RESULTANTS AND CHOW FORMS VIA EXTERIOR SYZYGIES

David Eisenbud and Frank-Olaf Schreyer

With an Appendix by Jerzy Weyman:
Homomorphisms of some vector bundles on the Grassmannian

Abstract. Given a sheaf on projective space $\mathbf{P}^n$ we define a sequence of canonical and easily computable *Chow complexes* on the Grassmannians of planes in $\mathbf{P}^n$, generalizing the well-known Beilinson monad on $\mathbf{P}^n$. If the sheaf has dimension k, then the Chow form of the associated k-cycle is the determinant of the Chow complex on the Grassmannian of planes of codimension $k+1$. Using the theory of vector bundles and the canonical nature of the complexes we are able to give explicit determinantal and Pfaffian formulas for resultants in some cases where no polynomial formulas were known. For example, the Horrocks-Mumford bundle gives rise to a polynomial formula for the resultant of five homogeneous forms of degree eight in five variables.

Let W be a vector space of dimension $n+1$ over a field K. The *Chow divisor* of a k-dimensional variety X in $\mathbf{P}^n = \mathbf{P}(W)$ is the hypersurface, in the Grassmannian $\mathbf{G} = \mathbf{G}_{k+1}$ of planes of codimension $k+1$ in $\mathbf{P}^n$, consisting of those planes meeting X. The *Chow form* is its defining equation. For example the resultant of $k+1$ forms of degree e in $k+1$ variables is the Chow form of $\mathbf{P}^k$ embedded by the e^{th} Veronese mapping in $\mathbf{P}^n$ with $n = \binom{k+e}{k} - 1$. In this paper we will give a new expression for the Chow divisor, closely related to Beilinson's monad for sheaves on projective space, and derive new polynomial formulas for Chow forms in a number of particular cases of the following types:

1. Bézout formulas for resultants. The classic formula of Bézout gives the resultant of two homogeneous forms in two variables as a determinant of linear forms in the Plücker coordinates of the space generated by the two forms. By analogy we will call any formula for the Chow form in Plücker coordinates a *Bézout expression* of the Chow form. Our simplest example of a new Bézout expression is for the resultant of three forms degree 2 in three variables (we also give corresponding formulas for any degree): it is the Pfaffian of the alternating matrix of linear forms in the Plücker coordinates

$$\begin{pmatrix}
0 & [245] & [345] & [135] & [045] & [035] & [145] & [235] \\
-[245] & 0 & -[235] & [035] & [025] & [015] & [125] & -[125]+[045] \\
-[345] & [235] & 0 & [134] & [035] & [034] & [135] & [234] \\
-[135] & -[035] & -[134] & 0 & [023] & [013] & [123]-[034] & -[123] \\
-[045] & -[025] & -[035] & -[023] & 0 & [012] & -[015] & -[024]+[015] \\
-[035] & -[015] & -[034] & -[013] & -[012] & 0 & [023]-[014] & -[023] \\
-[145] & -[125] & -[135] & -[123]+[034] & [015] & -[023]+[014] & 0 & -[124]+[035] \\
-[235] & [125]-[045] & -[234] & [123] & [024]-[015] & [023] & [124]-[035] & 0
\end{pmatrix}$$

Here the monomials in the three variables a, b, c are ordered $a^2, ab, ac, b^2, bc, c^2$ and the brackets $[ijk]$ denote the corresponding Plücker coordinates of the net of quadrics. Using

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the theory of rank two vector bundles on $\mathbf{P}^2$ we can construct many such formulas for ternary forms of any degree.

2. Stiefel formulas for resultants. The Grassmannian is a quotient of an open set in the variety of $(k+1) \times (n+1)$ matrices over K; the entries of these matrices are called *Stiefel coordinates* on the Grassmannian (or on the Stiefel manifold.) Pulling back the Chow divisor, we get a divisor whose ideal is generated by a polynomial in the Stiefel coordinates. For example if X is the rational normal curve this polynomial is the Sylvester determinant. Even when we cannot express the Chow form of a variety as the determinant or Pfaffian of a matrix in the Plücker coordinates, we can sometimes express it as the determinant or Pfaffian of a map of equivariant vector bundles on the Grassmannian. Such maps pull back to matrices in the Stiefel coordinates whose determinant or Pfaffian defines the (closure of the) preimage of the Chow divisor. We say that such a matrix gives a *Stiefel expression* for the Chow form. The classical Sylvester determinant is such a Stiefel expression.

Explicit polynomial expressions, in particular Stiefel expressions for the resultant of $k+1$ forms of degree $d \geq 2$ in $k+1$ variables (Chow form of the d-uple embedding of $\mathbf{P}^k$) have been known only for $k \leq 3$ (all d) and $k = 4, d = 2$ (see for example Gel'fand, Kapranov, and Zelevinsky [1994]. Using our method and constructions of vector bundles on $\mathbf{P}^k$ we give new Stiefel expressions. In particular, the Horrocks-Mumford bundle gives rise to Pfaffian Stiefel expressions for the resultants of 5 forms of degrees 4, 6, or 8 in 5 variables. The matrices involved are too large to exhibit here; but Macaulay2 programs for producing them and other new examples can be found at `http://www.msri.org/****`.

We next introduce the basic ideas of this paper, and then describe our main results. Let

$$\mathbf{P}^n \xleftarrow{\pi_1} \mathbf{F}_l \xrightarrow{\pi_2} \mathbf{G}_l$$

be the incidence correspondence; that is, let $\mathbf{F}_l$ be the set of flags consisting of a point $p \in \mathbf{P}^n$ and an plane $L \in \mathbf{G}$ of codimension l in $\mathbf{P}^n$ with $p \in L$. Taking $l = k+1$, the Chow divisor of a reduced irreducible k-dimensional subvariety $X \subset \mathbf{P}^n$ is by definition $D_X = \pi_2(\pi_1^{-1}X)$.

If $\mathcal{F}$ is any sheaf whose support is X, it follows that the Chow divisor in $\mathbf{G}$ is the codimension 1 part of the support of the sheaf $\mathcal{G} = \pi_{2*}(\pi_1^*\mathcal{F})$. If $\mathcal{F}$ is generically a vector bundle of rank r on X then $\mathcal{G}$ will be generically a vector bundle of rank r on D_X. Thus D_X can be recovered as the codimension 1 part of the (scheme-theoretic) support of $\mathcal{G}$.

If $\mathcal{G}$ happens to be presented by a square matrix with nonzero determinant in the Plücker coordinates on $\mathbf{G}$ (or more generally by a monomorphism of vector bundles on $\mathbf{G}$) then the Fitting lemma shows that the determinant is the r^{th} power of the Chow form of X. One of the central contributions of this paper is to give a simple characterization of a class of sheaves $\mathcal{F}$ for which $\mathcal{G}$ has such a presentation: they are the "Ulrich sheaves" described below.

More generally, one can use the determinant of a complex, first introduced (for this purpose!) by Arthur Cayley [1848]. This determinant is in general a rational function, the alternating product of certain minors in matrices representing the complex. If $\mathcal{C}$ is a complex of locally free sheaves on $\mathbf{G}$ whose only homology in codimension 1 is $\mathrm{H}^0(\mathcal{C}) = \mathcal{G}$, then the determinant of $\mathcal{C}$ is the r^{th} power of a form defining the codimension 1 part of the support of $\mathcal{G}$. Such complexes were produced from Koszul complexes by Cayley, F. S. Macaulay, Jouanolou and other authors who derived expressions for resultants as rational functions in the Chow or Stiefel coordinates. However, these complexes have been constructed explicitly for only a limited class of sheaves $\mathcal{F}$. For modern results, see Weyman and Zelevinsky [1994] as well as Jouanolou [1995]. An exposition may be found in the book of Gel'fand, Kapranov and Zelevinsky [1994]. Of course the Chow form is a polynomial: in these rational function expressions the denominator divides the numerator. However, it is not known how to make the quotient explicit. Refinements aimed at reducing the degree of the denominator are an active subject of research; see for example d'Andrea and Dickenstein [2000].

Grothendieck gave a conceptual framework for these constructions in an unpublished letter to David Mumford in 1962; the details are worked out by Knudsen and Mumford in [1976], where the letter is described: By general theory there always exist locally free complexes $\mathcal{C}$, well-defined up to homotopy equivalence, with

$$\mathcal{C} \simeq \mathbf{R}\pi_{2*}(\pi_1^*\mathcal{F})$$

in the derived category. If $\mathcal{F}$ is generically a vector bundle of rank r on X, then $\mathcal{C}$ satisfies the conditions above, and so the determinant of $\mathcal{C}$ is the r^{th} power of the Chow form. We will call such a complex $\mathcal{C}$ a *Chow complex for $\mathcal{F}$.* More generally, when $\mathcal{F}$ is any coherent sheaf on $\mathbf{P}^n$, whose support has dimension k, the determinant of a Chow complex for $\mathcal{F}$ is the Chow form of the k-cycle of $\mathcal{F}$, that is the sum of the Chow forms of the k-dimensional components of the support, each raised to the power equal to the multiplicity of $\mathcal{F}$ at the generic point of that component.

Our first main result gives a canonical Chow complex $\mathbf{U}_{k+1}(\mathcal{F})$ for each coherent sheaf $\mathcal{F}$, part of a sequence of complexes generalizing the Beilinson monad for $\mathcal{F}$. The construction is so explicit that it can be made on a computer. Recall that a plane of codimension l in $\mathbf{P}^n$ corresponds to an $(n+1-l)$-quotient of W, and thus to an l-dimensional subspace of W. We write U_l for the tautological l-subbundle on $\mathbf{G}_l$.

Theorem 0.1 *For any coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n$ there is a canonical complex $\mathbf{U}_l(\mathcal{F})$ of vector bundles on $\mathbf{G}_l$ with*

$$\mathbf{U}_l(\mathcal{F}) \simeq \mathbf{R}\pi_{2*}(\pi_1^*\mathcal{F}).$$

The e^{th} term of $\mathbf{U}_l(\mathcal{F})$ is $\sum_j \mathrm{H}^j(\mathcal{F}(e-j)) \otimes \wedge^{j-e} U_l$.

The complex $\mathbf{U}_n(\mathcal{F})$ on $\mathbf{P}^n$ itself is the Beilinson monad defined in Eisenbud, Fløystad and Schreyer [2000]. The sheaf $\mathcal{F}$ can be recovered from $\mathbf{U}_n(\mathcal{F})$ simply by taking homology. The sheaf $\mathcal{F}$ can be recovered from some of the other $\mathbf{U}_l(\mathcal{F})$ as well: just as one can recover a variety of dimension k from its Chow divisor in $\mathbf{G}_{k+1}$, so one can recover any sheaf $\mathcal{F}$ whose support has dimension at most k from the Chow complex $\mathbf{U}_l(\mathcal{F})$ as long as $l > k$. All these matters are explained in Section 1

Most significant in our treatment is that we can give an explicit and canonical description of the maps in the complex $\mathbf{U}_l$. Until now, in general, it has only been possible to write down the sheaves in such a complex (see for example Gel'fand-Kapranov-Zelevinsky [1994] section 3.4E, "Weyman's complexes"), or to approximate the maps via a spectral sequence. With enough vanishing of cohomology it was possible to write down the maps; but these cases were often not the ones of primary interest. Also, previous authors seem only to have considered formulas coming from the case where $\mathcal{F}$ is a line bundle. Our technique allows us to recover explicit expressions of the Chow form in all the previously known cases, and, using vector bundles as in the examples mentioned above, some new ones.

The most useful formulas for the Chow form occur when the complex $\mathbf{U}$ has just one nontrivial map Ψ

$$\mathbf{U} : \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow C^{-1} \xrightarrow{\Psi} C^0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots.$$

In this case the Chow form is the determinant of Ψ, and if the bundles C^i are direct sums of exterior powers of the tautological bundles, then one gets a determinantal expression for the Chow form in Stiefel coordinates.

An even better case occurs when $C^{-1} \cong \oplus \mathcal{O}_{\mathbf{G}}(-1)$, a direct sum of copies of $\mathcal{O}_{\mathbf{G}}(-1)$, and C^0 is a direct sum of copies of $\mathcal{O}_{\mathbf{G}}$: then the Chow form is given directly as a determinant in the Plücker coordinates—that is, we get a Bézout expression for the Chow form of $\mathcal{F}$ and thus for a power of the Chow form of the support of $\mathcal{F}$. If $\mathcal{F}$ has rank 1, or if $\mathcal{F}$ has rank 2 and the map Ψ is skew symmetric so that we can extract the square root of the determinant as the Pfaffian, then we get the Chow form of the support of $\mathcal{F}$ itself.

Such cases are considered in Section 2. Our second main result describes precisely the conditions on the sheaf $\mathcal{F}$ that are necessary for the Chow complex $\mathbf{U}_{k+1}(\mathcal{F})$ to degenerate to one of these special forms. For example:

Theorem 0.2 *The Chow complex $\mathbf{U}_{k+1}(\mathcal{F})$ above degenerates to a single map $\mathcal{O}_{\mathbf{G}}^d(-1) \to \mathcal{O}_{\mathbf{G}}^d$ if and only if the module of twisted global sections $\oplus_m \mathrm{H}^0(\mathcal{F}(m))$ is a Cohen-Macaulay module with a linear free resolution.*

Here by a linear free resolution we mean a free resolution of the form

$$\cdots \longrightarrow S^{r_2}(-2) \longrightarrow S^{r_1}(-1) \longrightarrow S^{r_0}.$$

Such Cohen-Macaulay graded modules M have been studied by Bernd Ulrich [1984] under the name "maximally generated maximal Cohen-Macaulay modules" and by others (see Brennan, Herzog, and Ulrich [1987], Herzog, Ulrich, and Backelin [1991], and the references given there) under the names "linear maximal Cohen-Macaulay modules" or simply "Ulrich modules" — we shall call the corresponding sheaves *Ulrich sheaves*. For example, a line bundle $\mathcal{F}$ on a curve X of genus g embedded in P^n is an Ulrich sheaf if and only if $\mathcal{F}(-1)$ has degree $g-1$ and no global sections; that is, $\mathcal{F}$ corresponds to a point in $\mathrm{Pic}^{g-1}(X)$ which lies outside the theta divisor $\Theta \subset \mathrm{Pic}^{g-1}(X)$.

In Section 3 we turn to the problem of giving determinantal and Pfaffian expression for the Chow form of an Ulrich sheaf $\mathcal{F}$. We can express them directly in terms of the free resolution of the corresponding module M by using a construction developed in Lejeune-Jalabert and Angeniol [1989] to describe Atiyah classes. Suppose that

$$0 \to F_c \xrightarrow{\phi_c} \cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0$$

is the linear free resolution of M as above. Regarding the ϕ_i as matrices of elements of W, we can compose them as if they were matrices of linear forms in the exterior algebra: we write $\Psi_{\mathcal{F}} := (1/c!)\phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_c$ for this product (defined in a slightly different way in positive characteristic), which is represented by a matrix of forms in $\wedge^c W$. We may identify $\wedge^c W$ with the the space of linear forms on $\mathbf{G}$ and we have:

Theorem 0.3 *If $\mathcal{F}$ is an Ulrich sheaf, then $\Psi_{\mathcal{F}}$ is the (only) nonzero map in the Chow complex $\mathbf{U}(\mathcal{F})$. In particular the Chow form of $\mathcal{F}$ is $\det \Psi_{\mathcal{F}}$. If $\mathcal{F}$ is a vector bundle on a k-dimensional variety X, and $\mathcal{F}$ is skew symmetric in an appropriate sense, then (in characteristic not 2) $\Psi_{\mathcal{F}}$ is skew-symmetric, and the square-root of the Chow form of $\mathcal{F}$ is the Pfaffian of $\Psi_{\mathcal{F}}$.*

Theorem 0.3 gives a new method for constructing resultants and Chow forms: find Ulrich sheaves (or weakly Ulrich sheaves, or Ulrich sheaves satisfying the skew-symmetry condition...) and then construct the map $\Psi_{\mathcal{F}}$. For this construction one can either use the product formula above or the definition of the canonical Chow complex $\mathbf{U}(\mathcal{F})$ from maps in a certain free resolution over the exterior algebra.

The second part of this paper gives a number of examples of this method. We can be completely explicit in only a small number of cases, and these sections leave open a multitude of theoretical and practical problems. Central to this pursuit is the

Problem. *Does every embedded variety $X \subset \mathbf{P}^n$ have an Ulrich sheaf? If X has an Ulrich sheaf, what is the smallest possible rank for such a sheaf?*

For example, Brennan, Herzog, and Ulrich [1987] showed that when X is an arithmetically Cohen-Macaulay curve over an infinite field, or a a complete intersection, or a linear determinantal variety, then X has an Ulrich sheaf. Doug Hanes showed in his Thesis under Hochster [1999] that the d-uple embeddings of $\mathbf{P}^k$ have Ulrich sheaves when $k \leq 2$ or $k = 3$ and $d = 2^r$ is a power of two.

Section 4 is devoted to the case of curves. We complete (and reprove) the result of Brennan, Ulrich, and Herzog by showing that every curve in $\mathbf{P}^n$ over an infinite field has skew-symmetric rank 2 Ulrich sheaves. If the field is algebraically closed it has rank 1 Ulrich sheaves; they are in one-to-one correspondence with the line bundles of degree $g - 1$ having no sections. Thus there are Bézout expressions for the Chow forms of such curves, generalizing the case of binary forms ($\mathbf{P}^1$ and the line bundle $\mathcal{O}_{\mathbf{P}^1}(-1)$) and the well-known result that the equation of any plane curve over an algebraically closed field can be written as the determinant of a matrix of linear forms.

Such Ulrich sheaves give rise, in principle, to continuous families of resultant formulas for sections of a line bundle on a curve of genus ≥ 1, but it is not easy to make such formulas explicit. We illustrate with the case of hyperelliptic curves, and provide a resultant formula for functions of the form $a + b\sqrt{f},\ c + d\sqrt{f}$ where a, b, c, d and f are polynomials in one variable. We carry out the proof completely only in case the degrees of the various polynomials are small. In the special case of elliptic curves, we get a resultant formula for doubly periodic functions written in terms of the Weierstrass $\wp$-function and its derivative.

We turn in Section 5 to the case where $X \subset \mathbf{P}^n$ is the d^{th} Veronese (d-uple) embedding of $X = \mathbf{P}^k$. This is the case that gives rise to resultant formulas for $k + 1$ forms of degree d in $k + 1$ variables. We give cohomological criteria for a bundle on $\mathbf{P}^k$ to be Ulrich for the d-uple embedding. Following a suggestion of Jerzy Weyman, we use this to show that every Veronese variety has an Ulrich sheaf, obtained by applying a certain (unique) Schur functor to the tautological quotient bundle. This gives a way of writing a power of the resultant as the determinant of a matrix of linear forms in the Plücker coordinates. This may be of computational significance: the use of resultants in computation is to determine whether or not a set of polynomials has a common zero; a power of the resultant does this just as well.

In this section we also find Ulrich modules of rank 2 for each Veronese embedding of $\mathbf{P}^2$. We prove a lower bound on the ranks of possible Ulrich modules and using this and a result of Hartshorne-Hirschowitz on the existence of mathematical instanton bundles we show that rank 2 Ulrich modules exist on the d-uple embedding of $\mathbf{P}^3$ if and only if d is not divisible by 3. On $\mathbf{P}^4$ we show that the Horrocks-Mumford bundle is weakly Ulrich for the 4,6, and 8-uple embeddings, and satisfies the skew-symmetry condition necessary for us to get a Pfaffian Stiefel formula for the corresponding resultants.

