On a Subfactor Construction of a Factor Not Anti–Isomorphic to Itself

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Abstract

Let α be a $\mathbb{Z}_3$–kernel of $\mathfrak{L}\left(F_{\frac{11}{3}}\right)$ with obstruction $\epsilon = e^{\frac{2\pi i}{3}}$ to lifting and β a $\mathbb{Z}_3$–kernel of the hyperfinite factor R with obstruction $\bar{\epsilon}$ to lifting. Define an action γ of $\mathbb{Z}_3$ on $\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R$ by $\alpha \otimes \beta$, up to an inner automorphism. The aim of this paper is to show that the factor $M = \left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R\right) \rtimes_\gamma \mathbb{Z}_3$, which is similar to Connes' example of a factor which is not anti–isomorphic to itself, is the enveloping algebra of an inclusion of II_1 factors $A \subset B$. Here A is a free group factor and B is isomorphic to the crossed product $A \rtimes_\theta \mathbb{Z}_9$, where θ is a $\mathbb{Z}_3$–kernel of A with non-trivial obstruction to lifting. By using a Connes' argument involving the invariant $\chi(M)$, we will verify that M is not anti–isomorphic to itself. Furthermore, we will prove that for one of the generator of $\chi(M)$, which we will denote by σ, the Jones invariant $\varkappa(\sigma)$ is equal to $e^{\frac{2\pi i}{9}}$.

1 Introduction

In this paper we are going to show that there exists an inclusion $A \subset B$ of free group factors, which by iteration of the Jones basic construction give us a II_1 factor which is not anti–isomorphic to itself. The II_1 factor we obtain is essentially the same example given by Connes in [Co3] and [Co4] of a II_1 factor not anti–isomorphic to itself.

An anti–isomorphism of a von Neumann algebra M is a vector space isomorphism $\Phi : M \to M$ with $\Phi(x^*) = \Phi(x)^*$ and $\Phi(xy) = \Phi(y)\Phi(x)$. Note

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that a von Neumann algebra M is anti–isomorphic to itself if and only if it is isomorphic to its conjugate algebra M^c (cf. Section 6).

In Section 3 we will first give a slightly different description of the Connes' type II_1 factor which is not anti–isomorphic to itself (see [Co3] or [Co4] for more details on the Connes factor). In our approach we will use the recently developed theory of free group factors. The II_1 factor M we are going to study is constructed from the tensor product of a free group factor and the hyperfinite II_1 factor R. We will use two $\mathbb{Z}_3$–kernels, $\alpha \in \mathrm{Aut}\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right)\right)$ and $\beta \in \mathrm{Aut}(R)$, which have conjugate obstructions to lifting, to generate an action of $\mathbb{Z}_3$ on $\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R$. The action is given, up to an inner automorphism, by $\alpha \otimes \beta$, and the factor M is equal to the crossed product $\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R\right) \rtimes_\gamma \mathbb{Z}_3$ (cf. Section 4).

The main result of this paper is Theorem 4.3, where we show that M is the enveloping algebra of an inclusion $A \subset B$ of free group factors, with A isomorphic to $\mathfrak{L}\left(F_{\frac{35}{27}}\right)$ and $B = A \rtimes_\theta \mathbb{Z}_9$, for a $\mathbb{Z}_3$–kernel θ of A with obstruction $e^{\frac{2\pi i}{3}}$ to lifting.

Though extensive work has been done both by V. Jones and A. Connes on examples of II_1 and III_λ factors non anti-isomorphic to itself, it does not seem that the possibility of realizing any of these factors through a subfactor construction has been really studied, and there is little literature on the subject. The result we will present here is a generalization of a similar result obtained by F. Rădulescu in the case of $\mathbb{Z}_2$–kernels. The proof is mainly based on Voiculescu's random matrix model introduced in [Vo].

In Section 5 we will use Connes' exact sequence (see [Co4]) to show that for the factor M the Connes invariant $\chi(M)$ is equal to $\mathbb{Z}_9$, a result due to Connes ([Co3] and [Co4]). Furthermore, if σ denotes the generator of $\chi(M)$ described in Remark 5.4, then the invariant $\varkappa(\sigma)$ defined by Jones in [Jo2] is equal to $e^{\frac{2\pi i}{9}}$.

In the last section, we will use an argument of Connes (see [Co3]) to show that $M = \left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R\right) \rtimes_\gamma \mathbb{Z}_3$ is not anti-isomorphic to itself. The main ingredients of this argument are the uniqueness (up to inner automorphism) of the decomposition of γ into the product of an approximately inner automorphism and a centrally trivial automorphism and the fact that the only subgroup of order 3 in $\chi(M)$ is generated by the dual action $\hat{\gamma}$ on $M = \left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R\right) \rtimes_\gamma \mathbb{Z}_3$. This will provide us with a canonical way to associate to a von Neumann algebra a complex number, which is invariant under isomorphism, and distinguishes M from its conjugate algebra M^c.

2 Definitions

Let M be a finite factor with trace τ. For a unitary u in M we denote by $\mathrm{Ad}_M\, u$ the inner automorphism of M defined by $\mathrm{Ad}_M\, u(x) = uxu^*$. We let $\overline{\mathrm{Int}(M)}$ be the closure of the group $\mathrm{Int}(M)$ of inner automorphisms of M in the topology of pointwise weak convergence.

We say that an automorphism $\alpha \in \mathrm{Aut}(M)$ is centrally trivial if for any central sequence $x_n \in M$ (i.e. a bounded sequence in M with the property $\lim_{n\to\infty} \|[x_n, y]\|_2 = 0$ for every $y \in M$) we have $\lim_{n\to\infty} \|\alpha(x_n) - x_n\|_2 = 0$. We denote by $\mathrm{Ct}(M)$ the set of centrally trivial automorphisms of M.

The Connes invariant is defined as follows:

$$\chi(M) = \frac{\mathrm{Ct}(M) \cap \overline{\mathrm{Int}(M)}}{\mathrm{Int}(M)}.$$

A central sequence (x_n) is said to be hypercentral if $\lim_{n\to\infty} \|[x_n, y_n]\|_2 = 0$ for every central sequence $(y_n) \in M$. Given a free ultrafilter ω we denote by C_ω the algebra of bounded sequence (x_n) in M with $\lim_{n\to\omega} \|[x_n, y]\|_2 = 0$ for all $y \in M$, and by $\mathfrak{I}_\omega$ the subalgebra of C_ω consisting of central sequences which tend $*$-strongly to zero. For $x = (x_n) \in C_\omega$, we set $\tau^\omega(x) = \lim_{n\to\omega} \tau(x_n)$. Then $M_\omega = C_\omega/\mathfrak{I}_\omega$ is a von Neumann algebra with trace τ^ω.

Remark 2.1 *By [McD] the existence of non-trivial hypercentral sequences is equivalent to the non-triviality of the center of M_ω for some (and then for all) free ultrafilter ω.*

Definition 2.2 *A $\mathbb{Z}_3$–kernel on a von Neumann algebra M is an automorphism $\alpha \in \mathrm{Aut}(M)$ such that there exists a unitary U in M with the property $\alpha^3 = \mathrm{Ad}_M\, U$.*

Note that if α is a $\mathbb{Z}_3$–kernel then $\alpha(U) = \lambda U$, for λ a cube root of unity. If $\lambda \neq 1$ we say that α has obstruction λ to lifting, meaning that such an automorphism cannot be lifted to an action of $\mathbb{Z}_3$ on M.

We conclude this section by defining an invariant $\varkappa(\sigma)$ for any element σ in $\chi(M)$. Define $\mathrm{Out}(M) = \frac{\mathrm{Aut}(M)}{\mathrm{Int}(M)}$, and let $\xi : \mathrm{Aut}(M) \to \mathrm{Out}(M)$ be the usual quotient map.

Definition 2.3 *Let M be a II_1 factor without non-trivial hypercentral sequences and $\sigma \in \chi(M)$. Let ϕ be any automorphism on M with $\xi(\phi) = \sigma$ and u_n a sequence of unitaries with the properties*

$$\phi = \lim_{n\to\infty} \mathrm{Ad}\ u_n \quad \textit{and} \quad \lim_{n\to\infty} \|\phi(u_n) - \lambda_\phi u_n\|_2 = 0$$

for some λ_ϕ of modulus one (such sequence always exists since $\phi \in \overline{\mathrm{Int}(M)}$). Define $\varkappa(\sigma) = \lambda_\phi$.

