(\underline{t}) = 2\frac{d^2}{dx^2}\log\vartheta\left(\Sigma_{j=1}^g \int_{P_0}^{P_j} \vec{\omega} + \Sigma t_i U_i - \Delta\right) + \text{const.}$$

where $U_j \in \mathbb{C}^g$ and Δ are constants that depend on the curve (as well as the additive const.).

This function could have one of two properties:

(a) $u(\underline{t})$ is elliptic (i.e., doubly periodic) in the first variable x;

(b) $u(\underline{t})$ is an algebraic expression in g elliptic functions $\wp_1, \dots, \wp_g, \wp_1', \dots, \wp_g'$ where each $(\wp_i, \wp_i')$ corresponds to a given lattice Λ_i.

The first question, where knowledge is much more advanced, also arises in an another way: suppose that one asks what is the dependence on time ($t = t_3$) of the poles in x of a solution of the KdV equation. It turns out [AMcKM] that they obey an "algebraically completely integrable hamiltonian system" (aci for short) called the Calogero-Moser system, with hamiltonian:

$$H = \frac{1}{2}\sum \dot{x}_i^2 - 2\sum_{i\neq j}\frac{1}{(x_i - x_j)^2}.$$

Inspired by this formula, Krichever [Kr] showed that the hamiltonian system with

$$H = \frac{1}{2}\sum \dot{x}_i^2 - 2\sum_{i\neq j}\wp(x_i - x_j)$$

likewise is aci, and moreover that any algebro-geometric KP solution that is elliptic in x has the form: $u(\underline{t}) = \sum_{j=1}^g \wp(x - x_j(\underline{t}))$, with x_j depending on $t_2, t_3, \dots, t_g$ according to this hamiltonian, which is called the Calogero-Moser-Krichever system. To link these problems with the previous part, one needs only to observe that the formula given for $u(\underline{t})$ linearizes the KP flows on Jac X, therefore periodicity in x is equivalent to requiring that an elliptic curve E, tangent at the origin of motion to U_1, be contained in Jac$X$. Following this point of view, Treibich and Verdier in a series of papers gave a geometric description of what they named "elliptic solitons"; Markman interpreted the locus as a generalized Hitchin system (cf. [DM] for a survey).

As regards question (b), however, nothing is known apart from a few examples. First of all, since the functions $\dfrac{d^k}{dx^k}\log\vartheta\left(\Sigma_{j=1}^g \int_{P_0}^{P_j} \vec{\omega} + xU_1 - \Delta\right)$, $k \geq 2$, generate the function field of Jac$X$ and the t-flows span the tangent space to Jac$X$, in order for such a basic function to be elliptic Jac$X$ would have to be splittable as an $E_1 \times \ldots \times E_g$, secondly, the time flows would have to be tangent to tori, which seems to be a more subtle question. We make this observation more precise in the following:

Remark. The locus of elliptic solitons of genus g has dimension g; while the locus of abelian varieties of dimension g isogenous to a product of g elliptic curves has the same dimension, it is very unlikely that the dimension of the Jacobian sublocus is the same. Thus, in principle the locus of elliptic solitons does not coincide with the locus of elliptic solutions.

We will give some concrete details on elliptic solitons in the next section. Then section 3 is related to both problems in yet another way, and harks back to classical questions. We conclude this section with the examples of (b) that we know of.

Example 1. In [BBM1,2] the authors implement a theorem of Appell and by suitable analytic conditions on the period matrix they produce elliptic solutions for integrable hierarchies; they give examples of hyperelliptic and trigonal curves where the condition is satisfied by choosing suitable homology bases. In [BBME], the more general reduction criterion due to Weierstrass is applied.

Example 2. In [BE1-3], the authors work out explicit expressions for elliptic solutions in special cases, as well as equations for the locus of genus-2 curves which allow reduction ("Humbert surface").

Example 3. In [Le] and in [MS] the authors calculate the expression of elliptic solutions for the KdV, Boussinesq equations, resp., coming from the curve: $y^2 = x(x^4 - \alpha^4)(x^4 - \beta^4)$, $y^4 = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$, resp.

More examples will be given in section 3.

