LOCATING ANISOTROPIES IN ELECTRICAL IMPEDANCE TOMOGRAPHY

ERKKI SOMERSALO

Abstract. In this article, we consider the problem of finding the support of an inhomogenous possibly anisotropic inclusion in a background of constant electric conductivity from the electrical impedance tomography data at the boundary of a bounded body. The article discusses the linear sampling method applied to this problem. A practical algorithm for solving this problem is suggested.

1. Introduction

In this article, the following electrical impedance tomography problem (EIT) is considered: On the surface of a body with unknown impedance distribution, one applies a set of prescribed electric currents and measures the corresponding voltages on the surface. From this information, one seeks to estimate the internal structure of the body. Potential application areas of the EIT range from medical imaging and monitoring to industrial process monitoring and nondestructive material testing.

Often, the materials encountered in applications are anisotropic, i.e., the electromagnetic properties of the medium depend on the direction. It is well known that in general, the anisotropic EIT problem allows no unique solution. Hence, in the presence of anisotropies one has either to use *a priori* information complementary to that obtained from the measurements or one has to confine to more modest goals than recovering the full information of the material parameters. In this article, the latter approach is taken. More precisely, the following inverse problem is studied: Assume that in an isotropic body with otherwise known electric properties, there is an unknown possibly anisotropic inclusion. Given the EIT data on the surface of the body, estimate the *support* of the unknown inclusion.

In recent years, a number of articles have been published, where the goal is to determine the shape of an inclusion based on either far-field or near field measurements with various probing modalities, see e.g. [1], [2], [4], [3], [7], [9], [10], [11], [6] and the references in these articles. The starting point of the present work is the linear sampling method originally introduced in the article [4]. The ideas here come close to those presented in [1] and [2].

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2. The inverse problem

Let $B \subset \mathbb{R}^n$, $n = 2, 3$, denote a bounded simply connected domain with a C^2 smooth connected boundary. When low-frequecy time-harmonic electromagnetic field is induced in the body, within the quasi-static approximation of Maxwell's equations the electric field is written in terms of the voltage potential u that satisfies

$$\nabla \cdot \gamma \nabla u = 0 \text{ in } B.$$

Here, $\gamma = \gamma(x) \in \mathbb{C}^{n\times n}$ is the admittance distribution that in terms of the conductivity $\sigma(x) \in \mathbb{R}^{n\times n}$, permittivity $\varepsilon(x) \in \mathbb{R}^{n\times n}$ and frequency $\omega > 0$ is given as

$$\gamma = \sigma - \mathrm{i}\omega\varepsilon.$$

Assume that $D \subset B$ is an open set that has a smooth boundary and consists of one or several simply connected components. Furthermore, assume that $\partial D \cap \partial B = \emptyset$. We assume that the permittivity of the material outside D is so low that it can be neglected, i.e., $\gamma \approx \sigma$ in $B \setminus \overline{D}$. Furthermore, we assume that the material is isotropic outisde D. To simplify the discussion, we assume that in fact, $\sigma = 1$ outside D. It is not hard to see that the results in this article can be generalized in a straightforward manner to cover the case where σ is a strictly positive scalar function outside D. These assumptions lead to the following model: By denoting by χ_D the characteristic function of the set D, the admittance γ is of the form

$$\gamma(x) = 1 + h(x)\chi_D(x).$$

Furthermore, we assume that the perturbation $h \in C^1(B, \mathbb{C}^{n\times n})$ is symmetric and satisfies in addition that for some positive constants α, $\beta > 0$ and for all $\zeta \in \mathbb{C}^n$,

$$\mathrm{Re}\big(z\overline{\zeta} \cdot \gamma(x)\zeta\big) \geq \alpha|\zeta|^2, \text{ for all } x \in B, \text{ for some } z \in \mathbb{C} \tag{1}$$

$$\mathrm{Im}\big(\overline{\zeta} \cdot h(x)\zeta\big) \leq -\beta|\zeta|^2, \ x \text{ in an open set } \mathcal{O} \subset D \tag{2}$$

To describe the measurement, we fix the following notation. By $H_0^s(\partial B)$ (with the notation $H_0^0(\partial B) = L_0^2(\partial B)$) we denote the class of Sobolev functions of smoothness index s over the boundary with the restriction

$$\tau f = \langle f, 1\rangle = 0,$$

the brackets denoting the natural pairing between $H^s(\partial B)$ and $H^{-s}(\partial B)$. since $\tau : H^s(\partial D) \to \mathbb{C}$ is continuous, $H_0^s(\partial B)$ is a closed subspace.