Jones proved in [Jo2] that this definition makes sense (i.e. $\varkappa(\sigma)$ does not depend on the choice of ϕ or u_n) and that $\varkappa$ is a conjugacy invariant, meaning that if $\alpha, \beta \in \mathrm{Ct}(M) \cap \overline{\mathrm{Int}(M)}$ and there exists $\psi \in \mathrm{Aut}(M)$ such that $\psi\alpha\psi^{-1} = \beta$, then $\varkappa(\xi(\alpha)) = \varkappa(\xi(\beta))$.

3 Preliminaries

A von Neumann algebra M is said to be full if $\overline{\mathrm{Int}(M)} = \mathrm{Int}(M)$. Obviously all type I factors are full, while the hyperfinite factor R provides an example of a II_1 factor which is not full since $\overline{\mathrm{Int}(R)} = \mathrm{Aut}(R) \neq \mathrm{Int}(R)$.

Remark 3.1 *For an arbitrary factor, being full is equivalent to having no non-trivial central sequence (see [Co1]).*

The following result is an easy consequence of Lemma 4.3.3 in [Sa].

Lemma 3.2 *Let G be a discrete group containing a non-abelian free group and let τ be the usual trace on $\mathfrak{L}(G)$. Then any central sequence in $\mathfrak{L}(G)$ satisfies* $\lim_{n\to\infty} \|x_n - \tau(x_n)\| = 0$.

Proof Set $E = G - \{e\}$, where e denotes the identity element in G. Let g_1, g_2 be two generators of the free group and $F = \{g \in E \,|\, g = g_1\tilde{g},\ \tilde{g} \in E\}$. Take $x \in \mathfrak{L}(G)$. Then x can be expressed as $x = \sum_{g\in G} \lambda_g \delta_g$ and the function $f : G \to \mathbb{C}$ defined by $f(g) = \lambda_g$ belongs to $l^2(G)$.

For such f we have that

$$\sum_{g\in E} |f(g)|^2 = \|x - \tau(x)\|_2^2 \quad \text{and} \quad \sum_{g\in G} |f(g_i g g_i^{-1}) - f(g)|^2 = \|[x, \delta_{g_i}]\|_2^2.$$

Now if (x_n) is a central sequence in $\mathfrak{L}(G)$ then $\|[x_n, \delta_{g_i}]\| \to 0$ for $n \to \infty$, hence we can apply Lemma 4.3.3 in [Sa] and conclude that $\|x_n - \tau(x_n)\|_2 \to 0$ for $n \to \infty$.

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By applying the previous lemma and Lemma 2 in [Jo1] (see also Lemma 15.42 in [EK] for a printed version) to the factor $\mathfrak{L}\left(F_{\frac{11}{3}}\right) \cong \mathfrak{L}(F_3 * \mathbb{Z}_3)$, we obtain the following corollary

Corollary 3.3 $\mathfrak{L}\left(F_{\frac{11}{3}}\right)$ *is full.*

Proposition 3.4 $\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R$ *has no non-trivial hypercentral sequences.*

Proof First we want to show that any central sequence in $\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R$ has the form $(1 \otimes x_n)$, for a central sequence (x_n) in R.

By the proof of Lemma 3.2, $\mathfrak{L}\left(F_{\frac{11}{3}}\right)$ satisfies the hypothesis of Lemma 2.11 in [Co1] with $Q_1 = \mathfrak{L}\left(F_{\frac{11}{3}}\right)$, $Q_2 = R$ and $b_i = \delta_{g_i}$, where g_i is a generator of F_2 for $i = 1, 2$ and $\mathfrak{L}(F_2) \subset \mathfrak{L}\left(F_{\frac{11}{3}}\right)$. Therefore we can apply Lemma 2.11 to any central sequence (X_n) in $\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R$ to obtain that $\|X_n - (\tau \otimes 1)(X_n)\|_2 \to 0$ for $n \to \infty$. Since $(\tau \otimes 1)(X_n) \in \mathbb{C} \otimes R$ this implies that $X_n = 1 \otimes x_n$ for a central sequence (x_n) in R.

Now suppose (Y_n) is a hypercentral sequence in $\mathfrak{L}\left(F_{\frac{11}{3}}\right)\otimes R$. Since (Y_n) is central it has the form $Y_n = 1\otimes y_n$, for a hypercentral sequence (y_n) in R. So we need only to prove that R has no non-trivial hypercentral sequences.

By Remark 3.1 it is enough to show that R_ω has trivial center. But this follows from Theorem 15.15 in [EK].

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Theorem 3.5 *If* $\alpha \in \mathrm{Ct}\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right)\otimes R\right)$ *then* $\alpha = \mathrm{Ad}\ z(\nu\otimes id)$ *for some unitary* $z\in\mathfrak{L}\left(F_{\frac{11}{3}}\right)\otimes R$ *and an automorphism* ν *of* $\mathfrak{L}\left(F_{\frac{11}{3}}\right)$.

Proof Let $(K_n)_{n\in\mathbb{N}}$ be an increasing sequence of finite dimensional subfactors of R generating R and $R_n = K_n^{'}\cap R$ the relative commutant of K_n in R.

Set $L_n = 1\otimes R_n \subset \mathfrak{L}\left(F_{\frac{11}{3}}\right)\otimes R$. Then there exists an n_0 such that for all $x\in L_{n_0}$, $\|x\|\le 1$ one has $\|\alpha(x)-x\|_2 \le \frac{1}{2}$. In fact, otherwise it would exist a uniform bounded sequence (x_n), $x_n\in L_n$ and $\|x_n\|\le 1$, such that $\|\alpha(x_n)-x_n\|_2 > \frac{1}{2}$. But (x_n) is a central sequence in $\mathfrak{L}\left(F_{\frac{11}{3}}\right)\otimes R$ because for each m and $n\ge m$, x_n commutes with $\mathfrak{L}\left(F_{\frac{11}{3}}\right)\otimes K_m$, so we get a contradiction.

By Lemma 3.3 in [Co2], up to inner automorphism, α is of the form $\alpha_1\otimes 1_{R_{n_0}}$ where α_1 is an automorphism of $\mathfrak{L}\left(F_{\frac{11}{3}}\right)\otimes K_{n_0}$. Set $F = 1\otimes K_{n_0}$. Then F is a type I subfactor of $\mathfrak{L}\left(F_{\frac{11}{3}}\right)\otimes K_{n_0}$.

Applying [Lemma 3.11, [Co2]] to α_1 we obtain that $\alpha_1|_{1\otimes K_{n_0}} = \mathrm{Ad}\ V|_{1\otimes K_{n_0}}$. This implies that $\alpha = \mathrm{Ad}\ z(\nu\otimes 1)$ for some automorphism ν of $\mathfrak{L}\left(F_{\frac{11}{3}}\right)$.

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4 The realization of the crossed product $\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right)\otimes R\right)\rtimes_\gamma \mathbb{Z}_3$ as the enveloping algebra of an inclusion of free group subfactors

In this section we will show, using Voiculescu's random matrix model for free group algebras (see [Vo]), that the crossed product $\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right)\otimes R\right)\rtimes_\gamma \mathbb{Z}_3$ can be realized as the enveloping algebra of an inclusion of free group factors. To that purpose we will first give an explicit construction of $\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right)\otimes R\right)\rtimes_\gamma \mathbb{Z}_3$, by giving models for $\mathfrak{L}\left(F_{\frac{11}{3}}\right)$ and R.

Let $\{X_1,X_2,X_3\}$ be a free semicircular family and $u=\sum_{j=1}^3 e^{\frac{2\pi i j}{3}}e_j$ a unitary whose spectral projections have trace $\frac{1}{3}$, such that $alg\{e_j\}_{j=1}^3$ is free with respect to $\{X_1,X_2,X_3\}''$. Then $\mathfrak{L}\left(F_{\frac{11}{3}}\right)$ can be thought as the von Neumann algebra generated by $\{X_1,X_2,X_3,u\}$, as in [Vo] and [Ra1].