Section 6 is concerned with the existence of skew-symmetric rank 2 Ulrich sheaves on various surfaces, and thus with Pfaffian resultant formulas generalizing the Bézout formula for $\mathbf{P}^2$ given at the beginning of this introduction. We use Mukai's construction of vector bundles on surfaces, and describe the necessary data. Our main result here is the existence of skew-symmetric rank 2 Ulrich modules for certain embeddings of the plane blown up at a set of points, leading to Pfaffian Bézout expressions for the resultant of 3 ternary forms of degree d with assigned simple base points, valid when the ideal defining the set of base points is generated in degree $< d$.

Throughout this paper we rely on a certain construction of homomorphisms between exterior powers of the tautological bundle on a Grassmannian, explained in Section 1. In Section 7 Jerzy Weyman proves—in all characteristics—that in fact every homomorphism arises from this construction.

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1 Chow complexes obtained from the Beilinson monad

As above we write $\mathbf{G}_l$ for the Grassmannian of planes of codimension l in $\mathbf{P} := \mathbf{P}^n = \mathbf{P}(W)$ and $\mathbf{F}_l$ for the flag variety of flags consisting of a point $p \in \mathbf{P}$ and an plane $L \in \mathbf{G}$ of codimension l in $\mathbf{P}$ containing p. Throughout this section we will consider the incidence correspondence

$$\mathbf{P} \xleftarrow{\pi_1} \mathbf{F}_l \xrightarrow{\pi_2} \mathbf{G}_l.$$

Let $0 \longrightarrow U \longrightarrow W \otimes \mathcal{O}_{\mathbf{G}_l} \longrightarrow Q \longrightarrow 0$ be the tautological sequence on the Grassmanian $\mathbf{G}_l$, so that $U = U_l$ is a bundle of rank l. We write E for the exterior algebra $\wedge V$, where $V = W^*$. Any element $a \in \wedge^p(V)$ induces a homomorphism $\wedge^p W \to K$ and thus a homomorphism of sheaves

$$\wedge^p U \hookrightarrow \wedge^p W \otimes \mathcal{O}_{\mathbf{G}_l} \to \mathcal{O}_{\mathbf{G}_l}.$$

Using the diagonal map $\wedge^q U \xrightarrow{\Delta_U} \wedge^{q-p} U \otimes \wedge^p U$ we get maps

$$\wedge^q U \xrightarrow{(1\otimes a)\Delta_U} \wedge^{q-p} U$$

for every p, q.

We will use the well-known part a) of the following lemma heavily. We include part b) for background.

Proposition 1.1 *Let $U = U_l$ be the tautological subbundle on $\mathbf{G}_l$.*

a) The maps above make $\wedge U$ into a module over $\wedge V$.

*b) (**J. Weyman**) The maps*

$$\wedge^p V \to \operatorname{Hom}(\wedge^q U, \wedge^{q-p} U).$$

are isomorphisms for all integers p, q such that $0 \le q - p, q \le l$.

Proof. a) With notation as above, the naturality of the diagonal maps shows that the diagrams

$$\begin{tikzcd} \wedge^q U \arrow[r] \arrow[d, "(1\otimes a)\Delta_U"'] & \wedge^q W \otimes \mathcal{O}_{\mathbf{G}_l} \arrow[d, "(1\otimes a)\Delta_W \otimes 1"] \\ \wedge^{q-p} U \arrow[r] & \wedge^{q-p} W \otimes \mathcal{O}_{\mathbf{G}_l} \end{tikzcd}$$

commute. Since $\wedge W$ is a naturally a module over $E = \wedge V$ by this action (see for example Eisenbud [1995, Appendix A2.4.1]), so is $\wedge U$.

b) This is proved in an appendix to this paper by J. Weyman. In characteristic 0 the result follows from Bott's vanishing theorem (see Jantzen [1987]). In arbitrary characteristic it is more delicate. ∎

We will grade E by the convention that the elements of V have degree -1. As usual we write $E(q)$ for the free graded E-module of rank 1, with generator in degree $-q$. Thus, for example, if $q > p$ then $\mathrm{Hom}(E(q), E(q-p)) = E_{-p} = \wedge^p V$. Recall from Eisenbud, Fløystad and Schreyer [2000] that a *Tate resolution* is a doubly infinite exact complex of finitely generated free graded E-modules which is minimal in the sense that each free module maps into V times the next one. If $\mathcal{F}$ is any coherent sheaf on $\mathbf{P}$ then there is a *Tate resolution* $\mathbf{T}(\mathcal{F})$ naturally associated to $\mathcal{F}$, which can be computed, using free resolutions over an exterior algebra, from the module of twisted global sections $\oplus_e \mathrm{H}^0\mathcal{F}(e)$. Its e^{th} term is isomorphic to

$$T^e(\mathcal{F}) = \oplus_j \mathrm{H}^j(\mathcal{F}(e-j)) \otimes E(j-e).$$

For all this see Eisenbud-Fløystad-Schreyer [2000].

We can define an the additive functor $\mathbf{U}_l$ from graded free modules over E to locally free sheaves on $\mathbf{G}_l$ by taking $\mathbf{U}_l(E(p)) = \wedge^p U$, where $U = U_l$ is the tautological subbundle, and sending a map $\eta : E(q) \to E(q-p)$ to the map $\mathbf{U}_l(\eta) : \wedge^q U \to \wedge^{q-p} U$ made from the element $\wedge^p V$ corresponding to η.

If $\mathbf{T}$ is any Tate resolution then for $e >> 0$ or $e << 0$ we have $\mathbf{U}_l(T^e) = 0$, so $\mathbf{U}_l(\mathcal{F}) := \mathbf{U}_l(\mathbf{T})$ is a bounded complex of locally free sheaves on $\mathbf{G}_l$.

For example, $\mathbf{U}_n(\mathcal{F})$ is shown by Eisenbud, Fløystad and Schreyer [2001] to be a *Beilinson monad* for the sheaf $\mathcal{F}$ in the sense that it has the terms above, and its only homology is $\mathcal{F}$, in degree 0 (the functor $\mathbf{U}_n$ is called Ω in that paper).

Theorem 1.2 *If $\mathcal{F}$ is a sheaf on $\mathbf{P}^n$ then the complex $\mathbf{U}_l(\mathcal{F})$ represents $\mathbf{R}\pi_{2*}(\pi_1^*\mathcal{F})$ in the derived category of sheaves on the Grassmannian $\mathbf{G}_l$.*

Proof. By Theorem 6.1 of Eisenbud, Fløystad and Schreyer [2000], $\mathbf{U}_n(\mathcal{F})$ represents $\mathcal{F}$ in $D^b(\mathrm{Coh}(\mathbf{P}^n))$. We will show first that $\mathbf{U}_l(\mathcal{F}) = \pi_{2*}(\pi_1^*\mathbf{U}_n(\mathcal{F}))$, and second that $\mathbf{R}^i\pi_{2*}(\pi_1^*(\wedge^p U_n)) = 0$ for $i > 0$. It follows that $\mathbf{R}\pi_{2*}\pi_1^*\mathbf{U}_n(\mathcal{F}) \cong \pi_{2*}(\pi_1^*\mathbf{U}_n(\mathcal{F})) = \mathbf{U}_l(\mathcal{F})$, as desired.

On $\mathbf{F}$ we have inclusions of the universal subbundles

$$\pi_2^*(U_l) \subset \pi_1^*(U_n) \subset W \otimes \mathcal{O}_{\mathbf{F}}.$$

Pushing the left hand inclusion forward we get a canonical map $U_l = \pi_{2*}\pi_2^* U_l \to \pi_{2*}\pi_1^* U_n$, and we deduce similar maps on the exterior powers. To show that these are isomorphisms we may compute fiber by fiber. If $u \in \mathbf{G}_l$ then we will also write $u \subset W$ for the corresponding l-dimensional linear subspace.

Setting $\mathbf{P}' = \mathbf{P}(W/u) \subset \mathbf{P}(W)$, we have the decomposition

$$\wedge^p U_n \mid_{\mathbf{P}'} \cong \oplus_{i=0}^{p} \wedge^i u \otimes \wedge^{p-i} U'_{n-l},$$

where U'_{n-l} denotes the tautological sub-bundle on $\mathbf{P}'$. Thus the map $\wedge^p u \to \mathrm{H}^0(\wedge^p U_n \mid_{\mathbf{P}'})$ is an isomorphism, and all other cohomology of $\wedge^p U_n \mid_{\mathbf{P}'}$ vanishes.

From the base change theorem (Hartshorne [1977], III,12) It follows that $\mathbf{R}^i \pi_{2*}(\pi_1^* \wedge^p U_n) = 0$ for $i > 0$ and $\pi_{2*}(\pi_1^* \wedge^p U_n) \cong \wedge^p U_l$. ∎

The sheaf $\mathcal{F}$ is determined from $\mathbf{U}_n(\mathcal{F})$, the Beilinson monad, by the formula $\mathcal{F} = \mathrm{H}^0(\mathbf{U}_n(\mathcal{F}))$. In general we have:

Proposition 1.3 *If $\mathcal{F}$ is a coherent sheaf of dimension k on $\mathbf{P}$ and $l > k$ then $\mathcal{F}$ is determined by the complex $\mathbf{U}_l(\mathcal{F})$.*

Proof. The Tate resolution $\mathbf{T}(\mathcal{F})$ is determined by any differential $\phi_i : T^i(\mathcal{F}) \to T^{i+1}(\mathcal{F})$, because $\mathbf{T}^{\geq i+1}(\mathcal{F})$ is the minimal injective resolutuion of $\operatorname{im} \phi_i$ and $\mathbf{T}^{\leq i}(\mathcal{F})$ the minimal projective resolution of $\operatorname{im} \phi_i$. Moreover $\mathbf{T}(\mathcal{F})$ determines the Beilinson monad and hence $\mathcal{F}$. Thus it suffices to reconstruct one of the differentials.

The degrees of the generators of the free module in $T^e(\mathcal{F})$ range (potentially) from $e-k$ to e. Thus the degrees of the generators of $T^{-1}(\mathcal{F})$ and $T^0(\mathcal{F})$ range at most from $-k-1$ to 0. It follows that these free modules can be recovered from $\mathbf{U}(\mathcal{F}) = \mathbf{U}(\mathbf{T}(\mathcal{F}))$. By Proposition 1.1 the map between $T^{-1}(\mathcal{F})$ and $T^0(\mathcal{F})$ can recovered from $\mathbf{U}(\mathcal{F})$ as well.∎

Now we come to the case involved in the Chow divisor. Given a finite complex of locally free sheaves on scheme

$$\mathcal{B} : 0 \longrightarrow \ldots \longrightarrow \mathcal{B}^j \longrightarrow \mathcal{B}^{j+1} \longrightarrow \ldots \longrightarrow 0$$

its determinant bundle is defined as

$$\det(\mathcal{B}) = \prod_{j \text{ even}} \det(\mathcal{B}^j) \otimes \prod_{j \text{ odd}} \det(\mathcal{B}^j)^*.$$

If $\mathcal{B}$ is generically exact, then there is a Cartier divisor called the determinant divisor of $\mathcal{B}$ which measures the part of the homology of $\mathcal{B}$ supported in codimension 1; see Knudsen and Mumford [1976] or Gel'fand, Kapranov and Zelevinsky [1994 Appendix A] for the general definition. If $\mathcal{F}$ is a coherent sheaf on $\mathbf{P}(W)$ with support of dimension k, then we define the Chow divisor of $\mathcal{F}$ to be the usual Chow divisor of the k-dimensional cycle associated to $\mathcal{F}$—the sum of the Chow divisors of the k-dimensional components of the support of $\mathcal{F}$, each with multiplicity equal to the multiplicity of $\mathcal{F}$ on that component. The Chow form $\mathrm{Chow}(\mathcal{F})$ is the equation of that divisor; it is a section of $\mathcal{O}_{\mathbf{G}_{k+1}}(\deg \mathcal{F})$ defined up to multiplication by a scalar. The following Theorem is a more explicit version the main result of Knudsen and Mumford [1976] Chapter II.

Theorem 1.4 *Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}(W)$. If $\dim \mathcal{F} = k$ then the Chow divisor of $\mathcal{F}$ is the determinant divisor of the complex $\mathbf{U}_{k+1}(\mathcal{F})$. Moreover, in codimension 1 the only homology of this complex is at the 0^{th} term.*

We give a proof for the reader's convenience:

Proof. We may assume that the ground field is algebraically closed. Since $\mathbf{U}(\mathcal{F})$ represents $\mathbf{R}\pi_2(\pi_1^*\mathcal{F})$ its divisor does not pass through any point u of the Grassmannian such that $\mathrm{supp}(\mathcal{F}) \cap \mathbf{P}(W/u) = \emptyset$. For a general point u of a component of the zero locus of $\mathrm{Chow}(\mathcal{F})$ the subspace $\mathbf{P}(W/u)$ meets the support of $\mathcal{F}$ in a single point which belongs to a unique component X of the support. We have

$$\dim_{\kappa(u)}(\pi_{2*}\pi_1^*\mathcal{F}) \otimes \kappa(u) = \dim_{\kappa(u)} \mathrm{H}^0(\mathcal{F} \otimes \mathcal{O}_{\mathbf{P}(W/U)}) = \mathrm{length}(\mathcal{F} \otimes \mathcal{O}_{\mathbf{P}(W/u),X})$$

and the higher direct images vanish. ∎

2 Ulrich Sheaves

If $\mathbf{U}(\mathcal{F})$ is a two term complex then the determinant section of $\mathbf{U}(\mathcal{F})$ is is the determinant of a morphism between bundles. This situation corresponds to the case where the Tate resolution of $\mathcal{F}$ has "betti diagram" of the form:

$$\begin{array}{cccccc}
h^k\mathcal{F}(-k-3) & h^k\mathcal{F}(-k-2) & h^k\mathcal{F}(-k-1) & h^k\mathcal{F}(-k) & 0 & 0 \\
0 & 0 & h^{k-1}\mathcal{F}(-k) & h^{k-1}\mathcal{F}(-k+1) & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & h^1\mathcal{F}(-2) & h^1\mathcal{F}(-1) & 0 & 0 \\
0 & 0 & h^0\mathcal{F}(-1) & h^0\mathcal{F} & h^0\mathcal{F}(1) & h^0\mathcal{F}(2)
\end{array}$$

Here, by the betti diagram of $\mathbf{T}(\mathcal{F})$ we mean the table whose (i,j) entry is the number of generators of degree $j-i$ required by the j^{th} free module T^i in $\mathbf{T}(\mathcal{F})$—by Eisenbud-Fløystad-Schreyer [2000] this is the dimension of $\mathrm{H}^i(\mathcal{F}(j-i))$. (This is almost the same as the betti diagram in the programs Macaulay of Bayer and Stillman, or Macaulay2 of Grayson and Stillman, except that we think of the arrows in the resolution as going from left to right. This change of convention is convenient because of the fact that the generators of E have negative degree.)

For reasons that will become clear in a moment, we will call a sheaf $\mathcal{F}$ with cohomology as above a *weakly Ulrich sheaf*.

An even better situation occurs when the Tate resolution has betti diagram of the form

$$\begin{array}{cccccccc}
\dots & h^k\mathcal{F}(-k-3) & h^k\mathcal{F}(-k-2) & h^k\mathcal{F}(-k-1) & 0 & 0 & 0 & \\
 & 0 & 0 & 0 & 0 & 0 & 0 & \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 & 0 & 0 & 0 & 0 & 0 & 0 & \\
 & 0 & 0 & 0 & h^0\mathcal{F} & h^0\mathcal{F}(1) & h^0\mathcal{F}(2) & \dots
\end{array}$$

In this case we see from the previous section that the Chow form of $\mathcal{F}$ is the determinant of the $h^0(\mathcal{F}) \times h^k\mathcal{F}(-k-1)$ matrix whose entries are linear forms in the Plücker coordinates on the Grassmannian $\mathbf{G}_{k+1}$. (It follows that $h^0\mathcal{F} = h^k\mathcal{F}(-k-1) = \deg(\mathcal{F})$, which one can easily see in other ways as well.)

Modules whose associated sheaf have this sort of Tate resolution were first studied by Bernd Ulrich in [1984]. We will call them *Ulrich sheaves.* Thus a k-dimensional sheaf $\mathcal{F}$ on $\mathbf{P}$ is an Ulrich sheaf if $\mathcal{F}$ has no intermediate cohomology—that is, $\mathrm{H}^q(\mathcal{F}(d)) = 0$ for $1 \le q \le k-1$ and all d—and $\mathrm{H}^0(\mathcal{F}(j)) = 0$ for $j < 0$ while $\mathrm{H}^k(\mathcal{F}(j)) = 0$ for $j \ge -k$. Since an Ulrich sheaf has no intermediate cohomology, its restriction to the nonsingular part of X is automatically a vector bundle.

We can characterize Ulrich sheaves without referring to all the cohomology in several elementary ways. Since every 0-dimensional sheaf is an Ulrich sheaf, we will henceforward ignore this case.

Proposition 2.1 *Let $\mathcal{F}$ be a coherent, k-dimensional sheaf on the projective space $\mathbf{P} = \mathbf{P}^n$ over K with $k > 0$. The following are equivalent:*

a) *$\mathcal{F}$ is an Ulrich sheaf.*

b) *$\mathrm{H}^i\mathcal{F}(-i) = 0$ for $i > 0$ and $\mathrm{H}^i\mathcal{F}(-i-1) = 0$ for $i < k$.*

c) *If the support of $\mathcal{F}$ is a scheme X, then for some (respectively all) finite linear projections $\pi : X \to \mathbf{P}^k$ the sheaf $\pi_*\mathcal{F}$ is the trivial sheaf $\mathcal{O}^t_{\mathbf{P}^k}$ for some t.*

d) *The module $M := \mathrm{H}^0_*(\mathcal{F}) := \oplus_d \mathrm{H}^0(\mathcal{F}(d))$ of twisted global sections is an Ulrich module, in the sense of Backelin and Herzog [1987]; that is, M is a Cohen-Macaulay module of dimension $k+1$ over the homogeneous coordinate ring $S = k[x_0, \cdots, x_n]$ of $\mathbf{P}$, whose number of generators is equal to $\deg \mathcal{F}$, or equivalently whose S-free resolution*

$$\mathbf{F} : 0 \longrightarrow F_{n-k} \xrightarrow{\varphi_{n-k}} \ldots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow M \longrightarrow 0$$

is linear in the sense that F_i is generated in degree i for every i.