2. Elliptic and abelian solitons

Perhaps the most remarkable fact about Krichever's aci is that for the moduli space of spectral curves Γ that gives the base of his system, Jac Γ are all splittable. We provide a different interpretation of such moduli space, due to Treibich and Verdier, which allows us briefly to survey the Hitchin-Markman system, a 'non-abelian' version of aci in the sense that the motion takes place on (the cotangent bundle to) a moduli space of vector bundles over a fixed curve X. To begin with, we take $X = E$, our fixed elliptic curve.

Definition. Let S be the projectivization of the total space of the semistable rank-2 bundle with trivial determinant over E. S is the ruled surface $\pi_S : S \to E$ characterized by the existence of a unique section C_0 with zero self-intersection. Let's denote by S_q the fibre, $q \in E$.

Definition. A finite pointed morphism $\pi : (\Gamma, p) \to (E, q)$ is called a tangential cover when $\pi^*(E)$ is tangent to $A_\Gamma(\Gamma)$, with $A_\Gamma : r \mapsto \mathcal{O}(r - p)$ the Abel map. It is called primitive if it does not factor nontrivially through another tangential cover.

Theorem. [TV1] *Given a tangential cover $\pi : (\Gamma, p) \to (E, q)$ of degree n, there exists a morphism $\rho : \Gamma \to S$ of degree 1 that sends p to $p = C_0 \cap S_q$ and Γ to a member of the linear system $|nC_0 + S_q|$. Any two such morphisms ρ_1, ρ_2 are related by $\rho_1 = \sigma \circ \rho_2$, $\sigma \in Aut(S/E) = \mathbb{C}$. Conversely, any irreducible member of $|nC_0 + S_q|$ with the natural projection to E is a tangential cover of degree n. The map $\Gamma \mapsto \rho(\Gamma)$ gives a 1:1 correspondence between primitive tangential covers and the open affine set $V(n, E)$ of irreducible divisors in the linear system $|nC_0 + S_q|$, which has dimension n and only one fixed point, p; the general member of $V(n, E)$ is smooth of genus n.*

Definition. [DM, 4.6-4.9] Let $\mathrm{Higgs}_X^{sm}(r, d, \omega(D))$ be the component of the moduli space parametrizing semistable Higgs pairs (E, ϕ) that have generically irreducible and reduced spectral curves. Here E is a rank-r, degree-d vector bundle on the curve X, ω is the canonical line bundle of X, D is a divisor on X (of sufficiently large degree) and $\phi : E \to E \otimes \omega(D)$ is a twisted endomorphism. Let $b_i = (-1)^i$ trace $(\Lambda^i \phi)$ and let the hamiltonian be the characteristic polynomial map:

$$H : \ \mathrm{Higgs}_X^{sm}(r, d, \omega(D)) \to B_\omega : \bigoplus_{i=1}^{r} H^0(X, \omega(D)^{\otimes i}), \quad \phi \mapsto (b_i).$$

Let also $y \in H^0(X, \pi^*\omega(D))$ be the tautological section of the total space $\pi : \omega(D) \to X$. Then the spectral curve of ϕ is the inverse image Γ under $\mathrm{char}(\phi) = y^r - tr(\phi) y^{r-1} + \ldots + (-1)^r \det(\phi) : \ \omega(D) \to \omega(D)^{\otimes r}$, in the total space of $\omega(D)$, of the zero section of $\omega(D)^{\otimes r}$.

Theorem. [DM, 4.9] *The Hamiltonian map H is an algebraically completely integrable system. The generic (Lagrangian) fibre is the Jacobian of a smooth spectral curve of genus $r^2(g-1) + 1 + (\deg D)\frac{r(r-1)}{2}$.*

Theorem. [DM, 6.14] *The variety of elliptic KP solitons of degree n corresponding to a fixed pointed elliptic curve (E, q) is birational to the divisor of traceless Higgs pairs in the ($2n$-dimensional) symplectic leaf of* $\mathrm{Higgs}_E^{sm}(n, 0, \omega(q))$ *that have residue*

$$\begin{bmatrix} -1 & 0 & \ldots & & 0 \\ 0 & -1 & \ldots & & \\ \vdots & & & & \vdots \\ & & & -1 & 0 \\ 0 & \ldots & & 0 & n-1 \end{bmatrix}$$

over q.