The model for R is outlined in the following lemma. It is analogous to the construction given in the case of $\mathbb{Z}_2$ by Rădulescu [Ra2] and some parts of the arguments are probably well-known in the literature on subfactors. We include them here for the sake of completeness.

Lemma 4.1 *Let $(U_k)_{k\in\mathbb{Z}}$ be a family of unitaries with nine point spectrum so that each spectral projection for U_k has trace $\frac{1}{9}$ and let $g = \sum_{j=1}^{3} e^{\frac{2\pi i j}{3}} g_j$ be a unitary whose spectral projections $\{g_j\}_{j=1}^{3}$ have trace $\frac{1}{3}$. We assume the following relations:*

(i) $U_k^9 = 1$ *for all* $k \in \mathbb{Z}$,

(ii) $U_k g U_k^* = e^{\frac{-2\pi i}{3}} g$ *if* $k = 0, -1$, *while* $U_k g U_k^* = g$ *if* $k \in \mathbb{Z}/\{0,-1\}$,

(iii) $U_k U_{k+1} U_k^* = e^{\frac{2\pi i}{9}} U_{k+1}$, *for* $k \in \mathbb{Z}$,

(iv) $U_i U_j = U_j U_i$ *if* $|i-j| \geq 2$.

We define a trace on the algebra generated by these unitaries requiring that each non-trivial monomial in the U_k's and g have zero trace. Let

$$R_{-1} = \{gU_0^3, U_1, U_2, \ldots\}'' \subset \{g, U_0, U_1, U_2, \ldots\}'' = R_0.$$

Then $R_{-1} \subset R_0$ is an inclusion of type II_1 factors of index 9 and the relative commutant $R_{-1}' \cap R_0$ is generated by the unitary g.

If $\theta = \mathrm{Ad}_{R_{-1}}(U_0)$ then θ is a outer automorphism of R_{-1} with $\theta^9 = Id$. Moreover, R_0 is equal to the crossed product $R_{-1} \rtimes_\theta \mathbb{Z}_9$ and θ defines a $\mathbb{Z}_3$–kernel on R_{-1} with obstruction $e^{\frac{-2\pi i}{3}}$ to lifting.

Also, the k-th step in the iterated Jones' basic construction for the inclusion $R_{-1} \subset R_0$ is given by

$$R_k = \{gU_{-1}^3 \ldots U_{-k}^3, U_{-k}, U_{-k+1}, \ldots\}'', \quad k \geq 1.$$

Proof The properties of the family $(U_k)_{k\in\mathbb{Z}}$ and of the unitary g imply immediately that the automorphism Ad U_0 normalizes R_{-1}. Furthermore g commutes with R_{-1} (and generates the relative commutant of R_{-1} in R_0) so that $Ad_{R_{-1}}(U_0^3) = Ad_{R_{-1}}(gU_0^3)$. Since $h = gU_0^3$ is a unitary in R_{-1} (with spectral projections of trace $\frac{1}{3}$) and $(\mathrm{Ad}_{R_{-1}} U_0)(h) = e^{\frac{-2\pi i}{3}} h$, we obtain that $\theta = \mathrm{Ad}_{R_{-1}}(U_0)$ is a $\mathbb{Z}_3$–kernel with obstruction $e^{\frac{-2\pi i}{3}}$ to lifting.

Therefore, at least algebraically, $\mathrm{Ad}_{R_{-1}}(U_0)$ implements the crossed product $R_{-1} \rtimes_\theta \mathbb{Z}_9$. But according to our definition, the trace on the algebra generated by the U_k's and g is compatible with the usual trace defined on the crossed product so $(R_{-1} \cup \{U_0\})'' = R_{-1} \rtimes_\theta \mathbb{Z}_9$.

Since $(\mathrm{Ad}_{R_0} U_{-1})(U_0) = e^{\frac{2\pi i}{9}} U_0$ while $\mathrm{Ad}_{R_0}(U_{-1})$ acts identically on R_{-1}, we conclude that $\mathrm{Ad}_{R_0}(U_{-1})$ implements the dual action of $\mathbb{Z}_9$ on the crossed product $R_{-1} \rtimes_\theta \mathbb{Z}_9$. Hence, the next step in the iterated basic construction of the inclusion $R_{-1} \subseteq R_0$ is

$$R_1 = \{U_{-1}, g, U_0, U_1, \ldots\}'' = \{gU_{-1}^3, U_{-1}, U_0, U_1, \ldots\}''.$$

The following steps in the basic constructions are realized by successively adjoining the unitaries U_{-2}, U_{-3}, etc.

With the above description for R_1 the notation becomes homogeneous, in the sense that gU_{-1}^3 is a unitary with the property that if we multiply it by U_{-1} or U_{-2} on the right or on the left, the two products differ by $e^{\frac{2\pi i}{3}}$. Therefore, adjoining the unitary U_{-2} to the algebra R_1 we obtain the next step in the Jones basic construction.

To construct unitaries with the properties in the statement one has just to look for the unitaries implementing the crossed product in the Jones tower for an inclusion of the form $R \subset R \rtimes_\beta \mathbb{Z}_9$, where β is a $\mathbb{Z}_3$–kernel (with obstruction $e^{\frac{-2\pi i}{3}}$ to lifting, $\beta^3 = \mathrm{Ad}\ w$, and w is a unitary with spectral projections of trace $\frac{1}{3}$). The unitary g of order 3 is chosen from the elements of the first relative commutant.

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The next step is to give a concrete realization of the crossed product $\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R\right) \rtimes_\gamma \mathbb{Z}_3$, for a $\mathbb{Z}_3$–action γ.

Using the model $\mathfrak{L}\left(F_{\frac{11}{3}}\right) = \{X_1, X_2, X_3, u\}''$ we define the automorphism α on $\mathfrak{L}\left(F_{\frac{11}{3}}\right)$ by:

$$\alpha(X_i) = X_{i+1} \text{ for } i = 1, 2 \quad \text{and} \quad \alpha(X_3) = uX_1u^*,$$
$$\alpha(u) = e^{\frac{2\pi i}{3}} u.$$

Note that $\alpha^3 = \mathrm{Ad}\ u$, so α is a $\mathbb{Z}_3$–kernel with obstruction $e^{\frac{2\pi i}{3}}$ to lifting.

For the automorphism β on the hyperfinite II_1 factor we use the model described in Lemma 4.1: $R \cong R_0 = \{g, U_0, U_1, \ldots\}''$ and $\beta = \mathrm{Ad}_{R_0}(U_{-1})$ with $\beta^3 = \mathrm{Ad}_{R_0}(g)$ and $\beta(g) = e^{\frac{-2\pi i}{3}} g$.

The action of $\mathbb{Z}_3$ on $\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0$ is defined by

$$\gamma = \left(\mathrm{Ad}_{\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right)\otimes R_0\right)} W^2\right)(\alpha \otimes \beta),$$

where W is any cube root of $u \otimes g$ belonging to the fixed point algebra of $\alpha \otimes \beta$ in $\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0$. For example, if $\delta = e^{\frac{2\pi i}{9}}$, we can choose W equal to

$$W = \delta E_1 + \delta^2 E_2 + E_3,$$

where $E_1 = e_1 g_3 + e_2 g_2 + e_3 g_1$, $E_2 = e_1 g_1 + e_2 g_3 + e_3 g_2$ and $E_3 = e_1 g_2 + e_2 g_1 + e_3 g_3$.