Proof. a) $\Rightarrow$ b) is trivial.

b) $\Rightarrow$ c): By the finiteness and linearity of π we have $\mathrm{H}^i(\mathcal{F}(j)) = \mathrm{H}^i((\pi_*)\mathcal{F}(j)$. The vanishing of cohomology of b) gives vanishing for $\pi_*\mathcal{F}$ which characterizes the trivial vector bundles on $\mathbf{P}^k$.

c) $\Rightarrow$ d): By *c)* $M = \mathrm{H}^0_*(\mathcal{F})$ is a free module over $K[x_0, \ldots, x_k] = \mathrm{H}^0_*(\mathcal{O}_{\mathbf{P}^k})$ generated in degree 0. Thus M is a linear Cohen-Macaulay module, that is an Ulrich module.

d) $\Rightarrow$ a): The equivalence of the two characterizations of Ulrich modules given in d) may be found in Brennan, Herzog, and Ulrich [1987, Prop. 1.5]. A graded S-module M is 0-regular if and only if the free resolution of $M_{\geq 0}$ is linear is proved in Eisenbud-Goto [1984] (see also Eisenbud [1995, Theorem 20.18]). If M is a $k+1$-dimensional Cohen-Macaulay module with linear resolution, then the associated sheaf $\mathcal{F}$ is also 0-regular. The Cohen-Macaulay property of M gives the vanishing of the intermediate cohomology of $\mathcal{F}$, and (since $\dim M = k+1 > 1$) also shows that $M = \mathrm{H}^0_*(\mathcal{F})$. Thus $\mathrm{H}^0(\mathcal{F}(j)) = 0$ for $j < 0$, and $\mathcal{F}$ is Ulrich. ∎

From the linearity of the resolution $\mathbf{F}$ of an Ulrich module M it follows, for example, that the rank of F_i is $\binom{n-k}{i} \cdot \operatorname{rank} F_0$; to see this reduce modulo a maximal regular sequence, and observe that M must reduce to a direct sum of copies of the residue field K. In particular, $\operatorname{rank} F_{n-k} = \operatorname{rank} F_0$, and this rank is equal to the degree of $\mathcal{F}$. (For more details, see for example Brennan, Herzog, and Ulrich [1987].) The same kind of argument gives:

Corollary 2.2 *If $\mathcal{F}$ is an Ulrich sheaf of dimension k on $\mathbf{P}^r$ then $\chi(\mathcal{F}(e)) = h^0(\mathcal{F})\binom{e+k}{k}$.* ∎

In Theorem 4.1 we will generalize this to sheaves on X that are Ulrich sheaves for the d-uple embedding of $\mathcal{F}$.

In our applications we will be particularly interested in the case where the Ulrich sheaf is a vector bundle on its support, and is self-dual up to a twist. In this case the criterion above can be simplified:

Corollary 2.3 *Let $\mathcal{F}$ be a vector bundle on a k-dimensional Gorenstein scheme $X \subset \mathbf{P}^r$. If $\mathcal{F} \cong \mathcal{F}^*(k+1) \otimes \omega_X$, then $\mathcal{F}$ is an Ulrich sheaf on $\mathbf{P}^r$ if and only if $\mathcal{F}$ is 0-regular.*

Proof. The 0-regularity implies that $\mathrm{H}^i(\mathcal{F}(j)) = 0$ for $j > -i$. The rest of the necessary vanishing follows from Serre duality. ∎

Brennan, Herzog and Ulrich discovered in [1987] that linear determinantal varieties have rank one Ulrich modules, so we can give Bézout expressions for their Chow forms using the ideas above. This series of examples includes rational normal scrolls, Bordiga-White surfaces and many more. We can give a different description of their Ulrich modules as follows:

Example 2.4 Let $\varphi: F \to G$ with $F = \oplus_1^f \mathcal{O}$ and $G = \oplus_1^g \mathcal{O}(1)$, $f \leq g$, a linear $f \times g$ matrix on $\mathbf{P}^n$ which drops rank in expected codimension $(f - g + 1)$. The Eagon-Northcott type complex

$$0 \to \Lambda^f F \otimes D_{f-g+1} G^* \to \ldots \to \Lambda^g F \otimes G^* \to \Lambda^{g-1} F \to \mathcal{F} \to 0,$$

see Eisenbud [1995, Theorem A2.10], is a linear resolution of a module annihilated by the maximal minors of φ and has length $f - g + 1$. It is thus the resolution of an Ulrich sheaf on $X = V(I_g(\varphi))$, and one can check that the sheaf has rank 1 (it is isomorphic, in the generic case, to $I_{g-1}\varphi'$, the ideal generated by the $g-1 \times g-1$ minors of the submatrix φ' obtained from φ by omitting one row.) Hence the Chow form of $Chow(X) = Chow(\mathcal{F})$ is polynomial of degree $\binom{f}{g-1}$ in the Plücker coordinates, and $\deg X = \binom{f}{g-1}$.

Example 2.5 Consider the scroll $S(2,1) \subset \mathbf{P}^4$ defined by

$$\varphi = \begin{pmatrix} x_0 & x_1 & x_3 \\ x_1 & x_2 & x_4 \end{pmatrix}.$$

Using the Ulrich sheaf $\mathcal{F}$ as above and Theorem 3.1 we obtain its Chow form as the determinant of the matrix

$$\begin{pmatrix} [034] & [013] & [023] \\ -[134] & [023] + [014] & -[123] - [024] \\ [234] & -[024] & [124] \end{pmatrix}.$$

The Chow forms of rational normal scrolls have further interpretations: Consider $(r+1)$-dimensional spaces α of sections of bundles $\oplus_{i=1}^r \mathcal{O}_{\mathbf{P}^1}(d_i)$. The Chow form of the scroll $S(d_1, \ldots, d_r) \subset \mathbf{P}^N$ with $N + 1 = \sum_i (d_i + 1)$ describes those α, where the minors of the corresponding morphism

$$\mathcal{O}_{P^1}^{r+1} \xrightarrow{\alpha} \oplus_{i=1}^r \mathcal{O}_{\mathbf{P}^1}(d_i)$$

have a common zero. (Such formulas were also worked out by Henri Lombardi and J.-P. Jouanolou (unpublished).)

In case of $S(2,1)$ there is also the interpretation for plane conics with one assigned base point: Since $S(2,1)$ is the image of $\mathbf{P}^2$ by the linear system of conics with a single assigned base point, say $(1:0:0)$, its Chow form describes those 3-dimensional subspaces of conics which have a further base point. In section 5 we will generalize this examples to forms of any degree on $\mathbf{P}^2$ with several simple assigned base points.

From the point of view of examples, it is interesting to note that if two schemes in projective spaces support (weakly) Ulrich sheaves, then so does their Segre product:

Proposition 2.6 *Let $\mathcal{F}_1$ be a coherent sheaf on $\mathbf{P}(W_1)$ and let $\mathcal{F}_2$ be a coherent sheaf on $\mathbf{P}(W_2)$. Set $d = \dim(\mathcal{F}_1)$. Let $\mathcal{G}$ be the Segre product of $\mathcal{F}_1$ with $F_2(d)$ on $\mathbf{P} = \mathbf{P}(W_1 \otimes W_2)$; that is, $\mathcal{G} = (\pi_1^* \mathcal{F}_1) \otimes (\pi_2^* \mathcal{F}_2(d))$ on the Segre variety $\mathbf{P}(W_1) \times \mathbf{P}(W_2) \subset \mathbf{P}$.*

a) If $\mathcal{F}_1, \mathcal{F}_2$ are weakly Ulrich, then $\mathcal{G}$ is weakly Ulrich.

b) If $\mathcal{F}_1, \mathcal{F}_2$ are Ulrich, then $\mathcal{G}$ is Ulrich.

Of course a similar result holds for the Segre product of $\mathcal{F}_1(\dim \mathcal{F}_2)$ and $\mathcal{F}_2$.

Proof. Both parts follow easily from the Künneth formula

$$\mathrm{H}^i(\mathcal{G}(d)) = \oplus_{i=j+k} H^j(\mathcal{F}_1(d)) \otimes \mathrm{H}^k(\mathcal{F}_2(d)).$$

For example, in part a) we need $\mathrm{H}^j(\mathcal{F}_1(-j-k-2)) \otimes \mathrm{H}^k(\mathcal{F}_2(d-j-k-2)) = 0$ when $j+k < d + \dim F_2$. If $j < d$ then the first factor vanishes since $\mathcal{F}_1$ is weakly Ulrich, while if $j = d$ then the second factor vanishes for the same reason. ∎

Corollary 2.7 *With notation as in Proposition 2.6, suppose that $\mathcal{F}_1, \mathcal{F}_2$ are Ulrich of dimensions d_1, d_2. The map over the exterior algebra $E_{\mathrm{Segre}} = \wedge((W_1 \otimes W_2)^*) = \wedge(W_1^* \otimes W_2^*)$*

$$\mathrm{H}^{d_1+d_2}(\mathcal{G}(-d_1-d_2-1)) \otimes \omega_{E_{\mathrm{Segre}}} \longrightarrow \mathrm{H}^0(\mathcal{G}) \otimes \omega_{E_{\mathrm{Segre}}}$$

is derived from the tensor product of the corresponding maps for $\mathcal{F}_1$ and $\mathcal{F}_2$ over $\wedge W_1^*$ and $\wedge W_2^*$ respectively via the canonical injection $\wedge W_1^* \otimes \wedge W_2^* \subset \wedge(W_1^* \otimes W_2^*)$. ∎

It follows that in situations where we can compute a Bézout expression for the Chow forms of $\mathcal{F}_1$ and $\mathcal{F}_2$ we can also compute a Bézout expression for the Chow form of the Segre product. Similar remarks and formulas hold in the case of weakly Ulrich sheaves and Stiefel expressions of the Chow form.

3 Chow forms as determinants and Pfaffians

Throughout this section we will work with a sheaf $\mathcal{F}$ of dimension k on $\mathbf{P}^n = \mathbf{P}(W)$. For simplicity, we will write $\mathbf{U}$ for the functor $\mathbf{U}_{k+1}$ defined in Section 1. We set $c = n - k$, the codimension of $\mathcal{F}$.

As we have seen in the previous section, if $\mathcal{F}$ is weakly Ulrich then the complex $\mathbf{U}(\mathcal{F})$ is given by a single map $\Psi_{\mathcal{F}} : \mathbf{U}(T^{-1}\mathcal{F}) \to \mathbf{U}(T^0)\mathcal{F}$ of vector bundles on the Grassmannian, and the Chow form of $\mathcal{F}$ is the determinant of Ψ_F. In this section we will make Ψ_F explicit. The tools we develop will allow us to show that if $\mathcal{F}$ is skew symmetrically self-dual in a natural sense then the complex $\mathbf{U}(\mathcal{F})$ is skew symmetric, and in particular $\Psi_{\mathcal{F}}$ is skew symmetric. When $\mathcal{F}$ is also weakly Ulrich, the square root of the the Chow form of $\mathcal{F}$ is the Pfaffian of $\Psi_{\mathcal{F}}$. In particular, when $\mathcal{F}$ is in addition a sheaf of rank 2 supported on a variety of X, the Chow form of X itself is the Pfaffian of $\Psi_{\mathcal{F}}$.

The matter is simplest in the Ulrich case, and we describe this first: Suppose $\mathcal{F} = \widetilde{M}$ is an Ulrich sheaf and let $c = n - k$ be its codimension. By Proposition 2.1 M has a linear free resolution

$$L : 0 \longrightarrow L^{-c} \xrightarrow{\alpha_{-c}} \cdots \longrightarrow L^{-1} \xrightarrow{\alpha_{-1}} L^0,$$

Where $L^{-i} = S \otimes P_i$ for some finite dimensional vector space P_i concentrated in degree i. Each map α_i in the resolution corresponds to a map $P_i \to W \otimes P_{i-1}$, and because the maps of free S-modules compose to 0, the composite $P_i \to W \otimes P_{i-1} \to W \otimes W \otimes P_{i-2}$ has image contained in $\wedge^2 W \otimes P_{i-2}$. More generally, each composite $P_i \to (\otimes_1^j W) \otimes P_{i-j}$ has image in $\wedge^j W \otimes P_{i-j}$.

In particular we get a map $\varphi_{-c,0} : P_c \to \wedge^c W \otimes P_0$. We may identify $\wedge^c W$ with the space of linear forms on the Grassmannian $\mathbf{G} = \mathbf{G}_{k+1}$ of planes of codimension $k+1$, and thus $\varphi_{-c,0}$ gives a map $\Psi_{\mathcal{F}} : \mathcal{O}_{\mathbf{G}}(-1) \otimes P_c \to \mathcal{O}_{\mathbf{G}} \otimes P_0$.

Theorem 3.1 *If $\mathcal{F} = \widetilde{M}$ is an Ulrich sheaf on $\mathbf{P}(W)$, then with notation as above, the ranks of P_c and P_0 are the same and the Chow form of $\mathcal{F}$ is the determinant of the map $\Psi_{\mathcal{F}}$.* ∎

Remark: Angeniol and Lejeune-Jalabert [1989] define maps generalizing the $P_{i+1} \to \wedge^j W \otimes P_{i-j} \subset (\otimes_1^j W) \otimes P_{i-j}$ for the free resolution of any module M and use them to construct the Atiyah classes of M. Because each L^i is a free module generated in a single degree, our situation is simpler. In particular, in our case the maps themselves—not just their cohomology classes—are well-defined.

To generalize Theorem 3.1 we replace free resolutions by the linear free monads studied in Eisenbud, Fløystad and Schreyer [2001]. Here is a summary of the theory (references to EFS refer to that paper): Let $\mathcal{F}$ be any coherent sheaf on $\mathbf{P}(W)$. By by EFS, Example 8.5 and Proposition 8.6, there is a unique complex

$$L = \mathbf{L}(\mathcal{F}): \quad \cdots \xrightarrow{\alpha_{-2}} L^{-1} \xrightarrow{\alpha_{-1}} L^0 \xrightarrow{\alpha_0} L^1 \xrightarrow{\alpha_1} \cdots$$

such that $L^{-i} = S(-i) \otimes P_i$ and $L^i = 0$ if $|i| > n$, with the property that the sheafification of the homology of L is zero except for $\widetilde{\mathrm{H}^0(L)} = \mathcal{F}$. The complex $\mathbf{L}(\mathcal{F})$ is called the linear free monad of $\mathcal{F}$. It is functorial in $\mathcal{F}$, and may be constructed from the Tate resolution

$$\mathbf{T}(\mathcal{F}): \cdots \to T^{-1}(\mathcal{F}) \to T^0(\mathcal{F}) \to T^1 \to \cdots;$$

in fact, $\mathbf{L}(\mathcal{F}) = \mathbf{L}(P)$, the complex corresponding to the graded E-modules $P := \mathrm{im}(T^{-1}(\mathcal{F}) \to T^0(\mathcal{F}))$ under the Bernstein-Gel'fand-Gel'fand correspondence. Thus $\mathbf{L}(F)$ has the form

$$\cdots \longrightarrow P_1 \otimes S = L^{-1} \longrightarrow P_0 \otimes S = L^0 \longrightarrow P_{-1} \otimes S = L^1 \longrightarrow \cdots.$$

For example, the linear free monad for an Ulrich sheaf is equal to the minimal free resolution.

Associated to $L = \mathbf{L}(\mathcal{F})$ are the maps

$$\varphi_{-i,-j}: \quad P_i \to \wedge^{i-j} W \otimes P_j.$$

adjoint to the multiplication map $\wedge^{i-j} V \otimes P_i \to P_{i-j}$ that defines the E-module structure on P. These may also be computed from the differentials of L, as above.

We can now describe the "middle" map $T^{-1}(\mathcal{F}) \to T^0(\mathcal{F})$ in the Tate resolution $\mathbf{T}(\mathcal{F})$. Let $\mathfrak{m}$ be the ideal of elements of negative degree in E (the augmentation ideal) and define graded vector spaces A and B by

$$A = P/\mathfrak{m}P \qquad B = \{p \in P \mid \mathfrak{m}p = 0\}.$$

A projective cover $F \to P$ is a minimal map from a free E-module F onto P. It follows from Nakayama's Lemma that $F \cong E \otimes A$. A projective cover is determined by the data of a splitting $\eta : A \to P$ (as graded vector spaces) of the natural projection map $P \to A$. Dually, an injective envelope $P \to G$ is uniquely determined by a splitting $\pi : P \to B$ of the inclusion $B \subset P$; we take $G \cong \omega_E \otimes B$, and the map from P is the unique map to $\omega_E \otimes B$ whose composition with the projection to $(\omega_E)_0 \otimes B = B$ is π.

The composition of $F \to P$ and $P \to G$ is a map

$$\varphi_P : E \otimes A \to \omega_E \otimes B$$

whose image is P. We define $\Psi_{\mathcal{F}} = \mathbf{U}(\varphi_P)$, which is a map of vector bundles on the Grassmanian $\mathbf{G}_{k+1}$. For example if L is a free resolution of an Ulrich sheaf then, by Eisenbud-Fløystad-Schreyer [2001], Proposition 8.7, $A = P_c$ and $B = P_0$, so in that case φ_P is the map induced by the map $\varphi_{-c,0}$ defined before Theorem 3.1, and the map $\Psi_{\mathcal{F}}$ is the same as the one given there. (No choice of η and π is involved because $A = P_{-c}$, $B = P_0$ in that case.)

Theorem 3.2 *If $\mathcal{F}$ is a weakly Ulrich sheaf of dimension k on $\mathbf{P}(W)$, with linear monad $\mathbf{L}(P)$, then the Chow form of $\mathcal{F}$ is* $\det \mathbf{U}(\Psi_{\mathcal{F}})$.

Proof. By Theorem 1.4 the Chow form of $\mathcal{F}$ is the determinant of the complex $\mathbf{UT}(\mathcal{F})$. Since $\mathcal{F}$ is weakly Ulrich, this complex consists of a single map:

$$\mathbf{U}(\mathcal{F}) = \mathbf{U}(T^{-1} \to T^0) = \mathbf{U}(\varphi_P) = \Psi_{\mathcal{F}}.$$

■

The skew symmetry of $\mathbf{U}(\mathcal{F})$

We now show that appropriate symmetry or skew symmetry of $\mathcal{F}$ makes $\mathbf{U}(\mathcal{F})$ symmetric or skew symmetric. The functor

$$D : \mathcal{F} \mapsto \mathcal{E}xt^c(\mathcal{F}, \omega_{\mathbf{P}^n})(k+1)$$

defines a duality on the category of k-dimensional Cohen-Macaulay sheaves on $\mathbf{P}^n$ and there is a canonical morphism $\iota : \mathcal{F} \to \mathbf{DD}(\mathcal{F})$. Let $\epsilon = \pm 1$. As with any duality, we say that a morphism $\sigma : \mathcal{F} \longrightarrow D(\mathcal{F})$ is ϵ-symmetric if

$$\begin{tikzcd}
 & DD(\mathcal{F}) \arrow[dd, "D\sigma"] \\
\mathcal{F} \arrow[ur, "\iota"] \arrow[dr, "\sigma"'] \arrow[r, phantom, "\epsilon"] & \\
 & D(\mathcal{F})
\end{tikzcd}$$

commutes up to the sign ϵ. In case $\epsilon = 1$ we say that $\mathcal{F}$ is symmetric; if $\epsilon = -1$ then $\mathcal{F}$ is called skew symmetric.

Theorem 3.3 *Suppose that $\mathcal{F}$ is a Cohen-Macaulay sheaf of dimension k on $\mathbf{P}^n$. Any ϵ-symmetric isomorphism $\mathcal{F} \to D(\mathcal{F})$, induces an ϵ-symmetric isomorphism*

$$\mathbf{U}(\mathcal{F}) \to \mathrm{Hom}_{\mathbf{G}_{k+1}}(\mathbf{U}(\mathcal{F}), \mathcal{O}_{\mathbf{G}_{k+1}}(-1))[1].$$

In particular the map $\mathbf{U}T^{-1}(\mathcal{F}) \xrightarrow{\Psi_{\mathcal{F}}} \mathbf{U}T^0(\mathcal{F})$ is ϵ-symmetric, and for $j > 1$ the map $\mathbf{U}T^{-j}(\mathcal{F}) \longrightarrow \mathbf{U}T^{-j+1}(\mathcal{F})$ is dual to $\mathbf{U}T^{j-1}(\mathcal{F}) \longrightarrow \mathbf{U}T^j(\mathcal{F})$.