Assume that one applies an electric current $f \in H_0^{-1/2}(\partial B)$ on the surface of the body B. Then the voltage potential u satisfies

$$\begin{aligned} \nabla \cdot \gamma \nabla u &= 0 \\ \partial u\big|_{\partial B} &= f, \end{aligned} \tag{3}$$

where we have denoted the normal derivative of u at the boundary as $\partial u|_{\partial B}$. One can see, by a standard argument using the Lax-Milgram lemma and Fredholm theory that the problem (3) has a unique solution $u \in H_0^1(B)$, where we have denoted

$$H_0^1(B) = \left\{ u \in H^1(B) \mid u\big|_{\partial B} \in H_0^{1/2}(\partial B) \right\}.$$

For later reference, let us denote by T the continuous solution operator

$$T : H_0^{-1/2}(\partial B) \to H_0^1(B), \quad f \mapsto u.$$

Further, we define the Neumann-to-Dirichlet map Λ as

$$\Lambda : H_0^{-1/2}(\partial B) \to H_0^{1/2}(\partial B), \quad f \mapsto u|_{\partial B},$$

where u is the unique solution of the problem (3). We shall also consider the forward problem when no inclusion is present, i.e., the boundary value problem

$$\begin{aligned} \Delta v &= 0 \\ \partial v\big|_{\partial B} &= f, \end{aligned} \tag{4}$$

and the corresponding solution operator and Neumann-to-Dirichlet maps,

$$T_0 : H_0^{-1/2}(\partial B) \to H_0^1(B), \quad f \mapsto v,$$

$$\Lambda_0 : H_0^{-1/2}(\partial B) \to H_0^{1/2}(\partial B), \quad f \mapsto v|_{\partial B}.$$

The inverse problem studied in this article can be formulated as follows:

Problem 2.1. *Given the Neumann-to-Dirichlet map Λ, determine the support D of the perturbation.*

In the following section, we discuss the linear sampling method that gives a practical way of estimating the support.

3. The linear sampling method

The linear sampling method discussed in this article is based on the use of certain singular solutions. Therefore, let us define the singular solution $B \ni x \mapsto \Phi(x, y, \hat{\alpha})$, where $y \in B$ is a parameter and $\hat{\alpha} \in \mathbb{R}^n$ is a unit vector, as a solution of the following homogenous Neumann problem,

$$\begin{aligned} \Delta \Phi &= \hat{\alpha} \cdot \nabla \delta(x - y) \\ \partial \Phi\big|_{\partial B} &= 0. \end{aligned} \tag{5}$$

Physically, the singular solution corresponds to the electromagnetic potential created by a dipole source at y pointing in the direction $\hat{\alpha}$.

In terms of the operator T_0, we may write the singular solution as

$$\Phi(x, y, \hat{\alpha}) = \hat{\alpha} \cdot \vec{\Psi}(x - y) - T_0(\vec{\alpha} \cdot \partial \vec{\Psi}(x - y)) - c_y,$$

where

$$\vec{\Psi}(x - y) = -\frac{1}{2^{n-1}\pi} \frac{x - y}{|x - y|^n}, \quad n = 2, 3,$$

and c_y is a constant equal to the integral of $x \mapsto \alpha \cdot \vec{\Psi}(x-y)$ over the boundary ∂B. We shall use the notation $\Phi_y(x) = \Phi(x, y, \hat{\alpha})$, suppressing the dependence on the direction as this plays little role in the discussion to ensue.

The starting point of the linear sampling method lies in the following observation that we formulate as a lemma for later reference. In the following, we shall use the notation $\phi_y(x) = \Phi_y(x)\big|_{\partial B}$.

Lemma 3.1. *Assume that $y \in B \setminus \overline{D}$. Then $\phi_y \notin \mathrm{Ran}(\Lambda - \Lambda_0)$.*

Proof. Assume on the contrary that $\phi_y \in \mathrm{Ran}(\Lambda - \Lambda_0)$, and let $\phi_y = (\Lambda - \Lambda_0)\psi_y$, $\psi_y \in H_0^{-1/2}(\partial B)$. Let $u_y = T\psi_y$ and $v_y = T_0\psi_y$ denote the solutions of the problems (3) and (4) with the boundary data ψ_y, respectively. Set $w_y = u_y - v_y$. Now we observe that

$$\partial w_y = 0, \quad w_y\big|_{\partial B} = u_y - v_y = (\Lambda - \Lambda_0)\psi_y = \phi_y,$$

i.e., the Cauchy data of w_y and Φ_y coincide on ∂B. By Holmgren's Uniqueness Theorem, we have $w_y = \Phi_y$ in $B \setminus (\overline{D} \cup \{y\})$. The claim of the lemma follows now, since w_y has no singularity at y while Φ_y is singular. □

Unfortunately, the converse is not in general true: When $y \in D$, there is no guarantee that $\phi_y \in \mathrm{Ran}(\Lambda - \Lambda_0)$. As in the case of inverse scattering problems, we have to confine to an approximate solution of the equation $(\Lambda - \Lambda_0)\psi_y = \phi_y$. To this end, we need to introduce some notations and definitions, and prove a number of auxiliary results.