Observation 4.2 *Note that W belongs to the center of the fixed point algebra of $\alpha \otimes \beta$ since for any element z in the fixed point algebra we have*

$$z\,(u \otimes g) = (\alpha \otimes \beta)^3 (z\,(u \otimes g)) = \mathrm{Ad}\,(u \otimes g)(z\,(u \otimes g)) = (u \otimes g)\,z.$$

In $M = \left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0\right) \rtimes_\gamma \mathbb{Z}_3$, we denote by v the unitary implementing the crossed product. Thus the spectral projection of v in M have trace $\frac{1}{3}$.
Using the relations $[W,u]=0$ and $[W,g]=0$ we conclude that

$$\text{Ad } v(g) = e^{\frac{-2\pi i}{3}} g, \tag{1}$$

and

$$\text{Ad } v(u) = e^{\frac{2\pi i}{3}} u. \tag{2}$$

In addition,

$$\text{Ad } v(x) = \text{Ad } W^2(\alpha(x)), \text{ for all } x \in \{X_1, X_2, X_3\}'' \tag{3}$$

$$\text{Ad } v(y) = \text{Ad } W^2(\beta(y)), \text{ for all } y \in \{U_0, U_1, U_2, \ldots\}''. \tag{4}$$

We can now prove our main theorem:

Theorem 4.3 *Let $M = \left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0\right) \rtimes_\gamma \mathbb{Z}_3$. Then there exist subalgebras $A \subset B$ in M such that A is isomorphic to $\mathfrak{L}(F_{\frac{35}{27}})$ and B is equal to the crossed product $A \rtimes_\theta \mathbb{Z}_9$, where θ is a $\mathbb{Z}_3$–kernel on A with non-trivial obstruction to lifting. Furthermore, M is the enveloping algebra for the inclusion $A \subset B$.*

Proof Let A denote the von Neumann subalgebra of M generated by $\{X_1, X_2, X_3, u, g, v\}$. We want to describe how Ad U_0 acts on the elements generating A.

Obviously

$$(\text{Ad } U_0)(X_i) = X_i \quad \text{ for } i = 1, \ldots, 3,$$

$$(\text{Ad } U_0)(u) = u \quad \text{and} \quad (\text{Ad } U_0)(g) = e^{\frac{-2\pi i}{3}} g.$$

Moreover, using (4) and the explicit expression for $W^2 = \delta^2 E_1 + \delta^4 E_2 + E_3$, where $\delta = e^{\frac{2\pi i}{9}}$, we obtain

$$\begin{aligned}(\text{Ad } U_0)(v) = U_0(vU_0^* v^*)v = U_0 \text{ Ad } W^2(\beta(U_0^*))v = \bar{\delta}\, (\text{Ad } U_0(W^2))(W^2)^* v \\ = \bar{\delta}\, (E_1 + \delta^2 E_2 + \delta^4 E_3)(W^2)^* v = e^{\frac{-2\pi i}{3}}\, (E_1 + E_2 + e^{\frac{-2\pi i}{3}} E_3)v.\end{aligned}$$

This implies that Ad U_0 leaves A invariant. Furthermore, $\text{Ad}_A(U_0^3)$ acts identically on $\{X_1, X_2, X_3, u, g\}''$, while $\text{Ad}_A(U_0^3)(v) = e^{\frac{-2\pi i}{3}}\, v$.

Set $\theta = \text{Ad}_A(U_0)$. Then, using (1) and the fact that $\text{Ad}_A(g^*)$ acts identically on X_1, X_2, X_3, u, g we conclude that

$$\theta^3 = \text{Ad}_A(g^*) \quad \text{and} \quad \theta(g^*) = e^{\frac{2\pi i}{3}}\, g^*.$$

Thus θ is a $\mathbb{Z}_3$–kernel on A with obstruction $e^{\frac{2\pi i}{3}}$ to lifting. Furthermore, at least algebraically, the algebra $B = (A \cup \{U_0\})''$ can be identified with the crossed product $A \rtimes_\theta \mathbb{Z}_9$ with U_0 implementing the crossed product.

To complete the proof that $B = A \rtimes_\theta \mathbb{Z}_9$ we need to check that any monomial in the variables $\{X_1, X_2, X_3, u, v, g, U_0\}$ can be written using only one occurrence of U_0 to some power. We also need to verify that any monomial containing U_0 has zero trace.

Since the crossed product $\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0\right) \rtimes_\gamma \mathbb{Z}_3$ is implemented by the unitary v, any monomial in the variables $\{X_1, X_2, X_3, u, g, v, U_0\}$ has the form $v^k m$, where m is an element in $\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0$ and $k = 0, 1, 2$. But by the construction of R_0, U_0 can appear at most once in m, so any monomial in B can be written using only one occurrence of U_0.

Furthermore, by the definition of the trace on the crossed product, the trace of any monomial of the form $v^k m$ for $k = 1, 2$, is zero. So we have only to look at the trace of a monomial of the form m containing U_0. But looking at the way the trace is defined on R_0 and on $\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0$, we can conclude immediately that m has zero trace. Therefore $B = A \rtimes_\theta \mathbb{Z}_9$.

In addition, using an argument similar to the one used to build the model for R (Lemma 4.1), one can easily verify that the unitaries $U_1, U_2, \ldots$ implement the consecutive terms in the iterated basic construction of $A \subset A \rtimes_\theta \mathbb{Z}_9$. This implies that $\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0\right) \rtimes_\gamma \mathbb{Z}_3$ is the enveloping algebra of the above inclusion of subfactors.

The last step of the proof is to show that A is isomorphic to $\mathfrak{L}\left(F_{\frac{35}{27}}\right)$. This will be the content of the next proposition.

∎

Proposition 4.4 *With the same notation as in the previous theorem, define $A = \{X_1, X_2, X_3, u, g, v\}''$ endowed with a trace with respect to which X_1, X_2, X_3, u are free, X_1, X_2, X_3 are semicircular and u, g, v are unitaries with spectral projections of trace $\frac{1}{3}$.*

In addition, the trace on the algebra A has the following properties

(i) Any monomial m in X_1, X_2, X_3, u, g, v which contains v has zero trace.

(ii) If m does not contain v then its trace in A is the same as its trace in $\{X_1, X_2, X_3, u\}'' \otimes \{g\}''$.

Under these conditions A is a free group factor with fractional number of generators equal to $\frac{35}{27}$.

Proof This proof is based on Voiculescu's random matrix model introduced in [Vo]. We intend to give an explicit random matrix model for the algebra A and then use it to show that A is a free group factor. This model will be a subalgebra of the algebra of 9×9 matrices with entries in a von Neumann algebra.

Let D be any von Neumann algebra with finite trace τ, containing free elements $\{a_i\}_{i=1,\ldots,18}$, where $a_1, a_4, a_7, a_{10}, a_{12}, a_{14}, a_{16}, a_{17}, a_{18}$ are semicircular, while $a_2, a_3, a_5, a_6, a_8, a_9, a_{11}, a_{13}, a_{15}$ are circular. Denote by $(e_{ij})_{i,j=1,\ldots,9}$ the canonical system of matrix units in $M_9(\mathbb{C})$. Note that by [Jo3] the unitaries $\{u, g, v\}$ generate a copy of $M_3(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus M_3(\mathbb{C}) \subseteq M_9(\mathbb{C})$, with minimal projections of trace $\frac{1}{9}$.

Set $\epsilon = e^{\frac{2\pi i}{3}}$. We let in $M_9(\mathbb{C}) \subseteq D \otimes M_9(\mathbb{C})$:

$$\begin{aligned}
u &= \epsilon\,(e_{11} + e_{44} + e_{77}) + \epsilon^2(e_{22} + e_{55} + e_{88}) + (e_{33} + e_{66} + e_{99}),\\
g &= \epsilon\,(e_{11} + e_{55} + e_{99}) + \epsilon^2(e_{33} + e_{44} + e_{88}) + (e_{22} + e_{66} + e_{77}),\\
v &= (e_{12} + e_{23} + e_{31}) + (e_{45} + e_{56} + e_{64}) + (e_{78} + e_{89} + e_{97})
\end{aligned}$$

or, in matrix notation

$$u = \begin{pmatrix}
\epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \epsilon^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \epsilon & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \epsilon^2 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \epsilon & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \epsilon^2 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

$$g = \begin{pmatrix}
\epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \epsilon^2 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \epsilon^2 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \epsilon & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \epsilon^2 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \epsilon
\end{pmatrix}$$

$$v = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}$$

With this choice of u, g, v and with $\delta = e^{\frac{2\pi i}{9}}$ we obtain

$$W = \delta^2 Id \oplus Id \oplus \delta\, Id,$$

where $Id : 2001 - 031.tex, v1.12001/10/1517 : 45 : 33levyExplevy$ denote the identity of $M_3(\mathbb{C})$. In matrix notation

$$W = \begin{pmatrix} \delta^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta \end{pmatrix}$$

