A SAMPLING METHOD FOR AN INVERSE BOUNDARY VALUE PROBLEM FOR HARMONIC VECTOR FIELDS

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ABSTRACT. Two variants of sampling methods have been recently suggested and developed by Colton and Kirsch for the approximate solution of inverse obstacle scattering problems for acoustic waves, i.e., for inverse boundary value problems for the Helmholtz equation. We present an analogue of the second variant of these methods for the solution of an inverse boundary value problem from electrostatics, i.e., for an inverse Dirichlet problem for harmonic vector fields.

1. Introduction

Inverse boundary value problems for partial differential equations, in principle, consist of determining the shape of an unknown object D from a knowledge of the type of the boundary condition on its boundary ∂D and of one or several solutions to the differential equation at locations away from the boundary. They are difficult to solve since they are both nonlinear and improperly posed. Recently new solution methods have been developed which circumvent the issue of nonlinearity by introducing a parameter point z and then solving a linear operator equation to determine whether or not $z \in D$. These methods are called sampling methods and two different variants were introduced first for inverse obstacle scattering problems, i.e., for inverse boundary value problems for the Helmholtz equation, by Colton and Kirsch [6] and by Kirsch [11] (see also the recent survey [5]). The second variant was extended to an inverse Dirichlet problem for the Laplace equation by Hähner [10] in three dimensions and by Kress [13, 14] in two dimensions. For the Laplace equation, the two-dimensional case is slightly more involved due to the behavior of the logarithmic fundamental solution at infinity. The inverse Neumann problem for the Laplace equation equation in two dimensions was considered by Kress and Kühn [16]. A corresponding approach for the inverse impedance tomography problem, i.e., for transmission problems for the Laplace equation, was initiated by Brühl and Hanke [3, 4].

The sampling method of Kirsch has also been extended to inverse obstacle scattering problems for elastic waves by Alves and Kress [1] and by Arens [2]. However, at the time this is being written, a corresponding analysis for the inverse obstacle scattering problem for electromagnetic waves has no yet been completed, although with factorization theorems some of the ingredients are already available (see [15]). Since the theoretical foundation of the sampling method in potential theory is conceptionally simpler than in scattering theory, in this paper we will consider the sampling

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method for an inverse obstacle scattering problem in electrostatics. We anticipate that investigating this inverse boundary value problem for harmonic vector fields, i.e., for solutions of

$$\operatorname{div} v = 0, \quad \operatorname{curl} v = 0 \tag{1.1}$$

might help to finally extend Kirsch's sampling method also to the case of the time-harmonic Maxwell equations.

If D is a bounded simply connected domain in $\mathbb{R}^3$ with a connected C^2 boundary ∂D and outward unit normal ν and y is a point in $\mathbb{R}^3 \setminus \bar{D}$ and p a vector in $\mathbb{R}^3$, we consider the exterior boundary value problem to find a harmonic vector field $v(\cdot, y, p) \in C^1(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ in $\mathbb{R}^3 \setminus D$ satisfying the Dirichlet boundary condition

$$\nu \times v(\cdot, y, p) = -\nu \times \operatorname{grad} \operatorname{div} \{\Phi(\cdot\,, y)p\} \quad \text{on } \partial D, \tag{1.2}$$

where Φ denotes the fundamental solution to the Laplace equation. Furthermore, we require that

$$\int_{\partial D} \nu \cdot v \, ds = 0 \tag{1.3}$$

and $v(x) \to 0$ as $|x| \to \infty$ uniformly for all directions. We may interpret $v + \operatorname{grad} \operatorname{div}\{\Phi p\}$ as the total electrostatic field of a dipole at the point y with polarization p in the exterior of a perfect conductor D. The condition (1.3) implies that the total charge of D vanishes. Since w depends linearly on the polarization vector p we may write

$$v(\cdot\,, \cdot\,, p) = wp,$$

where w is a 3×3–matrix depending on $x, y \in \mathbb{R}^3 \setminus \bar{D}$.

Let B be an additional bounded simply connected domain with a connected C^2 boundary ∂B that contains $\bar{D}$. Then the inverse problem we are interested in is to determine the shape of D, i.e., the boundary ∂D, from a knowledge of $w(x, y)$ for all x, y in ∂B. To solve this inverse problem, we will characterize D in terms of spectral data of the integral operator W defined by

$$(Wg)(x) := \int_{\partial B} w(x, y) g(y) \, ds(y), \quad x \in \partial B. \tag{1.4}$$

We will consider the two cases where the density g is either normal or tangential to the boundary ∂B. In the case, where $g = \tilde{g}\nu$ with some scalar function $\tilde{g}$ and ν denoting the outward normal to ∂B, we replace (1.4) by the integral operator $W_\nu : L^2(\partial B) \to L^2(\partial B)$ defined by

$$(W_\nu \tilde{g})(x) := \int_{\partial B} w_\nu(x, y) \tilde{g}(y) \, ds(y), \quad x \in \partial B, \tag{1.5}$$

with the scalar kernel

$$w_\nu(x, y) = \nu(x) \cdot w(x, y) \nu(y), \quad x, y \in \partial B,$$

i.e., related to the direction of the dipoles we only use the normal components of the field v as data. In the second case, where g is a tangential field, we view (1.4) as an

integral operator $W : L_t^2(\partial B) \to L_t^2(\partial B)$ mapping the space $L_t^2(\partial B)$ of tangential fields on ∂B into itself, i.e., we use only the tangential components of the field v as data. To distinguish between the two cases, in the sequel, we use a subscript ν for those operators that correspond to the first case.