It is now a natural question whether there can be a curve X of genus $g > 1$ in this "tangential position" to a family of solitons, more specifically an integrable system equivalent to a Lax pair whose entries are rational functions over a curve of genus > 1. The answer must be negative for dimensional reasons, as noted in [DKN, 2.1] (comment on Lax pairs whose coefficients are sections of a line bundle over a curve), and indeed Treibich showed that if X with $g > 1$ and $\phi : (\Gamma, p) \to (X, q)$ are in tangential position, then Γ and X have to be hyperelliptic with q a Weierstrass point while p is not [T1]. Instead, in [DP] we ask the following question: are there families of KP "abelian solitons," namely Jacobians $\mathrm{Jac}(\Gamma)$ containing an abelian subvariety P of dimension $k > 1$ in such a way that the first k KP flows (which span a hyperosculating space to the Abel image of Γ at the image of ∞) are tangent to P? We found only a very special case, namely P is "coelliptic" with $\mathrm{Jac}(\Gamma)/P$ isomorphic to a fixed elliptic curve E, but we were able to show that they form a completely integrable system:

Definition. Fix any two integers $n \geq g \geq 2$ and an elliptic curve E. We also fix the origin q of E, and define the $(g-1)$-dimensional family of coelliptic solitons to be the curves Γ with a map $\pi : \Gamma \to E$ of degree n, generically ramified simply over $g - 1$ points, with a branch point of multiplicity $(g-1)$ at q.

Theorem. [DP, Th. 1] *The family of degree n, genus g coelliptic solitons constructed above is birationally equivalent to an algebraically completely integrable system with $(g-1)$-dimensional Prym varieties P as Liouville tori.*

Note that by construction, the first $(g-1)$ KP flows are indeed tangent to the Prym variety $P = \mathrm{Prym}(\pi)$, namely the connected component through the origin of the kernel of the norm map: $\mathrm{Jac}\ \Gamma \to E$. Our integrable system is obtained by symplectic reduction from the Hitchin-Markman system of Higgs pairs $\mathrm{Higgs}_E(n, d, \omega(q + \max\{0, n-g\}q'))$ (for any $q \neq q' \in E$) over a subvariety of spectral curves cut out by hamiltonians. Therefore, the hamiltonian flows are compatible with the KP flows [DM §6].

In view of this result, for $g = 3$ e.g., one might still be able to write a Lax pair for the integrable systems, describing the motion of a divisor on P, a 2-dimensional abelian variety ("ultraelliptic" case) and corresponding to genus-3 KP flows and their Baker functions. To that end, in [EEP3, EP] we investigate the hyperelliptic Kleinian functions and the differential equations that they satisfy. These analogs of the $\wp$ function were developed in [BEL] and in genus 2 also introduced by D. Grant [G] along with an analog of Jacobi's derivative formula, and extended to genus 3 by Y. Ônishi [O]. We plan to devise a proper Ansatz for the Baker function of an ultraelliptic soliton, and follow Krichever [Kr] in showing that the Lax pair for the system can be derived from a Frobenius-Stickelberger formula.

3. Lamé

We begin with a different point of view, which in truth is closely related to the theory of elliptic solitons. The classical spectral problem for differential operators with elliptic coefficients:

$$L = \partial^n + u_{n-2}(x)\ \partial^{n-2} + \ldots + u_0(x),$$

where $\partial = \frac{d}{dx}$ and $u_j(x) \in \mathbb{C}(\wp, \wp') = \mathbb{C}(E)$, with the normalization $u_{n-1}(x) = 0$, can be viewed in the context of isospectral deformations of commutative algebras of differential operators, namely the KP flows:

Remark. If the ring of differential operators that commute with L is of (arithmetic) genus > 0, which is to say that L is finite-gap, then the corresponding KP solution is an elliptic soliton.

Indeed, the ring of operators that commute with L is given by the differential-operator part of the ring of formal pseudo-differential operators: $\{\sum_{j\le N} c_j(L^{1/n})^j,\ c_j \in \mathbb{C}\}$ so all the coefficients are periodic in x.