If $\mathcal{F}$ is skew symmetric we define the Pfaffian of the skew symmetric complex $\mathbf{U}(\mathcal{F})$ by taking an appropriate Pfaffian of the middle map $\mathbf{U}T^{-1}(\mathcal{F}) \xrightarrow{\Psi_{\mathcal{F}}} \mathbf{U}T^0(\mathcal{F})$ times the alternating product of those terms from the determinant of $\mathbf{U}(\mathcal{F})$ that are associated with the maps $\mathbf{U}T^{-j}(\mathcal{F}) \longrightarrow \mathbf{U}T^{-j+1}(\mathcal{F})$ for $j > 0$. The determinant of $\mathbf{U}(\mathcal{F})$ is then the square of the Pfaffian of $\mathbf{U}(\mathcal{F})$.

Corollary 3.4 *Assume that the characteristic of the ground field is not 2. If $\mathcal{F}$ is a skew-symmetric Cohen-Macaulay sheaf of rank 2 on a k-dimensional subscheme $X \subset \mathbf{P}^n$ such that $\wedge^2\mathcal{F} \cong \omega_X(k+1)$, then the Chow form of X is the Pfaffian of the complex $\mathbf{U}(\mathcal{F})$. In particular if $\mathcal{F}$ is weakly Ulrich, then the Chow form of X is the Pfaffian of the skew-symmetric map of vector bundles $\mathbf{U}(\Psi_{\mathcal{F}})$.*

Remark. In order to include the case of characteristic 2 we would have to add the condition that the duality D is alternating, not just skew symmetric, and then prove the corresponding result for $\Psi_{\mathcal{F}}$. We leave this task to the interested reader.

Proof of Corollary 3.4. The skew-symmetric pairing $\mathcal{F} \otimes \mathcal{F} \to \wedge^2\mathcal{F} \cong \omega_X(k+1)$ gives rise to a skew symmetric isomorphism

$$\mathcal{F} \to \mathrm{Hom}(\mathcal{F}, \omega_X(k+1) \cong \mathrm{Ext}^c(\mathcal{F}, \omega_{\mathbf{P}^n}) \cong \mathbf{D}(\mathcal{F}).$$

The rest follows from Theorem 3.3 and the discussion above. ∎

To prove Theorem 3.3 we will first analyze the map on linear monads induced by the (skew) symmetric isomorphism $\mathcal{F} \to D\mathcal{F}$. From this analysis will come a certain symmetry property of φ_P. The map φ_P may be represented by a matrix of elements of $\wedge W$. An element $\alpha \in \wedge^t W$ induces for any integer s a map (which we again call α) defined by

$$\wedge^s U \otimes \wedge^v V \xrightarrow{\alpha} \wedge^{s+t-v} U \qquad : \qquad u \otimes e \mapsto \alpha(e)(u)$$

where $\alpha(e) \in \wedge^{v-t}V$ acts on U as described in Section 1. The map $\Psi_{\mathcal{F}}$ is constructed from these pieces, so we will derive a symmetry property for Ψ_F.

The difficulty of proving Theorem 3.3 comes from the delicacy of the signs involved. For example, consider the case where the map $\varphi_P : E(k+1-i) \to \omega_E(i)$ in Theorem 3.2 is given by a 1×1 matrix whose entry is in $\wedge^{c+2i}W$. One might suppose that any 1×1 matrix would be symmetric, and correspond to a symmetric map of vector bundles $\Psi : (\wedge^i U)^* \cong \wedge^{k+1-i}U \otimes \wedge^v V \to \wedge^i U$ on the Grassmannian. But actually Ψ is symmetric if i is even and skew symmetric if i is odd. The general result we need is the following:

Lemma 3.5 *Set $c = v-k-1$ and let $\alpha \in \wedge^{c+i+j}W$. The dual into $\mathcal{O}_{\mathbf{G}_{k+1}}(-1)$ of the map*

$$\wedge^{k+1-i}U \otimes \wedge^v V \xrightarrow{\alpha} \wedge^j U$$

is the map

$$\wedge^{k+1-j}U \otimes \wedge^v V \xrightarrow{(-1)^{k(i+j)+ij} \cdot \alpha} \wedge^i U.$$

Proof of Lemma 3.5. We identify $\wedge^i U$ with $\mathrm{Hom}(\wedge^{k+1-i}U \otimes \wedge^v V, \mathcal{O}_{\mathbf{G}_{k+1}}(-1))$ via the map τ sending $\beta \otimes e \in \wedge^{k+1-i}U \otimes \wedge^v V$ to the functional

$$\tau :\ \wedge^i U \ni \chi \mapsto (\chi \wedge \beta)(e) \in \mathcal{O}_{\mathbf{G}_{k+1}}(-1).$$

We must show that the diagram

$$\begin{tikzcd} \wedge^{k+1-i}U \otimes \wedge^v V \arrow[r, "\alpha"] \arrow[d, "\tau"'] & \wedge^j U \arrow[d, "\tau^*"] \\ (\wedge^i U)^* \arrow[r, "\alpha^*"'] & (\wedge^{k+1-j}U \otimes \wedge^v V)^* \end{tikzcd}$$

commutes up to a sign of $(-1)^{k(i+j)+ij}$. Although this is a diagram of vector bundles, we may reduce the problem to one of vector spaces by working fiberwise. For each $p \in \mathbf{G}_{k+1}$ the fiber U_p of U is a subspace of W, and the action of V on U_p is induced by its action on W. Thus the annihilator of U_p in V acts as zero on U_p, and we may therefore replace W by U_p and V by U_p^*, and assume that $U = W$ so that $v = k+1$ and $c = 0$.

From the definitions we see that

$$\begin{aligned} \alpha^*\tau(\beta \otimes e): &\quad \gamma \otimes e \mapsto \left[\left((\alpha(e))(\gamma)\right) \wedge \beta\right](e), \\ \tau^*\alpha(\beta \otimes e): &\quad \gamma \otimes e \mapsto \left[\left((\alpha(e))(\beta)\right) \wedge \gamma\right](e). \end{aligned}$$

Since these expressions are multilinear in α, β, γ it suffices to check the case where α, β, γ are products of elements in some fixed basis $\{x_1, \ldots, x_v\}$ of W. Set $a = \alpha(e) \in \wedge^{k+1-i-j} V$. The expressions are both zero unless $a(\beta) \wedge \gamma$ is a scalar times the product of all the basis elements $\{x_1, \ldots, x_v\}$. Under this assumption, what we are trying to prove is equivalent to the statement that $a(\gamma) \wedge \beta = (-1)^{k(i+j)+ij} a(\beta) \wedge \gamma$.

Let $\overline{\alpha}$ be the element of $\wedge^{k+1-i-j} W$ such that $\alpha(e)(\overline{\alpha}) = a(\overline{\alpha}) = 1$. Our assumptions imply that we can can factorize γ and β as $\gamma = \gamma' \overline{\alpha}$ and $\beta = \overline{\alpha}\beta'$. With this notation

$$\begin{aligned} a(\gamma) \wedge \beta &= a(\gamma' \wedge \overline{\alpha}) \wedge \beta = (-1)^{\gamma' a} \gamma' \wedge \beta = (-1)^{\gamma' a} \gamma' \wedge \overline{\alpha} \wedge \beta', \\ a(\beta) \wedge \gamma &= a(\overline{\alpha} \wedge \beta') \wedge \gamma = \beta' \wedge \gamma = \beta' \wedge \gamma' \wedge \overline{\alpha} = (-1)^{(\gamma'+a)\beta'} \gamma' \wedge \overline{\alpha} \wedge \beta', \end{aligned}$$

where we have also written γ', a, and β' for the degrees of these elements. Thus the diagram commutes up to the sign $(-1)^{(\gamma'+a)\beta' + \gamma' a}$. But $(\gamma' + a)\beta' + \gamma' a = \gamma(\beta - a) + (\gamma - a)a = \gamma\beta - a^2 = (k+1-i)(k+1-j) + (k+1-i-j)^2$ and this is congruent modulo 2 to $k(i+j)+ij$ as required. ∎

Proof of Theorem 3.3. We will show that the “middle” differential

$$\mathbf{U}T^{-1}(\mathcal{F}) \xrightarrow{\Psi_{\mathcal{F}}} \mathbf{U}T^0(\mathcal{F})$$

is ϵ symmetric. This condition depends on an identification of $\mathbf{U}T^0(\mathcal{F})$ with the dual of $\mathbf{U}T^{-1}(\mathcal{F})$. Changing this identification is the same as multiplying $\Psi_{\mathcal{F}}$ by an automorphism of its source or target, so it suffices to show that $\Psi_{\mathcal{F}}$ times such an isomorphism is ϵ symmetric.

Once we know that the middle differential is ϵ symmetric, we can take the injective resolution of P, from which the positively indexed maps of $\mathbf{U}(\mathcal{F})$ are made, to be dual to the free resolution of $\mathbf{P}$ from which the negatively indexed maps of $\mathbf{U}(\mathcal{F})$ are made.

To analyze $\Psi_{\mathcal{F}}$ we will make use of the analysis of φ_P described before Theorem 3.2. We have decompositions

$$\begin{aligned} T^{-1}(\mathcal{F}) &= \sum_i A_{c+i} \otimes E(-c-i) \text{ and} \\ T^0(\mathcal{F}) &= \sum_j B_{-j} \otimes \omega_E(j). \end{aligned}$$

In terms of this decomposition, the (i,j) component of φ_P is $\pi_j \varphi_{-c-i,j} \eta_{-c-i}$. Applying the functor $\mathbf{U}$ we see that $\Psi_{\mathcal{F}}$ decomposes into maps

$$\begin{aligned} \mathbf{U}(A_{c+i} \otimes E(-c-i)) = \mathbf{U}(A_{c+i} \otimes \wedge^v V \otimes \omega_E(k+1-i)) &= A_{c+i} \otimes \wedge^v V \otimes \wedge^{k+1-i} U \\ \xrightarrow{(\Psi_{\mathcal{F}})_{i,j}} \quad \mathbf{U}(B_{-j} \otimes \omega_E(j)) &= B_{-j} \otimes \wedge^j U \end{aligned}$$

where U denotes the tautological sub bundle on the Grassmanian. With this indexing, we will show that the maps $(\Psi_{\mathcal{F}})_{i,j}$ and $(\Psi_{\mathcal{F}})_{j,i}$ are dual up to a certain sign.

By EFS Theorem 4.1 we may identify A_{c+i} with $\mathrm{H}^{k-i}(\mathcal{F}(i-k-1))$ and B_j with $\mathrm{H}^j(\mathcal{F}(-j))$. As we have assumed that $\mathcal{F} \cong \mathrm{Ext}^c(\mathcal{F}, \omega_{\mathbf{P}^n}(k+1))$, we have

$$B_j^* = \mathrm{H}^j(\mathcal{F}(-j))^* = \mathrm{H}^j(\mathrm{Ext}^c(\mathcal{F}, \omega_{\mathbf{P}^n}(k+1))(-j))^* = \mathrm{H}^{n-j}(\mathcal{F}(k+1-j)) = A_j$$

by Serre duality. With this identification it suffices to check the signs in the maps φ_P rather than in the maps $\varphi_{\bullet,\bullet}$.

The linear complex $\mathcal{H}om(L, \omega_{\mathbf{P}^n})(k+1)[c]$, is a linear free monad for the dual sheaf $D(\mathcal{F}) \cong \mathcal{F}$. By the uniqueness and functoriality of linear monads, the isomorphism σ induces an isomorphism $L \cong \mathcal{H}om(L, \omega_{\mathbf{P}^n})(k+1)[c]$.

To simplify notation we set $\check{L}^i = D(L^i) = \mathcal{H}om(L^i, \omega_{\mathbf{P}^n})(k+1)$. We follow standard sign conventions (see for example Iverson [1986]) and define the dual complex $\check{L} = \mathcal{H}om(L, \omega_{\mathbf{P}^n})(k+1)$ to have differentials $(-1)^i \check{\alpha}_i$. Shifting the complex c steps also introduces the sign $(-1)^c$. Thus the isomorphism $L \to \check{L}[c]$ consists of a sequence of isomorphisms $\sigma_j : L^j \to \check{L}^{-c-j}$ as in the following diagram:

$$\begin{tikzcd}
{} \arrow[r] & L^{-c-i} \arrow[r, "\alpha_{-c-i}"] \arrow[d, "\sigma_{-c-i}"'] & \cdots \arrow[r, "\alpha_{-c-1}"] & L^{-c} \arrow[r] \arrow[d, "\sigma_{-c}"'] & \cdots \arrow[r, "\alpha_{j-1}"] & L^j \arrow[r] \arrow[d, "\sigma_j"'] & {} \\
{} \arrow[r] & \check{L}^i \arrow[r, "(-1)^{c+i-1}\check{\alpha}_{i-1}"] & \cdots \arrow[r, "(-1)^c\check{\alpha}_0"] & \check{L}^0 \arrow[r] & \cdots \arrow[r, "(-1)^{-j}\check{\alpha}_{-c-j}"] & \check{L}^{-c-j} \arrow[r] & {}
\end{tikzcd}$$

From the diagram we see that

$$(*) \quad \begin{aligned} \varphi_{-c-i,j} &= \sigma_j \alpha_{j-1} \otimes \ldots \otimes \alpha_{-c-i} \\ &= (-1)^s \check{\alpha}_{-c-j} \otimes \ldots \otimes \check{\alpha}_{i-1} \sigma_{-c-i} \end{aligned}$$

with $s = c(c+i+j) + \binom{c+j+1}{2} + \binom{i}{2}$, where the $c(c+i+j)$ comes from the shift, and the rest is the contribution of the signs in the duality, separating the parts with positive and negative indices.

We next prove that the map σ_{-c-i} is, up to a sign we shall identify, the dual of σ_i. By the uniqueness and functoriality of linear monads and the ϵ symmetry of σ the induced map of complexes $\sigma' : L \to \check{L}[c]$ factors as the composite $\sigma' = \epsilon D(\sigma')\iota'$ where ι' is the canonical morphism of complexes

$$\iota' : L \longrightarrow \mathcal{H}om(\mathcal{H}om(L, \omega_{\mathbf{P}^n})[c], \omega_{\mathbf{P}^n})[c].$$

The components of ι' are given by

$$\iota'_\ell : L_\ell \xrightarrow{(-1)^{(c+1)(c+\ell)}\iota} \check{\check{L}}_\ell,$$

where ι denotes the canonical morphism $M \longrightarrow \check{\check{M}}$ of sheaves, c.f. Iverson [1981], p.73. Thus $\sigma_{-c-i} = \epsilon(-1)^{(c+1)i} \check{\sigma}_i \iota$.

Combining this equation with $(*)$ we get

$$\begin{aligned} \varphi_{-c-i,j} &= \epsilon(-1)^{s+(c+1)i} \check{\alpha}_{-c-j} \otimes \ldots \otimes \check{\alpha}_{i-1} \check{\sigma}_i \iota \\ &= \epsilon(-1)^{s+t} \mathrm{transpose}(\sigma_i \alpha_{i-1} \otimes \ldots \otimes \alpha_{-c-j}) \end{aligned}$$

with $t = (c+1)i + \binom{c+i+j}{2}$ since in the transpose matrix the tensor factors occur in the opposite order, and this tensor lies in $\wedge^{c+i+j}W$.

Now

$$\begin{aligned} s+t &= c(c+i+j) + \binom{c+j+1}{2} + \binom{i}{2} + (c+1)i + \binom{c+i+j}{2} \\ &= c^2 + c(i+j) + \frac{(c+j)^2+(c+j)}{2} + \frac{i^2-i}{2} + ci + i + \frac{(c+i+j)^2-(c+i+j)}{2} \\ &\equiv (c+1)(i+j) + ij \bmod 2. \end{aligned}$$

By Lemma 3.5, we see that all the diagonal blocks $(\Psi_{\mathcal{F}})_{i,i} = \mathbf{U}(\varphi_{-c-i,i})$ will be ϵ symmetric. Because $(c+1)(i+j)+ij+k(i+j)+ij \equiv v(i+j)$ we can multiply the block matrix $\Psi_{\mathcal{F}}$ by the diagonal matrix of signs $\Delta = \oplus_j(-1)^{vj}Id_{A_j}$ where Id_{A_j} is the identity map on A_j, to get a map which is ϵ symmetric; that is, setting $\Psi'_{\mathcal{F}} = \Psi_{\mathcal{F}}\Delta$ we will have $\epsilon\mathcal{H}om((\Psi'_{\mathcal{F}}), \mathcal{O}_{\mathbf{G}}(-1)) = \Psi'_{\mathcal{F}}$. ∎

4 Curves

Resultants of binary forms were the starting point for this subject (Leibniz [1692], Bézout [1779], Sylvester [1840], [1842], see Kline [1972] for some historical remarks), and they correspond to the simplest cases of Chow forms of curves. We begin by explaining how they fit into our theory.

Example 4.1 Binary Forms Consider the **rational normal curve** $\mathbf{P}^1 \hookrightarrow \mathbf{P}^d$, $(s:t) \mapsto (s^d, s^{d-1}t, \ldots, t^d)$ we use $[i,j]$ for the ij^{th} Plücker coordinate of $\mathbf{G}_2 = \mathbf{G}(2, \mathrm{H}^0(\mathbf{P}^d, \mathcal{O}(1)))$ with respect to the given basis. The following determinantal formula can be deduced directly by computation from the classic Bézout formula for the resultant, or, since the rational normal curve is a linear determinantal variety, it could be deduced from Theorem 3.1.

From our point of view the most direct method is the computation of a map in a Tate resolution.

Proposition 4.2 *The Chow form of the rational normal curve of degree d is the determinant of the $d \times d$ symmetric matrix $A = (a_{ij})$ with*

$$a_{ij} = \sum_{\substack{p<min(i,j)\\ p+q=i+j-1}} [p,q].$$

Proof. Consider $\mathcal{L} = \mathcal{O}_{\mathbf{P}^1}(-1)$. With respect to $H = \mathcal{O}_{\mathbf{P}^1}(d)$ the betti numbers of the Tate resolution of $\mathcal{L}$ are

$$\begin{matrix} 3d & 2d & d & - & - & - \\ - & - & - & d & 2d & 3d \end{matrix}$$

Let $y_0, \ldots, y_d$ denote the dual basis in V to the monomial basis in $W = \mathrm{H}^0\mathcal{O}_{\mathbf{P}^1}(d)$.

The $d \times 2d$ matrix comes from multiplication $\mathrm{H}^0(\mathbf{P}^1, \mathcal{O}(d-1)) \times \mathrm{H}^0(\mathbf{P}^1, \mathcal{O}(d)) \to \mathrm{H}^0(\mathbf{P}^1, \mathcal{O}(2d-1))$, hence is given by the Sylvester type matrix

$$B = (b_{kl}) = (y_{k-l}) = \text{ transpose } \begin{pmatrix} y_0 & y_1 & \ldots & y_d & 0 & \ldots & 0 \\ 0 & y_0 & \ldots & y_{d-1} & y_d & \ldots & 0 \\ \vdots & \ddots & \ddots & & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & y_0 & \ldots & y_{d-1} & y_d \end{pmatrix}.$$

To prove the formula we must show that the kernel of this matrix is the image of a matrix $A = (a'_{i,j})$ with $a'_{i,j} = \sum_{\substack{p<min(i,j)\\ p+q=i+j-1}} y_p \wedge y_q$.