In particular, we will show that W_ν and W are compact, self-adjoint, and positive semi-definite operators. Therefore their square roots $W_\nu^{1/2}$ and $W^{1/2}$ are well defined. If for some constant unit vector e and a parameter point $z \in \mathbb{R}^3$ we define the dipole field $H(\cdot\,, z) := \operatorname{grad}\operatorname{div}\{\Phi(\cdot\,, z)e\}$ in $\mathbb{R}^3 \setminus \{z\}$, then our main result is the characterization of the domain D by the property that the improperly posed linear operator equations

$$W_\nu^{1/2}\tilde{g} = \nu \cdot H(\cdot\,, z)|_{\partial B} \tag{1.6}$$

and

$$W^{1/2}g = \{\nu \times H(\cdot\,, z)|_{\partial B}\} \times \nu \tag{1.7}$$

have solutions $\tilde{g} \in L^2(\partial B)$ and $g \in L_t^2(\partial B)$, respectively, if and only if $z \in D$. With the aid of Picard's theorem this can be used numerically for a visualization of the unknown domain.

The plan of this paper is as follows. Since the study of an inverse problem always requires a solid foundation of the corresponding direct problem, in Section 2 we summarize the classical existence and uniqueness results for the exterior Dirichlet problem for harmonic vector fields extended by an investigation of the solution operator as needed in the following analysis. Then, in Section 3 we introduce the inverse problem with dipoles in normal direction. Our main result will be the theoretical foundation of the sampling method through Theorem 3.4 and its two corollaries. The final Section 4 is devoted to the inverse problem with dipoles in tangential directions and is kept shorter, since the analysis is analogous to that in Section 3.

For simplicity, we have assumed that the domain D is simply connected. In principle, the analysis can be extended to multiply connected domains. However, technical difficulties arise through nonuniqueness issues in connection with the Neumann problem for harmonic vector fields and the integral operator $I+M$ as used in the existence analysis. The case when D has a finite number of components can also be considered. Here, the flux condition (1.3) has to be imposed for each component of D to ensure uniqueness for the exterior Dirichlet problem.

2. Exterior Dirichlet problem

Recall that D is a bounded simply connected domain in $\mathbb{R}^3$ with a connected C^2 boundary ∂D and exterior unit normal ν. By $C_{t,div}^{0,\alpha}(\partial D)$ we denote the space of all $C^{0,\alpha}(\partial D)$ Hölder continuous tangential fields a that possess a weak surface divergence $\operatorname{Div} a \in C^{0,\alpha}(\partial D)$ equipped with the norm

$$\|a\|_{C_{t,div}^{0,\alpha}} := \|a\|_{C^{0,\alpha}} + \|\operatorname{Div} a\|_{C^{0,\alpha}}$$

(see [8], p. 169). Now we consider the exterior Dirichlet problem for harmonic vector fields: Given a tangential field $c \in C_{t,div}^{0,\alpha}(\partial D)$ with $\operatorname{Div} c = 0$, find a vector field

$v \in C^1(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ satisfying the differential equations

$$\operatorname{div} v = 0, \quad \operatorname{curl} v = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D}, \tag{2.1}$$

the boundary condition

$$\nu \times v = c \quad \text{on } \partial D, \tag{2.2}$$

and the flux condition

$$\int_{\partial D} \nu \cdot v \, ds = 0. \tag{2.3}$$

At infinity it is required that $v(x) \to 0$ for $|x| \to \infty$ uniformly for all directions. In order to introduce notations for the subsequent analysis we briefly recall the classical results on existence and uniqueness for (2.1)–(2.3). For a comprehensive study of boundary value problems for harmonic vector fields we refer to Martensen [17].

Since we assume that D and, consequently, $\mathbb{R}^3 \setminus \bar{D}$ are simply connected, from (2.1) we have that $v = \operatorname{grad} u$ for some harmonic function u. Then the homogeneous form $\nu \times v = 0$ on ∂D of the boundary condition (2.2) implies that $u = \text{const}$ on ∂D. Now, using Green's integral theorem and the behavior of harmonic vector fields at infinity, from (2.3) it can be deduced that $\operatorname{grad} u = 0$ and consequently $v = 0$ for any solution to the homogeneous boundary value problem (2.1)–(2.3). From the vector formula

$$\operatorname{Div}\{\nu \times v\} = -\nu \cdot \operatorname{curl} v \tag{2.4}$$

(see [8], p. 169) we observe that the condition $\operatorname{Div} c = 0$ is necessary for the solvability of (2.1)–(2.3).

If we denote by

$$\Phi(x, y) := \frac{1}{4\pi} \frac{1}{|x - y|}, \quad x \neq y,$$

the fundamental solution of Laplace's equation in $\mathbb{R}^3$, then the vector field

$$v(x) = \operatorname{curl} \int_{\partial D} \Phi(x, y) a(y) \, ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \tag{2.5}$$

with density $a \in C^{0,\alpha}_{t,div}(\partial D)$ is a solution of (2.1)–(2.3) if a solves the integral equation

$$a + Ma = 2c, \tag{2.6}$$

where the integral operator $M : C^{0,\alpha}_{t,div}(\partial D) \to C^{0,\alpha}_{t,div}(\partial D)$ is given by

$$(Ma)(x) := 2 \int_{\partial D} \nu(x) \times \operatorname{curl}_x\{\Phi(x, y) a(y)\} ds(y), \quad x \in \partial D. \tag{2.7}$$