The question of when L is finite-gap is surprisingly subtle:

Theorem. (cf. [I]) *$L = \partial^2 + c\wp(x)$ is finite gap if and only if the constant $c = -n(n+1)$, where n is a natural number.*

In fact, the first significant example of elliptic solition cited in section 1 arises this way for $n = 2$. A powerful and effective method was introduced recently by F. Gesztesy and his collaborators (cf. [GW] for a survey) by extending Picard's theorem to link the finite-gap property with the qualitative behavior of the eigenfunctions. One is then faced with the question of computing the equation for the isospectral curve Γ whose Jacobian splits along E. There are several recursive approaches, [Al], [DGU], [GRT], to finding a differential operator B of order n that commutes with L, so that the polynomial equation $B^2 = L^{2n+1}$ +(lower-order terms) that they satisfy gives the (possibly singular) curve, but a more classical one can be found in [He], namely an Ansatz for the expression of the eigenfunctions at the branched values of the spectral parameter. For example when n is odd and the branchpoint is of one of two possible types (the expressions for the other cases are analogous):

$$\psi(x) = ((\wp(x) - e_1)^{1/2}(\wp(x) - e_2))^{1/2} \sum_{j=0}^{(n-3)/2} b_j(\wp(x) - e_3)^{n/2-j-1}$$

(notice how this differs from the more general Ansatz for the KP elliptic solitons used by Krichever:

$$\psi = \sum_{j=1}^{n} a_i(t,k,\alpha)\phi(x - x_i, \alpha)e^{xk+tk^2}$$

where

$$\phi(x,\alpha) = -\frac{\sigma(x-\alpha)}{\sigma(\alpha)\sigma(x)}e^{\zeta(\alpha)x}$$

and σ, ζ are the classical Weierstrass functions).

By expanding the Lamé equation (using MAPLE, e.g.) and solving algebraic conditions of compatibility for the b_j, one can give examples of Lamé curves:

$n = 2$ $\qquad w^2 = (z^2 - 3g_2)\prod_{i=1}^3(z + 3e_i)$

$n = 3$ $\qquad w^2 = z\prod_{i=1}^3(z^2 - 6ze_i + 45e_i^2 - 15g_2)$

and the general-n case can be calculated recursively. There are several questions to be addressed:

• Which Lamé Jacobians are isogenous to a product of elliptic curves (or even isomorphic to products?)

• When that happens, what is an explicit expression for a KdV solution in terms of elliptic functions?

• In view of recent work on elliptic solitons, more general elliptic potentials are known to be finite-gap, as reviewed in the next proposition: the two questions above should be asked for these too.

Proposition. ([T1,2]). *For any $n \geq 1$ and $(a_i) \in \mathbb{N}^4$ such that $\sum_{i=0}^{3} a_i(a_i + 1) = 2n$, there is a unique hyperelliptic tangential cover $\Gamma \to E$ and a half period $\omega \in \{\omega_0 = q, \omega_1, \omega_2, \omega_3\}$ such that the potential*

$$u(x) = \sum_{i=0}^{3} a_i(a_i+1)\wp(x-\omega_i)$$

is the KdV initial value associated to the divisor $\pi^*(\omega - q)$, and the other normalizing data. For any other point ρ on the elliptic curve, the potential

$$u(x) = 2\wp(x-\rho) + 2\wp(x+\rho) + \sum_{i=0}^{3} a_i(a_i+1)\wp(x-\omega_i)$$

is finite-gap if and only if $\sum_{i=0}^{3}(2a_i+1)^2\wp'(\rho-\omega_i) = 0$.

Recall that a tangential cover is called primitive if it does not factor nontrivially through another one, and exceptional if the (geometric) genus of its canonical image on the surface S (defined in section 2) is 0.