To get a handle of the following definition, assume for a while that $\phi_y = (\Lambda - \Lambda_0)\psi_y$ for some $\psi_y \in H_0^{-1/2}(\partial B)$ and $y \in D$. By denoting again $u_y = T\psi_y$ and $v_y = T_0\psi_y$, we observe as in the proof of the previous lemma that in $B \setminus \overline{D}$, $u_y - v_y = \Phi_y$, and in D,

$$\nabla \cdot \gamma \nabla u_y = \Delta v_y = 0.$$

At the boundary ∂D, the solutions must satisfy

$$\begin{aligned} u_y\big|_{\partial D}^{-} - v_y\big|_{\partial D}^{-} &= \Phi_y\big|_{\partial D}, \\ \partial_\gamma u_y\big|_{\partial D}^{-} - \partial v\big|_{\partial D}^{-} &= \partial \Phi_y\big|_{\partial D}. \end{aligned}$$

Above, the notation $\partial_\gamma u_y|_{\partial D}^{-} = n \cdot \gamma \nabla u_y|_{\partial D}^{-}$ for the conormal derivative was used, and the subscript "-" indicates that the traces are from inside of ∂D.

Hence, in order to investigate the equation $(\Lambda - \Lambda_0)\psi_y = \phi_y$, it is natural to study the following interior transmission problem of impedance tomography.

Problem 3.2. *The interior transmission problem (ITP) of electrical impedance tomography with boundary data $(f, g) \in H^{1/2}(\partial D) \times H^{-1/2}(\partial D)$ is to find functions $(u, w) \in (H^1(D) \times H^1(D))/\mathbb{C}$ satisfying the equations*

$$\nabla \cdot \gamma \nabla u = \Delta v = 0 \textit{ in } D,$$

with

$$\begin{aligned} u\big|_{\partial D} - v\big|_{\partial D} &= f, \\ \partial_\gamma u\big|_{\partial D} - \partial\big|_{\partial D} v &= g. \end{aligned}$$

Observe that above, the solution of the interior transmission problem can be unique only up to an additive constant. The space $(H^1(D) \times H^1(D))/\mathbb{C}$ consists of equivalence classes of the relation

$$(u, v) \sim (u', v') \text{ if and only if } (u, v) = (u' - c, v' - c), \quad c \in \mathbb{C}.$$

It turns out that under the assumptions made about the admittance γ, the ITP has a unique solution. We formulate this as a lemma.

Lemma 3.3. *Assume that the admissivity γ satisfies the conditions (1) and (2). Then the interior transmission problem 3.2 has a unique solution.*

Proof: Tho show that there are at most one solution, assume that (u, v) satisfy the homogenous interior transmission problem, i.e., $(f, g) = (0, 0)$. By integrating by parts we obtain

$$\begin{aligned} 0 &= \int_{\partial D} (\overline{v}\partial v - v\overline{\partial v})dS = \int_{\partial D} (\overline{u}\partial_\gamma u - u\overline{\partial_\gamma u})dS \\ &= 2\mathrm{i} \int_D \mathrm{Im}(\overline{\nabla u} \cdot h\nabla u)dx. \end{aligned}$$

By the assumption (2), we obtain that u is constant in an open set of D, and by the unique continuation property it is a constant in the whole of D. Thus v has the same Cauchy data with a constant solution on ∂D and so also v is constant in D, which completes the proof of uniqueness.

To prove the existence, we refer to the article [2], where the interior transmission problem for the scattering case was studied. It is not hard to see, that the same argument goes through here as well. We leave the details out. □

Although the ITP is closely related to the existence of the solution of the equation $(\Lambda - \Lambda_0)\psi_y = \phi_y$, there is no equivalence, since we cannot in general extend the solutions (u, v) of the ITP from D to the whole domain B. Therefore, we consider only such solutions that have the extension property. This is the motivation for the following considerations.

Let $\Omega \subset \mathbb{R}^n$ denote a ball that contains $\overline{B}$ in its interior. Further, let $G(x, y)$ denote Green's function of the Laplacian,

$$G(x, y) = \begin{cases} \dfrac{1}{2\pi}\log\dfrac{1}{|x - y|}, & n = 2, \\ \\ \dfrac{1}{4\pi}\dfrac{1}{|x - y|}, & n = 3. \end{cases}$$

We define the potential operator

$$\mathcal{S}_D : L^2(\partial\Omega) \to C^\infty(D), \quad \omega \mapsto \int_{\partial\Omega} G(x,z)\omega(z)dS(z), \quad x \in D.$$

Further, we define a particular class of harmonic functions in D as

$$\mathbb{H}(D) = \{u \mid u = \mathcal{S}_D\omega,\ \omega \in L^2(\partial\Omega)\}.$$

We also define the operators

$$K : L^2(\partial\Omega) \to H^{1/2}(\partial D), \quad \omega \mapsto \int_{\partial\Omega} G(x,z)\omega(z)dS(z), \quad x \in \partial D,$$

and, finally

$$L : L^2(\partial\Omega) \to H_0^{-1/2}(\partial B), \quad \omega \mapsto \partial \int_{\partial\Omega} G(x,z)\omega(z)dS(z), \quad x \in \partial B.$$

Observe that by Green's formula, the integral of $L\omega$ over ∂B automatically vanishes. The class $\mathbb{H}(D)$ has the following approximation property of harmonic functions in D.