Clearly, v as defined by (2.5) satisfies $\operatorname{div} v = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ and, by Stokes' theorem, the flux condition (2.3). Using the relation

$$\operatorname{Div} Ma = -K^* \operatorname{Div} a \tag{2.8}$$

(see [8], p. 169), where $K^* : C(\partial D) \to C(\partial D)$ denotes the normal derivative of the single-layer potential operator, given by

$$(K^*\varphi)(x) := 2 \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(x)} \, \varphi(y) \, ds(y), \quad x \in \partial D,$$

from (2.6) and the solvability condition $\operatorname{Div} c = 0$ we deduce that

$$\operatorname{Div} a - K^* \operatorname{Div} a = 0.$$

This implies that $\operatorname{Div} a = 0$, since $I - K^*$ is injective (see Theorem 6.20 in [13]). Because of the vector identity $\operatorname{curl}\operatorname{curl} = -\Delta + \operatorname{grad}\operatorname{div}$ we have that

$$\operatorname{curl}\operatorname{curl} \int_{\partial D} \Phi(x,y) a(y) \, ds(y) = \operatorname{grad}\operatorname{div} \int_{\partial D} \Phi(x,y) a(y) \, ds(y) \tag{2.9}$$

for integrable densities a and $x \in \mathbb{R}^3 \setminus \partial D$. From this, with the aid of the Gauss surface divergence theorem, we can deduce that

$$\operatorname{curl}\operatorname{curl} \int_{\partial D} \Phi(x,y) a(y) \, ds(y) = \operatorname{grad} \int_{\partial D} \Phi(x,y) \operatorname{Div} a(y) \, ds(y) \tag{2.10}$$

for $a \in C^{0,\alpha}_{t,div}(\partial D)$ and $x \in \mathbb{R}^3 \setminus \partial D$ (see [8], p. 170). Therefore, $\operatorname{Div} a = 0$ now implies that $\operatorname{curl} v = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. Finally, by the potential theoretic jump relations (see Theorem 6.11 in [8]), the integral equation (2.6) implies that the boundary condition (2.2) is fulfilled.

The integral operator $M : C^{0,\alpha}_{t,div}(\partial D) \to C^{0,\alpha}_{t,div}(\partial D)$ is compact (see Theorem 6.16 in [8]). Since for a simply connected domain D the operator $I + M$ can be shown to be injective (see Theorem 5.4 in [7]), by the Riesz theory, it has a bounded inverse $(I + M)^{-1} : C^{0,\alpha}_{t,div}(\partial D) \to C^{0,\alpha}_{t,div}(\partial D)$. This concludes the classical uniqueness and existence analysis for the exterior Dirichlet problem (2.1)–(2.3).

Recall that $B \subset \mathbb{R}^2$ is an additional bounded, simply connected domain with connected C^2 boundary such that $\bar{D} \subset B$. We introduce an operator

$$A_\nu : \{\nu \times v|_{\partial D}\} \times \nu \mapsto \nu \cdot v|_{\partial B}$$

that, for solutions to (2.1)–(2.3), maps the tangential component on ∂D onto the normal component on ∂B. From the above existence analysis we have that

$$A_\nu = 2U_\nu (I + M)^{-1} R, \tag{2.11}$$

where $R : L^2_t(\partial D) \to L^2_t(\partial D)$ is defined by

$$Ra := \nu \times a$$

and $U_\nu : L^2_t(\partial D) \to L^2(\partial B)$ by

$$(U_\nu a)(x) := \nu(x) \cdot \operatorname{curl} \int_{\partial D} \Phi(x,y) a(y) \, ds(y), \quad x \in \partial B.$$

For the subsequent analysis we need to consider the operator A_ν in appropriate L^2 spaces, i.e., Hilbert spaces. By $L^2_t(\partial D)$ we denote the space of square integrable tangential fields on ∂D and, further, we denote

$$L^2_{t,div}(\partial D) := \{a \in L^2_t(\partial D) : \operatorname{Div} a \in L^2(\partial D)\}$$

equipped with the norm

$$\|a\|^2_{L^2_{t,div}} := \|a\|^2_{L^2_t} + \|\operatorname{Div} a\|^2_{L^2}.$$

Because its kernel is weakly singular, M is a compact operator from $L^2_t(\partial D)$ into itself. Using the Fredholm alternative it can be shown that the nullspace of $I + M$ with respect to $L^2_t(\partial D)$ coincides with the nullspace with respect to $C^{0,\alpha}_{t,div}(\partial D)$ (see also [8], p. 59). Therefore, the inverse operator $(I + M)^{-1} : L^2_t(\partial D) \to L^2_t(\partial D)$ also exists and is bounded. Since U_ν is an integral operator with continuous kernel, it is compact from $L^2_t(\partial D)$ into $L^2(\partial B)$ and, consequently, the operator A_ν as given by (2.11) is a compact operator from $L^2_t(\partial D)$ into $L^2(\partial B)$. We note that due to the solvability condition $\operatorname{Div} c = 0$ the operator A_ν given by (2.11) corresponds to the mapping of the tangential component of harmonic vector fields on ∂D onto the normal component on ∂B only on the subspace $R(L^{2,0}_{t,div}(\partial D))$, where

$$L^{2,0}_{t,div}(\partial D) := \left\{ a \in L^2_{t,div}(\partial D) : \operatorname{Div} a = 0 \right\}.$$

Since M is also compact from $L^2_{t,div}(\partial D)$ into itself (see [9]), using (2.8) it can be seen that

$$(I + M)^{-1}(L^{2,0}_{t,div}(\partial D)) \subset (L^{2,0}_{t,div}(\partial D)).$$

Theorem 2.1. *The compact operator $A_\nu : L^2_t(\partial D) \to L^2(\partial B)$ is injective on the subspace $R(L^{2,0}_{t,div}(\partial D))$.*