The equation $B \cdot A = 0$ holds since a term $y_{k-l} \wedge y_p \wedge y_q$ arising in the product $b_{kl}a'_{lj}$ is cancelled either by a term $y_p \wedge y_{k-l} \wedge y_q$ in the product $b_{k,k-p}a'_{k-p,j}$ or by a term $y_q \wedge y_p \wedge y_{k-l}$ in $b_{k,k-q}a'_{k-q,j}$, in case $k-l<j$ or $j \leq k-l$ respectively.

Since the d rows of A are linearly independent, and we know that the kernel of B is generated by d independent elements of degree 2, we see that the rows of A generate the kernel of B as required. ∎

If we want to obtain the Sylvester resultant formula, we apply $\mathbf{U}_2$ to the Tate resolution shifted. The resulting complex

$$\oplus^d \mathcal{U} \to \oplus^{2d} \mathcal{O},$$

written in Stiefel coordinates, gives the classical Sylvester formula for two polynomials $f = f_0 s^d + f_1 s^{d-1} t + \ldots + f_d t^d$ and $g = g_0 s^d + g_1 s^{d-1} t \ldots + g_d t^d$ of equal degree.

We will generalize these formulas to arbitrary curves over an algebraically closed field, and obtain partial results for more general ground fields.

By a *curve* we will mean a purely one-dimensional scheme X, projective over K. The theory of Ulrich sheaves on curves is significantly simpler than the theory for higher-dimensional varieties because it is essentially independent of the embedding. To state the result, we say that a sheaf $\mathcal{G}$ on a curve X has *no cohomology* if $\mathrm{H}^0(\mathcal{G}) = \mathrm{H}^1(\mathcal{G}) = 0$.

Theorem 4.3 *If X is a curve embedded in $\mathbf{P} = \mathbf{P}^{n+1}$ with hyperplane divisor H, then a sheaf $\mathcal{F}$ is an Ulrich sheaf for X in $\mathbf{P}$ if and only if $\mathcal{F} = \mathcal{G}(H)$ for some $\mathcal{G}$ with no cohomology.*

Proof. $\mathcal{G}(H)$ is 0-regular because $\mathrm{H}^1(\mathcal{G}(H)(-H)) = \mathrm{H}^1(\mathcal{G}) = 0$. Similarly, $\mathcal{E}xt^{n-1}_{\mathbf{P}}(\mathcal{G}(H), \mathcal{O}_{\mathbf{P}}(-n-1))$ is 2-regular because $\mathrm{H}^0(\mathcal{G}) = 0$. (One can also see the desired vanishing directly from the Tate resolution: for example, the vanishing of $\mathrm{H}^0(\mathcal{G})$ implies that the free module $T^0(\mathcal{G})$ has no generators in degree 0; and it follows that for $j < 0$ the module $T^{-j}(\mathcal{G})$ has no generators in degree $-j$. But by Eisenbud-Fløystad-Schreyer [2000, Thm. 7.1] the space of generators of $T^{-j}(\mathcal{G})$ in degree $-j$ is $\mathrm{H}^0(\mathcal{G}(-jH))$.) ∎

To find sheaves with no cohomology it suffices to look for sheaves on a single component of the reduced scheme X_{red} or even on its normalization. Thus we are led to ask: Given a nonsingular irreducible curve X over an arbitrary field K, what are the sheaves $\mathcal{G}$ over X with no cohomology? Such a sheaf $\mathcal{G}$ can have no torsion, so (since X is nonsingular) $\mathcal{G}$ is automatically locally free. From the vanishing of the cohomology we see that the Euler characteristic of $\mathcal{G}$ is 0, so by Riemann-Roch the degree of $\mathcal{G}$ is $\mathrm{rank}(\mathcal{G}) \cdot (1-g)$, where $g = \mathrm{genus}(X)$. Over an algebraically closed field, there are always line bundles of this type. This generalizes the fact that the equation of any plane curve can be written as the determinant of a linear matrix:

Proposition 4.4 *A line bundle L on a curve X has no cohomology if and only if $\deg(L) = \mathrm{genus}(X) - 1$ and L has no sections. If X contains infinitedly many $K-rational$ points, then such line bundles exist on X, and thus the Chow form of X, in any projective embedding, can be written as a determinant of linear forms in the Plücker coordinates.*

Proof. The first statement is immediate from the Riemann-Roch theorem. For the second, take $L = \mathcal{O}_X(p_1 + \cdots + p_g - q)$, where the p_i and q are general K-rational points. ∎

To arrive at explicit resultant formulas further work has to be done. We have to compute the appropiate differentials in the Tate complex explicitedly.

Example 4.5 Hyperelliptic resultant formulas Consider a fixed polynomial $f = f_0 + f_1 t + \ldots + f_{2g+2}t^{2g+2}$ with no multiple roots. To write explicit Stiefel and Bézout formulas for the resultant of two functions $a(t)+b(t)\sqrt{f(t)}$ and $c(t)+d(t)\sqrt{f(t)}$ with $a,b,c,d \in K[t]$ we consider them as functions on the hyperellipic curve C of genus g with function field $K(t,\sqrt{f})$. Let $k = \max\{\deg a, g+1+\deg b, \deg c, g+1+\deg d\}$ and consider the embedding of C given by $t \mapsto (1 : t : \ldots : t^k : \sqrt{f} : t\sqrt{f} : \ldots : t^{k-g-1}\sqrt{f})$. We want the Chow form of this embedding. By Theorem 4.3 and Theorem 3.3, a formula as the determinant of a symmetric matrix will arrize if choose as Ulrich sheaf $\mathcal{L}(H)$ with the line bundle $\mathcal{L}$ a "non vanishing theta characteristic" —that is, a line bundle $\mathcal{L}$ on C such that $\mathcal{L}\otimes\mathcal{L} = \omega_C$, the canonical bundle, and $\mathcal{L}$ has no cohomology. A non vanishing theta characteristic in turn corresponds to a factorization $f = f^{(1)}f^{(2)}$ of f into two polynomials of degree $g+1$. All of our formulas will depend on the choice of such factorization and we will obtain $\frac{1}{2}\binom{2g+2}{g+1}$ Bézout formulas.

Before we come to the Bézout formulas we will prove a Stiefel formula for the resultant that is highly parallel to the Sylvester formula for the ordinary resultant. We will then deduce a Bézout formula in a way that is analogous to our proof of Proposition 4.2. Let

$$syl(k,r) = \text{ transpose } \begin{pmatrix} r_0 & r_1 & \ldots & r_k & 0 & \ldots & 0 \\ 0 & r_0 & \ldots & r_{k-1} & r_k & \ldots & 0 \\ \vdots & \ddots & \ddots & & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & r_0 & \ldots & r_{k-1} & r_k \end{pmatrix}$$

be the $2k \times k$ "Sylvester block" of a polynomial r of degree k.

Proposition 4.6 *With notation as above, two functions $a+b\sqrt{f}$ and $c+d\sqrt{f}$ with $a,b,c,d \in K[t]$ have a common zero if and only if the determinant of the $4k \times 4k$ matrix*

$$\begin{pmatrix} syl(k,a) & syl(k,bf^{(2)}) & syl(k,c) & syl(k,df^{(2)}) \\ syl(k,bf^{(1)}) & syl(k,a) & syl(k,df^{(1)}) & syl(k,c) \end{pmatrix}$$

vanishes.

Proof. Let $\pi : C \to \mathbf{P}^1$ denote the double cover corresponding to the inclusion $K(t) \subset K(C) = K(t)[\sqrt{f}]$. We consider the embedding of C as a curve of degree $2k$ in projective space $\mathbf{P}^{2k+1-g}$ corresponding to the line bundle $\mathcal{O}_C(H) = \pi^*(\mathcal{O}_{\mathbf{P}^1}(k))$. The space of global sections of $\mathcal{O}_C(H)$ has basis corresponding to the functions $1, t, \ldots, t^k, \sqrt{f}, t\sqrt{f}, \ldots, t^{k-g-1}\sqrt{f}$, so the Chow form of C in this embedding is the resultant we seek. We write $e_0, \ldots, e_k, e_{k+1}, \ldots, e_{2k-g} \in V = \mathrm{H}^0(\mathcal{O}_C(H))^*$ for the dual basis.

Every line bundle $\mathcal{L}$ on C can be described as a rank 2 vector bundle $\mathcal{B} = \pi_*\mathcal{L}$ on $\mathbf{P}^1$ together with an action $\mathcal{B} \xrightarrow{y} \mathcal{B}(g+1)$ satisfying $y^2 = fid_{\mathcal{B}}$. For example $\pi_*\mathcal{O}_C = \mathcal{O}\oplus\mathcal{O}(-g-1)$ with the action defined by $y = \begin{pmatrix} 0 & f \\ 1 & 0 \end{pmatrix}$. The bundle $\mathcal{B} = \mathcal{O}(-1)\oplus\mathcal{O}(-1)$ with the action of $\begin{pmatrix} 0 & f^{(1)} \\ f^{(2)} & 0 \end{pmatrix}$ corresponds to a non vanishing theta characteristic $\mathcal{F}$ on C. In particular, $\mathcal{F}$ is a line bundle of degree $g-1$ with no cohomology. See Buchweitz and

Schreyer [2002] for a detailed exposition. The Stiefel formula above is obtained by applying the functor $\mathbf{U}$ to the line bundle $\mathcal{F}(2H)$.

The global sections of $\mathcal{F}(H)$ has a basis corresponding to the functions

$$\sqrt{f^{(1)}}, t\sqrt{f^{(1)}}, \ldots, t^{k-1}\sqrt{f^{(1)}}, \sqrt{f^{(2)}}, t\sqrt{f^{(2)}}, \ldots, t^{k-1}\sqrt{f^{(2)}},$$

while $\mathrm{H}^0(\mathcal{F}(2H))$ has a basis corresponding to

$$\sqrt{f^{(1)}}, t\sqrt{f^{(1)}}, \ldots, t^{2k-1}\sqrt{f^{(1)}}, \sqrt{f^{(2)}}, t\sqrt{f^{(2)}}, \ldots, t^{2k-1}\sqrt{f^{(2)}}.$$

Thus the map

$$\mathrm{Hom}(E, \mathrm{H}^0(\mathcal{F}(H))) \to \mathrm{Hom}(E, \mathrm{H}^0(\mathcal{F}(2H)))$$

in the Tate resolution is given by the $4k \times 2k$ matrix over the exterior algebra

$$B = \begin{pmatrix} syl(k, e_0 + e_1 t + \ldots + e_k t^k) & syl(k, (e_{k+1} + \ldots + e_{2k-g} t^{k-g-1}) f^{(2)}) \\ syl(k, (e_{k+1} + \ldots + e_{2k-g} t^{k-g-1}) f^{(1)}) & syl(k, e_0 + e_1 t + \ldots + e_k t^k) \end{pmatrix}.$$

The desired Sylvester formula follows by interpreting the induced map

$$\mathrm{H}^0(\mathcal{F}(H)) \otimes U \longrightarrow \mathrm{H}^0(\mathcal{F}(2H)) \otimes \mathcal{O}_{\mathbf{G}}$$

in terms of Stiefel coordinates. ∎

We now use these constructions as in Proposition 4.2 to derive **hyperelliptic Bézout formulas**. It suffices to compute the kernel of the map B. By Theorem 0.1 this will be a $2k \times 2k$ matrix with entries in $\Lambda^2 V$. Because $\mathcal{F}$ is a theta characteristic Theorem 3.3 show that the kernel will be represented in suitable bases by a symmetric matrix.

The final formula may be written in terms of the 2×2 minors of the $2 \times (2k+1-g)$ matrix

$$\begin{pmatrix} a_0 & \ldots & a_k & b_0 & \ldots & b_{k-g-1} \\ c_0 & \ldots & c_k & d_0 & \ldots & d_{k-g-1} \end{pmatrix}.$$

However we will work with the larger $2 \times 3(k+1)$ matrix

$$\begin{pmatrix} a_0 & \ldots & a_k & (bf^{(1)})_0 & \ldots & (bf^{(1)})_k & (bf^{(2)})_0 & \ldots & (bf^{(2)})_k \\ c_0 & \ldots & c_k & (df^{(1)})_0 & \ldots & (df^{(1)})_k & (df^{(2)})_0 & \ldots & (df^{(2)})_k \end{pmatrix}$$

whose minors are linear combinations of those of the matrix above with coefficients, which depend on the coefficients of $f^{(1)}$ and $f^{(2)}$.

If $0 \le p, q \le k$ then we denote by $[p, q]$ the minor formed by the columns with indices p and q. We write $p^{(1)}$ for the column with index $p + (k+1)$, and $q^{(2)}$ the column with index $q + 2(k+1)$. Thus brackets like $[p^{(1)}, q]$ and $[p^{(1)}, q^{(2)}]$ represent 2×2 minors of the large matrix.

Consider the $k \times k$ matrices $A^{11}, \ldots, A^{22}$ defined by

$$\begin{aligned}
A^{11}_{i,j} &= \sum_{\substack{0 \le p < q \le k \\ p < \min(i,j) \\ p+q = i+j-1}} [p^{(2)}, q] + [p, q^{(2)}], \\
A^{12}_{i,j} &= \sum_{\substack{0 \le p < q \le k \\ p < \min(i,j) \\ p+q = i+j-1}} [p, q] \quad + \sum_{\substack{0 \le p, q \le k \\ p < j \\ p+q = i+j-1}} [p^{(1)}, q^{(2)}], \\
A^{21}_{i,j} &= \sum_{\substack{0 \le p < q \le k \\ p < \min(i,j) \\ p+q = i+j-1}} [p, q] \quad + \sum_{\substack{0 \le p, q \le k \\ p < j \\ p+q = i+j-1}} [p^{(2)}, q^{(1)}], \\
A^{22}_{i,j} &= \sum_{\substack{0 \le p < q \le k \\ p < \min(i,j) \\ p+q = i+j-1}} [p^{(1)}, q] + [p, q^{(1)}].
\end{aligned}$$

The matrix A is actually symmetric. This becomes visible if we expand the expressions into brackets of the smaller $2 \times (2k+1-g)$ matrix.

Proposition 4.7 *Suppose $k \leq 12$. The functions $a + b\sqrt{f}$ and $c + d\sqrt{f}$ have a common zero if and only if the determinant of the matrix*

$$A = \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix}.$$

vanishes.

The formula should certainly hold for any k; but as noted in the proof we have performed the necessary computations only up to $k = 12$.

Notice that in case $b = d = 0$ the matrix reduces to twice the Bezout matrix for binary forms of degree k. This fits with the fact that two functions on $\mathbf{P}^1$ with a common zero have two common zeroes when pulled back to C.

Proof. As in the proof of Proposition 4.2 it suffices to check that $B \cdot A = 0$, when we regard A as a matrix over the exterior algebra, because the linear independence of the columns of A is visible from the specialization to the case of binary forms $b = d = 0$. For each specific value of g and k this can checked by Computer algebra, and we did this for all cases $1 \leq g+1 \leq k \leq 12$. ■

As a concrete application of Proposition 4.7 we do the case of an elliptic curve over the complex numbers.

Example 4.8 Resultant of doubly periodic functions Consider an elliptic curve $C = \mathbf{C}/\Gamma$ and the corresponding Weierstrass $\wp$-function, with functional equation

$$\wp'(z) = 4\wp^3(z) - g_2\wp(z) - g_3 = 4(\wp(z)-\rho_1)(\wp(z)-\rho_2)(\wp(z)-\rho_3)$$

where the $\rho_j's$ are the values of $\wp$ at the half periods. Two doubly periodic functions

$$f(z) = a_0 + a_1\wp(z) + a_2\wp^2(z) + b_0\wp'(z)/2$$

and

$$g(z) = c_0 + c_1\wp(z) + c_2\wp^2(z) + d_0\wp'(z)/2$$

have a common zero iff the determinant of

$$\begin{pmatrix} -\rho_1\rho_2[13] - (\rho_1+\rho_2)[03] & -\rho_1\rho_2[23] + [03] & [01] & [02] \\ -\rho_1\rho_2[23] + [03] & (\rho_1+\rho_2)[23] + [13] & [02] & [12] \\ [01] & [02] & \rho_3[13] + [03] & \rho_3[23] \\ [02] & [12] & \rho_3[23] & -[23] \end{pmatrix}$$

vanishes, where the bracket $[ij]$ denotes the minor made from the i^{th} and j^{th} columns of the matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & b_0 \\ c_0 & c_1 & c_2 & d_0 \end{pmatrix}.$$

This formula follows from Proposition 4.7, with one of the roots of f at infinity, and with the factorization given by $f^{(1)} = (\wp(z)-\rho_1)(\wp(z)-\rho_2)$.

Returning to our general discussion, we may ask whether it is possible to give a Bézout formula for the Chow form of a curve over a field K even if the curve does not have enough K-rational points to apply Theorem 4.3. In this case the curve may have no rank 1 Ulrich sheaf, as happens, for example, for a conic without real points in $\mathbf{P}^2_{\mathbf{R}}$. However, it may be that there are always rank 2 Ulrich sheaves. For example, assuming that X has genus 0, The structure sheaf $\mathcal{O}_X$ and the canonical bundle ω_X are defined over K, and there is a unique extension

$$\eta : \quad 0 \to \omega_X \to \mathcal{E} \to \mathcal{O}_X \to 0$$

corresponding to a nonzero element $\eta \in \mathrm{H}^1(\omega_X^{-1}) = K$. Over an algebraic closure of K the bundle $\mathcal{E}$ splits as $\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$ (the sequence above is the Koszul complex) and thus $\mathcal{E}$ has no cohomology.

The main theorem of Brennan, Herzog, and Ulrich [1987] generalizes this example and says that if K is algebraically closed and X is a 1-dimensional arithmetically Cohen-Macaulay subscheme of $\mathbf{P}$ then there exists a rank 2 sheaf $\mathcal{F}$ with no cohomology, which in addition satisfies $\mathcal{F} \cong \mathcal{H}om(\mathcal{F}, \omega_X)$. (Their statement does not include the separability hypothesis below; but they apply a result of Eisenbud [1988] which is proved only in the algebraically closed case. We do not see how to extend their proof beyond the separable case, as below.) A variation on their proof allows one to drop the "arithmetically Cohen-Macaulay" hypothesis. Here is a geometric version of the argument, developed in conversation with Joe Harris.

Proposition 4.9 *Let X be a projective curve, separable over the field K. If K is infinite then X has a coherent sheaf $\mathcal{E}$ with no cohomology which is a rank 2 vector bundle over the normalization of X_{red}, and satisfies* $\mathrm{Hom}(\mathcal{E}, \omega_X) = \mathcal{E}$.

Proof. Let $\pi : C \to X_0$ be the normalization. It is enough to find a rank 2 vector bundle without cohomology on C with $\mathrm{Hom}(\mathcal{E}, \omega_C)$, since we have $\mathrm{Hom}_X(\mathcal{E}, \omega_X) = \mathrm{Hom}_C(\mathcal{E}, \mathrm{Hom}(\mathcal{O}_C, \omega_X) = \mathrm{Hom}_C(\mathcal{E}, \omega_C)$. Since we have dealt with the case of $\mathbf{P}^1$ above, we will assume that the genus g of C is greater than 0. Let L be a line bundle on C of strictly positive degree

Any extension class $\eta \in \mathrm{Ext}^1(\omega_C \otimes L, L^{-1})$ gives rise to a short exact sequence

$$\eta : \quad 0 \to L^{-1} \to \mathcal{E} \to \omega_C \otimes L \to 0$$

where $\mathcal{E}$ is a vector bundle. For any such bundle $\wedge^2 \mathcal{E} = \omega_C$, whence $\mathrm{Hom}(\mathcal{E}, \omega_C) = \mathcal{E}$.