Theorem 3.4. *For each $\varepsilon > 0$ and $v \in H^1(D)$ satisfying $\Delta v = 0$ in the weak sense there is $v^\varepsilon \in \mathbb{H}(D)$ such that $\|v - v^\varepsilon\|_{H^1(D)} < \varepsilon$.*

The proof of this theorem is based on the following density result.

Lemma 3.5. *The operator K has dense range in both $L^2(\partial D)$ as in $H^1(\partial D)$.*

Proof: The proof is quite similar to the corresponding one in the article [8]. To prove the denseness in $L^2(\partial D)$, assume that $\eta \in L^2(\partial D)$ is such that for all $\omega \in L^2(\partial\Omega)$,

$$(K\omega, \eta)_{L^2(\partial D)} = (\omega, K^*\eta)_{L^2(\partial\Omega)} = 0,$$

implying that

$$K^*\eta(x) = \int_{\partial D} G(y,x)\eta(y)dS(y) = \int_{\partial D} G(x,y)\eta(y)dS(y) = 0$$

for $x \in \partial\Omega$. Then, the function w defined as

$$w(x) = \int_{\partial D} G(x,y)\eta(y)dS(y), \quad x \in \mathbb{R}^n,$$

is harmonic both in $\mathbb{R}^n \setminus \overline{D}$ and D. It has vanishing Dirichlet boundary data on $\partial\Omega$ and $w \to 0$ as $|x| \to \infty$, so $w = 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$, and by the unique continuation principle, $w = 0$ in $\mathbb{R}^n \setminus \overline{D}$. By the continuity of the single layer potential, this implies also that $w\big|_{\partial D}^- = 0$ and hence $w = 0$ in D. The conclusion $\eta = 0$ follows from the well known jump relation $\eta = \partial w\big|_{\partial D}^- - \partial w\big|_{\partial D}^+$ of the normal derivatives.

To prove the denseness in $H^1(\partial D)$, we equip it with the inner product

$$(\eta, \mu)_{H^1(\partial D)} = \int_{\partial D} \left(\overline{\eta(x)}\mu(x) + \overline{\mathrm{Grad}\eta(x)} \cdot \mathrm{Grad}\mu(x)\right) dS(x),$$

where Grad is the surface gradient on ∂D. By denoting by $K^\dagger$ the adjoint of K as a mapping from $L^2(\partial\Omega)$ to $H^1(\partial D)$, assume that we have

$$(K\omega, \eta)_{H^1(\partial D)} = (\omega, K^\dagger \eta)_{L^2(\partial\Omega)} = 0$$

for all $\omega \in L^2(\partial\Omega)$ or explicitly,

$$K^\dagger \eta(x) = \int_{\partial D} \Big(G(y,x)\eta(y) + \mathrm{Grad}_y G(y,x) \cdot \mathrm{Grad}\eta(y) \Big) dS(y) = 0,$$

when $x \in \partial\Omega$. To rewrite the second term in a more convenient form, assume, for a while that η is smooth. Then, by Gauss' surface divergence theorem and by using $G(x,y) = G(y,x)$, we obtain for $x \notin \partial D$,

$$\begin{aligned}\int_{\partial D} \mathrm{Grad}_y G(y,x) \cdot \mathrm{Grad}\eta(y) dS(y) &= -\int_{\partial D} G(y,x) \cdot \mathrm{Div}\mathrm{Grad}\eta(y) dS(y) \\ &= \int_{\partial D} \mathrm{Grad}_y G(x,y) \cdot \mathrm{Grad}\eta(y) dS(y),\end{aligned}$$

and by extension, this holds for all $\eta \in H^1(\partial D)$. Furthermore, since $\nabla_x G(x,y) = -\nabla_y G(x,y)$, we obtain

$$\begin{aligned}\int_{\partial D} \mathrm{Grad}_y G(x,y) \cdot \mathrm{Grad}\eta(y) dS(y) &= \int_{\partial D} \nabla_y G(x,y) \cdot \mathrm{Grad}\eta(y) dS(y) \\ &= -\nabla \int_{\partial D} G(x,y) \cdot \mathrm{Grad}\eta(y) dS(y).\end{aligned}$$