Proof. From above we already know that $(I + M)^{-1} : L^{2,0}_{t,div}(\partial D) \to L^{2,0}_{t,div}(\partial D)$ is injective and R trivially is injective. Therefore, to establish injectivity of A_ν it remains to show that $U_\nu : L^{2,0}_{t,div}(\partial D) \to L^2(\partial B)$ is injective. The property $\operatorname{Div} a$ for $a \in L^{2,0}_{t,div}(\partial D)$ implies that the vector field v defined by (2.5) is harmonic (see (2.10)). From $U_\nu a = 0$ we have $\nu \cdot v = 0 = 0$ on ∂B, that is, v solves the homogeneous Neumann problem for harmonic vector fields in $\mathbb{R}^3 \setminus B$. Since for harmonic vector fields in simply connected domains we can write $v = \operatorname{grad} u$ for some harmonic function u, from the uniqueness results for the Neumann problem for harmonic functions we can deduce that the homogeneous Neumann problem in simply connected domains only allows the trivial solution. Therefore we have $v = 0$ in $\mathbb{R}^3 \setminus B$, whence by analyticity $v = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ follows. From this, using the potential theoretic jump-relations in the L^2 sense due to Kersten [12] (see [13] p. 172) we obtain that $a + Ma = 0$. Hence $a = 0$, since $I + M$ is injective. □

Note that by the Gauss divergence theorem and the condition (1.3) we have that

$$A_\nu R(L^{2,0}_{t,div}(\partial D)) \subset L^2_0(\partial B), \tag{2.12}$$

where

$$L^2_0(\partial B) := \left\{ g \in L^2(\partial D) : \int_{\partial B} g \, ds = 0 \right\}.$$

The L^2 adjoint $A^*_\nu : L^2(\partial B) \to L^2_t(\partial D)$ is given by

$$A^*_\nu = 2R^*(I + M^*)^{-1}U^*_\nu. \tag{2.13}$$

Here, as can be seen by interchanging orders of integration, the adjoint $U_\nu^* : L^2(\partial B) \to L_t^2(\partial D)$ of U_ν is described by

$$(RU_\nu^* g)(x) = \nu(x) \times \operatorname{curl} \int_{\partial B} \Phi(x,y) g(y) \nu(y)\, ds(y), \quad x \in \partial D,$$

the adjoint $M^* : L_t^2(\partial D) \to L_t^2(\partial D)$ of M by $M^* = RMR$ (see [8], p. 171), and the adjoint of R by $R^* = -R$. This can be used to transform (2.13) into

$$A_\nu^* = 2(I - M)^{-1} R^* U_\nu^*,$$

i.e., for $g \in L^2(\partial B)$ we have that $a := A_\nu^* g$ solves

$$(I - M)a = 2R^* U_\nu^* g.$$

Hence, since M is compact from $L_{t,div}^2(\partial D)$ into itself, by the Fredholm alternative and $R^* U_\nu^* g \in L_{t,div}^2(\partial D)$ we can conclude that A_ν^* maps $L^2(\partial B)$ boundedly into $L_{t,div}^2(\partial D)$.

3. Inverse problem with normal dipoles

We now consider the special case of the exterior Neumann problem (2.1)–(2.3) with the boundary data given by a dipole with polarization vector p located in some point $y \in \mathbb{R}^3 \setminus \bar{D}$, i.e., the field $v = v(\cdot\,, y, p)$ is harmonic in $\mathbb{R}^3 \setminus \bar{D}$, satisfies the boundary condition (1.2), has zero flux and vanishes at infinity. We note that in view of (2.4) the solvability condition $\operatorname{Div} c$ is satisfied for the particular boundary data (1.2). The inverse problem we want to consider in this section is, given $\nu(x) \cdot v(x, y, \nu(y))$ for all $x, y \in \partial B$, determine the shape of D. We will develop an explicit characterization of the unknown domain D in terms of spectral data of the integral operator $W_\nu : L^2(\partial B) \to L^2(\partial B)$ with kernel w_ν as defined by (1.5).

In the sequel, by $(\cdot\,, \cdot)$ we denote the inner product in $L^2(\partial B)$ and $L^2(\partial D)$, respectively. We define a bounded operator $N : L_{t,div}^2(\partial D) \to L_t^2(\partial D)$ by

$$N = \operatorname{Grad} S \operatorname{Div}, \tag{3.1}$$

where $S : L^2(\partial D) \to L^2(\partial D)$ denotes the single-layer potential operator

$$(S\varphi)(x) := \int_{\partial D} \Phi(x,y)\, \varphi(y)\, ds(y), \quad x \in \partial D.$$

The operator N is well defined, since S maps $L^2(\partial D)$ boundedly into the Sobolev space $H^1(\partial D)$ (see Theorem 3.6 in [8]).

Theorem 3.1. *The operator N is negative semi-definite with respect to the L^2 inner product. The nullspace of N is given by $L_{t,div}^{2,0}(\partial D)$.*

Proof. This follows from the Gauss surface divergence theorem, i.e.,

$$(Na, b) = (\operatorname{Grad} S \operatorname{Div} a, b) = -(S \operatorname{Div} a, \operatorname{Div} b), \tag{3.2}$$

and the positive definiteness of the single-layer operator S. The latter is a consequence of the relation

$$(S\varphi, \varphi) = \int_{\mathbb{R}^3} |\operatorname{grad} u|^2 dx$$

for the single-layer potential u with density φ which follows by applying Green's theorem and using the jump relations. □

For the following theorem we note that $A_\nu N A_\nu^*$ is well defined on $L^2(\partial B)$, since A_ν^* is bounded from $L^2(\partial B)$ into $L^2_{t,div}(\partial D)$ and N is bounded from $L^2_{t,div}(\partial D)$ into $R(L^{2,0}_{t,div}(\partial D))$ as consequence of (2.4).

