Invariant Stein domains in Stein symmetric spaces and a non-linear complex convexity theorem

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Abstract

We prove a complex version of Kostant's non-linear convexity theorem. Applications to the construction of G-invariant Grauert tubes of Riemannian symmetric G/K spaces are given.

Introduction

Let $X = G/K$ be a semisimple non-compact Riemannian symmetric space. We may assume that G is semisimple with finite center and we write $G = NAK$ for an Iwasawa decomposition of G. By our assumption, G sits in its universal complexification $G_{\mathbb{C}}$ and so $X \subseteq X_{\mathbb{C}} := G_{\mathbb{C}}/K_{\mathbb{C}}$. Note that $X_{\mathbb{C}}$ is a Stein symmetric space. Observe that the group G does not act properly (i.e. with compact isotropy subgroups) on $G_{\mathbb{C}}/K_{\mathbb{C}}$ since $K_{\mathbb{C}}$ is not compact. Write U for the compact real form of $G_{\mathbb{C}}$. The maximal connected subdomain of $GUK_{\mathbb{C}}/K_{\mathbb{C}} \subseteq X_{\mathbb{C}}$ on which G acts properly and which contains X was introduced in [AkGi90]. It is given by

$$\Xi := G\exp(i\Omega)K_{\mathbb{C}}/K_{\mathbb{C}}$$

where Ω is a polyhedral convex domain in $\mathfrak{a} := \mathrm{Lie}(A)$ defined by

$$\Omega := \{X \in \mathfrak{a} : (\forall \alpha \in \Sigma)\ |\alpha(X)| < \frac{\pi}{2}\}.$$

Here Σ denotes the restricted root system with respect to $\mathfrak{a}$.

One of our principal aims is to construct a broad class of G-invariant Stein subdomains in Ξ. Let us remind the reader that it is a notoriously difficult problem to verify that certain unions of orbits of non-compact groups are Stein. For example the problem posed in [AkGi90] whether Ξ is Stein was unsolved until the last year.

The Iwasawa decomposition on G cannot be holomorphically extended to the whole group $G_{\mathbb{C}}$; we have that $N_{\mathbb{C}}A_{\mathbb{C}}K_{\mathbb{C}} \subsetneq G_{\mathbb{C}}$ is an open Zariski dense subset. The *Iwasawa domain* Ξ_I

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is defined as the maximal connected G-invariant subdomain in $X_\mathbb{C}$ which contains X and is contained in $N_\mathbb{C}A_\mathbb{C}K_\mathbb{C}/K_\mathbb{C}$:

$$\Xi_I := \Big(\bigcap_{g\in G} g(N_\mathbb{C}A_\mathbb{C}K_\mathbb{C}/K_\mathbb{C})\Big)_0,$$

where $(\cdot)_0$ refers to the connected component containing X. Since Ξ_I is the connected component of an open intersection of Stein domains, it is easy to see that Ξ_I is Stein (cf. proof of Proposition 3.4).

The domains Ξ and Ξ_I again became of recent interest. One has $\Xi = \Xi_I$. Here are the results in chronological order:

- $\Xi \subseteq \Xi_I$ for all classical groups G (cf. [KrSt01a]).
- $\Xi = \Xi_I$ for all classical groups G (cf. [GiMa01]).
- $\Xi_I \subseteq \Xi$ for all G (cf. [Ba01]).

In [KOS01] it will be shown that $\Xi \subseteq \Xi_I$ for the exceptional cases. Also it is announced in [Hu01] that $\Xi \subseteq \Xi_I$. This will then give us $\Xi = \Xi_I$ in the general case. In particular, Ξ is Stein. Also, in the preprint [BHH01] a complex-geometric proof of the Steinness of Ξ is given.

Geometrically, the domain Ξ is a rather complicated object. If G is a group of Hermitian type, then $\Xi = X \times \overline{X}$ (cf. [BHH01], [GiMa01] or [KrSt01b]). If $X = G/K$ is classical, then there exists a group of Hermitian type $S \supseteq G$ with maximal compact subgroup $U \supseteq K$ such that G/K is a real form of the Hermitian symmetric space S/U. In [BHH01] and [KrSt01b] a G-equivariant subdomain $\Xi_0 \subseteq \Xi$ was exhibited which is biholomorphic to S/U. Further one has $\Xi = \Xi_0 = S/U$ if and only if Σ is of type C_n or BC_n. In particular, if $\Sigma \neq C_n, BC_n$, then $S/U \subsetneq \Xi$ and the explicit geometric structure of Ξ is very intricate. For many exceptional spaces G/K the geometric structure is even more complicated.

