COMPLEX DIMENSIONS OF SELF-SIMILAR FRACTAL STRINGS AND DIOPHANTINE APPROXIMATION

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Abstract. In [LvF2], the authors develop a theory of complex dimensions of fractal strings (one-dimensional fractal drums). These complex dimensions enter into explicit formulas to describe the geometric features of fractal strings.

In this paper, we discuss the computational aspects of these explicit formulas for self-similar fractal strings. For these strings, the problem of finding the complex dimensions becomes accessible to computation and mathematical experimentation, as well as to detailed theoretical investigation, since it amounts to solving a Dirichlet polynomial equation. We distinguish the lattice case and the nonlattice case, and we show that the set of lattice strings is dense in the set of all self-similar strings. In the lattice case, the Dirichlet polynomial reduces to an ordinary polynomial. In the nonlattice case, finding the complex dimensions becomes a problem of Diophantine approximation. We study in detail the almost periodic structure of the complex dimensions in this case. Our results suggest, in particular, that a nonlattice string is fractal in a countable set of fractal dimensions that is dense in a connected interval. We illustrate our theory by means of several examples.

In the long run, this work is aimed at developing a Diophantine approximation theory of self-similar strings (and sets), both qualitatively and quantitatively.

1. Introduction

In the book [LvF2], we develop a theory of complex dimensions of fractal objects. For fractal strings (one-dimensional drums with fractal boundary, see [L, LP, LM]), the complex dimensions are defined as the poles of the corresponding *geometric zeta function*. This zeta function contains important geometric information about the fractal string.

To be precise, a fractal string $\mathcal{L}$ is a disjoint union of intervals, the length of which form a sequence

$$\mathcal{L} = l_1, l_2, l_3, \ldots, \tag{1.1}$$

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of finite total length $|\mathcal{L}| = \sum_{j=1}^{\infty} l_j$. The geometric zeta function of $\mathcal{L}$ is defined as

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} l_j^s. \tag{1.2}$$

We mention the Minkowski dimension as an example of the geometric information contained in the zeta function. Recall that the *Minkowski dimension* D of the boundary[1] $\partial\mathcal{L}$ of $\mathcal{L}$ is defined as the value $D \in [0,1)$ for which the *volume of the inner ε-tubular neighborhood of $\partial\mathcal{L}$*,

$$V(\varepsilon) = \sum_{j\colon l_j \geq 2\varepsilon} 2\varepsilon + \sum_{j\colon l_j < 2\varepsilon} l_j, \tag{1.3}$$

satisfies $\lim_{\varepsilon\downarrow 0} V(\varepsilon)\varepsilon^{\delta-1} = 0$ or ∞ according to whether $\delta > D$ or $\delta < D$. In other words, $D = 1 - \liminf_{\varepsilon\downarrow 0} \log V(\varepsilon)/\log\varepsilon$. It can be recovered as the abcissa of convergence of $\zeta_{\mathcal{L}}$.

The value $s = D$ is necessarily a singularity of $\zeta_{\mathcal{L}}$, and throughout this theory, we assume that it is a pole and that $\zeta_{\mathcal{L}}$ has a meromorphic continuation to some region

$$W = \{s = \sigma + it\colon \sigma \geq S(t)\}, \tag{1.4}$$

called the *window*, for some Lipschitz continuous function $S\colon \mathbb{R} \to (-\infty, D)$, called the *screen*.

If these conditions, and the hypotheses $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ to be stated in §2 about the growth behaviour of $\zeta_{\mathcal{L}}$, are satisfied, we have the explicit formula[2]

$$V(\varepsilon) = \sum_{\omega} \frac{(2\varepsilon)^{1-\omega}}{\omega(1-\omega)} \operatorname{res}\left(\zeta_{\mathcal{L}}(s); \omega\right) + 2\varepsilon\zeta_{\mathcal{L}}(0) + o(\varepsilon), \tag{1.5}$$

where ω runs through the set $\mathcal{D}_{\mathcal{L}}(W)$ of *complex dimensions* of $\mathcal{L}$ in W, defined as the set of poles of $\zeta_{\mathcal{L}}$ in W:

$$\mathcal{D}_{\mathcal{L}}(W) = \{\omega \in W\colon \zeta_{\mathcal{L}} \text{ has a pole at } \omega\}. \tag{1.6}$$

Since $\omega = D$ is a complex dimension, this formula expresses $V(\varepsilon)$ as a sum of the first term,

$$\frac{(2\varepsilon)^{1-D}}{D(1-D)} \operatorname{res}\left(\zeta_{\mathcal{L}}; D\right), \tag{1.7}$$

of order ε^{1-D}, and the *oscillatory terms* $C\varepsilon^{1-\omega}$, for coefficients $C = C_\omega$. These terms have order $\varepsilon^{1-\operatorname{Re}\omega}$, and exhibit multiplicative oscillations of period $e^{2\pi/\operatorname{Im}\omega}$.

From the explicit formula (1.5), we again recover the Minkowski dimension as the order of vanishing of $V(\varepsilon)$. Moreover, we also find heuristically, and this is fully explained and proved in [LvF2, Theorem 6.12], that $\partial\mathcal{L}$ is Minkowski measurable (i.e., $\lim_{\varepsilon\downarrow 0} V(\varepsilon)\varepsilon^{D-1}$ exists) if and only if every complex dimension except D lies

[1] The boundary $\partial\mathcal{L}$ of $\mathcal{L}$ is simply the topological boundary of the open set defining $\mathcal{L}$; see [LvF2, Chapter 1].

[2] We assume that $0 \in W$ and that the complex dimensions ω are simple poles of $\zeta_{\mathcal{L}}$. See (2.11) for the general formula. Throughout, $\operatorname{res}(g(s); \omega)$ denotes the residue of the meromorphic function $g = g(s)$ at ω.

strictly to the left of the line $\operatorname{Re}\omega = D$. In that case, the Minkowski content of $\partial\mathcal{L}$ is given by

$$\mathcal{M} = \mathcal{M}(D;\mathcal{L}) = \frac{2^{1-D}}{D(1-D)} \operatorname{res}\left(\zeta_{\mathcal{L}}; D\right). \tag{1.8}$$

More generally, a *dimension free region* is a region in the complex plane such that the intersection with the set of complex dimensions $\mathcal{D}_{\mathcal{L}}(W)$ is only D. If there are positive constants C and p such that

$$\left\{\omega \in \mathbb{C}\colon\ \operatorname{Re}\omega \geq D - C(1+|\operatorname{Im}\omega|)^{-p}\right\} \tag{1.9}$$

is a dimension free region, then (1.3) allows us to deduce, by techniques explained in [LvF3], that for every $\delta > 0$,

$$V(\varepsilon) = \mathcal{M}(D;\mathcal{L})\varepsilon^{1-D}\left(1 + O\left(|\log\varepsilon|\right)^{\delta-1/p}\right), \tag{1.10}$$

as $\varepsilon \to 0^+$.

Because of the explicit formula for $V(\varepsilon)$ and the other geometric (and spectral) information contained in the complex dimensions, the problem of finding the complex dimensions of a fractal string, and in general of any fractal object, is of fundamental importance.

For self-similar strings, to be introduced in §3, the complex dimensions are the roots of a Dirichlet polynomial

$$r_1^s + \cdots + r_N^s = 1.$$

Thus, the problem of finding the complex dimensions becomes accessible to numerical computation. We first solve this problem for lattice strings, in which case the associated equation is polynomial. Then the solution for nonlattice strings is found by an approximation procedure. Thus we find that the set of lattice strings is dense in the set of self-similar strings (§4 and §5). It follows that the set of complex dimensions of a nonlattice string has an almost periodic structure, in a very precise sense, amenable to computer experimentation; see, in particular, Theorem 5.6 and the comments following it.

In §6, we discuss the structure of the complex dimensions of a nonlattice string close to the line $\operatorname{Re} s = D$, the dimension of the string. In §7, we derive from this that nonlattice strings have a dimension free region, which is of the form (1.9) when the nonlattice string is badly approximable by lattice strings. In general, the dimension free region is much thinner, with a corresponding weaker form of (1.10).

The algorithms developed in §§6 and 7 are used—along with our theoretical investigations—to approximate and plot the complex dimensions of a variety of nonlattice self-similar strings, as well as to better understand their rich structure, in terms of Diophantine properties of their scaling ratios (or weights); see, e.g., Examples 7.3–7.5 and Figures 7–10.

We end this paper with a theorem and conjectures about the dimensions of fractality of a nonlattice string; see §8.

In closing this introduction, we note that although the theory of complex dimensions of self-similar sets has been developed so far mostly in the one-dimensional case in [LvF1, LvF2] (i.e., in the case of self-similar strings), we expect that once it will have been suitably extended to the higher-dimensional case, the methods developed in the present paper will also apply to that more general situation.

2. Explicit Formulas

For a function $\varphi(t)$ on $(0,\infty)$, we let

$$\widetilde{\varphi}(s)=\int_0^\infty \varphi(t)t^{s-1}\,dt \tag{2.1}$$

be the *Mellin transform of* φ.

We formulate the explicit formula in the framework of [LvF2, Chapter 4]. Let η be a measure on $(0,\infty)$, supported away from 0. The *geometric zeta function* of η is defined by

$$\zeta_\eta(s)=\int_0^\infty t^{-s}\eta(dt). \tag{2.2}$$

When we take $\eta=\sum_{j=1}^\infty \delta_{\{l_j^{-1}\}}$, where $\delta_{\{a\}}$ denotes the Dirac measure concentrated at the point a, we find $\zeta_\eta(s)=\sum_{j=1}^\infty l_j^s$, and we recover the theory of fractal strings.

We assume that ζ_η has a meromorphic continuation to a region W, given by (1.4), for some function $S\colon \mathbb{R}\to\mathbb{R}$, bounded from above and Lipschitz continuous. We denote by $\mathcal{D}_\eta(W)$ the set of *complex dimensions* of η in W, defined as the set of poles of ζ_η in W:

$$\mathcal{D}_\eta(W)=\{\omega\in W\colon \zeta_\eta \text{ has a pole at } \omega\}. \tag{2.3}$$

We assume the following growth conditions:

There exist real constants $\kappa\geq 0$ and $C>0$ and a sequence $\{T_n\}_{n\in\mathbb{Z}}$ of real numbers tending to $\pm\infty$ as $n\to\pm\infty$, with $T_{-n}<0<T_n$ for $n\geq 1$ and $\lim_{n\to+\infty} T_{-n}/T_n=-1$, such that

(H$_1$): For all $n\in\mathbb{Z}$ and all $\sigma\geq S(T_n)$,

$$|\zeta_\eta(\sigma+iT_n)|\leq C\cdot|T_n|^\kappa, \tag{2.4}$$

(H$_2$): For all $t\in\mathbb{R}$, $|t|\geq 1$,

$$|\zeta_\eta(S(t)+it)|\leq C\cdot|t|^\kappa. \tag{2.5}$$

With these conditions, we have the following explicit formula for the distribution η:

Theorem 2.1. *Let η be a generalized fractal string, satisfying hypotheses* (**H**$_1$) *and* (**H**$_2$). *Then, for suitable functions φ,*

$$\int_0^\infty \varphi(t)\,\eta(dt)=\sum_{\omega\in W}\operatorname{res}\left(\widetilde{\varphi}(s)\zeta_\eta(s);\omega\right)+R_\eta(\varphi), \tag{2.6}$$

where ω runs through $\mathcal{D}_\eta(W)$ and the poles of $\widetilde{\varphi}$, and $R_\eta(\varphi)$ is the error term,

$$R_\eta(\varphi)=\frac{1}{2\pi i}\int_S \widetilde{\varphi}(s)\zeta_\eta(s)\,ds. \tag{2.7}$$

It is of order $\sup S-1$, *in a distributional sense.*

See [LvF2, §4.4] for a proof and for a discussion of the class of functions φ to which this formula is applicable. The reader will also find there a discussion of the order of the error term; see, in particular, [LvF2, Theorems 4.12, 4.20 and 4.23].