Theorem 3.2. *The operators W_ν, A_ν, and N are related through*

$$W_\nu = -A_\nu N A_\nu^*. \tag{3.3}$$

Proof. Define operators $P_\nu : L^2_t(\partial D) \to L^2(\partial B)$ and $P_\nu^* : L^2(\partial B) \to L^2_t(\partial D)$ by

$$(P_\nu a)(x) := \nu(x) \cdot \operatorname{curl} \operatorname{curl} \int_{\partial D} \Phi(x,y) a(y)\, ds(y), \quad x \in \partial B,$$

and

$$(P_\nu^* g)(x) := -\operatorname{Grad} \int_{\partial B} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\, g(y)\, ds(y), \quad x \in \partial D.$$

Then, using (2.10), for $a \in L^2_{t,div}(\partial D)$ we can write

$$(P_\nu a)(x) = \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(x)} \operatorname{Div} a(y)\, ds(y), \quad x \in \partial B.$$

Now, we can interchange orders of integration and use the Gauss surface divergence theorem to see that P_ν and P_ν^* are adjoint with respect to the L^2 inner product, since $L^2_{t,div}(\partial D)$ is dense in $L^2_t(\partial D)$. Noting that

$$\operatorname{div}_x \{\Phi(\cdot\,, y)\, g(y) \nu(y)\} = -\frac{\partial \Phi(\cdot\,, y)}{\partial \nu(y)}\, g(y)$$

for $g \in L^2(\partial B)$ from the boundary condition (1.2) we have that

$$\nu(x) \cdot v(x, y, g(y)\nu(y)) = \left(A_\nu \operatorname{Grad} \frac{\partial \Phi(\cdot\,, y)}{\partial \nu(y)}\, g(y) \right)(x), \quad x, y \in \partial B.$$

Integrating this over ∂B and using the boundedness of A_ν we deduce that

$$W_\nu = -A_\nu P_\nu^*. \tag{3.4}$$

With the aid of (2.10), by the definition of the operators, we have $P_\nu = A_\nu N$ on $L^2_{t,div}(\partial D)$, whence

$$P_\nu^* = N A_\nu^* \tag{3.5}$$

on $L^2(\partial B)$ follows, since $L^2_{t,div}(\partial D)$ is dense in $L^2_t(\partial D)$. Inserting this into (3.4) completes the proof. □

As a first consequence of the factorization (3.3) we note that the operator $W : L^2(\partial B) \to L^2(\partial B)$ is compact, since $A : L^2(\partial D) \to L^2(\partial B)$ is compact and $N A_\nu^* : L^2(\partial B) \to L^2(\partial D)$ is bounded.

Theorem 3.3. *The compact operator $W_\nu : L^2(\partial B) \to L^2(\partial B)$ is positive semi-definite. The nullspace of W_ν is given by* $\operatorname{span}\{1\}$ *and there exists a complete orthonormal system g_n, $n = 1, 2, \ldots$, of $L_0^2(\partial B)$ of eigenelements of W_ν with positive eigenvalues λ_n, i.e.,*

$$W_\nu g_n = \lambda_n g_n, \quad n = 1, 2, \ldots . \tag{3.6}$$

Proof. From (3.3) and the self-adjointness of N we observe that W_ν is a self-adjoint operator. For $g \in L^2(\partial B)$ we have $(W_\nu g, g) = -(NA_\nu^* g, A_\nu^* g)$ and the negative semi-definiteness of N implies that W_ν is positive semi-definite. From (3.2) and the positive definiteness of S it follows that equality $(W_\nu g, g) = 0$ holds if and only if $NA_\nu^* g = 0$.

To characterize the nullspace of W_ν, as consequence of (3.5) we note that the nullspaces of NA_ν^* and P_ν^* coincide. If $P_\nu^* g = 0$ then the double-layer potential

$$u(x) := \int_{\partial B} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\, g(y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \partial B, \tag{3.7}$$

satisfies $\operatorname{Grad} u = 0$ on ∂D, that is, $u = \text{const}$ on ∂D. Consequently, by the maximum-minimum principle for harmonic functions and analyticity we have $u = \text{const}$ in B. The jump-relations for the double-layer potential in the L^2 sense (see [8] p. 45) imply that $g - K_B g = 0$ where

$$(K_B g)(x) := 2 \int_{\partial B} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\, g(y)\, ds(y), \quad x \in \partial B.$$

This implies that g is continuous, since with the help of the Fredholm alternative it can be seen that the nullspaces of $I - K_B$ with respect to $L^2(\partial B)$ and $C(\partial B)$ coincide (see [8] p. 59). Hence, because the normal derivative of the double layer potential with continuous density is continuous across ∂B, from the uniqueness for the exterior Neumann problem for Laplace's equation we have $u = 0$ in $\mathbb{R}^3 \setminus B$. Therefore, again the jump relations finally imply that $g = \text{const}$. Conversely, if $g = \text{const}$ then, by Green's theorem, the double-layer potential u defined by (3.7) is constant in B, whence $P_\nu^* g = -\operatorname{Grad} u|_{\partial D} = 0$ follows.