The domain Ξ (in [Gi98] it is called *complex crown of* X) is universal in the sense that many analytical and geometrical constructions on the Riemannian symmetric space X extend to Ξ. In [KrSt01a] it was shown that the inclusion $\Xi \subseteq \Xi_I$ implies that all eigenfunctions on X for the algebra of G-invariant differential operators $\mathbb{D}(X)$ extend holomorphically to Ξ. In [GiMa01] the problem $\Xi \subseteq \Xi_I$ is included in a broad class of geometrical problems connected with Matsuki duality. These geometrical problems include in particular the problem of the parametrization of compact complex cycles in flag domains (cf. [Wo92]).

Let us now come to the contents of this paper. Our first main result is a complex version of Kostant's non-linear convexity theorem (cf. [Kos73]). Write $G \to A,\ g \mapsto a(g)$ for the middle projection in $G = NAK$.

Theorem. (Kostant) *Let $X \in \mathfrak{a}$. Then*

$$a(K\exp(X)) = \exp(\mathrm{conv}(\mathcal{W}X)),$$

where $\mathcal{W} = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ denotes the Weyl group and $\mathrm{conv}(\cdot)$ *refers to the convex hull of* $(\cdot)$.

One shows that the middle projection $a\colon G \to A$ holomorphically extends to

$$a\colon G\exp(i\Omega)K_\mathbb{C} \to A_\mathbb{C}. \tag{1}$$

Then our main result is:

Theorem A. (Complex Convexity Theorem) *For all $X \in \Omega$ we have that*

$$a(G\exp(iX)) \subseteq A\exp(i\,\mathrm{conv}(\mathcal{W}X)).$$

Our convexity theorem features interesting applications to the geometry of the domain Ξ and its generic subdomains which are defined as follows: Let $\omega \subseteq \Omega$ be a non-empty convex $\mathcal{W}$-invariant open subset. Then we can form the domains

$$\Xi(\omega) := G \exp(i\omega) K_{\mathbb{C}} / K_{\mathbb{C}}.$$

Note that $\Xi = \Xi(\Omega)$ and that the Iwasawa projection (1) naturally factors to a holomorphic mapping $a: \Xi \to A_{\mathbb{C}}$. Finally we define for every $g \in G$ the *horospherical tube* $T(g,\omega) \subseteq X_{\mathbb{C}}$ by

$$T(g,\omega) := g\big(N_{\mathbb{C}} A \exp(i\omega) K_{\mathbb{C}} / K_{\mathbb{C}}\big).$$

As an application of Theorem A we now obtain:

Theorem B. *Let ω be an open convex Weyl group invariant subset of Ω. Then the following assertions hold:*

(i) $a(\Xi(\omega)) = A \exp(i\omega)$.

(ii) $\Xi(\omega) = \big(\bigcap_{g\in G} T(g,\omega)\big)_0$.

(iii) *The domain $\Xi(\omega)$ is Stein.*

Let us emphasize that in the the case of $\Xi(\omega) = \Xi$, the inclusion $\Xi \subseteq \Xi_I$ means only the existence of an Iwasawa projection $a: \Xi \to A_{\mathbb{C}}$; (i) in Theorem B gives a more precise information on the image of this projection. Further (ii) in the above theorem is a much stronger statement than (iii); in particular, (ii) implies (iii) since all horospherical tubes $T(g,\omega)$ are Stein.

Theorem B can be considered as an analogue of Lassalle's results for compact symmetric spaces (cf. [La78]). We can interpret Theorem A and Theorem B as statements for G-orbits in $X_{\mathbb{C}}$ intersecting $A \exp(i\Omega) K_{\mathbb{C}} / K_{\mathbb{C}}$ which in the case of compact symmetric spaces G/K are true for arbitrary G-orbits (cf. [La78]).