Furthermore, this choice of u, g, v satisfies the required conditions

$$ug = gu, \qquad vu = \epsilon\, uv, \qquad vg = \overline{\epsilon}\, gv.$$

Denote by $Re(a) = \frac{1}{2}(a + a^*)$, for $a \in D \otimes M_9(\mathbb{C})$. We set:

$$\begin{aligned} X_1 = & a_1 \otimes e_{11} + a_4 \otimes e_{22} + a_7 \otimes e_{33} + a_{10} \otimes e_{44} + a_{12} \otimes e_{55} + a_{14} \otimes e_{66} \\ & + a_{16} \otimes e_{77} + a_{17} \otimes e_{88} + a_{18} \otimes e_{99} + 2Re(a_2 \otimes e_{15}) + 2Re(a_3 \otimes e_{19}) \\ & + 2Re(a_5 \otimes e_{26}) + 2Re(a_6 \otimes e_{27}) + 2Re(a_8 \otimes e_{34}) + 2Re(a_9 \otimes e_{38}) \\ & + 2Re(a_{11} \otimes e_{48}) + 2Re(a_{13} \otimes e_{59}) + 2Re(a_{15} \otimes e_{67}). \end{aligned}$$

To find X_2 and X_3, use the relations:

$$X_2 = \mathrm{Ad}\,((W^2)^* v)(X_1) \quad \text{and} \quad X_3 = \mathrm{Ad}\,((W^2)^* v)(X_2)$$

Thus, using matrix notation we have:

$$X_1 = \begin{pmatrix} a_1 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & a_3 \\ 0 & a_4 & 0 & 0 & 0 & a_5 & a_6 & 0 & 0 \\ 0 & 0 & a_7 & a_8 & 0 & 0 & 0 & a_9 & 0 \\ 0 & 0 & a_8^* & a_{10} & 0 & 0 & 0 & a_{11} & 0 \\ a_2^* & 0 & 0 & 0 & a_{12} & 0 & 0 & 0 & a_{13} \\ 0 & a_5^* & 0 & 0 & 0 & a_{14} & a_{15} & 0 & 0 \\ 0 & a_6^* & 0 & 0 & 0 & a_{15}^* & a_{16} & 0 & 0 \\ 0 & 0 & a_9^* & a_{11}^* & 0 & 0 & 0 & a_{17} & 0 \\ a_3^* & 0 & 0 & 0 & a_{13}^* & 0 & 0 & 0 & a_{18} \end{pmatrix}$$

$$X_2 = \begin{pmatrix} a_4 & 0 & 0 & 0 & \bar{\delta}^4 a_5 & 0 & 0 & 0 & \bar{\delta}^2 a_6 \\ 0 & a_7 & 0 & 0 & 0 & \bar{\delta}^4 a_8 & \bar{\delta}^2 a_9 & 0 & 0 \\ 0 & 0 & a_1 & \bar{\delta}^4 a_2 & 0 & 0 & 0 & \bar{\delta}^2 a_3 & 0 \\ 0 & 0 & \delta^4 a_2^* & a_{12} & 0 & 0 & 0 & \delta^2 a_{13} & 0 \\ \delta^4 a_5^* & 0 & 0 & 0 & a_{14} & 0 & 0 & 0 & \delta^2 a_{15} \\ 0 & \delta^4 a_8^* & 0 & 0 & 0 & a_{10} & \delta^2 a_{11} & 0 & 0 \\ 0 & \delta^2 a_9^* & 0 & 0 & 0 & \bar{\delta}^2 a_{11}^* & a_{17} & 0 & 0 \\ 0 & 0 & \delta^2 a_3^* & \bar{\delta}^2 a_{13}^* & 0 & 0 & 0 & a_{18} & 0 \\ \delta^2 a_6^* & 0 & 0 & 0 & \bar{\delta}^2 a_{15}^* & 0 & 0 & 0 & a_{16} \end{pmatrix}$$

$$X_3 = \begin{pmatrix} a_7 & 0 & 0 & 0 & \delta a_8 & 0 & 0 & 0 & \bar{\delta}^4 a_9 \\ 0 & a_1 & 0 & 0 & 0 & \delta a_2 & \bar{\delta}^4 a_3 & 0 & 0 \\ 0 & 0 & a_4 & \delta a_5 & 0 & 0 & 0 & \bar{\delta}^4 a_6 & 0 \\ 0 & 0 & \bar{\delta} a_5^* & a_{14} & 0 & 0 & 0 & \delta^4 a_{15} & 0 \\ \bar{\delta} a_8^* & 0 & 0 & 0 & a_{10} & 0 & 0 & 0 & \delta^4 a_{11} \\ 0 & \bar{\delta} a_2^* & 0 & 0 & 0 & a_{12} & \delta^4 a_{13} & 0 & 0 \\ 0 & \delta^4 a_3^* & 0 & 0 & 0 & \bar{\delta}^4 a_{13}^* & a_{18} & 0 & 0 \\ 0 & 0 & \delta^4 a_6^* & \bar{\delta}^4 a_{15}^* & 0 & 0 & 0 & a_{16} & 0 \\ \delta^4 a_9^* & 0 & 0 & 0 & \bar{\delta}^4 a_{11}^* & 0 & 0 & 0 & a_{17} \end{pmatrix}$$

Obviously X_1, X_2, X_3 commute with g. Moreover, using Voiculescu's random matrix theory [Vo] and the assumption that the family $\{a_i\}_{i=1,\ldots,18}$ is free, it is easy to check that X_1, X_2, X_3, u are free with respect to the unique normalized trace on $D \otimes M_9(\mathbb{C})$. In addition, X_1, X_2, X_3, v are independent (with respect to the trace) of g.

To show that the normalized trace of $D \otimes M_9(\mathbb{C})$ has the properties claimed in the the statement let $v^k m$, $k = 1, 2$ be any monomial in X_1, X_2, X_3, u, g, v containing v. Since m commutes with g, it must be of the form

$$m = \begin{pmatrix} * & 0 & 0 & 0 & * & 0 & 0 & 0 & * \\ 0 & * & 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 & * & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 & * & 0 \\ * & 0 & 0 & 0 & * & 0 & 0 & 0 & * \\ 0 & * & 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & * & 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 & * & 0 \\ * & 0 & 0 & 0 & * & 0 & 0 & 0 & * \end{pmatrix}$$

If we multiply m by v or v^2 we obtain a matrix with zero on the diagonal and a few entries outside the diagonal. This implies that $v^k m$ has zero trace for $k = 1, 2$. Thus, we have built a matrix model for A which satisfies the conditions of the statement.

In order to show that A is a free group factor we will reduce A by one of the spectral projections of g of trace $\frac{1}{3}$, and show that the new factor we obtain is a free group factor. Reduce A by $g_3 = e_{22} + e_{66} + e_{77}$.

Then $g_3 A g_3$ is generated by:

(i) $a_4 \otimes e_{22} + 2Re(a_5 \otimes e_{26}) + 2Re(a_6 \otimes e_{27}) + a_{14} \otimes e_{66} + a_{16} \otimes e_{77} + 2Re(a_{15} \otimes e_{67})$

(ii) $a_7 \otimes e_{22} + 2Re(\bar{\delta}^4 a_8 \otimes e_{26}) + 2Re(\bar{\delta}^2 a_9 \otimes e_{27}) + a_{10} \otimes e_{66} + a_{17} \otimes e_{77} + 2Re(\delta^2 a_{11} \otimes e_{67})$

(iii) $a_1 \otimes e_{22} + 2Re(\delta a_2 \otimes e_{26}) + 2Re(\bar{\delta}^4 a_3 \otimes e_{27}) + a_{12} \otimes e_{66} + a_{18} \otimes e_{77} + 2Re(\delta^4 a_{13} \otimes e_{67})$

(iv) $\epsilon^2 \otimes e_{22} + 1 \otimes e_{66} + \epsilon \otimes e_{77}$,

which, by Voiculescu's random matrix model [VDN], is isomorphic to the free group factor $\mathfrak{L}(F_3 * \mathbb{Z}_3) \cong \mathfrak{L}\left(F_{\frac{11}{3}}\right)$. This implies that A is also a free group factor, and using the well-known formula (see [Dy] or [Ra1])

$$(\mathfrak{L}(F_s))_{\frac{1}{N}} = \mathfrak{L}(F_{N^2(s-1)+1}),$$

we obtain that $A = \mathfrak{L}\left(F_{\frac{35}{27}}\right)$.