By Serre duality $\chi(\mathcal{E}) = 0$, so $\mathcal{E}$ will be an Ulrich sheaf as long as $\mathrm{H}^0(\mathcal{E}) = 0$. Since $\mathrm{H}^0(\mathcal{L}^{-1}) = 0$, this condition is satisfied if and only if the connecting homomorphism

$$\delta_\eta : \mathrm{H}^0(\omega_C \otimes L) \to \mathrm{H}^1(\mathcal{L}^{-1}) = \mathrm{H}^0(\omega_C \otimes L)^*$$

is an isomorphism. But

$$\eta \in \mathrm{Ext}^1(\omega_C \otimes L, L^{-1}) \cong \mathrm{H}^1(L^{-2} \otimes \omega_C^{-1}) \cong \mathrm{H}^0(L^2 \otimes \omega_C^2)^*,$$

and δ_η is induced by the multiplication pairing

$$\mathrm{H}^0(L \otimes \omega_C) \otimes \mathrm{H}^0(L \otimes \omega_C) \xrightarrow{m} \mathrm{H}^0(L^2 \otimes \omega_C^2)$$

in the sense η goes to δ_η under the composite

$$\begin{aligned} \mathrm{H}^0(L^2 \otimes \omega_C^2)^* \xrightarrow{m^*} & \ \mathrm{H}^0(L \otimes \omega_C)^* \otimes \mathrm{H}^0(L \otimes \omega_C)^* \\ & \cong \mathrm{H}^0(L \otimes \omega_C)^* \otimes \mathrm{H}^1(L^{-1}) \\ & \cong \mathrm{Hom}(\mathrm{H}^0(L \otimes \omega_C),\ \mathrm{H}^1(L^{-1})). \end{aligned}$$

Now the ring $R = \oplus_d \mathrm{H}^0(L^d \otimes \omega_C^d)$ is an integral domain, by separability it splits into a product of integral domains over the algebraic closure of K. It follows that the multiplication pairing is a direct sum of 1-generic pairings in the sense of Eisenbud [1988]. The results of that paper show that δ_η is an isomorphism unless η lies in a certain proper hypersurface in $\mathrm{H}^0(L^2 \otimes \omega_C^2)$. If K is infinite then this hypersurface cannot contain all the K rational points of this vector space. ■

Example 4.10 A Conic without a point The conic $C \subset \mathbf{P}^2$ defined by $x^2+y^2+z^2=0$ has no line bundle of degree -1 defined over $\mathbf{R}$. However there are rank 2 Ulrich sheaves. The cokernel

$$\mathcal{F} = \mathrm{coker}(\mathcal{O}_{\mathbf{P}^2}^4(-2) \xrightarrow{M} \mathcal{O}_{\mathbf{P}^2}^4(-1))$$

given by the matrix

$$M = \begin{pmatrix} 0 & x & y & z \\ -x & 0 & z & -y \\ -y & -z & 0 & x \\ -z & y & -x & 0 \end{pmatrix}$$

is a rank 2 sheaf on C with no cohomology. An explicit formula can be derived from Pfaffian Bézout formula for the resultant of 3 quadratic forms in 3 variables given in the introduction, by specializing one of the three quadratic forms to $x^2+y^2+z^2$ and eliminating unnecessary variables.

5 Veronese embeddings and Resultant formulas

Consider the d-uple embedding

$$\mathbf{P}^k \hookrightarrow \mathbf{P}^N$$

with $N = \binom{d+k}{k} - 1$. The Chow form is the resultant of $k+1$ homogeneous forms of degree d in $k+1$ variables, hence is of particular interest. To find determinantal or Pfaffian formulas for powers of such Chow forms, we need to look for vector bundles on $\mathbf{P}^k$ that become Ulrich sheaves on the d-uple embedding; Stiefel formulas come from weakly Ulrich sheaves. By an argument shown to us by Jerzy Weyman, even Ulrich sheaves always exist! By way of comparison, the classical search for Bézout or Stiefel formulas was essentially a search for line bundles on $\mathbf{P}^k$ that become Ulrich or weakly Ulrich on the d-uple embedding. Weakly Ulrich line bundles exist (and were found classically) if and only if $k \le 4$ or $k = 5,\ d \le 3$ (Ulrich line bundles never exist except when $k \le 2$ or $d = 1$.) We get a few more Stiefel formulas for the resultants themselves (and not just powers) from the Horrocks-Mumford bundle in the case $k = 5,\ d = 4, 6$ or 8.

It turns out that the cohomology of a sheaf that becomes an Ulrich sheaf on the d-uple embedding is determined by the rank of the sheaf alone, and the same idea works for the d-uple embedding of any variety:

Theorem 5.1 *Let $\iota : \mathbf{P}^m \hookrightarrow \mathbf{P}^n$ be the d-uple embedding. Suppose $\mathcal{F}$ is a sheaf of dimension k on $\mathbf{P}^m$. The sheaf $\iota_*\mathcal{F}$ is an Ulrich sheaf on $\mathbf{P}^n$ if and only if*

$$h^i(\mathcal{F}(e)) \neq 0 \Leftrightarrow \begin{cases} i = 0, & -d < e \\ 0 < i < k, & -(i+1)d < e < -id \\ i = k, & e < -kd \end{cases} .$$

In particular, $\mathcal{F}$ then has natural cohomology as sheaf on $\mathbf{P}^m$. Thus all the $h^i(\mathcal{F}(e))$ are determined by the formula

$$\chi(\mathcal{F}(e)) = h^0(\mathcal{F})\binom{\frac{e}{d}+k}{k}.$$

If $\mathcal{F}$ is a vector bundle of rank r on $\mathbf{P}^m$, then we can rewrite this formula as $\chi(\mathcal{F}(e)) = \frac{r}{m!}(e+d)\cdots(e+md) = (\frac{r}{m!}e^m) + \cdots + rd$.

The vanishing and non-vanishing results in the first part of Theorem 5.1 have a very simple interpretation in terms of the betti diagram of the Tate resolution of $\mathcal{F}$: they say that the nonzero terms form a sequence of non-overlapping strands and that all of the strands representing intermediate cohomology have length precisely $d-1$. The formulas in the second part then give the values of the nonzero terms. For example, if $\mathcal{F}$ is a rank 2 vector bundle on $\mathbf{P}^2$ which is an Ulrich sheaf for the d-uple embedding, Theorem 5.1 says precisely that the Tate resolution of $\mathcal{F}$, considered as a sheaf on $\mathbf{P}^2$, has betti diagram

$$\begin{array}{cccccccccccc}
\cdots & 2(d{+}2) & 1(d{+}1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots \\
\cdots & 0 & 0 & 1(d{-}1) & 2(d{-}2) & \cdots & (d{-}2)2 & (d{-}1)1 & 0 & 0 & \cdots & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1(d{+}1) & 2(d{+}2) & \cdots & d(2d) & \cdots
\end{array}$$

where the zeroth term is the one occuring at the far right, (so that for example $h^0(\mathcal{F}) = 2d^2$). Further examples are given in the discussion of sheaves on $\mathbf{P}^3$ below.

To prove that the cohomology vanishes as we claim, we will repeatedly use the following elementary result, which is an easy case of Eisenbud-Fløystad-Schreyer [2000 Lemma 7.4]:

Lemma 5.2 *Suppose $\mathcal{G}$ is a sheaf on $\mathbf{P}^k$.*
a) If $\mathrm{H}^{i+j}(\mathcal{G}(-1-j)) = 0$ for all $j \geq 0$ then $\mathrm{H}^i(\mathcal{G}) = 0$.
b) If $\mathrm{H}^{i-j}(\mathcal{G}(1+j)) = 0$ for all $j \geq 0$ then $\mathrm{H}^i(\mathcal{G}) = 0$.

Note that the case $i = 1$ in part a) is Mumford's result showing that a (-1)-regular sheaf is 0-regular. For the reader's convenience we give a quick proof.

Proof of Lemma 5.2. a): Translating the condition in a) to a condition on the Tate resolution $T^\bullet(\mathcal{G})$ over the exterior algebra E, we see that the free summand $\mathrm{H}^i(\mathcal{G}) \otimes \omega_E$ in $T^0(\mathcal{G})$ maps injectively into $T^1(\mathcal{G})$. Since $T^\bullet(\mathcal{G})$ is a minimal complex and E is Artinian, this is impossible.

Part b) follows by applying the same argument to the dual of the Tate resolution. ∎

Proof of Theorem 5.1. We begin by showing by induction on i that for $i < k$ we have $\mathrm{H}^i(\mathcal{F}(e)) = 0$ if $e \leq -(i+1)d$. Since $\mathcal{F}$ becomes an Ulrich sheaf under the d-uple embedding we have $\mathrm{H}^0(\mathcal{F}(-d)) = 0$, and it follows that $\mathrm{H}^i(\mathcal{F}(e)) = 0$ for $e \leq -d$, which is the case $i = 0$. For $i > 0$ we proceed by descending induction on e. Again since $\mathcal{F}$ becomes Ulrich on the d-uple embedding we have $\mathrm{H}^i(\mathcal{F}(-(i+1)d)) = 0$, the initial case. Assuming that $\mathrm{H}^i(\mathcal{F}(e)) = 0$ for some $e < -(i+1)d$, the induction on i gives the hypothesis to apply part b) of Lemma 5.2 to show that $\mathrm{H}^i(\mathcal{F}(-e-1)) = 0$.

Similarly, $\mathrm{H}^i(\mathcal{F}(e)) = 0$ for $i > 0$ and $e \geq -id$ follows by induction and part b) of Lemma 5.2. The nonvanishing of the remaining cohomology follows, since otherwise the Tate resolution for $\mathcal{F}$ would contain terms equal to zero.

We next prove the formulas for $\chi(\mathcal{F}(e))$. If $\iota_*\mathcal{F}$ is an Ulrich sheaf, then Corollary 2.2 shows that $\chi(\mathcal{F}(dt)) = h^0(\mathcal{F})\binom{k+t}{k}$. Since $\chi(\mathcal{F}(t))$ is a polynomial, it is determined by this relation, yielding the first formula.

If in addition $\mathcal{F}$ is a bundle of rank r on $\mathbf{P}^k$, then part c) of Proposition 2.1 shows that $\mathrm{h}^0(\mathcal{F}) = \deg \iota_*(\mathcal{F})$, which is r times the degree of the d-uple embedding of $\mathbf{P}^k$, that is, $\mathrm{h}^0(\mathcal{F}) = rd^k$. Substituting this in the first formula we get the last formulas. (One could also argue directly from the fact that the last formula must be a polynomial of degree k which vanishes at $-nd$ for $n = 1, \ldots, k$.) ■

Corollary 5.3 *Suppose there exists a rank r sheaf on $\mathbf{P}^k$ which is an Ulrich sheaf for the d-uple embedding. If a prime p divides d and p^t divides $k!$, then p^t divides r. For example, any Ulrich sheaf on the $k!$-uple embedding of $\mathbf{P}^k$ has rank a multiple of $k!$.*

Proof. In Theorem 5.1, note that $\chi(\mathcal{F}(1))$ is an integer. ■

The general problem of finding (weakly) Ulrich sheaves for the Veronese embeddings of a given variety can be reduced to problem for projective spaces by using a finite projection map (that is, a finite map such that $\pi^*(\mathcal{O}_{\mathbf{P}^k}(1) = \mathcal{O}_X(1)$—such things always exist by "Noether normalization") onto a projective space. This result and the following Corollary were inspired by the proof of the existence of rank 4 Ulrich sheaves on the 4-uple embedding of $\mathbf{P}^3$ given by Douglas Hanes in his thesis [1999].

Proposition 5.4 *Let $X \subset \mathbf{P}^n$ be a purely k-dimensional scheme, and let $\mathcal{F}$ be an Ulrich sheaf whose support is X. Suppose that $\pi : X \to \mathbf{P}^k$ is a finite projection. If $\mathcal{E}$ is a sheaf on projective space that is (weakly) Ulrich for the d-uple embedding of projective space, then $\mathcal{F} \otimes \pi^*\mathcal{E}$ is (weakly) Ulrich for the d-uple embedding of X.*

Proof. Since the cohomology of $\pi_*\mathcal{F}(n)$ is the same as the cohomology of $\mathcal{F}(n)$, we see from the cohomological characterization of Ulrich sheaves that $\pi_*\mathcal{F}$ is a trivial bundle $\mathcal{O}^t_{\mathbf{P}^k}$ on $\mathbf{P}^k$. Since

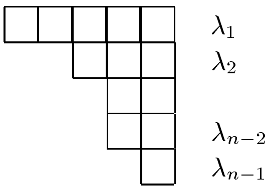
$$\mathrm{H}^q(\mathcal{F} \otimes \pi^*\mathcal{E}(d)) = \mathrm{H}^q(\pi_*\mathcal{F} \otimes \mathcal{E}(d)),$$

this group vanishes for exactly the same values of q, d as does $\mathrm{H}^q(\mathcal{E}(d))$, and this determines the weakly Ulrich and Ulrich properties. ■

If we apply Proposition 5.4 in the case where $X \cong \mathbf{P}^k$, embedded by the e-uple embedding, we get a weak converse to Corollary 5.3.

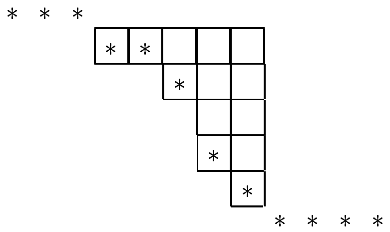
Corollary 5.5 *If $\mathbf{P}^k$ has Ulrich sheaves of ranks a and b on its d-uple and e-uple embeddings respectively, then it has an Ulrich sheaf of rank ab on its de-uple embedding.* ■

If our ground field K has characteric zero then any indecomposable homogenous bundle on $\mathbf{P}^n$ can be obtained by applying a Schur functor S_λ to the universal rank n quotient bundle $Q = \operatorname{coker} \mathcal{O}_{\mathbf{P}^n}(-1) \to \mathcal{O}^{n+1}_{\mathbf{P}^n}$ of $\mathbf{P}^n$ (the tangent bundle tensor $\mathcal{O}_{\mathbf{P}^n}(-1)$). Here $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a partition into at most n parts. Note $(S_\lambda Q)(1) = S_{\lambda+(1,1,\ldots,1)}Q$ and $\mathrm{H}^0(S_\lambda Q) = S_\lambda V$ with $V = H^0(\mathcal{O}(1))^*$. Thus up to twist we may assume that $\lambda_n = 0$. The theorem below implies that $S_\lambda Q$ has Castelnuovo-Mumford regularity precisely zero iff $\lambda_n = 0$. For our purposes it is convenient to visualize the partition as a Ferrers diagram whose row lengths are given by the λ_i, as follows:



The following result was pointed out to us by J. Weyman.

Theorem 5.6 *Suppose that K has characteristic zero. Let $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$ be a partition and Q the universal rank n quotient bundle on $\mathbf{P}^n$. The Tate resolution of the homogeneous bundle $\mathcal{F} = S_\lambda Q$ has nonzero terms only where there are $*$s in the following diagram, in which the Ferrers diagram has shape λ as above:*



More precisely, for $1 \leq i \leq n-1$ the cohomology group $\mathrm{H}^i((S_\lambda Q)(m))$ is nonzero if and only if $\lambda_{n-i+1} < -m - i \leq \lambda_{n-i}$, $\mathrm{H}^0 S_\lambda Q(m) = 0$ iff $m < 0$ and $\mathrm{H}^n S_\lambda Q(m) = 0$ iff $m \geq -n - \lambda_1 - 1$.

Proof. The cohomology of a homogeneous bundle on the homogeneous space $\mathbf{P}^n = \mathrm{GL}(n+1)/\begin{pmatrix} \mathrm{GL}(n) & * \\ 0 & \mathrm{GL}(1) \end{pmatrix}$ is determined by Bott's formula, see Jantzen [1981]. In particular $\mathrm{H}^i S_\lambda Q(m) \neq 0$ at most one i and

$$\mathrm{H}^i S_\lambda Q(m) = 0 \text{ for all } i \Leftrightarrow -m \in \{\lambda_i + n + 1 - i \mid i = 1, \ldots, n\}.$$

Thus the Hilbert polynomial $\chi S_\lambda Q(m)$ has precisely n integral zeroes and the Tate resolution "steps down" precisely at these n values by Lemma 5.2. ∎

Corollary 5.7 *Suppose that K has characteristic zero. The unique indecomposable homogeneous bundle on $\mathbf{P}^n$ that is an Ulrich sheaf for the d-uple embedding is $S_\lambda Q$ with $(\lambda = ((d-1)(n-1), (d-1)(n-2), \ldots, (d-1), 0)$. It has rank $d^{\binom{n}{2}}$.*

Proof. The first statement follows easily from the previous Theorem. The rank of $S_\lambda Q$ is given by the hook formula(see Stanley [1971] or Fulton [1997])

$$\operatorname{rank} S_\lambda Q = \prod_{(i,j)\in\lambda} \frac{n+i-j}{h(i,j)},$$

where $h(i,j)$ denotes the hook length of the $(i,j)^{th}$ box. The largest hook lenght is $h(1,1) = (d-1)(n-1) + (n-2) = d(n-1) - 1$. The denominators of the first row contribute with $\prod_j h(1,j) = (d(n-1)-1)(d(n-1)-2) \cdot \ldots \cdot (d(n-1)-d+1)(d(n-2)-1) \cdot \ldots \cdot 1 = \frac{[d(n-1)]!}{d^{n-1}(n-1)!}$. The numinators give $\frac{[d(n-1)]!}{(n-1)!}$. Thus the first row contributes with d^{n-1} and the total poduct yields $d^{\binom{n-1}{2}+n-1} = d^{\binom{n}{2}}$ by induction. ∎

Chow forms from line bundles on projective spaces

All the classically known formulas (and no new ones) for the resultant of $k+1$ forms of degreed d in $k+1$ variables come from applying these ideas to line bundles on projective spaces. We get Bézout formulas in this way only for binary forms of any degree or linear forms in any number of variables by Corollary 5.7.