Because in view of (2.12) we may consider W_ν as a compact and positive definite operator from $L_0^2(\partial B)$ into itself, the statement on the eigenvalues and eigenfunctions are straightforward consequences of the spectral theory for such operators (see Theorem 15.12 in [13]). □

Since the single-layer potential operator $S : L^2(\partial D) \to L^2(\partial D)$ is compact and positive definite, there exists a well defined compact and positive definite operator $S^{1/2} : L^2(\partial D) \to L^2(\partial D)$ that satisfies $S^{1/2} S^{1/2} = S$ (see (3.13) below). Because S is a bounded operator from the Sobolev space $H^{-1/2}(\partial D)$ into $H^{1/2}(\partial D)$ (see [18]) we have that

$$\sup_{\|\psi\|_{H^{-1/2}}=1} \|S^{1/2}\psi\|_{L^2}^2 = \sup_{\|\psi\|_{H^{-1/2}}=1} (S\psi, \psi) \le c^2$$

for some constant $c > 0$. Therefore

$$\|S^{1/2}\varphi\|_{H^{1/2}} = \sup_{\|\psi\|_{H^{-1/2}}=1} |(S^{1/2}\varphi, \psi)| = \sup_{\|\psi\|_{H^{-1/2}}=1} |(\varphi, S^{1/2}\psi)| \le c\|\varphi\|_{L^2}$$

for all $\varphi \in L^2(\partial D)$, that is, $S^{1/2}$ is a bounded operator from $L^2(\partial D)$ into $H^{1/2}(\partial D)$.

Using the eigenfunctions g_n of the operator W_ν we are now in a position to define the functions

$$\varphi_n := -\frac{1}{\sqrt{\lambda_n}}\, S^{1/2}\,\mathrm{Div}\, A_\nu^* g_n, \quad n = 1, 2, \ldots . \tag{3.8}$$

Since $A_\nu^* g_n \in L^2_{t,div}(\partial D)$ we have $\varphi_n \in H^{1/2}(\partial D)$. Using the Gauss surface divergence theorem, (3.3), (3.6), and the orthonormality of the g_n we find that

$$(\varphi_n, \varphi_m) = -\frac{1}{\sqrt{\lambda_n \lambda_m}}\, (g_n, A_\nu\,\mathrm{Grad}\, S\,\mathrm{Div}\, A_\nu^* g_n) = (g_n, g_m),$$

i.e., the φ_n form an orthonormal system in $L^2(\partial D)$.

For a constant unit vector e and $z \in B$ we define

$$\Psi(x, z) := \nu(x) \cdot \mathrm{grad}_x\, \mathrm{div}_x\{\Phi(x, z)e\}, \quad x \in \partial B.$$

In view of

$$\mathrm{grad}\,\mathrm{div}\{\Phi(\cdot\,, z)e\} = \mathrm{curl}\,\mathrm{curl}\{\Phi(\cdot\,, z)e\}, \tag{3.9}$$

by Stokes' theorem, we have $\Psi(\cdot\,, z) \in L^2_0(\partial B)$. Hence the Fourier series

$$\Psi(\cdot\,, z) = \sum_{n=1}^{\infty} (\Psi(\cdot\,, z), g_n) g_n \tag{3.10}$$

converges in $L^2(\partial B)$, because the g_n are complete in $L^2_0(\partial B)$.

Theorem 3.4. *The point $z \in B$ belongs to D if and only if*

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n}\, |(\Psi(\cdot\,, z), g_n)|^2 < \infty. \tag{3.11}$$

Proof. Let $z \in D$. Then as consequence of of (3.9) the field

$$H(\cdot, z) := \mathrm{grad}\,\mathrm{div}\{\Phi(\cdot\,, z)e\}$$

is harmonic in $\mathbb{R}^3 \setminus \{z\}$, vanishes at infinity, and, by Stokes' theorem, has vanishing flux through ∂D. Therefore, setting

$$b(\cdot\,, z) := \mathrm{Grad}\,\mathrm{div}\{\Phi(\cdot\,, z)e\}|_{\partial D}$$

we have $A_\nu b = \Psi$. Define $f \in H^1(\partial D)$ by $f(\cdot\,, z) := \mathrm{div}\{\Phi(\cdot\,, z)e\}|_{\partial D}$. Then there exists a function $\psi \in L^2(\partial D)$ such that $S\psi = f$, since S is an isomorphism from $L^2(\partial D)$ into $H^1(\partial D)$ (see [18]). Hence, we have

$$\Psi = A_\nu b = A_\nu\,\mathrm{Grad}\, f = A_\nu\,\mathrm{Grad}\, S\psi$$

and consequently $\Psi = A_\nu\,\mathrm{Grad}\, S^{1/2}\varphi$, where $\varphi = S^{1/2}\psi$ is in $H^{1/2}(\partial D)$. Now, from (3.3) and (3.8) we find that

$$\sqrt{\lambda_n}\,(\varphi, \varphi_n) = -(\varphi, S^{1/2}\,\mathrm{Div}\, A_\nu^* g_n) = (A_\nu\,\mathrm{Grad}\, S^{1/2}\varphi, g_n) = (\Psi, g_n)$$

for $n = 1, 2, \ldots$ and inserting this into Bessel's inequality for φ in terms of the orthonormal system (3.8) yields the convergence of the series (3.11).

Conversely, assume that the series (3.11) converges. Then,

$$\varphi := \sum_{n=1}^{\infty} \frac{1}{\sqrt{\lambda_n}} \, (\Psi(\cdot\,, z), g_n)\varphi_n$$

defines a function $\varphi \in L^2(\partial D)$. Using (3.3), (3.6), and (3.8) we obtain the relation

$$A_\nu \operatorname{Grad} S^{1/2}\varphi_n = \sqrt{\lambda_n}\, g_n, \quad n = 1, 2, \ldots . \tag{3.12}$$