■

5 The Connes invariant for the crossed product $\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0\right) \rtimes_\gamma \mathbb{Z}_3$

The arguments in this section are the same ones used by Jones in [Jo2] to compute the Connes invariant for his example of a factor which is anti-isomorphic to itself but has no involutory antiautomorphism. For the sake of completeness we will present them here for the factor we are considering. The definition of K, $K^\perp$ and L given below are due to Connes (see [Co4]).

Let N be a factor without non-trivial hypercentral sequences and G a finite subgroup of $\mathrm{Aut}(N)$ such that $G \cap \overline{\mathrm{Int}(N)} = \{Id\}$. Let $K = G \cap \mathrm{Ct}(N)$ and $K^\perp = \{f : G \to \mathbb{T} \,|$ f is a homomorphism and $f|_K \equiv 1\}$.

Let $\xi : \mathrm{Aut}(N) \to \mathrm{Out}(N)$ be the usual quotient map and Fint be the subgroup $\{\mathrm{Ad}\; u \,|\, u \in N$ is fixed by $G\} \subseteq \mathrm{Aut}(N)$. Denote by $\overline{\mathrm{Fint}}$ its closure in $\mathrm{Aut}(N)$ and by $G \vee \mathrm{Ct}(N)$ the subgroup of $\mathrm{Aut}(N)$ generated by $\{G \cup \mathrm{Ct}(N)\}$. Let $L = \xi((G \vee \mathrm{Ct}(N)) \cap \overline{\mathrm{Fint}}) \subseteq \mathrm{Out}(N)$.

Connes shows in [Co4] that there exists an exact sequence

$$0 \to K^\perp \xrightarrow{\partial} \chi(M) \xrightarrow{\Pi} L \to 0$$

where $M = W^*(N, G)$. The map ∂ is defined in the following way: write an element in the crossed product $W^*(N, G)$ as $\sum_{g \in G} a_g u_g$, with $a_g \in N$ and $\mathrm{Ad}\; u_g = g$ on N. For each $\eta \in K^\perp$, $\eta : G \to \mathbb{T}$, define

$$\Delta(\eta)\left(\sum_{g \in G} a_g u_g\right) = \sum_{g \in G} \eta(g) a_g u_g.$$

Then one can show that $\Delta(\eta) \in \mathrm{Ct}(M) \cap \overline{\mathrm{Int}(M)}$ and define ∂ as the composition $\varepsilon \circ \Delta$.

The map $\Pi : \chi(M) \to L$, is defined as follows. For $\sigma \in \chi(M)$ choose $\alpha \in \mathrm{Ct}(M) \cap \overline{\mathrm{Int}(M)}$ such that $\xi(\alpha) = \sigma$.

From the hypothesis $G \cap \overline{\mathrm{Int}(N)} = \{Id\}$, it follows (see Corollary 6 and Lemma 2 in [Jo1], or Lemma 15.42 in [EK]) that there exists a unitary $z \in M$

and a sequence u_n of unitaries in N^G such that $\alpha = \mathrm{Ad}\ z \lim_{n\to\infty} \mathrm{Ad}\ u_n$. Let $\psi_\sigma = (\mathrm{Ad}\ z^*\alpha)|_N$.

By construction $\psi_\sigma \in \overline{\mathrm{Fint}}$ and one can prove that $\psi_\sigma \in G \vee \mathrm{Ct}(N)$. Moreover, the image of ψ_σ in $\mathrm{Out}(N)$ does not depend on the particular choice of α. Therefore we can define a map $\Pi : \chi(M) \to L$ by $\Pi(\sigma) = \xi(\psi_\sigma)$.

To show that Π is surjective, consider an element μ in L and let $\alpha_\mu \in \overline{\mathrm{Fint}}$ be such that $\varepsilon(\alpha_\mu) = \mu$. α_μ commutes with G since it is the limit of automorphisms with this property, thus we may define an automorphism β_μ of M by

$$\beta_\mu\left(\sum_{g\in G} a_g u_g\right) = \sum_{g\in G} \alpha_\mu(a_g) u_g. \tag{5}$$

Then $\beta_\mu \in \mathrm{Ct}(M) \cap \overline{\mathrm{Int}(M)}$ and $\Pi(\xi(\beta_\mu)) = \mu$.

Remark 5.1 *Note that if u_n is a sequence of unitaries left invariant by G, with the property $\alpha_\mu = \lim_{n\to\infty} \mathrm{Ad}\ u_n$ in $\mathrm{Aut}(N)$, then $\beta_\mu = \lim_{n\to\infty} \mathrm{Ad}\ u_n$ in $\mathrm{Aut}(M)$. It follows that*

$$\varkappa(\varepsilon(\beta_\mu)) = \lim_{n\to\infty} (u_n)^* \beta_\mu(u_n) = \lim_{n\to\infty} (u_n)^* \alpha_\mu(u_n).$$

Our next goal is to show that if $N = \mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0$ and G is the subgroup $\langle\gamma\rangle \subset \mathrm{Aut}\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0\right)$ then $\chi(M) \cong \mathbb{Z}_9$, as has been observed by Connes in [Co3].

For all the rest of this paper we will identify $\langle\gamma\rangle$ with $\mathbb{Z}_3$. Note that by Proposition 3.4, $\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0$ has no non-trivial hypercentral sequences, so in order to use the exactness of the Connes sequence described above we only need to show that $\mathbb{Z}_3 \cap \overline{\mathrm{Int}\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0\right)} = \{Id\}$. Obviously it is enough to check that $\gamma \notin \overline{\mathrm{Int}\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0\right)}$.

This is equivalent to show that $\alpha \otimes \beta \notin \overline{\mathrm{Int}\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0\right)}$. By Corollary 3.3 in [Co5] $\alpha \otimes \beta \in \overline{\mathrm{Int}\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0\right)}$ if and only if $\alpha \in \overline{\mathrm{Int}\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right)\right)}$ and $\beta \in \overline{\mathrm{Int}(R)}$. But $\mathfrak{L}\left(F_{\frac{11}{3}}\right)$ is full (Corollary 3.3) and $\alpha \notin \mathrm{Int}\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right)\right)$, thus we can conclude that $\mathbb{Z}_3 \cap \overline{\mathrm{Int}\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0\right)} = \{Id\}$.

Hence by Connes [Co4], the sequence

$$0 \to K^\perp \overset{\partial}{\to} \chi(M) \overset{\Pi}{\to} L \to 0$$

is exact. In order to compute $\chi(M)$ we intend to show that

$$K^\perp = \mathbb{Z}_3 \quad \text{and} \quad L = \mathbb{Z}_3,$$

and that the lifting to $\chi(M)$ of the generator of L produces an element of order 9.

Lemma 5.2 *The group $K = \mathbb{Z}_3 \cap \mathrm{Ct}\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0\right)$ is trivial.*

Proof Obviously it is enough to show that the automorphism $\gamma = Ad\, W^2(\alpha \otimes \beta)$ is not in $\mathrm{Ct}\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0\right)$. We have already shown in the proof of Proposition 3.4 that any central sequence in $\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0$ has the form $(1 \otimes x_n)_{n\in\mathbb{N}}$ for a central sequence $(x_n)_{n\in\mathbb{N}}$ in R_0. It follows that $\gamma \in \mathrm{Ct}\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0\right)$ if and only if $\beta \in \mathrm{Ct}(R_0)$. But for $\epsilon = e^{\frac{2\pi i}{3}}$, β is outer conjugate to the automorphism $s_3^{\bar{\epsilon}}$ described by Connes in [Co6], and $s_3^{\bar{\epsilon}} \notin \mathrm{Ct}(R_0)$ by Proposition 1.6 in the same paper. Therefore $\beta \notin \mathrm{Ct}(R_0)$.