We discuss two applications, the geometric counting function and the volume of the tubular neighborhood.

2.1. The Geometric Counting Function.

Let

$$N_{\mathcal{L}}(x) = \#\{j : l_j^{-1} \leq x\} \tag{2.8}$$

be the counting function of reciprocal lengths, the *geometric counting function.* More generally,

$$N_\eta(x) = \int_0^x \eta(dt). \tag{2.9}$$

We take $\varphi(t) = 1$ or 0 according to whether $t < x$ and $t > x$, with Mellin transform $\widetilde{\varphi}(s) = x^s/s$, in formula (2.6) to obtain the explicit formula for the geometric counting function:

$$N_\eta(x) = \sum_{\omega \in \mathcal{D}_\eta(W)} \operatorname{res}\left(\frac{x^s \zeta_\eta(s)}{s}; \omega\right) + \{\zeta_\eta(0)\} + R_\eta(x), \tag{2.10}$$

where the term in braces is included if $0 \in W$ but is not a complex dimension of η. The error term $R_\eta(x)$ is of order $x^{\sup S}$.

2.2. The Volume of the Tubular Neighborhood.

For

$$\varphi(t) = \begin{cases} 2\varepsilon & \text{if } t \leq 1/(2\varepsilon), \\ 1/t & \text{if } t > 1/(2\varepsilon), \end{cases}$$

with $\widetilde{\varphi}(s) = (2\varepsilon)^{1-s}/(s(1-s))$, we obtain the volume of the tubular neighborhood

$$V(\varepsilon) = \sum_\omega \operatorname{res}\left(\frac{(2\varepsilon)^{1-s}\zeta_\eta(s)}{s(1-s)}; \omega\right) + \{2\varepsilon\zeta_\eta(0)\} + R_V(\varepsilon), \tag{2.11}$$

where the term in braces is included if $0 \in W$ but is not a complex dimension of η. When the complex dimensions are simple, this reduces to (1.5). The error term in (2.11) is of order $\varepsilon^{1-\sup S}$.

Remark 2.2. There is an interesting analogy between the above formula (or its special case (1.5)) and H. Weyl's formula for the volume of the tubular neighborhoods of a submanifold of $\mathbb{R}^m$. The latter formula is a polynomial in ε whose coefficients are expressed in terms of the Weyl curvature in different (integer) dimensions [BerG, 6.9.8 and Theorem 6.9.9]. In the above explicit formula, $V(\varepsilon)$ is expressed as an expansion in $\varepsilon^{1-\omega}$, where ω ranges over the 'visible' complex dimensions. This suggests that the complex dimensions of fractal strings (and the associated residues[3]) could have a direct geometric interpretation. See [LvF2, Chapter 6] for more information.

[3] or more generally, the Laurent expansions at these poles if they are not simple.

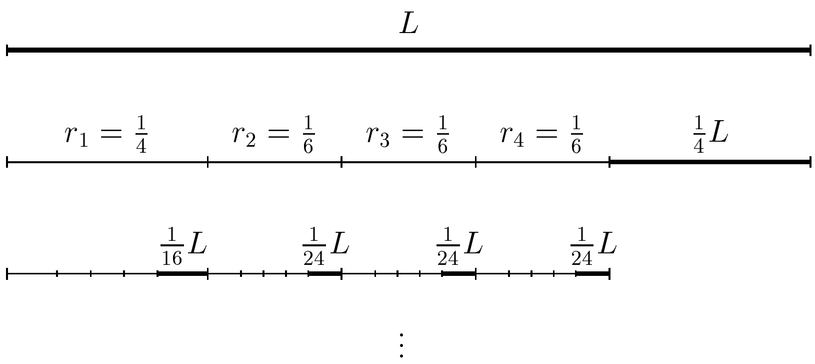

FIGURE 1. An example of a self-similar string (with $N = 4$ and scaling ratios $r_1 = \frac{1}{4}$, $r_2 = r_3 = r_4 = \frac{1}{6}$).

3. Self-Similar Fractal Strings

Given an open interval I of length L, we construct a self-similar fractal string $\mathcal{L}$ with scaling ratios $r_1, r_2, \ldots, r_N$ (see [L, LvF1, 2]). The following construction is reminiscent of the construction of the Cantor set. Let N scaling factors $r_1, r_2, \ldots, r_N$ be given ($N \geq 2$), with

$$1 > r_1 \geq r_2 \geq \ldots \geq r_N > 0. \tag{3.1}$$

Assume that

$$R := \sum_{j=1}^{N} r_j < 1. \tag{3.2}$$

Subdivide I into intervals of length $r_1 L, \ldots, r_N L$ (cf. Figure 1). The remaining piece of length $(1 - R)L$ is the first member of the string, denoted by l_1, also called the first length in Remark 3.4 below. Repeat this process with the remaining intervals, to obtain N new lengths $l_2, \ldots, l_{N+1}$ in the next step, and N^{k-1} new lengths in the k-th step. As a result, we obtain a self-similar fractal string $\mathcal{L}$ consisting of intervals of length

$$L(1 - R) r_{\nu_1} r_{\nu_2} \ldots r_{\nu_k}, \tag{3.3}$$

for all choices of $k \in \mathbb{N}$ and $\nu_1, \ldots, \nu_k \in \{1, \ldots, N\}$.

Remark 3.1. Throughout this paper, *we will always assume that a self-similar string is nontrivial; that is, we exclude the trivial case when $\mathcal{L}$ is composed of a single interval.* This will permit us to avoid having to consider separately this obvious exception to some of our theorems.

Theorem 3.2. *Let $\mathcal{L}$ be a self-similar string, constructed as above with scaling ratios $r_1, \ldots, r_N$. Then the geometric zeta function of this string has a meromorphic continuation to the whole complex plane, given by*

$$\zeta_{\mathcal{L}}(s) = \frac{(L(1-R))^s}{1 - \sum_{j=1}^{N} r_j^s}, \quad \textit{for } s \in \mathbb{C}. \tag{3.4}$$

Here, $L = \zeta_{\mathcal{L}}(1)$ is the total length of $\mathcal{L}$, which is also the length of I, the initial interval from which $\mathcal{L}$ is constructed, and R is given by (3.2).

Proof. Indeed, we have

$$\sum_{\nu_1=1}^{N} \cdots \sum_{\nu_k=1}^{N} (r_{\nu_1} \dots r_{\nu_k})^s = \sum_{\nu_1=1}^{N} \cdots \sum_{\nu_k=1}^{N} r_{\nu_1}^s \dots r_{\nu_k}^s = \left(\sum_{j=1}^{N} r_j^s \right)^k .$$

Let $s = D$ be the unique real solution of $\sum_{j=1}^N r_j^s = 1$. For $\operatorname{Re} s > D$, we have $\left|\sum_{j=1}^N r_j^s\right| < 1$. Hence, in view of (3.3), we deduce that

$$\begin{aligned} \zeta_{\mathcal{L}}(s) &= \sum_{k=0}^{\infty} \left(\sum_{\nu_1=1}^{N} \cdots \sum_{\nu_k=1}^{N} (L(1-R) r_{\nu_1} \dots r_{\nu_k})^s \right) \\ &= (L(1-R))^s \sum_{k=0}^{\infty} \left(\sum_{j=1}^{N} r_j^s \right)^k . \end{aligned}$$

Thus we obtain (3.4) for $\operatorname{Re} s > D$. Since the right-hand side of (3.4) is meromorphic on $\mathbb{C}$, we deduce the theorem. ☐

Corollary 3.3. *The set of complex dimensions $\mathcal{D}_{\mathcal{L}}$ of the self-similar string $\mathcal{L}$ is the set of solutions of the equation*

$$\sum_{j=1}^{N} r_j^{\omega} = 1, \qquad \omega \in \mathbb{C}. \tag{3.5}$$

Moreover, the multiplicity of the complex dimension ω in $\mathcal{D}_{\mathcal{L}}$ (that is, the multiplicity of ω as a pole of $\zeta_{\mathcal{L}}$) is equal to the multiplicity of ω as a solution of (3.5).

Remark 3.4. For a self-similar string, the total length of $\mathcal{L}$ is also the length of the initial interval I in the above construction. Note that we can always normalize a self-similar string in such a way that the numerator in formula (3.4) equals 1 (equivalently, that the first length of $\mathcal{L}$ is 1), by choosing

$$L = \frac{1}{1-R}. \tag{3.6}$$

This will not affect the complex dimensions of the string.

Remark 3.5. In [LvF3, §2], we found an Euler product for $\zeta_{\mathcal{L}}$, coming from a dynamical system associated with the string. The l_j's correspond to the periodic orbits of this dynamical system, see [LvF3, Remark 2.15].

3.1. **The Lattice and Nonlattice Case.**

Write

$$w_j = -\log r_j \tag{3.7}$$

for $j = 1, \dots, N$. In [LvF3], the numbers w_j are interpreted as the weights of a self-similar flow associated with $\mathcal{L}$. Consider the additive subgroup

$$G = \mathbb{Z} w_1 + \cdots + \mathbb{Z} w_N \tag{3.8}$$

of the group of real numbers.

Definition 3.6. If G is dense in $\mathbb{R}$, then $\mathcal{L}$ is a *nonlattice* self-similar string, and if G is discrete in $\mathbb{R}$, then $\mathcal{L}$ is a *lattice* self-similar string.

In the lattice case, there exist a positive real number w and natural numbers $k_1, \ldots, k_N$ without common factor such that $w_j = k_j w$ for $j = 1, \ldots, N$. The number $r = e^{-w}$ is called the *multiplicative generator* of $\mathcal{L}$. Thus

$$r_j = r^{k_j} \tag{3.9}$$

for $j = 1, \ldots, N$.