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1. Notation

Let G be a connected semisimple Lie group sitting inside a complexification $G_{\mathbb{C}}$. We denote by $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}$ the Lie algebras of G and $G_{\mathbb{C}}$, respectively. Let $K < G$ be a maximal compact subgroup and $\mathfrak{k}$ its Lie algebra. Denote by $\theta: G \to G$ a Cartan involution which has K as a fixed point set.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition attached to $\mathfrak{k}$. Take $\mathfrak{a} \subseteq \mathfrak{p}$ a maximal Abelian subspace and let $\Sigma = \Sigma(\mathfrak{g},\mathfrak{a}) \subseteq \mathfrak{a}^*$ be the corresponding root system. Related to this root system is the root space decomposition according to the simultaneous eigenvalues of $\mathrm{ad}(H), H \in \mathfrak{a}$:

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}^\alpha,$$

here $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ and $\mathfrak{g}^\alpha = \{X \in \mathfrak{g}: (\forall H \in \mathfrak{a})\ [H,X] = \alpha(H)X\}$. For the choice of a positive system $\Sigma^+ \subseteq \Sigma$ one obtains the nilpotent Lie algebra $\mathfrak{n} = \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}^\alpha$. Then one has the Iwasawa decomposition on the Lie algebra level

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}.$$

We write A, N for the analytic subgroups of G corresponding to $\mathfrak{a}$ and $\mathfrak{n}$. For these choices one has for G the Iwasawa decomposition, namely, the multiplication map

$$N \times A \times K \to G, \quad (n, a, k) \mapsto nak$$

In particular, every element $g \in G$ can be written uniquely as $g = n(g)a(g)\kappa(g)$ with each of the maps $\kappa(g) \in K$, $a(g) \in A$, $n(g) \in N$ depending analytically on $g \in G$. The last piece of structure theory we shall recall is the little Weyl group. We denote by $\mathcal{W} = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ the *Weyl group* of $\Sigma(\mathfrak{a}, \mathfrak{g})$.

Finally we define the domain

$$\Omega = \{X \in \mathfrak{a} \colon (\forall \alpha \in \Sigma)\ |\alpha(X)| < \frac{\pi}{2}\}.$$

Clearly Ω is convex and $\mathcal{W}$-invariant.

2. The complex convexity theorem

Let us first give the relevant notation. Denote by $K_{\mathbb{C}}$, $A_{\mathbb{C}}$, $N_{\mathbb{C}}$ the complexifications of K, A and N. Then $N_{\mathbb{C}}A_{\mathbb{C}}K_{\mathbb{C}}$ is a proper Zariski-open, hence dense subset of $G_{\mathbb{C}}$. Throughout this paper we will assume that $\Xi \subseteq \Xi_I$ holds, i.e.,

$$G \exp(i\Omega) \subseteq N_{\mathbb{C}}A_{\mathbb{C}}K_{\mathbb{C}}$$

(see the introduction for proofs of this statement). We now set $T_\Omega := A\exp(i\Omega) \subseteq A_{\mathbb{C}}$. Then $GT_\Omega \subseteq N_{\mathbb{C}}A_{\mathbb{C}}K_{\mathbb{C}}$.

One can show that one has a well defined holomorphic middle projection $N_{\mathbb{C}}A_{\mathbb{C}}K_{\mathbb{C}} \to A_{\mathbb{C}}/(A_{\mathbb{C}} \cap K_{\mathbb{C}})$. Due to the simple connectedness of T_Ω, this projection restricted to GT_Ω lifts to $A_{\mathbb{C}}$ (cf. [KrSt01a, proof of Th. 1.8(iii)]) and we obtain an analytic mapping

$$G \times T_\Omega \to A_{\mathbb{C}}, \quad (g, a) \mapsto a(ga)$$

holomorphic in the second variable such that

$$ga \in N_{\mathbb{C}}a(ga)K_{\mathbb{C}}$$

holds.

If V is a vector space and $E \subseteq V$ is a subset, then we denote by $\operatorname{conv}(E)$ the convex hull of E in V.

Theorem 2.1. (Complex Convexity Theorem) *Assume that G is classical. Then we have for all $X \in \Omega$ that*

$$a(G\exp(iX)) \subseteq A\exp(i\operatorname{conv}(\mathcal{W}X)). \qquad \blacksquare$$

The proof of Theorem 2.1 will be given in several steps.