In [T1] Treibich proves some characterizing properties that allow him to compute the tangential polynomials, that is to say the equations as covers of the curve E, for the primitive exceptional hyperelliptic tangential covers of degree $n(= deg\ \pi)$ up to 6, for example when:

$n = 3, \qquad S_3(T) = T^3 - 3\wp T + \wp', \ u(x) = 6\wp(x), \quad g = 2$

$n = 4, \qquad S_4(T) = T^4 + 3(e_i - 2\wp)T^2 + 4\wp' T - 3\prod_{j\neq i}(\wp - e_j),$
$u(x) = 2(3\wp(x) + \wp(x-\omega_i)), \ g = 2$ (with $i = 1, 2, 3$)

$n = 5$ also corresponds to genus 2;

$n = 6, \qquad S_6(T) = T^6 - 15\wp T^4 + 20\wp' T^3 - \frac{9}{4}(20\wp^2 - 3g_2)T^2 + 12\wp\wp' T - \frac{5}{4}\wp'^2,$
$u(x) = 12\wp(x), \quad g = 3.$

I posed these questions above rather than in the next section because this is work in progress, in collaboration with J.C. Eilbeck and V.Z. Enolskii [EEP2].

These are some (very) preliminary observations which we are pursuing in order to give a theory of Lamé tangential covers. The hyperelliptic curve Γ_n associated to the potential $-n(n+1)\wp$, for some n and some special elliptic curves will be singular, though the arithmetic genus is always n. In coordinates (z, w) as above, the "KP basis" of holomorphic differentials is $\{\omega_j : z^j \frac{dz}{w}\}$, $0 \le j \le g-1$, in the sense that the KP flows on $\mathrm{Jac}(\Gamma_n)$ are parallel to the directions $(0, \dots, \int \omega_j, \dots 0)$.

Observation 1. Aside from the cover that defines the elliptic soliton, corresponding to the differential $z^{g-1}\frac{dz}{w}$ (cf. [BC]), the time direction t_3 also is elliptic for $n = 2, 3$ and, in the "equianharmonic" case (i.e. when $\wp$ is associated to the curve with an automorphism of order 3) also for $n = 4$ and 5.

Observation 2. Jac (Γ_3) splits completely in the equianharmonic case. For $n = 2$, $\mathrm{Jac}(\Gamma_2)$ splits into two non-isomorphic elliptic curves; for $n = 3$, the curves along the $x = t_1, t_5$ direction are both equianharmonic, while the $t = t_3$ curve is not.

The next insight into differential equations with periodic coefficients was provided by Halphen. He generalized Ince's result and proved [H]:

The operator $L = \partial^3 - (n^2-1)\wp\partial - \frac{(n^2-1)}{2}\wp' = 0$ is finite gap.

The resulting elliptic solitons will be solutions of the Boussinesq equation, because the third KP time evolution corresponds to the equation

$$\partial_{t_3} L = [(L^{3/3})_+, L] = 0.$$

In this situation, to find the equation of the spectral curve one uses the Halphen Ansatz for the eigenfunction:

$$\psi(x;\alpha) = e^{kx} \sum_{j=0}^{n-1} a_j(x,\mu,k) \frac{d^j}{dx^j} \phi(x;\alpha)$$

where α is a complex number viewed as a point of the elliptic curve $\nu^2 = 4\mu^3 - g_2\mu - g_3$; by expanding the eigenvalue problems $L\psi = z\psi$ and $B\psi = w\psi$, where B is the lowest-order differential operator that commutes with L (not a polynomial in L) and imposing compatibility, one obtains the equation $F(w, z) = 0$ of a (trigonal) $(3, N)$ curve.

Halphen also pointed out that the equation $\partial^3 - \frac{4}{3}n^2\wp\partial + \frac{2n(n-3)(4n+3)}{27}\wp' = 0$ has solutions that can be expressed in terms of elliptic functions, so it would be interesting to check whether the operator $L = \partial^3 - \frac{4}{3}n^2\wp\partial + \frac{2n(n-3)(4n+3)}{27}\wp'$ is finite-gap. A Gröbner-basis approach together with new examples of third-order finite-gap operators is worked out in [Br].

In [EEP1], we showed that the $(3, 4)$ Halphen curve, which has genus 3, in the case of the equianharmonic $\wp$-function splits into three elliptic curves, and what's more, again the three elliptic curves are tangent to the three KP flows. In this case, all three curves are isomorphic.