Remark 3.7. The dichotomy lattice/nonlattice—also referred to as arithmetic/non-arithmetic in probability theory—comes from renewal theory (see [Fe, Chapter X]) and was used in this and in a related context in, for example, [Lal1–2, Str1–2, L, KiL, LvF1–3] and the relevant references therein.

Remark 3.8. In a suitable sense, the nonlattice case is the generic case. (See [LvF2, §2.3.1] for a more precise discussion.) A key objective of the rest of this paper is to demonstrate the relationship between lattice and nonlattice strings as well as to understand the qualitative and quantitative differences between various nonlattice strings in terms of the Diophantine properties of their scaling ratios (or weights).

3.2. **Examples of Self-Similar Strings.**

3.2.1. *The Cantor String.* The *Cantor string* is the self-similar string CS with total length 3 and scaling factors $r_1 = r_2 = 1/3$. Hence its weights are $w_1 = w_2 = \log 3$. It consists of lengths 3^{-n} with multiplicity 2^n, $n = 0, 1, 2, \ldots$. The geometric zeta function of this string is

$$\zeta_{\mathrm{CS}}(s) = \frac{1}{1 - 2 \cdot 3^{-s}}. \tag{3.10}$$

The complex dimensions are found by solving the equation

$$2 \cdot 3^{-\omega} = 1 \qquad (\omega \in \mathbb{C}). \tag{3.11}$$

We find

$$\mathcal{D}_{\mathrm{CS}} = \{D + in\mathbf{p} \colon n \in \mathbb{Z}\}, \tag{3.12}$$

with $D = \log_3 2$ and $\mathbf{p} = 2\pi/\log 3$. (See Figure 2.) All complex dimensions are simple. Further, the residue at each pole is $1/\log 3$.

3.2.2. *The Fibonacci String.* The *Fibonacci string* is the string Fib with total length 4 and scaling factors $r_1 = 1/2$, $r_2 = 1/4$, weights $w_1 = \log 2$, $w_2 = 2\log 2$ (see [LvF2, Chapter 2]). Its lengths are $1, 1/2, 1/4, \ldots, 1/2^n, \ldots$, with multiplicity respectively $1, 1, 2, \ldots, F_{n+1}, \ldots$, the Fibonacci numbers. Recall that these numbers are defined by the recursive equation $F_{n+1} = F_n + F_{n-1}$, and $F_0 = 0$, $F_1 = 1$.

The geometric zeta function of the Fibonacci string is

$$\zeta_{\mathrm{Fib}}(s) = \frac{1}{1 - 2^{-s} - 4^{-s}}. \tag{3.13}$$

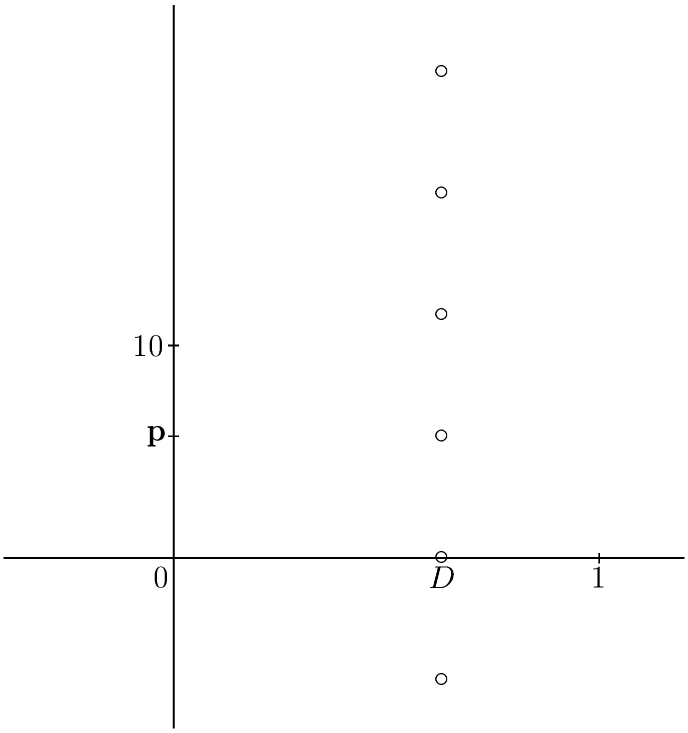

FIGURE 2. A diagram of the complex dimensions of the Cantor string. $D = \log_3 2$ and $\mathbf{p} = 2\pi/\log 3$.

The complex dimensions are found by solving the quadratic equation

$$(2^{-\omega})^2 + 2^{-\omega} = 1 \qquad (\omega \in \mathbb{C}). \tag{3.14}$$

We find the two solutions $2^{-\omega} = \left(-1+\sqrt{5}\right)/2 = \phi^{-1}$ and $2^{-\omega} = -\phi$, where

$$\phi = \frac{1+\sqrt{5}}{2} \tag{3.15}$$

is the golden ratio. Hence

$$\mathcal{D}_{\mathrm{Fib}} = \{D + in\mathbf{p} \colon n \in \mathbb{Z}\} \cup \{-D + i(n+1/2)\mathbf{p} \colon n \in \mathbb{Z}\}, \tag{3.16}$$

with $D = \log_2 \phi$ and $\mathbf{p} = 2\pi/\log 2$. (See Figure 3.) Again, all the poles are simple. Further, for all $n \in \mathbb{Z}$, the residue at the poles $D + in\mathbf{p}$ is $\frac{\phi+2}{5\log 2}$ and the residue at $-D + i(n+1/2)\mathbf{p}$ is $\frac{3-\phi}{5\log 2}$.

3.2.3. *A String with Multiple Poles.* Let $\mathcal{L}$ be the self-similar string with scaling factors $r_1 = r_2 = r_3 = 1/9$ and $r_4 = r_5 = 1/27$ and with total length $L = \frac{27}{16}$ (so that its first length is 1). The weights are $w_1 = w_2 = w_3 = 2\log 3$ and $w_4 = w_5 = 3\log 3$. The geometric zeta function of $\mathcal{L}$ is given by

$$\zeta_{\mathcal{L}}(s) = \frac{1}{1 - 3\cdot 9^{-s} - 2\cdot 27^{-s}}. \tag{3.17}$$

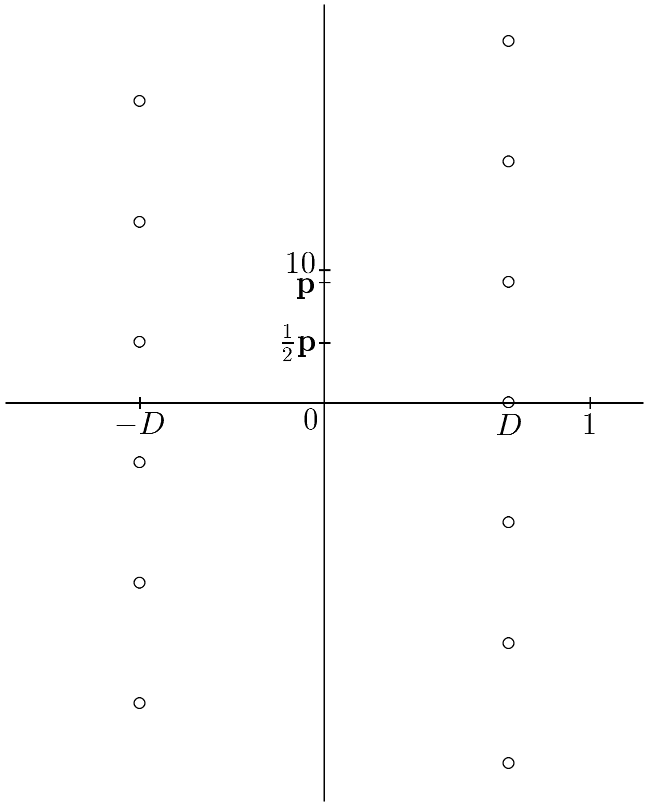

FIGURE 3. The complex dimensions of the Fibonacci string. $D = \log_2 \phi$ and $\mathbf{p} = 2\pi/\log 2$.

The complex dimensions are found by solving the cubic equation

$$2z^3 + 3z^2 = 1, \quad z = 3^{-\omega} \qquad (\text{with } z, \omega \in \mathbb{C}). \tag{3.18}$$

This factors as $(2z-1)(z+1)^2 = 0$. Thus we see that there is one line of simple poles $\omega = D + in\mathbf{p}$ ($D = \log_3 2$ and $\mathbf{p} = 2\pi/\log 3$, $n \in \mathbb{Z}$), with residue $\frac{4}{9\log 3}$, corresponding to the solution $z = 1/2$, and another line[4] of double poles $\omega = \frac{1}{2}i\mathbf{p} + in\mathbf{p}$ ($n \in \mathbb{Z}$) corresponding to the double solution $z = -1$. (See Figure 4.) The Laurent series at a double pole is

$$\frac{1}{3(\log 3)^2}\left(s - \tfrac{1}{2}i\mathbf{p} - in\mathbf{p}\right)^{-2} + \frac{5}{9\log 3}\left(s - \tfrac{1}{2}i\mathbf{p} - in\mathbf{p}\right)^{-1} + O(1),$$

as $s \to \frac{1}{2}i\mathbf{p} + in\mathbf{p}$.

3.2.4. *The 2-3 Nonlattice String.* The above examples are all self-similar lattice strings, as defined in §3.1. The reader may get the mistaken impression that in general, it is easy to find the complex dimensions of a self-similar string. However, in the nonlattice case, it is practically impossible to obtain complete information about the complex dimensions. Nevertheless, in §§4–7, we will obtain some partial information

[4]When we talk about a 'line of poles' in this context, we mean a discrete line, as depicted in Figures 2–4. In the following, for convenience, we will continue using this abuse of language.

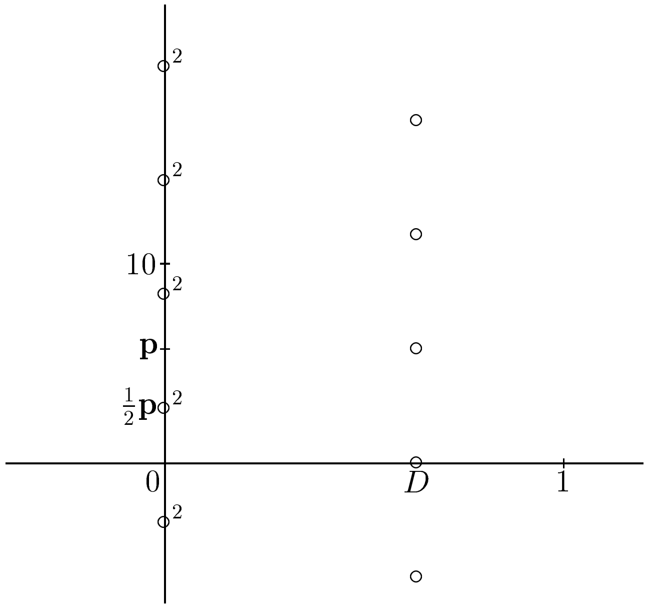

FIGURE 4. The complex dimensions of a string with multiple poles. $D = \log_3 2$ and $\mathbf{p} = 2\pi/\log 3$; here, the symbol $\circ^2$ denotes a multiple pole of order two.

about the location and the density of the complex dimensions of a nonlattice string. (See, in particular, Theorem 4.1 below.)