Fix $a \in T_\Omega$ and consider the function

$$f_a \colon K \to \mathfrak{a}_{\mathbb{C}}, \quad k \mapsto \log a(ka).$$

Further we write $p_{\mathfrak{a}_{\mathbb{C}}} \colon \mathfrak{g}_{\mathbb{C}} \to \mathfrak{a}_{\mathbb{C}}$ for the projection along $\mathfrak{k}_{\mathbb{C}} + \mathfrak{n}_{\mathbb{C}}$. For $x \in GT_\Omega K_{\mathbb{C}} \subseteq N_{\mathbb{C}}A_{\mathbb{C}}K_{\mathbb{C}}$ we write $b(x) = n(x)a(x)$ for the *triangular part* of x.

Lemma 2.2. *For any $a \in T_\Omega$, $k \in K$ and $X \in \mathfrak{k}$ we have*

$$\frac{d}{dt}\Big|_{t=0} f_a(\exp(tX)k) = p_{\mathfrak{a}_{\mathbb{C}}}(\mathrm{Ad}(b(ka))^{-1}X).$$

Proof. This result can be easily deduced from the known case for $a \in A$ by analytic continuation. However, for conveniene for the reader, we briefly recall the proof. We have

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} f_a(\exp(tX)k) &= \frac{d}{dt}\Big|_{t=0} \log a(\exp(tX)ka) \\
&= \frac{d}{dt}\Big|_{t=0} \log a(\exp(tX)b(ka)) \\
&= \frac{d}{dt}\Big|_{t=0} [\log a(b(ka)^{-1}\exp(tX)b(ka)) + \log a(ka)] \\
&= \frac{d}{dt}\Big|_{t=0} \log a(\exp(t\,\mathrm{Ad}(b(ka))^{-1}X)) = p_{\mathfrak{a}_{\mathbb{C}}}(\mathrm{Ad}(b(ka))^{-1}X),
\end{aligned}$$

proving the lemma. ■

Write $\mathfrak{g}_{\mathbb{C}}^{\mathbb{R}}$ for $\mathfrak{g}_{\mathbb{C}}$ considered as a real Lie algebra and $\kappa_{\mathbb{R}}$ for the Cartan-Killing form on $\mathfrak{g}_{\mathbb{C}}^{\mathbb{R}}$. Let κ be the Cartan-Killing form on $\mathfrak{g}$ and recall the following relation between κ and $\kappa_{\mathbb{R}}$:

$$(\forall X, X', Y, Y' \in \mathfrak{g}) \qquad \kappa_{\mathbb{R}}(X+iY, X'+iY') = 2(\kappa(X,X') - \kappa(Y,Y')).$$

For every $\lambda \in (\mathfrak{g}_{\mathbb{C}}^{\mathbb{R}})^*$ we define $H_\lambda \in \mathfrak{g}_{\mathbb{C}}^{\mathbb{R}}$ by $\lambda(X) = \kappa_{\mathbb{R}}(X, H_\lambda)$ for all $X \in \mathfrak{g}_{\mathbb{C}}^{\mathbb{R}}$.

For every $\lambda \in (\mathfrak{a}_{\mathbb{C}}^{\mathbb{R}})^*$ and $a \in T_\Omega$ we now define the function

$$f_{a,\lambda}: K \to \mathbb{R}, \quad k \mapsto \lambda(f_a(k)).$$

Lemma 2.3. *Let $a \in T_\Omega$ and $k \in K$. Then the following assertions hold:*

(i) *For all $X \in \mathfrak{k}$ one has $\frac{d}{dt}\Big|_{t=0} f_{a,\lambda}(\exp(tX)k) = \kappa_{\mathbb{R}}(X, \mathrm{Ad}(n(ka))H_\lambda)$.*

(ii) *We have $df_{a,\lambda}(k) = 0$ if and only if*

$$\mathfrak{k} \bot_{\kappa_{\mathbb{R}}} \mathrm{Ad}(n(ka))H_\lambda.$$

Proof. (i) is immediate from Lemma 2.2 and the notations introduced from above.
(ii) follows from (i). ■

Write $i\mathfrak{a}_{\mathbb{R}}^*$ for the subspace of $(\mathfrak{a}_{\mathbb{C}}^{\mathbb{R}})^*$ which vanishes on $\mathfrak{a}$. Our next goal is to determine the critical set of $f_{a,\lambda}$ for $\lambda \in i\mathfrak{a}_{\mathbb{R}}^*$.

We also denote by θ the holomorphic extension of the Cartan involution to $G_{\mathbb{C}}$. Further we write $G_{\mathbb{C}} \to G_{\mathbb{C}}$, $x \mapsto \overline{x}$ for the complex conjugation with respect to the real form G.