Hence, we define

$$w(x) = \int_{\partial D} G(x,y)\eta(y) dS(y) - \nabla \cdot \int_{\partial D} G(x,y) \mathrm{Grad}\eta(y) dS(y), \quad x \in \mathbb{R}^n \setminus \partial D,$$

which is harmonic both inside and outside of D, and by the above considerations, $w\big|_{\partial\Omega} = K^\dagger \eta = 0$. As above, we conclude that $w = 0$ in $\mathbb{R}^n \setminus \overline{D}$. On ∂D, the jump relations for vector potentials (see [5], Theorem 6.12),

$$w\big|_{\partial D}^{\pm}(x) = \int_{\partial D} \Big(G(x,y)\eta(y) - \nabla_x G(x,y) \cdot \mathrm{Grad}\eta(y)) \Big) dS(y).$$

This expression is the adjoint of the single layer operator

$$S : L^2(\partial D) \to H^1(\partial D), \quad \eta \mapsto \int_{\partial D} G(x,y)\eta(y) dS(y),$$

see e.g. [5], (pp. 43–44) or [8]. Hence, for all $\psi \in L^2(\partial D)$, we have

$$(S\psi, \eta)_{H^1(\partial D)} = (\psi, w\big|_{\partial D})_{L^2(\partial D)} = 0.$$

By the uniqueness of the interior Dirichlet problem for the Laplacian, we now deduce that S is injective, so choosing $\psi = S^{-1}\eta$ we obtain that $\eta = 0$. $\square$

With the aid of the above lemma, we prove Theorem 3.4.

Proof of Theorem 3.4: Clearly, since $H^1(\partial D)$ is dense in $H^{1/2}(\partial D)$, the range of K is also dense in $H^{1/2}(\partial D)$. Let $v \in H^1(D)$ be harmonic and $\varepsilon > 0$ be given. We choose first $\omega \in L^2(\partial\Omega)$ such that

$$\|K\omega - v\big|_{\partial D}\|_{H^{1/2}(\partial D)} < \delta$$

for some $\delta = \delta(\varepsilon) > 0$ to be defined later. Let

$$v^\varepsilon = \mathcal{S}_D\omega \in \mathbb{H}(D).$$

Then, by the continuity of the Dirichlet problem with respect to the boundary data, we have

$$\|v - v^\varepsilon\|_{H^1(D)} \leq C\|K\omega - v\big|_{\partial D}\|_{H^{1/2}(\partial D)} < C\delta,$$

and the claim follows by choosing $\delta = \varepsilon/C$. □

It is clear that if we define the space $\mathbb{H}(B)$ as functions of the form $u = \mathcal{S}_B\omega$ using obvious notations, the harmonic functions in B can be approximated by $\mathbb{H}(B)$–functions. This observation gives us the following density result.

Lemma 3.6. *The range of the operator L is dense in $H_0^{-1/2}(\partial B)$.*

Proof: Let $\varepsilon > 0$ and $\psi \in H_0^{-1/2}(\partial B)$ be given. Define a harmonic function v in B as $v = T_0\psi$. By the observation above, we can find $v^\varepsilon = \mathcal{S}_B\omega \in \mathbb{H}(B)$ with

$$\|v - v^\varepsilon\|_{H^1(B)} < \delta,$$

where $\delta = \delta(\varepsilon)$ is fixed later. But by the trace theorem,

$$\|\partial v - \partial v^\varepsilon\|_{H^{-1/2}(\partial B)} = \|\psi - L\omega\|_{H^{-1/2}(\partial B)} \leq C\|v - v^\varepsilon\|_{H^1(B)} < C\delta,$$

so by choosing $\delta = \varepsilon/C$ the desired approximation follows. □

The counterpart of Lemma 3.1 that we want to prove for $y \in D$ is the following.

Lemma 3.7. *Assume that $y \in D$. Then the interior transmission problem with the boundary data $(f, g) = (\Phi_y\big|_{\partial D}, \partial\Phi_y\big|_{\partial D})$ has a unique solution (u, v) with $v \in \mathbb{H}(D)$ if and only if $\phi_y = (\Lambda - \Lambda_0)L\omega$ for some $\omega \in L^2(\partial\Omega)$.*

Proof: Assume that (u, v) is the unique solution of the ITP with $v = \mathcal{S}_D\omega$. We can extend v to the whole B by setting $v = \mathcal{S}_B\omega$, and extend u to whole B by defining $u = v + \Phi_y$ in $B \setminus \overline{D}$. From the ITP boundary conditions, it follows now that u thus defined satisfies the equation $\nabla \cdot \gamma\nabla u = 0$ in the weak sense in B, and at the boundary ∂B,

$$\partial u\big|_{\partial B} = \partial(v + \Phi_y)\big|_{\partial B} = \partial v\big|_{\partial B} = L\omega.$$