From the well-posedness of the exterior Dirichlet problem for Laplace's equation with $H^{1/2}(\partial D)$ boundary data it can be seen that the operator A_ν maps $\operatorname{Grad}(H^{1/2}(\partial D))$ boundedly on $L^2(\partial B)$. Therefore, with the aid of the boundedness of the operator $S^{1/2} : L^2(\partial D) \to H^{1/2}(\partial D)$, (3.10), and (3.12) we deduce that φ satisfies $A_\nu \operatorname{Grad} S^{1/2}\varphi = \Psi(\cdot\,, z)$. By the uniqueness for the Neumann problem for harmonic vector fields in the exterior of B and analyticity, this implies that for the weak solution $v \in L^2_{loc}(\mathbb{R}^3 \setminus \bar{D})$ to the Dirichlet problem for harmonic vector fields in the exterior of D with tangential component $\operatorname{Grad} S^{1/2}\varphi$ we have $v = H(\cdot\,, z)$. Therefore the point z cannot belong to $B \setminus D$, since $H(\cdot\,, z) \notin L^2_{loc}(\mathbb{R}^3 \setminus \bar{D})$ if $z \in B \setminus D$. □

Since the operator W_ν is compact, self-adjoint, and positive semi-definite the square root operator $W_\nu^{1/2} : L^2(\partial B) \to L^2(\partial B)$ is well defined by

$$W_\nu^{1/2} g = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\, (g, g_n) g_n. \tag{3.13}$$

In terms of this operator, by using Picard's theorem (see Theorem 15.18 in [13]), we can reformulate Theorem 3.4 as the following corollary.

Corollary 3.5. *The point $z \in B$ belongs to D if and only if the operator equation*

$$W_\nu^{1/2} g = \Psi(\cdot\,, z)$$

has a solution $g \in L^2(\partial B)$.

Corollary 3.6. *The domain D is uniquely determined by a knowledge of $\nu(x) \cdot v(x, y, \nu(y))$ for $x, y \in \partial B$.*

Proof. This is an immediate consequence of Theorem 3.4. □

4. Inverse problem with tangential dipoles

We now turn to the case of the inverse problem using tangential dipole fields $g \in L^2_t(\partial D)$ and the tangential components of $v(x, y, g(y))$ as data. Again we will develop an explicit characterization of the unknown domain D in terms of spectral data of the integral operator $W : L^2_t(\partial B) \to L^2_t(\partial B)$ with kernel w as defined by (1.4). For doing so, we need to replace the operator A_ν mapping the tangential components of harmonic vector fields on ∂D onto the normal components on ∂B by the operator

A that maps the tangential components on ∂D onto the tangential components on ∂B, i.e.,

$$A : \{\nu \times v|_{\partial D}\} \times \nu \mapsto \{\nu \times v|_{\partial B}\} \times \nu.$$

Then (2.11) has to be replaced by

$$A = 2U(I + M)^{-1}R, \tag{4.1}$$

where $U : L_t^2(\partial D) \to L_t^2(\partial B)$ now is given by

$$(Ua)(x) := \left[\nu(x) \times \operatorname{curl} \int_{\partial D} \Phi(x, y)a(y)\, ds(y)\right] \times \nu(x), \quad x \in \partial B.$$

For the analogue of Theorem 2.1 we only need to be concerned with the injectivity of U. However, the injectivity of $U : L_{t,div}^{2,0}(\partial D) \to L_t^2(\partial B)$ is obtained as in the proof of Theorem 2.1 by using the uniqueness for the Dirichlet problem for harmonic vector fields with zero flux through ∂B in the exterior of B rather than the uniqueness for the Neumann problem. Hence we can state the following theorem.

Theorem 4.1. *The compact operator $A : L_t^2(\partial D) \to L_t^2(\partial B)$ is injective on the subspace $R(L_{t,div}^{2,0}(\partial D))$.*

As in Section 2 we note that due to the solvability condition $\operatorname{Div} c = 0$ the operator A given by (4.1) corresponds to the mapping of the tangential components of harmonic vector fields on ∂D onto the tangential components on ∂B only on the subspace $R(L_{t,div}^{2,0}(\partial D))$.

The L^2 adjoint $A^* : L_t^2(\partial B) \to L_t^2(\partial D)$ is given by

$$A^* = 2R^*(I + M^*)^{-1}U^*, \tag{4.2}$$

where

$$(U^*g)(x) = \left[\nu(x) \times \operatorname{curl} \int_{\partial B} \Phi(x, y)g(y)\, ds(y)\right] \times \nu(x), \quad x \in \partial D,$$

is the adjoint $U^* : L_t^2(\partial B) \to L_t^2(\partial D)$ of U. This, as in Section 2, can be used to show that A^* maps $L_t^2(\partial B)$ boundedly into $L_{t,div}^2(\partial D)$.

For the sequel, the spaces $L_{t,div}^2(\partial B)$ and $L_{t,div}^{2,0}(\partial B)$ are defined analogously to the corresponding spaces on ∂D.

Theorem 4.2. *The operators W, A, and N are related through*

$$W = -ANA^*. \tag{4.3}$$

Proof. Define operators $P : L_t^2(\partial D) \to L_t^2(\partial B)$ and $P^* : L_t^2(\partial B) \to L_t^2(\partial D)$ by

$$(Pa)(x) := \operatorname{Grad} \operatorname{div} \int_{\partial D} \Phi(x, y)a(y)\, ds(y), \quad x \in \partial B,$$

and

$$(P^*g)(x) := \operatorname{Grad} \operatorname{div} \int_{\partial B} \Phi(x, y)g(y)\, ds(y), \quad x \in \partial D.$$