On the other hand $\mathcal{L} = \mathcal{O}(j)$ on $\mathbf{P}^k$ gives rise to a 2 term complex, and hence a Stiefel formula for the Chow form of the d-uple image, iff

$$\mathrm{H}^0\mathcal{L}(-H) = 0 \text{ and } \mathrm{H}^k\mathcal{L}(-(k-2)H) = 0,$$

equivalently, iff

$$d-1 \geq j \geq -k + (k-2)d = (k-2)(d-1) - 2.$$

Thus the Chow forms of $\mathbf{P}^1, \mathbf{P}^2, \mathbf{P}^3$ for arbitrary d-uple embeddings, on $\mathbf{P}^4$ for quadrics and cubics, and on $\mathbf{P}^5$ for quadrics, can be written as determinants of maps of vector bundles on the Grassmannian, or as determinantal formulas in the Stiefel coordinates. This is precisely the list of Gel'fand, Kapranov and Zelevinski, [1994] Chap. 13, Prop. 1.6, and the formulas are the same. For instance in the case of three ternary quadrics we have:

Example 5.8 For the 2-uple embedding (quadrics) of $\mathbf{P}^2$ the line bundle $\mathcal{O}_{\mathbf{P}^2}(1)$ is weakly Ulrich, and we see that the Chow form is the determinant of a canonical map on the Grassmannian $\mathbf{G}$

$$\mathcal{O}_{\mathbf{G}}(-1)^6 \to \mathcal{U} \oplus \mathcal{O}_{\mathbf{G}}(-1)^3.$$

The map is easy to calculate, and in Stiefel coordinates it has matrix

$$\begin{pmatrix} a_0 & b_0 & c_0 & [0,1,5] & 0 & [0,1,2] \\ a_1 & b_1 & c_1 & [0,3,5] & [0,3,4] & [0,1,4]-[0,2,3] \\ a_2 & b_2 & c_2 & [0,4,5]-[1,2,5] & [0,3,5] & [0,1,5] \\ a_3 & b_3 & c_3 & 0 & [1,3,4] & [0,3,4] \\ a_4 & b_4 & c_4 & [2,3,5] & [2,3,4]+[1,3,5] & [0,3,5] \\ a_5 & b_5 & c_5 & [2,4,5] & [2,3,5] & 0 \end{pmatrix}.$$

Thus the determinant of this matrix is the resultant of three quadratic forms $d = d_0x^2 + d_1xy + d_2xz + d_3y^2 + d_4yz + d_5z^2$ for $d = a, b, c$ with $(i,j,k)^{\text{th}}$ Plücker coordinate

$$[i,j,k] = det \begin{pmatrix} a_i & b_i & c_i \\ a_j & b_j & c_j \\ a_k & b_k & c_k \end{pmatrix}.$$

Ulrich sheaves on $\mathbf{P}^2$

To get new formulas for resultants, we replace line bundles with vector bundles of higher rank. The Chow forms of these bundles are the desired resultants raised to a power equal to the rank of the bundle. But if the rank the bundle is 2, then its natural symplectic structure allows us to find a polynomial square root by taking a Pfaffian in place of a determinant, so we get formulas for the resultant itself.

Proposition 5.9 *If α is a $(d+1) \times (d-1)$ matrix of linear forms on $\mathbf{P}^2$ whose minors of order $d-1$ generate an ideal of codimension 3 (the generic value), then*

$$\operatorname{coker}\big(\mathcal{O}_{\mathbf{P}^2}(d-2)^{d-1} \xrightarrow{\alpha} \mathcal{O}_{\mathbf{P}^2}(d-1)^{d+1}\big)$$

is an Ulrich sheaf on the d-uple embedding of $\mathbf{P}^2$.

For example, we may take

$$\alpha = \begin{pmatrix} x_0 & x_1 & x_2 & 0 & \dots & 0 \\ 0 & x_0 & x_1 & x_2 & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \\ 0 & \dots & & x_0 & x_1 & x_2 \end{pmatrix}.$$

Proof of Proposition 5.9. Setting $\mathcal{F} = \operatorname{coker}\big(\mathcal{O}_{\mathbf{P}^2}(d-2)^{d-1} \xrightarrow{\alpha} \mathcal{O}_{\mathbf{P}^2}(d-1)^{d+1}\big)$ we see that $\wedge^2\mathcal{F} \cong \mathcal{O}_{\mathbf{P}^2}(3d-3) = \mathcal{O}_{\mathbf{P}^2}(3d) \otimes \omega_{\mathbf{P}^3}$. Since $\mathcal{F}$ is a rank 2 vector bundle,

$$\mathcal{F} = \mathcal{F}^* \otimes \wedge^2\mathcal{F} = \mathcal{F}^* \otimes \mathcal{O}_{\mathbf{P}^2}(3d) \otimes \omega_{\mathbf{P}^3},$$

so, *as a sheaf on the ambient space of the d-uple embedding of* $\mathbf{P}^2$, $\mathcal{F}$ satisfies the duality hypothesis of Corollary 2.3. Further, the given presentation that $\mathcal{F}$ shows that $\mathcal{F}$ is $(d-1)$-regular as a sheaf on $\mathbf{P}^2$, and thus it is 0-regular on the ambient space of the e-uple embedding for any $e \geq d-1$. Thus Corollary 2.3 shows that $\mathcal{F}$ is an Ulrich sheaf on the d-uple embedding. ∎

The betti diagram of the Tate resolution of such a rank 2 sheaf $\mathcal{F}$ is given just after Theorem 5.1. Instead of specifying α, we could define $\mathcal{F}$ by giving the $(d-1) \times 2(d-2)$ matrix β of linear forms over E that occurs at the end of the middle strand of the Tate resolution. For the choice of α above we get

$$\beta = \begin{pmatrix} e_0 & e_1 & 0 & 0 & & \dots & & 0 \\ e_1 & e_2 & e_0 & e_1 & & & & \\ 0 & 0 & e_1 & e_2 & & & & \vdots \\ \vdots & & & & \ddots & \ddots & & 0 \\ & & & & & & e_0 & e_1 \\ 0 & & & \dots & & 0 & e_1 & e_2 \end{pmatrix},$$

and the vector bundle $\mathcal{E}$ has a conic of maximal order jumping lines. One can show by semi-continuity that β can be taken to be any sufficiently general $(d-1) \times 2(d-2)$ matrix of linear forms over E, but unlike for the matrix α, we do not know how to recognize when β is sufficiently general to give rise to a Tate resolution of the right form.

Bundles on $\mathbf{P}^3$

Proposition 5.10 *Suppose $d \geq 2$. There exist rank 2 Ulrich sheaves for the d-uple embedding of $\mathbf{P}^3$ if and only if $d \not\equiv 0 \pmod 3$.*

Proof. By Hartshorne and Hirschowitz [1982] there exist rank 2 vector bundles $\mathcal{F}$ with $c_1 = 0$ and and natural cohomology on $\mathbf{P}^3$ for any given c_2. For $d \not\equiv 0 \pmod 3$ and $c_2 = (d^2-1)/3$ the sheaf $\mathcal{F}(d-2)$ is Ulrich for the d-uple embedding. The converse follows from Corollary 5.3. ∎

Remark 5.11 The bundles $\mathcal{F}$ in the proof of the proposition are called "instanton bundles", see Tikhomirov [1997], because they satisfy the instanton conditions

$$\mathcal{F} \text{ is stable of rank 2, } c_1(\mathcal{F}) = 0 \text{ and } \mathrm{H}^1(\mathcal{F}(-2)) = 0.$$

Equivalently their linear monad $\mathbf{L}(\mathcal{F})$ has shape

$$0 \longrightarrow \mathcal{O}(-1)^{c_2} \longrightarrow \mathcal{O}^{2c_2+2} \longrightarrow \mathcal{O}(1)^{c_2} \longrightarrow 0.$$

Except for the 2-uple embedding, it is an open problem us to find an explicit expression for these rank 2 Ulrich sheaves.

For the 2-uple embedding the rank 2 Ulrich sheaf is essentially unique:

Proposition 5.12 *If $\mathcal{E}$ is the Null-correlation bundle on $\mathbf{P}^3$, then $\mathcal{F} := \mathcal{E}(-2)$ is, up to automorphisms of $\mathbf{P}^3$, the unique rank 2 Ulrich sheaf on the 2-uple embedding on $\mathbf{P}^3$,*

Proof. By Theorem 5.1, $\mathcal{F}$ is an rank 2 Ulrich sheaf if and only if the betti diagram of the Tate resolution of $\mathcal{F}$ has the form

$$\begin{array}{cccccccccccc}
*** & 64 & 35 & 16 & 5 & . & . & . & . & . & . & \dots \\
\dots & . & . & . & . & 1 & . & . & . & . & . & \dots \\
\dots & . & . & . & . & . & 1 & . & . & . & . & \dots \\
\dots & . & . & . & . & . & . & 5 & 16 & 35 & 64 & ***
\end{array}$$

with Hilbert polynomial $\chi\mathcal{F}(t) = \frac{1}{3}(t+2)(t+4)(t+6)$. (Here and henceforward, we replace each zero in a Betti diagram by a "." to improve legibility.) As proved in Okonek-Schneider-Spindler [1980], the Null correlation bundle is determined (up to twist) by its intermediate cohomology $\mathcal{F}$ and the choice of a nondegenerate 2-form, here given by the map in the middle of the Tate resolution. Thus $\mathcal{F}$ must be a twist of the null correlation bundle, the twist is determined by a comparison of Hilbert polynomials. ∎

By Corollary 5.3 there is no rank 2 bundle on $\mathbf{P}^3$ that is an Ulrich sheaf for the 3-uple embedding. Corollary 5.7 gives a homogeneous bundle of rank 9. The following example gives a whole family of rank 3 Ulrich bundles for this case. These bundles give determinantal Bézout formulas for the cube of the resultant of 4 forms of degree 3 in 4 variables.

Example 5.13 *A family of rank 3 vector bundles on $\mathbf{P}^3$ which are Ulrich sheaves for the 3-uple embedding.*

By Theorem 5.1 $\mathcal{F}$ is an Ulrich sheaf for the 3-uple embedding if and only if the betti diagram of its Tate resolution has the form

$$\begin{array}{cccccccccccc}
\dots & 81 & 40 & 14 & & . & . & . & . & . & . & . \\
. & . & . & . & 5 & 4 & . & . & . & . & . & . \\
. & . & . & . & . & . & 4 & 5 & . & . & . & . \\
. & . & . & . & . & . & . & . & 14 & 40 & 81 & \dots
\end{array}$$

Calculation shows that if we take a sufficiently general 5×4 matrix over the exterior algebra in 4 variables, then its Tate resolution has this form. ∎

It follows at once from the definitions that a sheaf on $\mathbf{P}^k$ becomes weakly Ulrich on the d-uple embedding if and only if

$$\begin{aligned}
&h^0\mathcal{F}(-2d) = 0; \\
&h^i\mathcal{F}((-i-2)d) = 0 = h^i\mathcal{F}((-i+1)d) \quad 0 < i < k-1; \text{ and} \\
&h^k\mathcal{F}((-k+1)d) = 0.
\end{aligned}$$

From the form of the cohomology diagram of the "null correlation bundle" on $\mathbf{P}^3$ given in the proof of Proposition 5.12 we see that a twist of this bundle becomes weakly Ulrich on each d-uple embedding, and thus gives a Pfaffian Stiefel formula for the of the resultant of 4 forms in 4 variables of any degree. For any $d \geq 2$ the corresponding 2-term complex on $\mathbf{G}(4, \mathrm{H}^0\mathcal{O}_{\mathbf{P}^3}(d))$ has the form

$$0 \to \mathcal{O}(-1)^b \oplus \mathcal{U}^a \to \mathcal{O}^b \oplus (\Lambda^3\mathcal{U})^a \to 0$$

with $a = d(d^2-4)/3$ and $b = 2d(4d^2-4)/3$.

Bundles on $\mathbf{P}^4$

Example 5.14 The Horrocks-Mumford bundle on $\mathbf{P}^4$ has rank 2 and Tate resolution

...	100	35	4	.	.	.	.	.	.	.	.	.
.	.	2	10	10	5	.	.	.	.	.	.	.
.	.	.	.	.	.	2	.	.	.	.	.	.
.	.	.	.	.	.	.	5	10	10	2	.	.
.	.	.	.	.	.	.	.	.	4	35	100	...

It gives rise to Pfaffian Stiefel formulas for $d = 4, 6, 8$.

Example 5.15 Suppose again that $k = 4$, and take $d = 2$. By Corollary 5.3 any Ulrich sheaf on the 2-uple embedding of $\mathbf{P}^4$ has rank divisible by 8. Consider a general map $E^3 \to E^5(-2)$. Its Tate resolution is

...	128	35	.	.	.	.	.	.
.	.	.	5	.	.	.	.	.
.	.	.	.	3	.	.	.	.
.	.	.	.	.	5	.	.	.
.	.	.	.	.	.	35	128	...

This gives a rank 8 Ulrich sheaf.

6 Surfaces

Throughout this section, X denotes a nonsingular projective surface over K, and we assume that K has characteristic 0. We study Ulrich sheaves on X. We write H for a hyperplane divisor and K for a canonical divisor on X.

In general it is rare to find an Ulrich line bundle on a surface; for example is easy to see that there are none on the d-uple embedding of $\mathbf{P}^2$ when $d > 1$. Thus we turn to rank 2 bundles. By Corollary 2.3, If $\mathcal{F}$ is a rank 2 vector bundle on X such that $c_1\mathcal{F} = 3H + K$ and $\mathcal{F}$ is 0-regular then $\mathcal{F}$ is Ulrich by Corollary 2.3. We can obtain a Pfaffian Bézout expression for the Chow form from $\mathcal{F}$ on X. We will call such a rank 2 bundle a *special rank 2 Ulrich bundle*.

Many surfaces have no rank 2 Ulrich bundles. For example one can see by considering the dimensions of the families that the general surface X of degree $d \geq 16$ in $\mathbf{P}^3$ is not defined by the Pfaffian of a $2d \times 2d$ skew symmetric linear matrix (see Beauville [2000]). Thus such a surface has no special rank 2 Ulrich bundle, and because Pic $X = \mathbf{Z}$ for a general surface, every rank 2 Ulrich sheaf would be special.

We are particularly interested in the case when $\mathcal{F}$ is a blow-up of $\mathbf{P}^2$; then the Chow form is the resultant of some ternary forms with some assigned base points (that is, the vanishing of the Chow form determines when these forms have an extra zero in common.)

Proposition 6.1

a) *Let C be a smooth curve on X of class $3H + K$ and let $\mathcal{L}$ be a line bundle on C with*

$$\deg \mathcal{L} = \frac{1}{2}H.(5H + 3K) + 2\chi\mathcal{O}_X.$$

If $\sigma_0, \sigma_1 \in \mathrm{H}^0(\mathcal{L})$ define a base point free pencil and $\mathrm{H}^1\mathcal{L}(H + K) = 0$ then the bundle $\mathcal{F}$ defined by the "Mukai exact sequence"

$$0 \longrightarrow \mathcal{F}^* \longrightarrow \mathcal{O}_X^2 \xrightarrow{(\sigma_0,\sigma_1)} \mathcal{L} \longrightarrow 0$$

is a special rank 2 Ulrich bundle.

b) *Every special rank 2 Ulrich bundle on X can be obtained from a Mukai sequence as in part a).*

Proof. a): We begin by proving that, under the hypotheses of part a), the map

$$(*) \qquad (\mathrm{H}^0\mathcal{O}(H+K))^2 \xrightarrow{(\sigma_0,\sigma_1)} \mathrm{H}^0\mathcal{L}(H+K)$$

is an isomorphism. Using Riemann-Roch on X and on C, and the given degree of $\mathcal{L}$, we immediately compute $\chi(\mathcal{L}(H+K)) = 2\chi(\mathcal{O}_X(H+K)$ and $\chi(\mathcal{L}(2H+K)) = 2\chi(\mathcal{O}_X(2H+K)$. Our hypothesis that $\mathcal{L}(H+K)$ is nonspecial implies that $\mathcal{L}(2H+K)$ is also nonspecial. With this and the Kodaira vanishing theorem on X, we see that χ is equal to h^0 for all four of these bundles. Thus it suffices to show that the map $(*)$ is injective.

Since $C \sim 3H+K$ there is an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-2H) \longrightarrow \mathcal{O}_X(H+K) \longrightarrow O_C(H+K) \longrightarrow 0,$$

from which we see that the restriction map $\mathrm{H}^0\mathcal{O}_X(H+K) \cong \mathrm{H}^0\mathcal{O}_C(H+K)$ is an injection. By the base point free pencil trick there is a left exact sequence

$$0 \longrightarrow \mathrm{H}^0\mathcal{L}^*(H+K) \longrightarrow (\mathrm{H}^0\mathcal{O}_C(H+K))^2 \xrightarrow{(\sigma_0,\sigma_1)} \mathrm{H}^0\mathcal{L}(H+K),$$

By the adjunction formula $K_C = (3H+2K)\mid_C$, so our hypothesis and Serre duality give $0 = \mathrm{h}^1\mathcal{L}(H+K) = \mathrm{h}^0\mathcal{L}^*(2H+K)$, whence $\mathrm{h}^0\mathcal{L}^*(H+K) = 0$ as well. Thus (*) is an injection.

We can now prove that $\mathcal{F}$ is Ulrich. The Mukai sequence implies that $\wedge^2\mathcal{F} = \mathcal{O}_X(3H+K)$, so by Corollary 2.3 it suffices to show that $\mathcal{F}$ is 0-regular. Twisting the Mukai sequence by $H+K$ and using the preceding result together with Kodaira vanishing, we see that $\mathrm{H}^1(\mathcal{F}^*(H+K)) = \mathrm{H}^2(\mathcal{F}^*(H+K)) = 0$. Serre duality now gives $\mathrm{H}^1(\mathcal{F}(-H)) = 0$ and $\mathrm{H}^0(\mathcal{F}(-H)) = 0$. Since $\wedge^2\mathcal{F} = \mathcal{O}_X(3H+K)$, we have $\mathcal{F}(-H) = \mathcal{F}^*(2H+K)$. By Serre duality $\mathrm{h}^2(\mathcal{F}(-2H)) = \mathrm{h}^0(\mathcal{F}^*(2H+K)) = \mathrm{h}^0(\mathcal{F}(-H)) = 0$, and $\mathcal{F}$ is Ulrich as claimed.

b): Conversely, if $\mathcal{F}$ is a special Ulrich bundle of rank 2, then two general sections τ_0, τ_1 of $\mathcal{F}$ become dependent on a smooth curve C of class $3H+K$. The cokernel of the induced map $0 \to \mathcal{F}^* \to \oplus_1^2\mathcal{O}$ is a line bundle $\mathcal{L}$ on C, generated by global sections, so we obtain the Mukai sequence

$$0 \to \mathcal{F}^* \to \oplus_1^2\mathcal{O} \xrightarrow{(\sigma_0,\sigma_1)} \mathcal{L} \to 0.$$

By Serre duality, $\chi(\mathcal{F}^*(H+K)) = \chi(\mathcal{F}(-H))$, which is 0 since $\mathcal{F}$ is Ulrich. Thus $\chi(\mathcal{L}(H+K)) = 2\chi(O_X(H+K))$. Applying the Riemann-Roch theorems on X and C again, we obtain the desired formula for the degree of $\mathcal{L}$. ∎

Corollary 6.2 *Suppose that the base field k is algebraically closed. If $X \subset \mathbf{P}^r$ is a del Pezzo surface, then X has a special rank 2 Ulrich Bundle. Thus there is a Pfaffian Bézout formula for the resultant of 3 ternary cubics with d basepoints in general position.*

Proof. In this case $K = -H$ and $C \sim 3H+K$ is a canonical curve of genus $g = H^2+1 = r+1$. Any general line bundle of degree

$$\deg \mathcal{L} = \frac{1}{2}H.(5H+3K) + 2\chi\mathcal{O}_X = g+1.$$

defines a nonspecial pencil. Thus we can apply Proposition 6.1 to get a special rank 2 Ulrich bundle on X.

The space of ternary cubics with d general base points has dimension $10-d$, so it suffices to treat the case of seven or fewer points. The linear series of cubics with 7 assigned base points maps the plane two-to-one onto itself, and the condition that three such cubics meet in an extra point is the condition that three lines in the plane meet in a point—a determinantal condition.