Hence, we have

$$\psi_y = (u - v)\big|_{\partial B} = (\Lambda - \Lambda_0)L\omega.$$

To prove the converse, let $\psi_y = (\Lambda - \Lambda_0)L\omega$ for some $\omega \in L^2(\partial\Omega)$. We define $u = TL\omega$ and $v = T_0L\omega$. As in the proof of Lemma 3.1, we see that $u - v = \Phi_y$ in $B \setminus \overline{D}$, and hence (u, v) satisfy the ITP with the boundary data $(f, g) = (\Phi_y\big|_{\partial D}, \partial\Phi_y\big|_{\partial D})$. On the other hand, the function

$$v_0 = v - \mathcal{S}_B\omega$$

is harmonic in B and at the boundary,

$$\partial v_0 = \partial v - L\omega = 0,$$

implying that v_0=constant. By the definition of T_0, we also see that the integral of v and thus v_0 over the boundary vanishes, so $v_0 = 0$ and the claim follows. □

The above lemma does not help us much since in general, the unique solution of the ITP is not such that $v \in \mathbb{H}(D)$. However, we can always find an approximate solution, as the following theorem states.

Theorem 3.8. *Assume that $y \in D$. Then for any $\varepsilon > 0$ the equation $(\Lambda - \Lambda_0)L\omega = \phi_y$ has an approximate solution in $\omega_y^\varepsilon \in L^2(\partial\Omega)$, i.e., ω_y^ε is the satisfies the estimate*

$$\|(\Lambda - \Lambda_0)L\omega_y^\varepsilon - \phi_y\|_{H^{1/2}(\partial B)} < \varepsilon. \tag{6}$$

Furthermore, when y approaches the boundary ∂D, $\|\omega_y^\varepsilon\|_{L^2(\partial\Omega)} \to \infty$.

Proof: Let (u_y, v_y) be the solution of the interior transmission problem with the transmission data $(f, g) = (\Phi_y\big|_{\partial D}, \partial\Phi_y\big|_{\partial D})$.

First, let $v_y^\varepsilon \in \mathbb{H}(D) = \mathcal{S}_D\omega$ be an approximation of v_y such that

$$\|v_y - v_y^\varepsilon\|_{H^1(D)} < \delta,$$

where $\delta = \delta(\varepsilon)$ is fixed later. We extend v_y^ε to the whole of B as $v = \mathcal{S}_B\omega$.

Having v_y^ε in B, we define u_y^ε in B as

$$u_y^\varepsilon = \chi_D u_y + (1 - \chi_D)(\Phi_y + v_y^\varepsilon).$$

We observe that on ∂B, we have $(\partial u_y^\varepsilon - \partial v_y^\varepsilon)\big|_{\partial B} = \partial\Phi_y\big|_{\partial B} = 0$. Let us denote

$$\partial u_y^\varepsilon\big|_{\partial B} = \partial v_y^\varepsilon\big|_{\partial B} = L\omega_y^\varepsilon.$$

Further, let us denote $w_y^\varepsilon = TL\omega_y^\varepsilon$. We show that when δ is small, w_y^ε and u_y^ε are close to each other. To this end, let us define

$$r_y^\varepsilon = w_y^\varepsilon - u_y^\varepsilon.$$

This residual satisfies the equations

$$\begin{aligned} \Delta r_y^\varepsilon &= 0 \text{ in } B \setminus \overline{D}, \\ \nabla \cdot \gamma \nabla r_y^\varepsilon &= 0 \text{ in } D, \end{aligned}$$

the boundary condition

$$\partial r_y^\varepsilon\big|_{\partial B} = 0,$$

as well as the transmission conditions

$$\begin{aligned} r_y^\varepsilon\big|_{\partial D}^+ - r_y^\varepsilon\big|_{\partial D}^- &= (v_y - v_y^\varepsilon)\big|_{\partial D}, \\ \partial r_y^\varepsilon\big|_{\partial D}^+ - \partial_\gamma r_y^\varepsilon\big|_{\partial D}^+ &= (\partial v_y - \partial v_y^\varepsilon)\big|_{\partial D}^{\cdot} \end{aligned}$$

By using Green's formula and the trace theorem, it is not hard to see that the function r^ε satisfies the estimate

$$\begin{aligned}\|r_y^\varepsilon\|_{H^1(B)} &\leq C(\|v_y - v_y^\varepsilon\|_{H^{1/2}(\partial D)} + \|\partial v_y - \partial v_y^\varepsilon\|_{H^{-1/2}(\partial D)} \\ &\leq C\|v_y - v_y^\varepsilon\|_{H^1(D)} \leq C\delta,\end{aligned}$$

and in particular,

$$\|r_y^\varepsilon\|_{H^{1/2}(\partial B)} \leq C\|r_y^\varepsilon\|_{H^1(B)} \leq C\delta.$$

Now we have the estimate

$$\begin{aligned}\|\Phi_y - (\Lambda - \Lambda_0)L\omega_y^\varepsilon\|_{H^{1/2}(\partial B)} &\leq \|\Phi_y - (w_y^\varepsilon - v_y^\varepsilon)\|_{H^{1/2}(\partial B)} \\ &\leq \|r_y^\varepsilon\|_{H^{1/2}(\partial B)} \leq C\delta,\end{aligned}$$

so by choosing $\delta = \varepsilon/C$ the claim follows.