In view of (2.9) and (2.10), for $a \in L^2_{t,div}(\partial D)$ and $g \in L^2_{t,div}(\partial B)$, we can write

$$(Pa)(x) := \operatorname{Grad} \int_{\partial D} \Phi(x,y) \operatorname{Div} a(y)\, ds(y), \quad x \in \partial B,$$

and

$$(P^* g)(x) := \operatorname{Grad} \int_{\partial B} \Phi(x,y) \operatorname{Div} g(y)\, ds(y), \quad x \in \partial D,$$

and from this interchanging integrations and employing a denseness argument it can be seen that P and P^* are adjoint with respect to the L^2 inner product. Integrating

$$v(x,y,g(y)) = -\left(A \operatorname{Grad} \operatorname{div}\{\Phi(\cdot\,,y)g(y)\}\right)(x)$$

over ∂B and using the boundedness of A we can deduce that

$$W = -AP^*. \tag{4.4}$$

Using (2.10), by the definition of the operators, we have $P = AN$, whence

$$P^* = NA^* \tag{4.5}$$

follows. Inserting this into (4.4) completes the proof. □

Theorem 4.3. *The compact operator $W : L^2_t(\partial B) \to L^2_t(\partial B)$ is positive semi-definite. The nullspace of W is given by $L^{2,0}_{t,div}(\partial B)$ and there exists an orthonormal system $g_n \in L^2_t(\partial B)$ of eigenelements of W with positive eigenvalues λ_n, i.e.,*

$$W_\nu g_n = \lambda_n g_n, \quad n = 1, 2, \ldots . \tag{4.6}$$

The eigenelements are complete in the orthogonal complement of $L^{2,0}_{t,div}(\partial B)$.

Proof. As in the proof of Theorem 3.3, from (4.3) we deduce that W is self-adjoint and that equality holds in $(Wg, g) \geq 0$ if and only if $P^* g = 0$. To characterize the nullspace of P^* let $P^* g = 0$. Then

$$u(x) := \operatorname{div} \int_{\partial B} \Phi(x,y) g(y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \partial B, \tag{4.7}$$

satisfies $\operatorname{Grad} u = 0$ on ∂D. As in the proof of Theorem 3.3 this implies that $u = \text{const}$ in B. Using the jump relations in the L^2 sense for the gradient of a single-layer potential (see [8] p. 45), the transformation (2.9) and Stokes' theorem, arguing as in Section 2 for the uniqueness of the solution to the exterior Dirichlet problem we can conclude that $u = 0$ in $\mathbb{R}^3 \setminus B$. In view of the jump relations this implies that $u = 0$ everywhere. Now let $f \in H^1(\partial B)$ and denote by S_B the single-layer operator for the domain B. Since S_B is an isomorphism from $L^2(\partial B)$ onto $H^1(\partial B)$, there exists $\varphi \in L^2(\partial B)$ such that $f = S_B \varphi$. Then we can interchange orders of integration to obtain that

$$(\operatorname{Grad} f, g) = (\operatorname{Grad} S_B \varphi, g) = -(\varphi, u) = 0$$

for the inner product in $L^2(\partial B)$. Hence $g \in L^2_{t,div}(\partial B)$ with $\operatorname{Div} g = 0$. Conversely from (2.9), (2.10), and the definition of P^* it is obvious that for $g \in L^2_{t,div}(\partial B)$ with $\operatorname{Div} g = 0$ we have that $P^* g = 0$. Now as in the proof of Theorem 3.3 the statement

on the eigenelements and eigenvalues follows from the spectral theorem. □

Using the eigenelements g_n of the operator W we can define the functions

$$\varphi_n := -\frac{1}{\sqrt{\lambda_n}}\, S^{1/2}\,\mathrm{Div}\, A^* g_n, \quad n = 1, 2, \ldots . \tag{4.8}$$

As in Section 3 it can be seen that $\varphi_n \in H^{1/2}(\partial D)$ and that the φ_n form an orthonormal system in $L^2(\partial D)$.

For a constant unit vector e and $z \in B$ we define

$$\chi(x,z) := [\nu(x) \times \mathrm{grad}_x \,\mathrm{div}_x\{\Phi(x,z)e\}] \times \nu(x), \quad x \in \partial B.$$

With the aid of the Gauss surface divergence theorem it can be seen that $\chi(\cdot\,, z)$ is orthogonal to the nullspace $L^{2,0}_{t,div}(\partial B)$ of W. Therefore, we have the Fourier series

$$\chi(\cdot\,, z) = \sum_{n=1}^{\infty} (\Psi(\cdot\,, z), g_n) g_n. \tag{4.9}$$

Theorem 4.4. *The point $z \in B$ belongs to D if and only if*

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} \,|(\chi(\cdot\,, z), g_n)|^2 < \infty. \tag{4.10}$$

Proof. If $z \in D$ then, obviously, we can write $\chi = A\,\mathrm{Grad}\, S^{1/2}\varphi$ with φ given as in the proof of Theorem 3.4 and from this convergence of the series (4.10) follows. Conversely, if the series (4.10) converges then

$$\varphi := \sum_{n=1}^{\infty} \frac{1}{\sqrt{\lambda_n}}\, (\chi(\cdot\,, z), g_n)\varphi_n$$

defines a function $\varphi \in L^2(\partial D)$. Again proceeding as in the proof of Theorem 3.4, with the aid of the expansion (4.9) it can be seen that φ satisfies $A\,\mathrm{Grad}\, S^{1/2}\varphi = \chi(\cdot\,, z)$. Now, using the uniqueness for the exterior Dirichlet problem for harmonic vector fields instead of the uniqueness for the exterior Neumann problem continuing as in the proof of Theorem 3.4, it can be shown that z cannot belong to $B \setminus D$. □

Corollary 4.5. *The point $z \in B$ belongs to D if and only if the operator equation*

$$W^{1/2} g = \chi(\cdot\,, z)$$

has a solution $g \in L^2(\partial B)$.

Corollary 4.6. *The domain D is uniquely determined by a knowledge of $\nu(x) \times v(x,y,p)$ for all $x, y \in \partial B$ and $p \in \mathbb{R}^3$ with $p \cdot \nu(y) = 0$.*

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