We take two scaling factors $r_1 = 1/2$, $r_2 = 1/3$. The self-similar string $\mathcal{L}$ with total length 6 and these scaling factors is nonlattice. Note that the weights $w_1 = \log 2$, $w_2 = \log 3$ are not a multiple of a common generator. It consists of lengths $2^{-m}3^{-n}$ with multiplicity the binomial coefficient $\binom{m+n}{m}$ for $m, n = 0, 1, \ldots$. The geometric zeta function of this string is

$$\zeta_{\mathcal{L}}(s) = \frac{1}{1 - 2^{-s} - 3^{-s}}. \tag{3.19}$$

The complex dimensions are found by solving the *transcendental* equation

$$2^{-\omega} + 3^{-\omega} = 1 \qquad (\omega \in \mathbb{C}). \tag{3.20}$$

In particular, we find by numerical approximation that $D \approx .78788\ldots$. See Figure 5 for a diagram of the complex dimensions.

In Example 7.3 below, we discuss the *Golden string,* which is the nonlattice string with scaling ratios $r_1 = 2^{-1}$ and $r_2 = 2^{-\phi}$, where ϕ is the golden ratio of formula (3.15). Its weights are $w_1 = \log 2$, $w_2 = \phi \log 2$.

Remark 3.9. The computations for Figure 5, and every computation for this paper, was done with Maple. The programs can be obtained from the second author.

For Figure 5, we actually computed the roots of $1 - e^{-ws} - e^{-w\alpha s} = 0$, with $w = \log 2$ and α is the rational approximation $485/306$ of $\log 3/\log 2$. This equation reduces to a polynomial equation of degree 485.

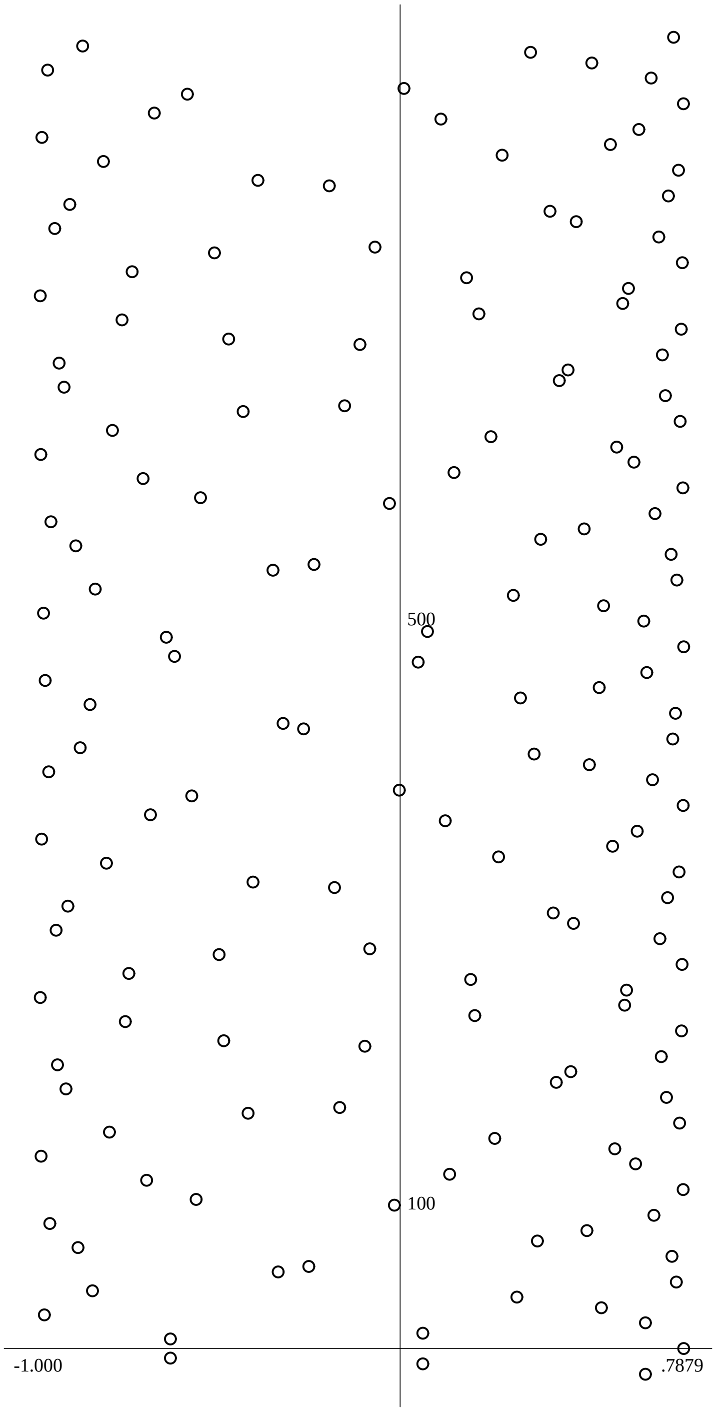

FIGURE 5. The complex dimensions of the nonlattice string with $r_1 = 1/2$ and $r_2 = 1/3$.

4. The Structure of the Complex Dimensions

We introduce a real number D_0 as follows: Let $N_0 \leq N$ be the number of integers j in $1, \ldots, N$ such that $r_j = r_N$, and let $D_0 \in \mathbb{R}$ be defined by

$$1 + \sum_{j=1}^{N-N_0} r_j^{D_0} = N_0 r_N^{D_0}. \tag{4.1}$$

The complex dimensions of a self-similar fractal string are described in the following theorem.

Theorem 4.1. *Let $\mathcal{L}$ be a self-similar string of dimension D and with scaling ratios $1 > r_1 \geq \cdots \geq r_N > 0$. Then the value $s = D$ is the only complex dimension of $\mathcal{L}$ on the real line, and it is simple; further, $D > 0$. The set of complex dimensions in $\mathbb{C}$ of $\mathcal{L}$ is contained in the bounded strip $D_0 \leq \operatorname{Re} s \leq D$:*

$$\mathcal{D}_{\mathcal{L}} = \mathcal{D}_{\mathcal{L}}(\mathbb{C}) \subset \{s \in \mathbb{C}\colon\ D_0 \leq \operatorname{Re} s \leq D\}. \tag{4.2}$$

It is symmetric with respect to the real axis and infinite, with density $w_N/(2\pi)$. More precisely,

$$\#\left(\mathcal{D}_{\mathcal{L}} \cap \{\omega \in \mathbb{C}\colon\ 0 \leq \operatorname{Im} \omega \leq T\}\right) = \frac{w_N}{2\pi} T + O(1), \tag{4.3}$$

as $T \to \infty$. Here, the elements of $\mathcal{D}_{\mathcal{L}}$ are counted according to multiplicity.

In the lattice case, *$\zeta_{\mathcal{L}}(s)$ is a rational function of $r^s = e^{-ws}$, where r is the multiplicative generator of $\mathcal{L}$. So, as a function of s, it is periodic with period $2\pi i/w$. The complex dimensions ω are obtained by finding the complex solutions z of the polynomial equation (of degree k_N)*

$$\sum_{j=1}^{N} z^{k_j} = 1, \quad \text{with } e^{-w\omega} = z. \tag{4.4}$$

Hence there exist finitely many poles $\omega_1 (= D), \omega_2, \ldots, \omega_q$, such that

$$\mathcal{D}_{\mathcal{L}} = \{\omega_u + 2\pi i n/w\colon\ n \in \mathbb{Z}, u = 1, \ldots, q\}. \tag{4.5}$$

In other words, the complex dimensions lie periodically on finitely many vertical lines, and on each line they are separated by $2\pi/w$. The multiplicity of the complex dimensions corresponding to one value of $z = e^{-w\omega}$ is the multiplicity of z as a solution of (4.4).

In the nonlattice case, *D is a simple complex dimension and it is the unique pole of $\zeta_{\mathcal{L}}$ on the line $\operatorname{Re} s = D$. Further, there is an infinite sequence of complex dimensions of $\mathcal{L}$ coming arbitrarily close (from the left) to the line $\operatorname{Re} s = D$. In addition, there exists a screen S to the left of the line $\operatorname{Re} s = D$, such that $\zeta_{\mathcal{L}}$ satisfies $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ with $\kappa = 0$ (see Equations (2.4) and (2.5) above), and the complex dimensions of $\mathcal{L}$ in W are simple. Finally, the complex dimensions of $\mathcal{L}$ can be approximated (via an explicit procedure specified in Theorem 5.6 below) by the complex dimensions of a sequence of lattice strings with larger and larger oscillatory period. Hence the complex dimensions of a nonlattice string have an almost periodic structure.*

Corollary 4.2. *Every self-similar string has infinitely many complex dimensions with positive real part.*

Proof of Theorem 4.1. For a proof, see [LvF2, Theorem 2.13, pp. 37–40]. There, D_0 was not introduced, but its property (4.2) can be deduced by an argument similar to that for D. See also Remark 4.3 below. The density estimate (4.3) gives the asymptotic density of the number of poles of $\zeta_{\mathcal{L}}$, counted with multiplicity. We will prove it here, since the $O(1)$ term improves [LvF2, Theorem 2.22, p. 47].

We need to estimate the winding number of the function $f(s) = 1 - \sum_{j=1}^N r_j^s$ when s runs around a contour $C_1 + C_2 + C_3 + C_4$, where C_1 and C_3 are vertical segments $c_1 - iT \to c_1 + iT$ and $c_3 + iT \to c_3 - iT$, with $c_1 > D$ and $c_3 < D_0$, respectively, and C_2 and C_4 are horizontal line segments $c_1 + iT \to c_3 + iT$ and $c_3 - iT \to c_1 - iT$, with $T > 0$.

For $\operatorname{Re} s = c_1$, $|\sum_{j=1}^N r_j^s| < 1$, so the winding number along C_1 is at most $1/2$. Likewise, for $\operatorname{Re} s = c_3$, we have $|1 - \sum_{j=1}^{N-m} r_j^s| \leq 1 + \sum_{j=1}^{N-m} r_j^{c_3} < m r_N^{c_3}$, so the winding number along C_3 is that of $m r_N^s$, up to at most $1/2$. Hence the winding number along $C_1 + C_3$ is $(T/\pi)\log r_N^{-1} = w_N T/\pi$, up to at most 1.