To prove the second claim of the theorem, assume that $y \in D$, and let $(v_y, w_y) \in (H^1(D) \times H^1(D))/\mathbb{C}$ be the solution of the interior transmission problem with the transmission data $(f, g) = (\Phi_y\big|_{\partial D}, \partial\Phi_y\big|_{\partial D})$. We show first that as y approaches the boundary ∂D, then $\|v_y\|_{H^1(D)} \to \infty$. to show this, assume first the contrary, $\sup_{y\in D}\|v_y\|_{H^1(D)} < \infty$. In particular, it follows that $\|v_y\|_{H^{1/2}(\partial D)} \leq C$ and $\|\partial v_y\|_{H^{-1/2}(\partial D)} \leq C$. We define in B the function W_y as

$$W_y = \chi_D w_y + (1 - \chi_D)\Phi_y,$$

satisfying the equations

$$\begin{aligned}\nabla\cdot\gamma\nabla W_y &= 0 \text{ in } D, \\ \Delta W_y &= 0 \text{ in } B \setminus \overline{D}\end{aligned}$$

with the transmission data

$$\begin{aligned}W_y\big|_{\partial D}^{+} - W_y\big|_{\partial D}^{-} &= -v\big|_{\partial D}^{-}, \\ \partial W_y\big|_{\partial D}^{+} - \partial_\gamma W_y\big|_{\partial D}^{-} &= -\partial v\big|_{\partial D}^{-},\end{aligned}$$

and the boundary condition

$$\partial W_y\big|_{\partial B} = 0.$$

An application of Green's formula leads now to the conlusion that

$$\|W_y\|_{H^1(B)} \leq C(\|v_y\|_{H^{1/2}(\partial D)} + \|\partial v_y\|_{H^{-1/2}(\partial D)}) \leq C$$

for all $y \in D$. In particular, we see that

$$\sup_{y\in D}\|\Phi_y\|_{H^1(B\setminus D)} < \infty,$$

which is a contradiction. Thus, we must have $\|v_y\|_{H^1(D)} \to \infty$ as claimed.

Let $v_y^\varepsilon \in \mathbb{H}(D)$ be an approximation of v_y in $H^1(D)$. It follows now that also $\|v_y^\varepsilon\|_{H^1(D)} \to \infty$. By the construction, $\|L\omega_y^\varepsilon\|_{H^{-1/2}(\partial B)} \to \infty$ whis is possible only if $\|\omega_y^\varepsilon\|_{L^2(\partial\Omega)} \to \infty$ as y appraches the boundary ∂D. The proof is complete. $\square$

Finally, let us briefly discuss the case when we try to find the an approximate solution when $y \notin D$. As it is customary in the linear sampling approach, we consider the Tikhonov regularized approximation of the solution to the equation $(\Lambda - \Lambda_0)\psi_y = \phi_y$. To this end, we need the following result.

Lemma 3.9. *Under the assumptions about γ made in Section 2, the operator $\Lambda - \Lambda_0 : H_0^{-1/2}(\partial B) \to H^{1/2}(\partial B)$ is injective and has a dense range.*

Proof: To show the injectivity, assume that $(\Lambda - \Lambda_0)\psi = 0$. Set, as usual, $u = T\psi$ and $v = T_0\psi$, yielding that $w = u - v$ is harmonic in $B \setminus \overline{D}$ and has vanishing Cauchy data on ∂B. Hence, $u = v$ in $B \setminus \overline{D}$. It follows then that the pair $(u\big|_D, v\big|_D)$ is a solution of the interior transmission problem 3.2 with vanishing boundary data, and so Lemma 3.3 implies that $u = v = 0$ in D and consequently in the whole of B. Hence, we deduce that also $\psi = 0$.