For six or fewer assigned base points the resultant is exactly the Chow form of the corresponding del Pezzo surface. ∎

Corollary 6.3 *Let C be a smooth curve of class $3H+K$ and let $\mathcal{L}$ be a line bundle on C such that $\mid \mathcal{L} \mid$ is a base point free pencil of degree $\deg \mathcal{L} = \frac{1}{2}H.(5H+3K)+2\chi\mathcal{O}_X$. The conditions of Proposition 6.1 are satisfied iff $\mid \mathcal{L} \mid$ does not arise as a projection from $\mid \mathcal{O}_C(2H+K) \mid$.*

Proof. To say that $\mid \mathcal{L} \mid$ arises as a projection from $\mid \mathcal{O}_C(2H+K) \mid$ means that of $\mathrm{H}^0(\mathcal{L}^*(2H+K)) \neq 0$. This space is Serre dual to $\mathrm{H}^1(\mathcal{L}(H+K)=0$. ∎

Remark 6.4 Pencils which arise as projections correspond to codimension 2 planes that are $\frac{7}{2}H.(H+K)+K^2-2\chi\mathcal{O}_X-$secant to $C \subset \mathbf{P}H^0\mathcal{O}(2H+K)$. Every component of the variety of such secants has dimension at least

$$\frac{1}{2}H.(H-K)+4\chi\mathcal{O}_X-4-K^2,$$

and we might expect equality. On the other hand the variety of pencils $\mid \mathcal{L} \mid$ has dimension at least

$$\rho(\mathcal{L}) = 2\deg \mathcal{L} - g_C - 2 = \frac{1}{2}H.(H-3K)+4\chi\mathcal{O}_X-1-K^2.$$

Thus we would expect the existence of an $\mathcal{L}$ which is not a projection, and thus of a special rank 2 Ulrich bundle, in case $H.K<3$.

Resultants of ternary forms with base points

Consider $X=\mathbf{P}^2(p_1,\ldots,p_e)$ the blow up of the plane in e distinct points and a very ample divisor class $H=dL-\sum_{i=1}^e E_i$. Here L denotes the class of a line and the E_i the exceptional divisors. The Chow form of X can be interpreted as the resultant of ternary forms of degree d with e assigned base points.

Theorem 6.5 *Let the ground field be infinite. Let $E=\{p_1,\ldots,p_e\}$ be a collection of e distinct points in $\mathbf{P}^2$ and let $X=\mathbf{P}^2(p_1,\ldots,p_e)$ be the blow up of $\mathbf{P}^2$ in these points, embedded by the linear system $|dL-\sum E_i|$. If the homogenous ideal I_E of the points is generated in degree $d-1$ then X has a special rank 2 Ulrich sheaf.*

Proof. Let $\eta : X \to \mathbf{P}^2$ be the blow up. By Proposition 6.1 we have to construct a pencil $|\mathcal{L}|$ of degree $\frac{(d-1)(5d-4)}{2}-e$ on a smooth curve of class $(3d-3)L-2\sum_i E_i$ on X which satisfies $\mathrm{H}^1\mathcal{L}(H+K)=\mathrm{H}^1(\mathcal{L}((d-3)L)=0$. Let $C'=\eta(C) \subset \mathbf{P}^2$ be the plane model. Every pencil on C corresponds to a pencil of adjoint curves of degree a, say, with assigned base points $F=q_1+\ldots+q_f$ on C', that is a pencil $\{\lambda A_0+\mu A_1\} \subset \mathrm{H}^0(\mathbf{P}^2,\mathcal{I}_{E\cup F}(a))$. The pencil of plane curves might have additional base points $G=r_1+\ldots+r_g$ away from C'. We have

$$a^2=e+f+g.$$

In order that $|\mathcal{L}|$ is not a projection from $|2H+K|$ we need $a > 2d-3$. We choose $a = 2d-2$ so that we can deal with the fewest number of additional points F and G. To complete the construction we will choose C' and $\mathcal{L}' = \eta_*\mathcal{L}$ simultaneously.

Take $G = r_1 + \ldots r_g$ as $g = \binom{d}{2}$ general points in the plane disjoint from E. By the Hilbert-Burch theorem, see [Eisenbud, 1995, 20.4], I_G is generated by the $d-1$ minors of a $d \times (d-1)$ matrix $\varphi_1 : \mathcal{O}_{\mathbf{P}^2}(-1)^{d-1} \to \mathcal{O}^d_{\mathbf{P}^2}$ with linear entries, since G impose independent conditions on forms of degree $d-2$. Since I_E and I_G are generated by forms of degree $d-1$ the sheaf $\mathcal{I}_{E\cup G}(2d-2))$ is globally generated. Choose a general pencil

$$A_0, A_1 \in \mathrm{H}^0(\mathbf{P}^2, \mathcal{I}_{E\cup G}(2d-2)).$$

Then in our construction F has to be the scheme defined by the ideal

$$I_F = (A_0, A_1) : I_{E\cup G}$$

and will consist of f simple points disjoint from $E \cup G$ by Bertini's theorem. C' and $\mathcal{L}'$ are then presented as follows: First note that the Hilbert-Burch matrix of $I_{E\cup F} = (A_0, A_1) : I_G$ is the $d \times (d+1)$ matrix $\varphi_2 : \mathcal{O}_{\mathbf{P}^2}(-1)^{d-1} \oplus \mathcal{O}_{\mathbf{P}^2}(-d+1)^2 \to \mathcal{O}^d$ obtained from φ_1 by writing A_0 and A_1 as a linear combination of the generators of I_G, c.f [Peskine-Szpiro, 1974]. Since $I_{2E\cup F} \subset I_{E\cup F}$ we can obtain the equation of C' as a determinant of a matrix φ_3 : $\mathcal{O}_{\mathbf{P}^2}(-1)^{d-1} \oplus \mathcal{O}_{\mathbf{P}^2}(-d)^2 \to \mathcal{O}^{d+1}_{\mathbf{P}^2}$ obtained from φ_2 by adding a column. The transposed matrix twisted gives $\mathcal{L}'$:

$$0 \to \mathcal{O}_{\mathbf{P}^2}(-d+1)^{d+1} \xrightarrow{\varphi_3^t} \mathcal{O}_{\mathbf{P}^2}(-d+2)^{d-1} \oplus \mathcal{O}^2_{\mathbf{P}^2} \longrightarrow \mathcal{L}' \to 0.$$

For a general choice $C' \in |(3d-3)L - 2E - F|$ the curve C' will have only ordinary double points in E. For example we could simply take all entries of degree $d-1$ in the matrix φ_3 general elements in $(I_E)_{d-1}$ and all the linear forms general.

Locally around a point p_i of E the sheaf $\mathcal{L}'$ defined by the sequence above has a stalk $\mathcal{L}_{p_i}$ which is minimally generated by two elements, since A_0 and A_1 intersect transversally at p_i. So $\mathcal{L}' = \eta_*\mathcal{L}$ for some line bundle $\mathcal{L}$ on C.

Since the additional generators of $\sum_m \mathrm{H}^0(\eta_*\mathcal{L}(m))$ are of degree $d-2$ and $\mathrm{H}^0(X, \mathcal{O}_X(H+K) \cong \mathrm{H}^0(\mathbf{P}^2, \mathcal{O}(d-3)$ we see that the desired isomorphism $(*)$ $\mathrm{H}^0(\mathcal{O}_X(H+K))^2 \cong \mathrm{H}^0(\mathcal{L}(H+K))$ from the proof of Proposition 6.1 holds. This completes the proof. ∎

Corollary 6.6 *There exists a Pfaffian Bézout formula for ternary forms of degree d with e assigned base points if the ideal of the points is generated in degree $d-1$.* ∎

Remark 6.7 Our computations suggest that the construction of the rank 2 Ulrich sheaf above, and hence the construction of a Bézout formula for forms with base points, works for a set of points E even under the weaker hypothesis that I_E is generated in degree d. For example, if E consists of $e \leq \binom{d+2}{2} - 6$ general points, there should be plenty of room to arrive at a nodal C' in the construction.

7 Appendix by Jerzy Weyman: $\mathrm{Ext}(\wedge^q U, \wedge^p U)$

In this appendix we will prove part b of Proposition 1.1, and also prove a complementary statement about the higher cohomology. In characteristic 0 these statements follow from the Bott vanishing theorem, but we prove them over a field of arbitrary characteristic:

Theorem 7.1 *Let $\mathbf{G}_l$ be the Grassmannian of codimension l planes in a vector space W with dual $V = W^*$ over a field K of arbitrary characteristic, and let U be the tautological l-sub bundle of $W \times \mathbf{G}_l$. For $0 \le p, q, \le l$ we have*

$$\mathrm{Hom}(\wedge^q U, \wedge^p U) = \begin{cases} 0; & \text{if } p > q \\ \bigwedge^{q-p} V; & \text{otherwise.} \end{cases}$$

Moreover $\mathrm{Ext}^i(\wedge^q U, \wedge^p U) = 0$ for $i > 0$ and all p, q.

Let $GL = GL_K(W)$ be the general linear group. We write Q for the tautological quotient bundle $Q = W/U$ on $\mathbf{G}_l$. If $\lambda = (\lambda_1, \ldots, \lambda_v)$ is a nondecreasing sequence of positive integers (a highest weight for GL) then we write $L_\lambda W$ for the Schur module corresponding to the highest weight λ. We may extend this notation to any nondecreasing sequence of integers λ (dominant integral weight) using the formula $L_\lambda W = L_{\mu'} W \otimes (\bigwedge^v W)^{\otimes \lambda_v}$ where μ' is the partition conjugate to $\mu = (\lambda_1 - \lambda_v, \ldots, \lambda_{v-1} - \lambda_v, 0)$. The proof of Theorem 7.1 rests on the following facts:

Lemma 7.2 *The tensor product $\bigwedge^p U \otimes \bigwedge^q U^*$ has a filtration with the associated graded object*

$$\oplus_{a+b=p-q, 0\le a \le p, 0 \le b \le q, a+b \le l} L_{(1^a, 0^{l-a-b}, (-1)^b)} U.$$

Lemma 7.3 *a) If if $a > 0$ then all cohomology groups of the vector bundles $L_{(1^a, 0^{l-a-b}, (-1)^b)} U$ are zero.*
b) All higher cohomology groups of the bundle $L_{(0^{l-b}, (-1)^b)} U$ are zero and

$$H^0(\mathbf{G}_l, L_{(0^{l-b}, (-1)^b)} U) = \bigwedge^b W^*.$$

Proof of Lemma 7.2 This is a standard fact on good filtrations (see Donkin [1985]) that the tensor product of Schur modules has the filtration with associated graded being a direct sum of Schur modules. The multiplicities of the Schur modules occurring are the same as in characteristic zero, and we can get the result by Littlewood-Richardson rule, using the isomorphism $\bigwedge^q U^* = \bigwedge^{l-q} U \otimes \bigwedge^l U^*$. ∎

Proof of Lemma 7.3 Let $\lambda = (\lambda_1, \ldots, \lambda_v)$ be an v-tuple of integers. Consider the full flag variety and the tautological subbundles U_i of rank i on it. We denote by $\mathcal{L}(\lambda) = \otimes_{1 \le i \le v} (U_i/U_{i-1})^{-\lambda_i}$ the line bundle on the full flag variety GL/B, where B is the Borel subgroup. Then we have

Lemma 7.4 *a) If λ is a dominant integral weight, then the higher cohomology groups of $\mathcal{L}(\lambda)$ vanish and*

$$H^0(GL/B, \mathcal{L}(\lambda)) = L_\lambda W.$$

b) Let us assume that for some i we have $\lambda_i = \lambda_{i-1} + 1$. Then all cohomology groups of $\mathcal{L}(\lambda)$ vanish.

Now part b) of Lemma 7.3 follows from part a) of Lemma 7.4. To prove part a) of Lemma 7.3 we consider the natural projection $\eta : GL/B \to \mathbf{G}_l$. We observe that by Kempf's Vanishing Theorem (see Jantzen [1987]) in the relative setting we have $L_{(1^a,0^{l-a-b},(-1)^b)}U = \eta_*(\mathcal{L}(0^{v-l},1^a,0^{l-a-b},(-1)^b))$ with higher direct images $\mathcal{R}^i\eta_*(\mathcal{L}(0^{v-l},1^a,0^{l-a-b},(-1)^b))$ being zero for $i>0$. Since by lemma 3 b) we know that all cohomology groups of $\mathcal{L}(0^{v-l},1^a,0^{l-a-b},(-1)^b)$ are zero, by the spectral sequence of the composition we are done. ∎

Proof of Lemma 7.4 The part a) is just Kempf's Vanishing Theorem. Part b) follows from the following consideration. Let $P(i)$ be a parabolic subgroup such that the corresponding homogeneous space is a flag variety of flags of dimensions $(1,2,\dots,i-1,i+1,\dots,v-1,v)$. The projection $\rho : GL/B \to GL/P(i)$ allows to identify GL/B with the projectivization $\mathbf{P}(U_{i+1}/U_{i-1})$. The bundle $\mathcal{L}(\lambda)$ is of the form $\rho^*(\mathcal{M})\otimes\mathcal{O}_{\mathbf{P}(U_{i+1}/U_{i-1})}(-1)$ because all the factors in the definition of $\mathcal{L}(\lambda_1,\dots,\lambda_v)$ except of the i-th and $i+1$-st are induced from $GL/P(i)$. Therefore by the Serre's Theorem (in relative setting) and by the projection formula we see that all higher direct images $\mathcal{R}^i\rho_*(\mathcal{L}(\lambda_1,\dots,\lambda_v))$ are zero. This implies part b). ∎

References

C. d'Andrea and A. Dickenstein: Explicit formulas for the multivariate resultant. Preprint, 2000.

B. Angéniol and M. Lejeune-Jalabert: *Calcul différentiel et classes caractéristiques en géométrie algébrique.* Travaux en Cours 38, Hermann, Paris, 1989.

A. Beauville: Determinantal hypersurfaces. Dedicated to William Fulton on the occasion of his 60th birthday. Mich. Math. J. 48 (2000) 39–64.

A. Beilinson: Coherent sheaves on $\mathbf{P}^n$ and problems of linear algebra. Funct. Anal. and its Appl. 12 (1978) 214–216. (Trans. from Funkz. Anal. i. Ego Priloz 12 (1978) 68–69.)

I.N. Bernstein, I.M. Gel'fand and S.I. Gel'fand: Algebraic bundles on $\mathbf{P}^n$ and problems of linear algebra. Funct. Anal. and its Appl. 12 (1978) 212–214. (Trans. from Funkz. Anal. i. Ego Priloz 12 (1978) 66–67.)

E. Bézout: *Théorie générale des équation algébraiques,* Pierres, Paris 1779.

J. Brennan, J. Herzog, and B. Ulrich: Maximally generated Cohen-Macaulay modules. Math. Scand. 61 (1987) 181–203.

R.-O. Buchweitz, D. Eisenbud, and J. Herzog: Cohen-Macaulay modules on quadrics, by In *Singularities, representation of algebras, and vector bundles (Lambrecht, 1985),* Springer-Verlag Lecture Notes in Math. 1273 (1987) 96–116.

R.-O. Buchweitz: Maximal Cohen-Macaulay modules and Tate-cohomolgy over Gorenstein rings, unpublished manuscript, 1987.

R.-O. Buchweitz and F.-O. Schreyer: Complete intersections of two quadrics, hyperelliptic curves and their Clifford Algebras, manuscript, 2002.

A. Cayley: On the theory of elimination. Cambridge and Dublin Mathematical J. 3 (1848) 116–120.

S. Donkin: *Rational representations of algebraic groups. Tensor products and filtration.* Lecture Notes in Math. 1140. Springer-Verlag, Berlin, 1985.

D. Eisenbud: *Commutative Algebra with a View Toward Algebraic Geometry.* Springer Verlag, 1995.

D. Eisenbud: Linear sections of determinantal varieties. American J. Math. 110 (1988) 541–575.

D. Eisenbud and S. Goto: Linear free resolutions and minimal multiplicity. J. Alg. 88 (1984) 89–133.

D. Eisenbud, G. Fløystad and F.-O. Schreyer: Sheaf cohomology and free resolutions over the exterior algebra. Preprint (2000).

W. Fulton: Young tableaux : with applications to representation theory and geometry. London Mathematical Society student texts 35, Cambridge University Press 1997.

I. M. Gelfand, M. Kapranov, and A. Zelevinsky: *Discriminants, resultants, and multidimensional determinants* Birkhäuser, Boston, 1994.

D. Grayson and M. Stillman: Macaulay2.
`http://www.math.uiuc.edu/Macaulay2/` (1993–).

D. Hanes: Special condition on maximal Cohen-Macaulay modules, and applications to the theory of multiplicities. Thesis, Michigan, 2000.

R. Hartshorne: *Algebraic Geometry.* Springer Verlag, 1977.

R. Hartshorne and A. Hirschowitz: Cohomology of the general instanton bundle. Ann. scient. Éc. Norm. Sup. a série 15 (1982) 365–390.

J. Herzog, B. Ulrich, and J. Backelin: Linear maximal Cohen-Macaulay modules over strict complete intersections. J. Pure and Appl. Alg. 71 (1991) 187–202.

G. Horrocks, D. Mumford: A rank 2 vector bundle on $\mathbf{P}^4$ with $15,000$ symmetries. Topology 12 (1973) 63–81.

B. Iversen: *Cohomology of Sheaves.* Springer-Verlag, New York, 1986.

J. C. Jantzen: *Representations of algebraic groups.* Pure and Applied Mathematics, 131, Academic Press, Boston, MA, 1987.

J.-P. Jouanolou: Aspects invariants de l'limination. Adv. Math. 114 (1995) 1–174.

M. Kline: *Mathematical thought from ancient to modern times* Oxford University Press, New York 1972.

F. Knudsen and D. Mumford: The projectivity of the moduli space of stable curves I: Preliminaries on “det” and “div”. Math. Scand. 39 (1976) 19–55.

G.W. Leibniz: *Mathematische Schriften* herausgegeben von C.I. Gerhardt, Letter to l'Hospital 28 April 1693, volume II, pp. 239; Vorwort volume VII, pp. 5. Georg Olms Verlag Hildesheim, New York 1971.

Ch. Okonek, M. Schneider, and H. Spindler: *Vector Bundles on Complex Projective Spaces.* Birkhäuser, Boston 1980.

C. Peskine and L. Szpiro: Liaison des variétiés algébriques I. Invent. Math. 26 (1974) 271–302

R. Stanley: Theory and application of plane partitions. Studies in Appl. Math. 1 (1971) 167–87 and 259–279.

R. G. Swan: K-theory of quadric hypersurfaces. Ann. of Math. 122 (1985) 113–153.

J.J. Sylvester: A method of determining by mere inspection the derivatives from two equations of any degree. Philosophical Magazine XVI (1840) 132–135. Memoir on the dialytic method of elimination. Part I. Philsophical Magazine XXI (1842) 534–539. Reprinted in: *The collected Mathematical Papers of James Joseph Sylvester, Volume I.* Chelsea New York 1973, reprint of the Cambridge 1904 edition.

A. A. Tikhomirov: The main component of the moduli space of mathematical instanton vector bundles on $\mathbf{P}^3$. Journal of the Mathematical Sciences 86 (1997) 3004–3087. (Translation from the Russian of *Itogi Nauki i Tkhniki. Seriya Sovremennaya Matematika i Ee Prilozheniya. Tematischeskie Obzory* Vol 25, Algebraic Geometry-2, 1995.)

B. Ulrich: Gorenstein rings and modules with high numbers of generators. Math. Z. 188 (1984) 23–32.

J. Weyman and A. Zelevinsky: Determinantal formulas for multigraded resultants. J. Alg. Geom. 3 (1994) 569–598.

David Eisenbud, Department of Mathematics, University of California, Berkeley, Berkeley CA 94720,
de@msri.org

Frank-Olaf Schreyer, FB Mathematik, Universität Bayreuth, D-95440 Bayreuth, Germany,
schreyer@btm8x5.mat.uni-bayreuth.de

Jerzy Weyman, Department of Mathematics, Northeastern University, Boston MA 02115,
weyman@neu.edu

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