Lemma 2.4. *Suppose that $X \in \Omega$ is regular. Set $a = \exp(iX)$ and let $k \in K$. Then the following implication holds*

$$ka \in NA_{\mathbb{C}}K_{\mathbb{C}} \Rightarrow k \in N_K(\mathfrak{a}).$$

Proof. We write $ka = nbk'$ for $n \in N$, $b \in A_{\mathbb{C}}$ and $k' \in K_{\mathbb{C}}$. Then

$$ka^2k^{-1} = ka(\theta(ka)^{-1}) = nbk'(\theta(nbk'))^{-1} = nb^2\theta(n)^{-1}.$$

Set $x := ka^2k^{-1}$. Then

$$\overline{x} = ka^{-2}k^{-1} = x^{-1}.$$

Thus we obtain that

$$\overline{x} = n\overline{b}^2\theta(n)^{-1} = \theta(n)b^{-2}n^{-1}.$$

Therefore $x = \theta(n)\overline{b}^{-2}n^{-1}$ and so

$$x^2 = \theta(n)\overline{b}^{-2}n^{-1}nb^2\theta(n)^{-1} \in A_{\mathbb{C}}\theta(N_{\mathbb{C}}).$$

Now let $0 < t \leq 1$ and write $a_t := \exp(itX)$ and accordingly we define x_t. Then we obtain that

$$\log x_t^2 = \log(k\exp(i4tX)k^{-1}) = \mathrm{Ad}(k)(i4tX) \in \mathfrak{a}_{\mathbb{C}} + \theta(\mathfrak{n}_{\mathbb{C}})$$

for small t and hence for all t by analyticity. But $\mathrm{Ad}(k)(i4tX) \in i\mathfrak{p}$. Hence by the regularity of X we obtain that $k \in N_K(\mathfrak{a})$, as was to be shown. ■

Proposition 2.5. *Suppose that $X \in \Omega$ and set $a = \exp(iX)$. Let $\lambda \in i\mathfrak{a}_{\mathbb{R}}^*$ be such that $H_\lambda \in i\mathfrak{a}$ is regular. Then we have*

$$df_{a,\lambda}(k) = 0 \iff k \in N_K(\mathfrak{a}).$$

Proof. The implication "⇐" follows immediately from Lemma 2.3 (ii).
"⇒": Suppose that $df_{a,\lambda}(k) = 0$. Then Lemma 2.3 (ii) implies that $\mathfrak{k}\perp_{\kappa_{\mathbb{R}}} \mathrm{Ad}(n(ka))H_\lambda$, or equivalently $\mathrm{Ad}(n(ka))H_\lambda \in i\mathfrak{g} + \mathfrak{p}$. Assume that $k \notin N_K(\mathfrak{a})$. By Lemma 2.4 we have $n(ka) = \exp(Y)$ for some $Y \in \mathfrak{n}_{\mathbb{C}}\backslash\mathfrak{n}$. Hence

$$\mathrm{Ad}(n(ka))H_\lambda = H_\lambda + \underbrace{[Y, H_\lambda]}_{\in\mathfrak{n}_{\mathbb{C}}} + \frac{1}{2}\underbrace{[Y,[Y,H_\lambda]]}_{\in[\mathfrak{n}_{\mathbb{C}},\mathfrak{n}_{\mathbb{C}}]} + \ldots.$$

Thus the fact that $H_\lambda \in i\mathfrak{a}$ is regular implies that $\mathrm{Ad}(n(ka))H_\lambda \in i\mathfrak{a} + (\mathfrak{n}_{\mathbb{C}}\backslash i\mathfrak{n})$. But this contradicts $\mathrm{Ad}(n(ka))H_\lambda \in i\mathfrak{g} + \mathfrak{p}$, proving our proposition. ■

Proof of Theorem 2.1. First we observe that it is sufficient to prove

$$a(K\exp(iX)) \subseteq A\exp(i\,\mathrm{conv}(\mathcal{W}X))$$

for all $X \in \Omega$. By a simple density/continuity argument we may further assume that X is regular. Suppose that there exists a $k \in K$ such that $a(k\exp(iX)) \notin A\exp(i\,\mathrm{conv}(\mathcal{W}X))$, or equivalently

$$\mathrm{Im}\log a(k\exp(iX)) \notin \mathrm{conv}(\mathcal{W}X).$$

Then we find a regular element $\lambda \in i\mathfrak{a}_{\mathbb{R}}^*$ such that

$$f_{\exp(iX),\lambda}(k) > \max_{Y\in i\,\mathrm{conv}(\mathcal{W}X)} \lambda(Y).$$