To prove the density, assume the contrary. Then there is an element $0 \neq \eta \in H_0^{1/2}(\partial B)^*$ with

$$\langle (\Lambda - \Lambda_0)\psi, \eta \rangle = 0$$

for all $\psi \in H_0^{-1/2}(\partial B)$. Since $\langle (\Lambda - \Lambda_0)\psi, c \rangle = 0$ for all constants c, we may assume that $\eta \in H_0^{-1/2}(\partial B)$. Further, since $\Lambda - \Lambda_0$ is symmetric, we deduce that $(\Lambda - \Lambda_0)\eta = 0$, and the injectivity implies $\eta = 0$. This contradction proves the claim. $\square$

The above lemma guarantees that we may apply the standard theory of minimum-norm solutions. In particular (see e.g. [5]), for every $\delta > 0$ there is a unique $\psi_y^\delta \in H^{-1/2}(\partial B)$ that minimizes the functional

$$F_\alpha(\psi) = \|(\Lambda - \Lambda_0)\psi - \phi_y\|^2_{H^{1/2}(\partial B)} + \alpha \|\psi\|^2_{H^{-1/2}(\partial B)}, \tag{7}$$

with the Morozov discrepancy constraint

$$\|(\Lambda - \Lambda_0)\psi - \phi_y\|_{H^{1/2}(\partial B)} \leq \delta, \tag{8}$$

used to fix the parameter $\alpha = \alpha(\delta)$. By Lemma 3.1, we observe that as $\delta \to 0+$, we must have $\|\psi\|_{H^{-1/2}(\partial B)} \to \infty$. In terms of the regularization parameter, we have $\alpha \to 0$ as $\delta \to 0$. What is more, by Lemma 3.6, for every $\varepsilon > 0$ we can always find an $\omega_y^{\delta,\varepsilon} \in L^2(\partial\Omega)$ such that

$$\|(\Lambda - \Lambda_0)(\psi_y^\delta - L\omega_y^{\delta,\varepsilon})\|_{H^{1/2}(\partial B)} < \varepsilon.$$

We can summarize these results in the following theorem that is the counterpart of Theorem 3.8 when $y \notin D$.

Theorem 3.10. *Assume that $y \in B \setminus \overline{D}$ Then for every $\delta > 0$ and $\varepsilon > 0$ there is an $\omega_y^{\delta,\varepsilon} \in L^2(\partial\Omega)$ such that*

$$\|(\Lambda - \Lambda_0)L\omega_y^{\delta,\varepsilon} - \phi_y\|_{H^{1/2}(\partial B)} < \delta + \varepsilon,$$

for which $\|\omega_y^{\delta,\varepsilon}\|_{L^2(\partial\Omega)} \to \infty$ as $\delta \to 0+$.

By comparing Theorems 3.8 and 3.10 it is not obvious how the linear sampling algorithm should be implemented. In the articles [3] and [7] (see also the review article [6] for further references) the linear sampling method in inverse scattering has been studied numerically. Based on those works, one can suggest the following procedure. Given a 'noise level' $\delta > 0$, one seeks to minimize the functional (7) under the constraint (8), with the parameter y varying in a given grid inside B. The norm of the solution ψ or alternatively, the size of the regularization parameter $\alpha = \alpha(\delta)$ is used then as a cut-off indicator.

References

[1] Brühl, M. and Hanke, M.: Numerical implementation of two non-iterative methods locating inclusions in impedance tomography. Inverse Problems **16** (2000) 1029–1042.

[2] Cakoni, F., Colton, D. and Haddar, H.: The linear sampling method for anisotropic media. Preprint

[3] Colton, D., Piana, M. and Potthast, R.: A simple method using Morozov's discrepancy principle for solving inverse scattering problems. Inverse Problems **13** (1997) 1477–1493.

[4] Colton, D. and Kirsch, A.: A simple method for solving inverse scattering problems in the resonance region. Inverse Problems **12** (1996) 383–393.

[5] Colton, D. and Kress, R.: *Inverse Acoustic and Electromagnetic Scattering Theory.* 2nd ed., Springer-Verlag, Berlin Heidelberg New York 1998.

[6] Colton, D., Coyle, J. and monk, P.: Recent development in inverse acoustic scattering theory. SIAM Review **42** (2000) 369–414.

[7] Colton, D., Giebermann, K. and Monk, P.: A regularized sampling method for solving three-dimensional inverse scattering problems. SIAM J. Comput. **21** (2000) 2316–2330.

[8] Colton, D. and Kress, R.: On the Denseness of Herglotz Wave Functions and Electromagnetic Herglotz Pairs in sobolev Spaces. Preprint, 2001.

[9] Ikehata, M.: Enclosing a polygonal cavity in a two-dimensional bounded domain from Cauchy data. Inverse Problems **15** (1999), 1231–1241.

[10] Ikehata, M. and Siltanen, S.: Numerical method for finding convex hull of an inclusion in conductivity from boundary measurements. Inverse Problems **16** (2000) 1043–1052.

[11] Kirsch, A.: Factorization of the far field operator for the inhomogenous medium case and an application in inverse scattering theory. Inverse Problems **15** (1999) 413–429.

Erkki Somersalo, Helsinki University of Technology, Institute of Mathematics, P.O. Box 1100, FIN–02015 HUT, Finland

E-mail address: erkki.somersalo@hut.fi