We will now show that the winding number along $C_2 + C_4$ is bounded, by a classical argument, originally applied to the Riemann zeta function (see [I, p. 69]). Let n be the number of distinct points on C_2 at which $\operatorname{Re} f(s) = 0$. For real values of z, $2\operatorname{Re} f(z+iT) = f(z+iT) + f(z-iT)$. Hence, putting $g(z) = f(z+iT) + f(z-iT)$, we see that n is bounded by the number of zeros of g in a disc containing the interval $[D_0, D]$. We take the disc centered at $D+1$, with radius $D - D_0 + 2$. We have $|g(D+1)| \geq 2f(D+1)$. Furthermore, let M be the maximum of g on the disc with the same center and radius $e \cdot (D - D_0 + 2)$, so $M \leq 2 + 2\sum_{j=1}^N r_j^{D+1-e(D-D_0+2)}$. By [I, Theorem D, p. 49], it follows that $n \leq \log|M/g(D+1)|$. This gives a uniform bound on the winding number over C_2. The winding number over C_4 is estimated in the same manner.

The approximation of a nonlattice string by lattice strings—along with the almost periodic structure of the complex dimensions of a nonlattice string mentioned at the end of the statement of the theorem—is discussed in §5 below; see especially Theorem 5.6 (and the remark following it) that provides more detailed qualitative and quantitative information than in the earlier results obtained in [LvF2, §2.6]. ☐

Remark 4.3. If the weights w_j $(j = 1, \ldots, N)$ are independent over the rationals, then

$$D_0 = \inf\{\operatorname{Re}\omega \colon \omega \text{ is a complex dimension}\}.$$

Otherwise, the infimum of the real parts of the complex dimensions may be larger.

Remark 4.4. For later reference, we point out the following strip that contains the complex dimensions. The weights are listed in increasing order, $w_1 \leq w_2 \leq \cdots \leq w_N$. Recall that the last N_0 weights are equal to w_N, and that $w_{N-N_0} < w_N$ is the largest weight less than w_N. Then the strip

$$\left\{ s \in \mathbb{C} \colon -\frac{\log(N+1)}{w_N - w_{N-N_0}} \leq \operatorname{Re} s \leq \frac{\log 2N}{w_1} \right\} \tag{4.6}$$

contains all the complex dimensions of $\mathcal{L}$.[5]

This can be seen as follows: Every value d for which $r_1^d + \cdots + r_N^d \leq 1/2$ is positive and larger than D. Furthermore, $r_j \leq r_1$, so $r_1^d + \cdots + r_N^d \leq N r_1^d$. The right bound is the value for which $N r_1^d = 1/2$. Similarly, every negative value d for which $N_0 r_N^d - \sum_{j=1}^{N-N_0} r_j^d \geq 2$ is less than D_0. Furthermore, $N_0 \geq 1$ and $r_j^d \leq r_{N-N_0}^d$ for $j = 1, \ldots, N - N_0$ and $d \leq 0$, so $N_0 r_N^d - \sum_{j=1}^{N-N_0} r_j^d \geq N_0 r_N^d - (N - N_0) r_{N-N_0}^d \geq r_N^d - (N-1) r_{N-N_0}^d$. The left bound is the value for which $r_N^d - (N-1) r_{N-N_0}^d = 2 r_{N-N_0}^d$, which is ≥ 2.

Remark 4.5. A self-similar string $\mathcal{L}$ defines a self-similar set in $\mathbb{R}$, namely its boundary $\partial\mathcal{L}$. As is well-known (see, e.g., [Fa, Theorems 9.1, 9.3, pp. 114, 118], the Minkowski dimension D of a self-similar set (satisfying the so-called 'open set condition') coincides with its 'similarity dimension' s [Mn], defined as the unique real solution of equation (3.5).

5. Approximating a Nonlattice String by Lattice Strings

We begin by stating several definitions and a result regarding the convergence of a sequence of fractal strings (not necessarily self-similar) and of the associated complex dimensions. Then we study in more detail the particular situation of self-similar strings.

In the following, it will be helpful to keep in mind that a meromorphic function with values in $\mathbb{C}$ can be viewed as a holomorphic function with values in the Riemann sphere $\mathbb{P}^1(\mathbb{C})$, by defining the value at a pole to be $\infty \in \mathbb{P}^1(\mathbb{C})$. Moreover, $\mathbb{P}^1(\mathbb{C})$ will be equipped with the chordal metric

$$\|a, b\| = \frac{|a - b|}{\sqrt{1 + |a|^2}\sqrt{1 + |b|^2}},$$

for $a, b \in \mathbb{P}^1(\mathbb{C})$.

Definition 5.1. Let $\mathcal{L}$ be a fractal string with window W, and let $\left\{\mathcal{L}^{(n)}\right\}_{n=1}^{\infty}$ be a sequence of fractal strings with windows $W^{(n)}$. We say that the *sequence* $\mathcal{L}^{(n)}$ *converges to* $\mathcal{L}$ (and write $\mathcal{L}^{(n)} \to \mathcal{L}$) if for every compact set $K \subset W$, we have that $K \subset W^{(n)}$ for all sufficiently large n, and $\zeta_{\mathcal{L}^{(n)}}(s) \to \zeta_{\mathcal{L}}(s)$ in the topology of $\mathbb{P}^1(\mathbb{C})$ uniformly on K.

Definition 5.2. Let $\mathcal{L}$ be a fractal string with complex dimensions $\mathcal{D} = \mathcal{D}(W)$ and let $\mathcal{L}^{(n)}$ be a sequence of fractal strings, with complex dimensions $\mathcal{D}^{(n)} = \mathcal{D}(W^{(n)})$. We say that the *complex dimensions of* $\mathcal{L}^{(n)}$ *converge locally to those of* $\mathcal{L}$ (*and write* $\mathcal{D}^{(n)} \to \mathcal{D}$), if for every compact set $K \subset W$ and every $\varepsilon > 0$, there is an integer n_0 such that for all integers $n \geq n_0$, there exists a bijection

$$b_n \colon K \cap \mathcal{D} \to \mathcal{D}^{(n)} \tag{5.1}$$

[5]If $\mathcal{L}$ is a lattice string and all the weights are equal, then there is only one line of complex dimensions, at $\operatorname{Re} s = (\log N)/w_1$. In that case, $N_0 = N$, and we put $w_{N-N_0} = w_0 = 0$.

that respects multiplicities and such that

$$|\omega - b_n(\omega)| < \varepsilon \text{ for all } \omega \in K \cap \mathcal{D}.$$

More precisely, b_n is a set-valued map from $K \cap \mathcal{D}$ to finite subsets of $\mathcal{D}^{(n)}$, such that the multiplicities of the elements of $b_n(\omega)$ add up to the multiplicity of ω and the distance from ω to each of the elements of $b_n(\omega)$ is bounded by ε.

Theorem 5.3. *Let $\mathcal{L}$ be a fractal string and let $\mathcal{L}^{(n)}$ be a sequence of fractal strings. Then* (with the notation of Definitions 5.1 and 5.2), *$\mathcal{L}^{(n)} \to \mathcal{L}$ implies $\mathcal{D}^{(n)} \to \mathcal{D}$.*

For the proof, see [LvF2, Theorem 2.26].

We now focus our attention on the case of self-similar strings. Let scaling ratios $r_1, \ldots, r_N$ be given that generate a nonlattice string $\mathcal{L}$; i.e., by Definition 3.6, the dimension of the $\mathbb{Q}$-vector space generated by the numbers $w_j = -\log r_j$ $(j = 1, \ldots, N)$ is at least 2.

The following lemma on simultaneous Diophantine approximation can be found in [Sch], Theorem 1A and the remark following Theorem 1E.

Lemma 5.4. *Let $w_1, w_2, \ldots, w_N$ be weights* (*see* (3.7)) *such that at least one ratio w_j/w_1 is irrational. Then for every $Q > 1$, there exist integers $1 \leq q < Q^{N-1}$ and $k_1, \ldots, k_N$ such that*

$$|qw_j - k_j w_1| \leq w_1 Q^{-1}$$

for $j = 1, \ldots, N$. In particular, $|qw_j - k_j w_1| < w_1 q^{-1/(N-1)}$ for $j = 1, \ldots, N$.

Remark 5.5. Note that $|qw_j - k_j w_1| \neq 0$ when w_j/w_1 is irrational, so $q \to \infty$ when $Q \to \infty$.

Theorem 5.6. *Let $\mathcal{L}$ be a nonlattice string with weights $w_1, \ldots, w_N$. Let $Q > 1$, and let q and k_j be as in Lemma 5.4. Let $\tilde{w} = w_1/q$. Then the lattice string $\widetilde{\mathcal{L}}$ with weights $\tilde{w}_j = k_j \tilde{w}$ $(j = 1, \ldots, N)$ approximates $\mathcal{L}$ in the sense of Definition 5.1. This approximation is such that the complex dimensions of $\mathcal{L}$ are approximated up to ε for εCQ periods of $\widetilde{\mathcal{L}}$, where*

$$C = \pi^{-1}(2N)^{-1-2w_N/\min\{w_1, w_N - w_{N-N_0}\}}.$$

Remark 5.7. Since the number of periods for which this approximation is valid tends to infinity, this shows that the complex dimensions of a nonlattice string exhibit an almost periodic behaviour. This almost periodic behaviour evolves in the sense that after following a periodic pattern for a certain (large) number of periods, this pattern starts to disappear, and gradually, a new periodic pattern, belonging to the next value of q, begins to emerge. The number of periods for which the pattern belonging to q persists increases with q. Since the denominators q increase exponentially as $Q \to \infty$, every old pattern is only a small fraction (less than one new period) of the new pattern.