The other task is to express explicitly the KP solution in terms of elliptic functions, which is achieved by the general formula once the evolution vectors U_j are computed. In [EEP1] we worked out the solution to the Krichever system, from which the KP flows are deduced. The method we used involves knowledge of the "Kleinian functions" $\wp_{ij}(\underline{z}) = -\frac{\partial^2}{\partial z_i \partial z_j} \log \sigma(\underline{z})$, and their use in giving an algebraic solution of Jacobi's inversion problem. Our final result is the following:

Theorem. *The 5-particle Calogero-Moser-Krichever system for the equianharmonic elliptic curve, with trigonal spectral curve*

$$w^3 = \left(z^2 + \frac{25}{4} g_3\right)\left(z^2 - \frac{135}{4} g_3\right)$$

has evolution divisor $\sum_{i=1}^{3}(z_i, w_i)$ *depending on the particles* $x_i(t_1, t_2, t_4),\ 1 \le i \le 5$ *according to the equations*

$$\left(\frac{df}{dy}\right)^2 - f^6 - 4\left(\widetilde{\wp}\left(\frac{2}{3}y\right) + 2g(y)\right) f^2 = 0$$

where

$$f = \frac{dx_i(y)}{dy}, \quad g(y) = \left.\frac{\partial x_i(t, y)}{\partial t}\right|_{t=0},$$

denoting $y = t_1,\ t = t_2,\ \widetilde{\wp}'^2 = 4\widetilde{\wp}^3 + (40 g_3)^2$ *(the curve tangent to the* t_4 *direction). In turn, the algebraic inversion problem is solved in terms of* f, g.

4. Problems

Kowalevski also gave the separation of variables for a dynamical problem, which is now known as the "Kowalevski top" [Ko2]. In this part, we looked at some links between her work on reduction and the theory of integrable systems; we have seen that it is quite exceptional that 'reduction persist' through an integrable system, and that this exceptional situation is related to the motion of the poles of rational/elliptic KP solutions. One area of KP theory which has so far eluded explicit integration is that of the higher-rank solutions, which will be parametrized by rank-r vector bundles over a curve as well as $(r-1)$ arbitrary functions, and I pose a problem in that area even if I don't have concrete suggestions, mainly because I believe Veselov's little-advertised example deserves further investigation.

4.1 As part of his thesis, Veselov [V] produced a hamiltonian system for the poles of certain solutions of the KP equation which turned out to be strictly of higher rank [La,LP]. Can one use the poles of the elliptic KP solutions of higher rank [KN, PW2] to construct an integrable system? Also, Veselov's system does not have a spectral-curve or aci model yet, to the best of my knowledge; the data however may be truly infinite-dimensional, because of the presence of the arbitrary function [PW1].

4.2 As for the abelian solitons of section 2, any matter of explicit expression (hamiltonians, Lax pairs, reduced KP solution) is open, and would give an application of the Kleinian (σ)-functions [BEL].

4.3 The two problems, reduction on one hand, and aci systems on the other, have come in contact in a fairly mysterious way, through elliptic/abelian solitons mainly. I do not know of an example of elliptic soliton that can be proved to be non-reducible to elliptic functions, although probably this is the generic situation. More significantly, at least two questions could be asked for the moduli space of curves: firstly, from experimental evidence [EEP2], as mentioned in section 3, when the elliptic soliton has lattice with extra symmetries there seem to be a better chance at reduction. Secondly, the question of finding equations for the subvarieties of $\mathcal{M}_g$ where reduction occurs has been asked already [BE1], and the answer in genus 2 degree N, is a Humbert surface $H_{N^2} \subset \mathcal{M}_2$, but one more nontrivial condition to detect would be whether the second KP flow is always tangent to the 'second elliptic curve' in the splitting of Jac (Γ). Again, this is unlikely, but the KP equation has a way of surprising us. Also, periodicity in t has been calculated for 2-dimensional elliptic solitons [S] but it might make sense to ask for periodicity of one of the higher flows $t_k,\ k > 1$, for any solution, and for a moduli space of curves with this property.

I would close by saying that the classical theory of elliptic operators with regular-singular points still needs to be brought out in the deeper context of algebraic geometry; one of Verdier's ideas was indeed to embed the elliptic solitons in a moduli space of local systems; this somewhat has been furthered by the theory of Higgs fields, and Hitchin systems, where finally a dictionary between completely integrable equations and monodromy problems may be found. Sophie Kowalevski's thesis was one early place where the two theories appeared together.

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