But Proposition 2.5 implies that $f_{\exp(iX),\lambda}$ takes its maximum at an element $k \in N_K(\mathfrak{a})$. Hence $f_{\exp(iX),\lambda}(k) = \lambda(\mathrm{Ad}(k)iX)$; a contradiction to our inequality from above. ■

3. Applications

Let now $\omega \subseteq \Omega$ be an open Weyl group invariant convex set. Then we define the domain

$$\Xi(\omega) := G \exp(i\omega) K_{\mathbb{C}}/K_{\mathbb{C}}.$$

If $\omega = \Omega$, then we set $\Xi := \Xi(\Omega)$. Write $\partial\Xi(\omega)$ for the toplogical boundary of $\Xi(\omega)$ in $G_{\mathbb{C}}/K_{\mathbb{C}}$. Note the following properties of $\Xi(\omega)$:

- $\Xi(\omega)$ is open in $G_{\mathbb{C}}/K_{\mathbb{C}}$ (cf. [AkGi90]).
- $\Xi(\omega)$ is connected and G-invariant.
- G acts properly on Ξ (cf. [AkGi90]).
- One has $G\exp(i\partial\omega)K_{\mathbb{C}}/K_{\mathbb{C}} \subseteq \partial\Xi(\omega)$. Moreover if $\overline{\omega} \subseteq \Omega$, then we have $\partial\Xi(\omega) = G\exp(i\partial\omega)K_{\mathbb{C}}/K_{\mathbb{C}}$ (cf. [KrSt01b, Prop. 4.1]).

From our discussions in the previous section it is clear that we have a holomorphic projection

$$a: \Xi_\omega \to A_{\mathbb{C}}, \quad x \mapsto a(x)$$

with $x \in N_{\mathbb{C}} a(x) K_{\mathbb{C}}/K_{\mathbb{C}}$ for all $x \in \Xi$. Finally we define the Abelian tube domain

$$T_\omega := A \exp(i\omega) \subseteq A_{\mathbb{C}}.$$

An immediate consequence of our complex convexity theorem then is:

Lemma 3.1. *We have*

$$a(\Xi(\omega)) = T_\omega.$$ ∎

Remark 3.2. From Lemma 3.1 we obtain in particular that $a(\Xi) \subseteq T_\Omega$. It is interesting to observe that this inclusion for $G = \mathrm{Sp}(n, \mathbb{R})$ extends a result of Siegel.

Consider the vector space $V := \mathrm{Symm}(n, \mathbb{R})$ of real symmetric matrices with its subcone of positive definite matrices V_+. Then we have the symmetric Siegel domain

$$S^+ := V + iV_+ \subseteq V_{\mathbb{C}}.$$

Recall that $S^+ \cong \mathrm{Sp}(n, \mathbb{R})/U(n)$.

If we write Δ_j for the j-th principal minor on $V_{\mathbb{C}}$, then Siegel's Lemma says

$$\Delta_j(z) \neq 0 \qquad \text{for } z \in S^+ \tag{3.1}$$

and all $1 \leq j \leq n$. Now consider the rational functions

$$\chi_j(z) = \frac{\Delta_j(z)}{\Delta_{j-1}(z)}$$

on $V_{\mathbb{C}}$. Then the inclusion $a(\Xi) \subseteq T_\Omega$ for $G = \mathrm{Sp}(n, \mathbb{R})$ implies

$$\mathrm{Im}\, \chi_j(z) > 0 \qquad \text{for } z \in S^+ \tag{3.2}$$

and for all j. Note that (3.2) implies (3.1). To see this one identifies S^+ with the symmetric space $\mathrm{Sp}(n, \mathbb{R})/U(n)$. Then with $S^- := \overline{S^+}$ one has a biholomorphism $\Xi \cong S^+ \times S^-$ (cf. [GiMa01] or [KrSt01b]). Realizing A as diagonal matrices in G, one then easily shows that $a(\Xi) \subseteq T_\Omega$ implies (3.2).