Proof of Theorem 5.6. Let $r_j = e^{-w_j}$,

$$f(s) = 1 - \sum_{j=1}^{N} r_j^s, \tag{5.2}$$

and, with $\tilde{r} = e^{-\tilde{w}}$,

$$\tilde{f}(s) = 1 - \sum_{j=1}^{N} \tilde{r}^{k_j s}. \tag{5.3}$$

To show that f is well approximated by $\tilde{f}$, we consider the expression

$$r_j^s - \tilde{r}^{k_j s} = -s \int_{k_j \tilde{w}}^{w_j} e^{-sx}\, dx.$$

Taking absolute values, we obtain

$$\begin{aligned} \left| r_j^s - \tilde{r}^{k_j s} \right| &\le |s| |w_j - \tilde{w}_j| e^{-\sigma w_j} \max\{1, e^{-\sigma(\tilde{w}_j - w_j)}\} \\ &\le |s| \frac{w_1}{qQ} e^{|\sigma|(w_N + w_1/(qQ))} \end{aligned}$$

for $j = 1, \ldots, N$. We simplify this bound further using $w_1/(qQ) < w_N$ in the exponent to find

$$\begin{aligned} \left| f(s) - \tilde{f}(s) \right| &\le \sum_{j=1}^{N} \left| r_j^s - \tilde{r}^{k_j s} \right| \\ &\le |s| \frac{w_1}{qQ} N e^{2 w_N |\sigma|}. \end{aligned}$$

By Remark 4.4, we may restrict $s = \sigma + it$ to $-\frac{\log 2N}{w_N - w_{N-N_0}} \le \sigma \le \frac{\log 2N}{w_1}$. For such s,

$$\left| f(s) - \tilde{f}(s) \right| \le |s| \frac{w_1}{2qQ} (2N)^{1 + 2w_N / \min\{w_1, w_N - w_{N-N_0}\}} = |s| \frac{w_1}{2qQ} (\pi C)^{-1}.$$

Thus, if $|s| < \varepsilon C Q \frac{2\pi q}{w_1}$ then $|f(s) - \tilde{f}(s)| < \varepsilon$. Since $\frac{2\pi q}{w_1}$ is the oscillatory period of $\widetilde{\mathcal{L}}$, the theorem follows. $\square$

Conversely, we have

Theorem 5.8. *Let $\mathcal{L}$ be a self-similar string, with scaling ratios $1 > r_1 \ge \ldots \ge r_N > 0$, and let $\mathcal{L}^{(n)}$ be a sequence of self-similar strings, with scaling ratios $1 > r_1^{(n)} \ge \ldots \ge r_{N^{(n)}}^{(n)} > 0$. Let $W = W^{(n)} = \mathbb{C}$ for all n. If $\mathcal{L}^{(n)} \to \mathcal{L}$, then $N^{(n)} = N$ for all sufficiently large n, and the scaling ratios of $\mathcal{L}^{(n)}$ converge to those of $\mathcal{L}$.*

See [LvF2, Theorem 2.30, p. 51].

6. Complex Dimensions of a Nonlattice String

A nonlattice string has weights $w_1 \le \cdots \le w_N$, where at least one ratio w_j/w_1 is irrational. Let

$$f(s) = 1 - \sum_{j=1}^{N} e^{-w_j s}. \tag{6.1}$$

Recall from Corollary 3.3 and Theorem 4.1 that D is the unique real solution of the equation $f(s)=0$. Moreover, the derivative

$$f'(s)=\sum_{j=1}^{N} w_j e^{-w_j s} \tag{6.2}$$

does not vanish at D. We first consider the case $N=2$.

6.1. **Continued Fractions.** Let α be an irrational real number with a continued fraction expansion $\alpha=[[a_0,a_1,a_2,\dots]]=a_0+1/(a_1+1/(a_2+\dots))$. We recall that the two sequences $a_0,a_1,\dots$ and $\alpha_0,\alpha_1,\dots$ are defined by $\alpha_0=\alpha$ and, for $n\geq 0$, $a_n=[\alpha_n]$, the integer part of α_n, and $\alpha_{n+1}=1/(\alpha_n-a_n)$. The *convergents* of α,

$$\frac{p_n}{q_n}=[[a_0,a_1,a_2,\dots,a_n]], \tag{6.3}$$

are successively computed by

$$\begin{aligned} p_{-2}=0, p_{-1}=1,\ p_{n+1}=a_{n+1}p_n+p_{n-1},\\ q_{-2}=1, q_{-1}=0,\ q_{n+1}=a_{n+1}q_n+q_{n-1}. \end{aligned} \tag{6.4}$$

We also define $q_n'=\alpha_1\cdot\alpha_2\cdot\dots\cdot\alpha_n$, and note the formula $q_{n+1}'=\alpha_{n+1}q_n+q_{n-1}$. Then

$$q_n\alpha-p_n=\frac{(-1)^n}{q_{n+1}'}. \tag{6.5}$$

For all $n\geq 1$, we have $q_n\geq\phi^{n-1}$, where $\phi=(1+\sqrt{5})/2$ is the golden ratio.

Let $n\in\mathbb{N}$ and choose l such that $q_{l+1}>n$. We can successively apply division with remainder to compute (see [O])

$$n=d_lq_l+n_l,\ n_l=d_{l-1}q_{l-1}+n_{l-1},\dots,n_1=d_0q_0,$$

where d_ν is the quotient and $n_\nu<q_\nu$ is the remainder of the division of $n_{\nu+1}$ by q_ν. We set $d_{l+1}=d_{l+2}=\ldots=0$. Then

$$n=\sum_{\nu=0}^{\infty} d_\nu q_\nu. \tag{6.6}$$

We call this the *α-adic expansion* of n. Note that $0\leq d_\nu\leq a_{\nu+1}$ and that if $d_\nu=a_{\nu+1}$, then $d_{\nu-1}=0$. Also $d_0<a_1$. It is not difficult to show that these properties uniquely determine the sequence $d_0,d_1,\dots$ of *α-adic digits* of n.

Lemma 6.1. *Let n be given by (6.6). Let $k\geq 0$ be such that $d_k\neq 0$ and $d_{k-1}=\dots=d_0=0$. Put $m=\sum_{\nu=k}^\infty d_\nu p_\nu$. Then $n\alpha-m$ lies strictly between*

$$\frac{(-1)^k}{q_{k+1}'}\left(d_k-1+\alpha_{k+2}^{-1}\right) \qquad \text{and} \qquad \frac{(-1)^k}{q_{k+1}'}\left(d_k+\alpha_{k+2}^{-1}\right).$$

In particular, $n\alpha-m$ lies strictly between $(-1)^k/q_{k+2}'$ and $(-1)^k/q_k'$.

Proof. We have $n\alpha - m = \sum_{\nu=k}^{\infty} d_\nu(\alpha q_\nu - p_\nu)$. Since $\alpha q_\nu - p_\nu = (-1)^\nu / q'_{\nu+1}$, the terms in this sum are alternately positive and negative, and it follows that $n\alpha - m$ lies between the sum of the odd numbered terms and the sum of the even numbered terms. To bound these sums, we use $d_\nu \leq a_{\nu+1}$. Moreover, $d_k \geq 1$, hence $d_{k+1} \leq a_{k+2} - 1$. It follows that $n\alpha - m$ lies strictly between

$$d_k(\alpha q_k - p_k) + a_{k+3}(\alpha q_{k+2} - p_{k+2}) + a_{k+5}(\alpha q_{k+4} - p_{k+4}) + \dots$$

and

$$d_k(\alpha q_k - p_k) + (a_{k+2} - 1)(\alpha q_{k+1} - p_{k+1}) + a_{k+4}(\alpha q_{k+3} - p_{k+3}) + \dots .$$

Now $a_{\nu+1}(\alpha q_\nu - p_\nu) = (\alpha q_{\nu+1} - p_{\nu+1}) - (\alpha q_{\nu-1} - p_{\nu-1})$, so both sums are telescopic. The first sum evaluates to $d_k(\alpha q_k - p_k) - (\alpha q_{k+1} - p_{k+1}) = (-1)^k (d_k + \alpha_{k+2}^{-1})/q'_{k+1}$. The second sum equals $d_k(\alpha q_k - p_k) - (\alpha q_{k+1} - p_{k+1}) - (\alpha q_k - p_k) = (-1)^k (d_k - 1 + \alpha_{k+2}^{-1})/q'_{k+1}$.

The cruder bounds follow on noting that $1 \leq d_k \leq a_{k+1}$, and using $q'_{k+2} = \alpha_{k+2} q'_{k+1}$ and $a_{k+1} + \alpha_{k+2}^{-1} = \alpha_{k+1}$. □

6.2. **Two Generators.** In the terminology of dynamical systems, this case corresponds to Bernoulli flows, see [LvF3]. Assume that $N = 2$, and let f be defined as in (6.1) with weights w_1 and $w_2 = \alpha w_1$, for some irrational number $\alpha > 1$. We want to study the complex solutions to the equation $f(\omega) = 0$ that lie close to the line $\operatorname{Re} s = D$. First of all, such solutions must have $e^{-w_1\omega}$ close to $e^{-w_1 D}$, so ω will be close to $D + 2\pi i q/w_1$, for an integer q. We write Δ for the difference $\omega - D - 2\pi i q/w_1$, so that

$$\omega = D + \frac{2\pi i q}{w_1} + \Delta.$$

Then we write $\alpha q = p + x(2\pi i)^{-1}$, hence

$$x = 2\pi i(q\alpha - p),$$

for an integer p, which we will specify below. With these substitutions, the equation $f(\omega) = 0$ transforms to

$$1 - e^{-w_1 D} e^{-w_1 \Delta} - e^{-w_2 D} e^{-x} e^{-w_2 \Delta} = 0.$$

This equation defines Δ as a function of x.

Lemma 6.2. *Let $w_1, w_2 > 0$ and $\alpha = w_2/w_1 > 1$; let D be such that $e^{-w_1 D} + e^{-w_2 D} = 1$, and let $\Delta = \Delta(x)$ be the function of x, defined implicitly by*

$$e^{-w_1 D} e^{-w_1 \Delta} + e^{-w_2 D} e^{-x} e^{-w_2 \Delta} = 1, \tag{6.7}$$

and $\Delta(0) = 0$. Then Δ is analytic in x, in a disc of radius at least π around $x = 0$, with power series

$$\Delta(x) = -\frac{e^{-w_2 D}}{f'(D)} x + \frac{w_1^2 e^{-w_1 D} e^{-w_2 D}}{2 f'(D)^3} x^2 + O(x^3), \qquad \text{as } x \to 0.$$

The coefficients in this power series are real. The coefficient of x^2 is positive.

Proof. Write $e^{-w_1\Delta} = y(x)$, so y is defined by $e^{-w_1 D}y + e^{-w_2 D}e^{-x}y^\alpha = 1$ and $y(0) = 1$. Since y does not vanish, it follows that if $y(x)$ is analytic in a disc centered at $x = 0$, then Δ will be analytic in that same disc. Moreover, y is real-valued and positive when x is real, so Δ is real-valued as well when x is real. Further, $y(x)$ is locally analytic in x, with derivative

$$y'(x) = \frac{e^{-w_2 D}y^\alpha e^{-x}}{e^{-w_1 D} + \alpha e^{-w_2 D}y^{\alpha-1}e^{-x}}.$$

Hence there is a singularity at those values of x at which the denominator vanishes, which is at $y = (\alpha/(\alpha-1))e^{w_1 D}$ and $e^{-x} = -\alpha^{-\alpha}(\alpha-1)^{\alpha-1}$. Since this value is negative, the disc of convergence of the power series for $y(x)$ is

$$|x| < |-\alpha\log\alpha + (\alpha-1)\log(\alpha-1) + \pi i|.$$

This is a disc of radius at least π. The first two terms of the power series for $\Delta(x)$ are now readily computed. $\square$

Substituting this in $\omega = D + 2\pi iq/w_1 + \Delta$, we find

$$\omega = D + 2\pi i\frac{q}{w_1} - \frac{e^{-w_2 D}}{f'(D)}\,x + \frac{w_1^2 e^{-w_1 D}e^{-w_2 D}}{2f'(D)^3}\,x^2 + O(x^3), \tag{6.8}$$

as $x = 2\pi i(q\alpha - p) \to 0$. We view this formula as expressing ω as an initial approximation $D + 2\pi iq/w_1$, which is corrected by each additional term in the power series. The first corrective term is in the imaginary direction, as are all the odd ones, and the second corrective term, along with all the even ones, are in the real direction. The second term decreases the real part of ω.