■

Lemma 5.3 *$L \cong \mathbb{Z}_3$, and a generator of L is given by*

$$\mu = \xi\left(Id \otimes (\mathrm{Ad}\ U_0^*\, \beta)\right)),$$

where U_0 is the unitary in R_0 defined in Lemma 4.1 with the property $\beta(U_0) = e^{\frac{2\pi i}{9}} U_0$.

Proof We want to show that

$$Id \otimes (\mathrm{Ad}\ U_0^*\, \beta) \in \left(\mathbb{Z}_3 \vee \mathrm{Ct}\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0\right)\right) \cap \overline{\mathrm{Fint}}.$$

Let's look first at $\mathbb{Z}_3 \vee \mathrm{Ct}\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0\right)$: multiplying $Id \otimes (\mathrm{Ad}\ U_0^*\, \beta)$ by the automorphism $\gamma^{-1} = (\mathrm{Ad}\ W^2(\alpha \otimes \beta))^{-1}$ we obtain

$$\mathrm{Ad}\ \left((W^2)^*(1 \otimes U_0^*)\right)(\alpha^{-1} \otimes Id),$$

which is in $\mathrm{Ct}\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0\right)$ since, by Remark 3.1, $\mathfrak{L}\left(F_{\frac{11}{3}}\right)$ does not have non-trivial central sequences. Thus $Id \otimes (\mathrm{Ad}\ U_0^*\, \beta) \in \mathbb{Z}_3 \vee \mathrm{Ct}\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0\right)$.

To prove that $Id \otimes (\mathrm{Ad}\ U_0^*\, \beta) \in \overline{\mathrm{Fint}}$ we have to show that there exists a sequence $(\tilde{u}_n)$ of unitaries in $\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0$ invariant respect to γ, such that $Id \otimes (\mathrm{Ad}\ U_0^*\, \beta) = \lim_{n\to\infty} \mathrm{Ad}\ \tilde{u}_n$. First of all observe that the sequence (x_n) of unitaries in R_0

$$x_n = \begin{cases} U_0 U_1^* U_2 U_3^* \dots U_n^*, & \text{if } n \text{ is odd,} \\ U_0 U_1^* U_2 U_3^* \dots U_n, & \text{if } n \text{ is even} \end{cases}$$

has the properties

$$\beta = \lim_{n\to\infty} \mathrm{Ad}\ x_n \quad \text{ and } \quad \beta(x_n) = e^{\frac{2\pi i}{9}}\, x_n.$$

Define $u_n = U_0^* x_n$. Obviously

$$(\mathrm{Ad}\ U_0^*)\,\beta = \lim_{n\to\infty} \mathrm{Ad}\ u_n \quad \text{and} \quad \beta(u_n) = u_n$$

so that

$$(\alpha\otimes\beta)(1\otimes u_n) = 1\otimes u_n. \tag{6}$$

Since by Observation 4.2 W belongs to the center of the fixed point algebra of $\alpha\otimes\beta$, using (6) we obtain

$$\gamma(1\otimes u_n) = \mathrm{Ad}\ W^2(\alpha\otimes\beta)(1\otimes u_n) = 1\otimes u_n$$

and $Id\otimes(\mathrm{Ad}\ U_0^*\,\beta) = \lim_{n\to\infty}\mathrm{Ad}\,(1\otimes u_n)$. Therefore $\tilde{u}_n = 1\otimes u_n$ is the desired sequence and $Id\otimes(\mathrm{Ad}\ U_0^*\,\beta) \in \left(\mathbb{Z}_3\vee \mathrm{Ct}\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right)\otimes R_0\right)\right)\cap\overline{\mathrm{Fint}}$.

So far we know that $\mu = \xi\,(Id\otimes(\mathrm{Ad}\ U_0^*\,\beta))$ is an element of order 3 of $L = \xi\left(\left(\mathbb{Z}_3\vee \mathrm{Ct}\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right)\otimes R_0\right)\right)\cap\overline{\mathrm{Fint}}\right)$. We want to show that it generates L. This means that if φ is any element in $\left(\mathbb{Z}_3\vee \mathrm{Ct}\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right)\otimes R_0\right)\right)\cap\overline{\mathrm{Fint}}$, then it differs from a power of $Id\otimes(\mathrm{Ad}\ U_0^*\,\beta)$ by an inner automorphism.

By Theorem 3.5 any automorphism in $\mathrm{Ct}\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right)\otimes R_0\right)$ can be written as $\mathrm{Ad}\ z(\nu\otimes id)$ for some unitary $z\in\mathfrak{L}\left(F_{\frac{11}{3}}\right)\otimes R_0$ and an automorphism $\nu\in\mathrm{Aut}\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right)\right)$. Thus, since $\varphi\in\mathbb{Z}_3\vee\mathrm{Ct}\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right)\otimes R_0\right)$ there is an $n\in\{0,1,2\}$ such that $\varphi\,\mathrm{Ad}\,(W^2)^n(\alpha\otimes\beta)^n$ is of the form $\mathrm{Ad}\ z(\nu\otimes Id)$. Solving for φ we obtain

$$\varphi = \mathrm{Ad}\ x(\nu\alpha^{-n}\otimes\beta^{-n}),$$

where $x = z(\nu\otimes Id)(W^{2n})^* \in\mathfrak{L}\left(F_{\frac{11}{3}}\right)\otimes R_0$.

Since $\varphi\in\overline{\mathrm{Int}\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right)\otimes R_0\right)}$ and by Corollary 3.3 $\mathfrak{L}\left(F_{\frac{11}{3}}\right)$ is full, we may apply Corollary 3.3 in [Co5] to conclude that $\nu\alpha^{-n} = \mathrm{Ad}\ w$ for some unitary w. This implies that $\varphi = \mathrm{Ad}\ x'(Id\otimes\beta^{-n})$, where $x' = x\,(w\otimes 1)$. Thus φ differs from $Id\otimes(\mathrm{Ad}\ U_0^*\,\beta)$ only by an inner automorphism. Hence $Id\otimes(\mathrm{Ad}\ U_0^*\,\beta)$ generates L.

∎

Note that if u_n is the sequence of unitaries defined in the previous lemma

$$(Id\otimes(\mathrm{Ad}\ U_0^*\,\beta))(1\otimes u_n) = 1\otimes U_0^* u_n U_0.$$

But $\lim_{n\to\infty}\mathrm{Ad}\ u_n = \mathrm{Ad}\ U_0^*\,\beta$ and $\beta(U_0) = e^{\frac{2\pi i}{9}}\,U_0$, so

$$\lim_{n\to\infty} u_n U_0 u_n^* = \mathrm{Ad}\ U_0^*(\beta(U_0)) = e^{\frac{2\pi i}{9}}\,U_0$$

and

$$\lim_{n\to\infty} (1\otimes u_n^*)(Id\otimes (\mathrm{Ad}\ U_0^*\,\beta))(1\otimes u_n) = e^{\frac{2\pi i}{9}}.$$

Thus, in view of Remark 5.1 we obtain the following:

Remark 5.4 *If $\mu = Id\otimes(\mathrm{Ad}\ U_0^*\,\beta)$, $\beta_\mu \in \mathrm{Ct}(M)\cup\overline{\mathrm{Int}(M)}$ is as described in equation [5] and $\sigma = \varepsilon(\beta_\mu)$, then $\varkappa(\sigma) = e^{\frac{2\pi i}{9}}$.*

Theorem 5.5 *Let M denote the crossed product $\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right)\otimes R_0\right)\rtimes_\gamma \mathbb{Z}_3$. Then $\chi(M) = \mathbb{Z}_9$.*

Proof. By Connes [Co4] the sequence

$$0\to\mathbb{Z}_3\to\chi(M)\to\mathbb{Z}_3\to 0$$

is exact and a lifting to $\chi(M)$ of a generator of $L\cong\mathbb{Z}_3$ is defined on $M = \left(\mathfrak{L}\left(F_{\frac{11}{3}}\right)\otimes R_0\right)\rtimes_\gamma\mathbb{Z}_3$ by $\sigma = \xi(\beta_\mu)$, where

$$\beta_\mu\left(\sum_{k=0}^{2} a_k v^k\right) = \sum_{k=0}^{2}(Id\otimes(\mathrm{Ad}\ U_0^*\,\beta))(a_k)v^k.$$

Since 3 is the only non-trivial divisor of 9, to prove that $\chi(M)\cong\mathbb{Z}_9$ it suffices to show that $\sigma^3\neq 1$, i.e. $\beta_\mu^3\notin\mathrm{Int}(M)$.