The example discussed above admits a natural generalization to all tube domains $V + iV_+$ associated to an Euclidean Jordan algebra V and cone V_+. ∎

For every $g \in G$ we define the *horospherical tube* associated to ω by

$$T(g,\omega) := g\big(N_{\mathbb{C}}T_\omega K_{\mathbb{C}}/K_{\mathbb{C}}\big) \subseteq G_{\mathbb{C}}/K_{\mathbb{C}}.$$

Note that $T(\mathbf{1},\omega)$ is biholomorphic to $N_{\mathbb{C}} \times T_\omega$. In particular, all horospherical tubes $T(g,\omega)$ are Stein.

Theorem 3.3. *For any non-empty open convex $\mathcal{W}$-invariant set $\omega \subseteq \Omega$ the domain $\Xi(\omega)$ is the connected component of the intersection of horospherical tubes:*

$$\Xi(\omega) = \Big(\bigcap_{g\in G} T(g,\omega)\Big)_0.$$

Proof. From Lemma 3.1 we obtain that

$$\Xi(\omega) \subseteq N_{\mathbb{C}}T_\omega K_{\mathbb{C}}/K_{\mathbb{C}} = a^{-1}(T(\mathbf{1},\omega)).$$

Thus the fact that $\Xi(\omega)$ is G-invariant and connected implies that $\Xi(\omega) \subseteq \big(\bigcap_{g\in G} T(g,\omega)\big)_0$. On the other hand we obtain from $\Xi = \Xi_I$ (cf. the discussion in the introduction) and Lemma 3.1 that $\Xi = \big(\bigcap_{g\in G} T(g,\Omega)\big)_0$. From this and Lemma 3.1 we hence get

$$\Big(\bigcap_{g\in G} T(g,\omega)\Big)_0 \subseteq \Xi \cap T(\mathbf{1},\omega) = \Xi(\omega),$$

completing the proof of the theorem. ■

An interesting application of Theorem 3.3 is the following:

Proposition 3.4. *Let $\omega \subseteq \Omega$ be an open convex $\mathcal{W}$-invariant set. Then the intersection*

$$I(\omega) := \bigcap_{g\in G} T(g,\omega)$$

is open. In particular, $I(\omega)$ and every connected component of $I(\omega)$ is Stein. In particular, $\Xi(\omega)$ is Stein.

Proof. (following a suggestion of Dmitri Akhiezer) Since $G = KAN$ and since AN leaves $T(\mathbf{1},\omega)$ invariant, we obtain

$$I(\omega) = \bigcap_{k\in K} T(k,\omega).$$

Hence $I(\omega)$ is an intersection of open sets over the "compact parameter space" K. In particular $I(\omega)$ is open. Since all horospherical tubes $T(k,\omega)$ are Stein, we obtain that $I(\omega)$ is Stein. With $I(\omega)$ all its connected components are Stein, concluding the proof of the theorem. ■

As a final application of the complex convexity theorem we prove a result on the characterization of the boundary of Ξ.

Proposition 3.5. *Let $\omega \subseteq \Omega$ be an open convex $\mathcal{W}$-invariant set. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence with $z_n \to z_0 \in \partial\Xi(\omega)$. Then*

$$(\operatorname{Im}\log a(z_n))_{n\in\mathbb{N}}$$

is a sequence in ω and every accumulation point of this sequence lies in $\partial\omega$.

Proof. Set $X_n := \operatorname{Im}\log a(z_n)$. By Lemma 3.1 we have $X_n \in \omega$. Since $\overline{\omega}$ is compact in $\mathfrak{a}$, we may assume that $X_n \to X_0$ with $X_0 \in \overline{\omega}$.

If $X_0 \notin \partial\omega$, then we find a convex Weyl group invariant open set ω_1 such that $\overline{\omega_1} \subseteq \omega$ and $X_n \in \omega_1$. Then Theorem 3.3 implies that $z_n \in \Xi(\omega_1)$ and so $z_0 \in \overline{\Xi(\omega_1)}$. Now $\overline{\omega}_1 \subseteq \Omega$ and so $\partial\Xi(\omega_1) = G\exp(i\partial\omega_1)K_{\mathbb{C}}/K_{\mathbb{C}}$. Thus $z_0 \in \Xi(\omega)$, contradicting the assumption $z_0 \in \partial\Xi(\omega)$. This concludes the proof of the proposition. ■

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