Theorem 6.3. *Let α be irrational with convergents p_ν/q_ν defined by* (6.4). *Let q be a positive integer, and let $q = \sum_{\nu=k}^\infty d_\nu q_\nu$ be the α-adic expansion of q, as in Lemma 6.1. Assume $k \geq 2$, or $k = 1$ and $a_1 \geq 2$, and put $p = \sum_{\nu=k}^\infty d_\nu p_\nu$. Then there exists a complex dimension of $\mathcal{L}$ at*

$$\begin{aligned}\omega = {} & D + 2\pi i\frac{q}{w_1} - 2\pi i\frac{e^{-w_2 D}}{f'(D)}(q\alpha - p) \\ & - 2\pi^2\frac{w_1^2 e^{-w_1 D}e^{-w_2 D}}{f'(D)^3}(q\alpha-p)^2 + O\left((q\alpha-p)^3\right).\end{aligned} \tag{6.9}$$

The imaginary part of this complex dimension is approximately $2\pi iq/w_1$, and its distance to the line $\operatorname{Re} s = D$ is at least $C/q_{k+2}'^2$, where $C = 2\pi^2 w_1^2 e^{-(w_1+w_2)D}/f'(D)^3$ depends only on w_1 and w_2.

Moreover, $|\zeta_{\mathcal{L}}(s)| \ll q_{k+2}'^2$ around $s = D + 2\pi iq/w_1$ on the line $\operatorname{Re} s = D$, and $|\zeta_{\mathcal{L}}(s)|$ reaches a maximum of size $C'(q\alpha - p)^{-2}$, where C' depends only on the weights w_1 and w_2.

Proof. By Lemma 6.1, the quantity $q\alpha - p$ lies between $(-1)^k/q_{k+2}'$ and $(-1)^k/q_k'$. Under the given conditions on k, $q_k' > q_k \geq 2$. Hence $x = 2\pi i(q\alpha - p)$ is less than π in absolute value. Then (6.8) gives the value of ω. The estimate for the distance of ω to the line $\operatorname{Re} s = D$ follows from this formula.

Since the derivative of f is bounded on the line $\operatorname{Re} s = D$, and f does not vanish on this line except at $s = D$, this also implies that $f(s)$ reaches a minimum of order $(q\alpha - p)^2$ on an interval around $s = D + 2\pi iq/w_1$ on the line $\operatorname{Re} s = D$. It follows that $|\zeta_{\mathcal{L}}(s)| \ll q'^2_{k+2}$ on the line $\operatorname{Re} s = D$, with a maximum of order $(q\alpha - p)^{-2}$. $\square$

Remark 6.4. An analogous theorem holds for any complex dimension ω_0 (rather than for D). Indeed, if $f(\omega_0) = 0$, then $f(\omega_0 + 2\pi iq/w_1)$ will be small, and we will find a complex dimension

$$\omega = \omega_0 + 2\pi i \frac{q}{w_1} + O(q\alpha - p)$$

close to $\omega_0 + 2\pi iq/w_1$. This is illustrated in Figures 7, 9 and 10 below for the repetitions of D and one other complex dimension.

We obtain more precise information when $q = q_k$ and ω is close to $D + 2\pi iq_k/w_1$:

Theorem 6.5. *For every $k \geq 0$ (or $k \geq 1$ if $a_1 = 1$), there exists a complex dimension ω of $\mathcal{L}$ of the form*

$$\omega = D + 2\pi i \frac{q_k}{w_1} - 2\pi i(-1)^k \frac{e^{-w_2 D}}{f'(D) q'_{k+1}} - 2\pi^2 w_1^2 \frac{e^{-(w_1+w_2)D}}{f'(D)^3 q'^2_{k+1}} + O\left(q'^{-3}_{k+1}\right), \tag{6.10}$$

as $k \to \infty$.

Moreover, $|\zeta_{\mathcal{L}}(s)| \ll q'^2_{k+1}$ around $s = D + 2\pi iq_k/w_1$ on the line $\operatorname{Re} s = D$, *and $|\zeta_{\mathcal{L}}(s)|$ reaches a maximum of size $C' q'^2_{k+1}$, where C' is as in Theorem 6.3.*

Proof. In this case, $q = q_k$ is the α-adic expansion of q. Put $p = p_k$. Then $x = 2\pi i(-1)^k / q'_{k+1}$, which is less than π in absolute value. The rest of the proof is the same as the proof of Theorem 6.3. $\square$

Remark 6.6. Theorem 6.3 implies that the density of complex dimensions in a small strip around $\operatorname{Re} s = D$ is $w_1/(2\pi)$. For 'Cantor-like' strings, with $w_1 = w_N$ (i.e., such that all weights coincide), there is only one line of complex dimensions, and this density coincides with formula (4.3) of Theorem 4.1. However, it is unclear how wide the strip around $\operatorname{Re} s = D$ should be. For example, in Figure 7, the strip extends to the left of $\operatorname{Re} s = 0$.

Theorem 6.7. *Let*

$$C = \frac{f'(D)^{3/2}}{\pi w_2 \sqrt{2}} e^{(w_1+w_2)D/2}. \tag{6.11}$$

The density function of the real parts of the complex dimensions,

$$\frac{\#\{\operatorname{Re}\omega \colon \omega \in \mathcal{D}, 0 \leq \operatorname{Im}\omega \leq T, \operatorname{Re}\omega \leq x\}}{\#\{\operatorname{Re}\omega \colon \omega \in \mathcal{D}, 0 \leq \operatorname{Im}\omega \leq T\}}, \tag{6.12}$$

is for large T and for values of x close to D asymptotically given by

$$1 - C\sqrt{D - x}, \tag{6.13}$$

as $x \to D$, for $x \leq D$.

Proof. By Theorem 4.1, there are approximately $\frac{w_2}{2\pi}T$ complex dimensions with $0 \leq \operatorname{Im}\omega \leq T$. Given $x < D$, we will count the number of them with $\operatorname{Re}\omega > x$. By Theorem 6.3, for every q with $0 \leq q < \frac{w_1}{2\pi}T$, we find a complex dimension with $0 \leq \operatorname{Im}\omega \leq T$ and real part approximately $D - C_1(q\alpha - p)^2$, where

$$C_1 = 2\pi^2 w_1^2 e^{-(w_1+w_2)D} f'(D)^{-3}.$$

By Lemma 6.1, for $q = \sum_{\nu=k}^{\infty} d_\nu q_\nu$, $|q\alpha - p|$ is roughly equal to d_k/q'_{k+1}, so we need to count the number of q such that (roughly) $d_k/q'_{k+1} < \sqrt{(D-x)/C_1}$. Determine l such that $1/q'_{l+1} < \sqrt{(D-x)/C_1} \leq 1/q'_l$. Thus we want those q with $k \geq l$, and if $k = l$, we want $d_l < q'_{l+1}\sqrt{(D-x)/C_1}$.

We find $k \geq l$ with a frequency of $(a_{l+1}+1)$ times in every q_{l+1} numbers. Indeed, of all q with $0 \leq q < q_{l+1}$, we have $k \geq l$ only for $q = 0, q_l, 2q_l, \dots, a_{l+1}q_l$. Of these multiples of q_l, moreover, we want those where the multiple d_l is at most $q'_{l+1}\sqrt{(D-x)/C_1}$. Since $q'_{l+1} \approx q_{l+1}$, the fraction of such numbers q is $\sqrt{(D-x)/C_1}$. Hence in $\frac{w_1}{2\pi}T$ numbers, we find approximately $\frac{w_1}{2\pi}T\sqrt{(D-x)/C_1}$ values of q for which $\operatorname{Re}\omega > x$. In a total of $\frac{w_2}{2\pi}T$ complex dimensions, this is a density of

$$(w_1/w_2)\sqrt{(D-x)/C_1},$$

which is $C\sqrt{D-x}$, where C is given by (6.11). The density of ω with $\operatorname{Re}\omega \leq x$ is then asymptotically given by $1 - C\sqrt{D-x}$. $\square$

Remark 6.8. This theorem is illustrated in the diagrams at the bottom of Figures 7 and 9. These diagrams show in one figure the graph of the accumulated density function (6.12) and the graph of the function (6.13). The function (6.13) approximates the accumulative density only in a small neighborhood of D. Figure 6 gives a graph of the difference of the two graphs in Figure 7 for the complex dimensions with real parts between $1/2$ and D.

6.3. **More than Two Generators.** In this case, the construction of approximations p_j/q of w_j is much less explicit than for $N = 2$, since there does not exist a continued fraction algorithm for simultaneous approximation. Lemma 5.4 replaces this algorithm. The number Q plays the role of q'_{k+1} in Theorem 6.5 above. In particular, if q is often much smaller than Q, then $w_1, \dots, w_N$ is well approximable by rationals, and we find a small dimension free region.

Remark 6.9. The L^3-algorithm of [LLL] allows one to find good denominators. However, the problem of finding the best denominator is NP-complete [Lag, Theorem C]. See also [RS] and the references therein, in particular [HJLS]. Also, it may be possible to adapt the algorithm in [El] to solve Dirichlet polynomial equations.

Again, we are looking for a solution of $f(\omega) = 0$ close to $D + 2\pi i q/w_1$, where f is defined by (6.1). We write $\omega = D + 2\pi i q/w_1 + \Delta$ and

$$w_j q = w_1 p_j + \frac{w_1}{2\pi i} x_j,$$

for $j = 1, \dots, N$. For $j = 1$, we take $p_1 = q$ and consequently $x_1 = 0$. In general, $x_j = 2\pi i(q w_j/w_1 - p_j)$. Then $f(\omega) = 0$ is equivalent to $\sum_{j=1}^{N} e^{-w_j D} e^{-x_j - w_j \Delta} = 1$.

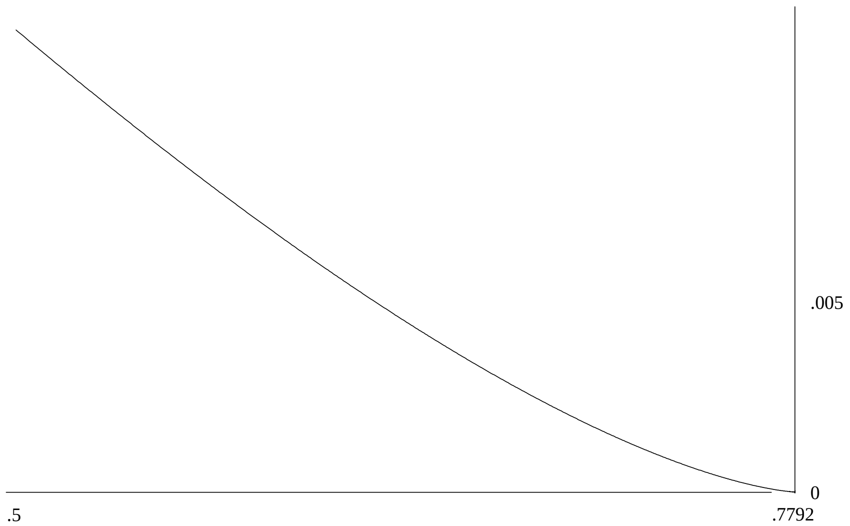

FIGURE 6. The error in the prediction of Theorem 6.7, for the Golden string.