From the relations

$$(\alpha\otimes\beta)(1\otimes(U_0^*)^3 g) = e^{\frac{2\pi i}{3}}\,(1\otimes(U_0^*)^3)\,g$$

and

$$\mathrm{Ad}\ W^2(1\otimes(U_0^*)^3 g) = 1\otimes(U_0^*)^3 g$$

we obtain that

$$\gamma(1\otimes(U_0^*)^3 g) = e^{\frac{2\pi i}{3}}(1\otimes(U_0^*)^3 g).$$

But $\mathrm{Ad}\ v = \gamma$ on $\mathfrak{L}\left(F_{\frac{11}{3}}\right)\otimes R_0$, so that $\mathrm{Ad}\ v(1\otimes(U_0^*)^3 g) = e^{\frac{2\pi i}{3}}(1\otimes(U_0^*)^3\,g)$ implies

$$\mathrm{Ad}\,(1\otimes(U_0^*)^3\,g)(v) = e^{\frac{-2\pi i}{3}}v.$$

Using the definition of β_μ and what we have just observed we obtain

$$\begin{aligned}\beta_\mu^3\left(\sum_{k=0}^{2} a_k\, v^k\right) &= \sum_{k=0}^{2}\left(Id\otimes(\mathrm{Ad}\,(U_0^*)^3\,\beta^3)\right)(a_k)v^k\\ &= \sum_{k=0}^{2}\left(Id\otimes\mathrm{Ad}(U_0^*)^3\ \mathrm{Ad}\ g\right)(a_k)v^k = \sum_{k=0}^{2}\mathrm{Ad}(1\otimes(U_0^*)^3 g)(a_k)v^k\\ &= \mathrm{Ad}\,(1\otimes(U_0^*)^3 g)\left(\sum_{k=0}^{2} e^{\frac{2\pi i k}{3}} a_k\, v^k\right).\end{aligned}$$

Thus, being the product of a dual action and an inner automorphism β_μ^3 must be outer. Hence $\chi(M) \cong \mathbb{Z}_9$.

■

6 $\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0\right) \rtimes_\gamma \mathbb{Z}_3$ is not anti–isomorphic to itself.

In this section we are going to show that $M = \left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0\right) \rtimes_\gamma \mathbb{Z}_3$ is not anti-isomorphic to itself by using the dual action $\widehat{\mathbb{Z}_3} \to \mathrm{Aut}(M)$, which gives rise to the only subgroup of order 3 in $\chi(M)$. This argument has been described by Connes in [Co3] and [Co4].

First of all note that the action γ can be decomposed as $\gamma = \mathrm{Ad}\, w\, \gamma_1\gamma_2$ where $\gamma_1 \in \mathrm{Ct}\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0\right)$ and $\gamma_2 \in \overline{\mathrm{Int}\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0\right)}$ and w is a unitary in $\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0$.

In fact, $\gamma = \mathrm{Ad}\ W^2(\alpha \otimes Id)(Id \otimes \beta)$ and $\gamma_1 = \alpha \otimes Id$ is centrally trivial, since any central sequence in $\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0$ has the form $(1 \otimes X_n)$, for a central sequence (X_n) in R_0. Furthermore in the proof of Lemma 4.1 we showed that $\beta = \lim_{n\to\infty} \mathrm{Ad}\, x_n$ so that $\gamma_2 = Id \otimes \beta = \lim_{n\to\infty} \mathrm{Ad}\,(1 \otimes x_n)$ belongs to $\overline{\mathrm{Int}\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0\right)}$.

Note also that this decomposition of γ into an approximately inner automorphism and a centrally trivial automorphism is unique up to inner automorphisms since $\chi\left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0\right) = 1$.

Now let M be an arbitrary von Neumann algebra. Define the conjugate M^c of M as the algebra whose underlying vector space is the conjugate of M (for $\lambda \in \mathbb{C}$, $x \in M$ the product λ by x in M^c is equal to $\bar{\lambda}x$) and whose ring structure is the same as in M. The opposite M^o of M is by definition the algebra whose underlying vector space is the same as for M while the product of x by y is equal to yx instead of xy. M^c and M^o are clearly isomorphic through the map $x \to x^*$. For $\psi \in \mathrm{Aut}(M)$ we denote by ψ^c the automorphism of M^c induced by ψ.

The proof of the following theorem is due to Connes ([Co3] and [Co4]).

Theorem 6.1 $M = \left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0\right) \rtimes_\gamma \mathbb{Z}_3$ *is not anti–isomorphic to itself.*

Proof We proved in the previous section (Theorem 5.5) that $\chi(M) \cong \mathbb{Z}_9$ and that the dual action $\widehat{\mathbb{Z}_3} \to \mathrm{Aut}(M)$ produces the only subgroup of order 3 in $\chi(M)$, namely $\langle\sigma^3\rangle = \langle\xi(\beta_\mu^3)\rangle$.

Since $\chi(M)$ is an invariant of the von Neumann algebra M, it follows that $\widehat{\mathbb{Z}_3}$ is an invariant of M. This implies that $M \rtimes_{\hat{\gamma}} \widehat{\mathbb{Z}_3}$ is an invariant of M as it is the dual action $\mathbb{Z}_3 \to \mathrm{Aut}(M \rtimes_{\hat{\gamma}} \widehat{\mathbb{Z}_3})$ on $M \rtimes_{\hat{\gamma}} \widehat{\mathbb{Z}_3}$.

By Takesaki's duality theory [Theorem 4.5, [Ta]]

$$M \rtimes_{\hat{\gamma}} \widehat{\mathbb{Z}_3} = \left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0\right) \otimes B(\ell^2(\mathbb{Z}_3)).$$

Moreover, the dual action $\hat{\hat{\gamma}}$ of $\hat{\gamma}$ corresponds to the action $\gamma \otimes \mathrm{Ad}\,(\lambda(\cdot)^*)$, where $\lambda(\cdot)$ is the regular representation of $\mathbb{Z}_3$ on $\ell^2(\mathbb{Z}_3)$ defined by

$$\lambda(h)\varepsilon(k) = \varepsilon(k-h) \quad \text{for} \quad h, k \in \mathbb{Z}_3,\ \varepsilon \in \ell^2(\mathbb{Z}_3).$$

Since $\langle\hat{\hat{\gamma}}\rangle \cong \mathbb{Z}_3$ is an invariant of M, it follows that the action γ of $\mathbb{Z}_3$ on $\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0$ is an invariant of M. Consequently the approximately inner automorphism and the centrally trivial automorphism γ_1 and γ_2 which appear in the decomposition of γ are invariants of M.

Note that

$$\gamma_1^3 = \mathrm{Ad}\,(u \otimes 1) \quad \text{and} \quad \gamma_1(u \otimes 1) = e^{\frac{2\pi i}{3}}\,(u \otimes 1)$$

so that γ_1 is a $\mathbb{Z}_3$–kernel of $\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0$ with obstruction $e^{\frac{2\pi i}{3}}$ to lifting.

Observe also that in the above argument we have only used the abstract group $\widehat{\mathbb{Z}_3}$ and the dual action defined on $M \rtimes_{\hat{\gamma}} \widehat{\mathbb{Z}_3}$. Hence we have found a canonical way to associate to the von Neumann algebra M a number (equal to $e^{\frac{2\pi i}{3}}$ in our case) which is invariant under isomorphisms.

Now if $M = \left(\mathfrak{L}\left(F_{\frac{11}{3}}\right) \otimes R_0\right) \rtimes_\gamma \mathbb{Z}_3$ was anti–isomorphic to itself then M and M^c would be isomorphic. But the obstruction associated to γ_1^c, and therefore to M^c is equal to $e^{\frac{-2\pi i}{3}}$.

■

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