The following lemma is the several variable analogue of Lemma 6.2. However, in this case we do not know the radius of convergence with respect to the variables $x_2, \dots, x_N$.

Lemma 6.10. *Let $w_1 \leq w_2 \leq \dots \leq w_N$, let D be such that $\sum_{j=1}^N e^{-w_j D} = 1$, and let $\Delta = \Delta(x_2, \dots, x_N)$ be implicitly defined by*

$$\sum_{j=1}^N e^{-w_j D} e^{-x_j - w_j \Delta} = 1, \tag{6.14}$$

and $x_1 = 0$. Then Δ is analytic in $x_2, \dots, x_N$, with power series

$$\begin{aligned} \Delta = &- \sum_{j=2}^N \frac{e^{-w_j D}}{f'(D)} x_j + \frac{1}{2} \sum_{j=2}^N \frac{e^{-w_j D}}{f'(D)} x_j^2 \\ &- \frac{1}{2} \sum_{j,k=2}^N \left(\frac{f''(D)}{f'(D)^3} + \frac{w_j + w_k}{f'(D)^2} \right) e^{-(w_j + w_k) D} x_j x_k + O\left(\sum_{j=2}^N |x_j|^3 \right). \end{aligned} \tag{6.15}$$

This power series has real coefficients. Moreover, the terms of degree two yield a positive definite quadratic form.

Proof. The proof is analogous to the proof of Lemma 6.2. The positive definiteness of the quadratic form follows from the fact that the complex dimensions lie to the left of $\operatorname{Re} s = D$, see Theorem 4.1. It can also be verified directly. $\square$

We substitute formula (6.15) into $\omega = D + 2\pi iq/w_1 + \Delta$ to find

$$\begin{aligned}\omega = D + 2\pi i\frac{q}{w_1} - \sum_{j=2}^{N} \frac{e^{-w_jD}}{f'(D)} x_j + \frac{1}{2}\sum_{j=2}^{N} \frac{e^{-w_jD}}{f'(D)} x_j^2 \\ - \frac{1}{2}\sum_{j,k=2}^{N} \left(\frac{f''(D)}{f'(D)^3} + \frac{w_j + w_k}{f'(D)^2}\right) e^{-(w_j+w_k)D} x_j x_k + O\left(\sum_{j=2}^{N} |x_j|^3\right),\end{aligned} \tag{6.16}$$

where $x_j = 2\pi i(qw_j/w_1 - p_j)$. Again, this formula expresses ω as an initial approximation $D + 2\pi iq/w_1$, which is corrected by each term in the power series. The corrective terms of degree one are again in the imaginary direction, as are all the odd degree ones, and the corrective terms of degree two, along with all the even ones, are in the real direction. The degree two terms decrease the real part of ω.

Remark 6.11. As in Remark 6.4, we have a formula analogous to (6.16) corresponding to any complex dimension ω_0. Thus every complex dimension ω_0 gives rise to a sequence of complex dimensions close to the points $\omega_0 + 2\pi iq/w_1$.

Theorem 6.12. *Let $N \geq 2$ and let $w_1, \dots, w_N$ be weights of a nonlattice string. Let Q and q be as in Lemma 5.4. Then $\mathcal{L}$ has a complex dimension close to $D + 2\pi iq/w_1$ at a distance of at most $O(Q^{-2})$ from the line* $\operatorname{Re} s = D$, *as $Q \to \infty$. The function $|\zeta_{\mathcal{L}}|$ reaches a maximum of order Q^2 on the line* $\operatorname{Re} s = D$ *around the point $s = D + 2\pi iq/w_1$.*

Proof. Again, the numbers x_j are purely imaginary, so the corrective terms of degree 1 (and of every odd degree) give a correction in the imaginary direction, and only the corrective terms of even degree will give a correction in the real direction. Since $|x_j| < 2\pi/Q$, the theorem follows. □

Remark 6.13. In analogy with Theorem 6.7, we find $1 - C(D - x)^{(N-1)/2}$ as an approximation of the density function of the complex dimensions close to $x = D$, for some positive constant C. However, in this case we do not know the value of C. It may depend on properties of Diophantine approximation of the weights $w_1, \dots, w_N$.

Figure 10 suggests that the accumulated density function of the 'Two-Three-Five string' is approximated by $1 - C(D - x)$, for some positive constant C.

7. Existence of Dimension Free Regions

Definition 7.1. An open domain in the complex plane is a *dimension free region* for the string $\mathcal{L}$ if it contains the line $\operatorname{Re} s = D$ and the only pole of $\zeta_{\mathcal{L}}$ in that domain is $s = D$.

From Theorem 6.5, and with the notation of §6.1 and §6.2, we deduce in the case $N = 2$,

Corollary 7.2. *Assume that the partial quotients $a_0, a_1, \dots$ of the continued fraction of α are bounded by M. Put $B = \pi^4 e^{-(w_1+w_2)D}/(2f'(D)^3)$. Then $\mathcal{L}$ has a dimension*

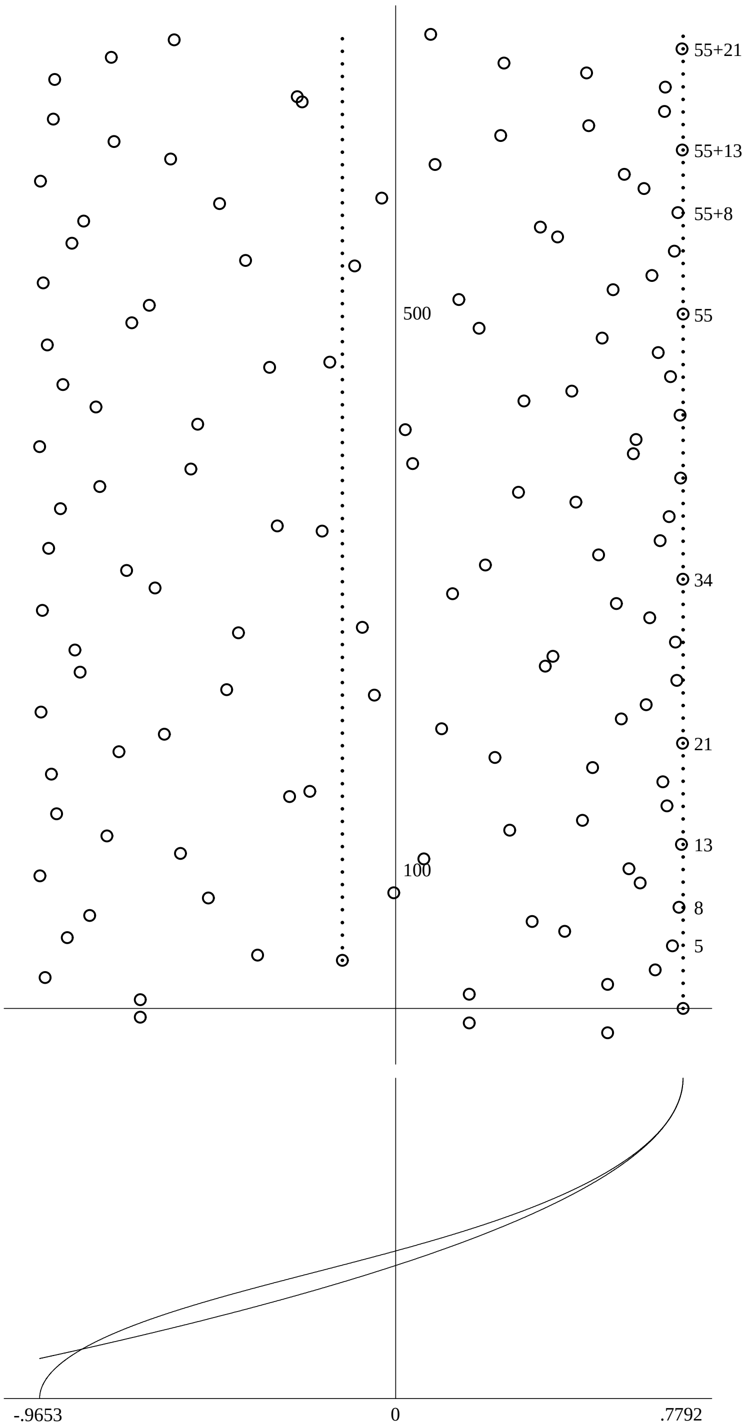

FIGURE 7. The complex dimensions of the Golden string; the accumulative density of their real parts, compared with the theoretical prediction of Theorem 6.7.

free region of the form

$$\left\{\sigma + it \in \mathbb{C}\colon \sigma > D - \frac{B}{M^2t^2}\right\}. \tag{7.1}$$

The function $\zeta_{\mathcal{L}}$ satisfies hypotheses $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *with $\kappa = 2$; see* (2.4) *and* (2.5).

More generally, let $a\colon \mathbb{R}^+ \to [1,\infty)$ be a function such that the partial quotients $\{a_k\}_{k=0}^{\infty}$ of the continued fraction of α satisfy $a_{k+1} \le a(q_k)$ for every $k \ge 0$. Then $\mathcal{L}$ has a dimension free region of the form

$$\left\{\sigma + it \in \mathbb{C}\colon \sigma > D - \frac{B}{t^2a^2(tw_1/(2\pi))}\right\}. \tag{7.2}$$

If $a(q)$ grows at most polynomially, then $\zeta_{\mathcal{L}}$ satisfies hypotheses $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *with κ such that $t^\kappa \ge t^2a^2(tw_1/(2\pi))$.*

Proof. We have $q'_{k+1} = \alpha_{k+1}q'_k \le 2a(q_k)q'_k \le 4a(q_k)q_k$. By Theorem 6.5, the complex dimension close to $D + it$ for $t = 2\pi q_k/w_1$, is located at $D + i(t + O(q'^{-1}_{k+1})) - (w_1^2/\pi^2)Bq'^{-2}_{k+1} + O(q'^{-4}_{k+1})$, where the big-$O$'s denote real-valued functions. The real part of this complex dimension is less than $D - Bt^{-2}a^{-2}(tw_1/(2\pi))$. □

Example 7.3. One of the simplest nonlattice strings is the *Golden string,* introduced in [LvF2, p. 33]. It is the nonlattice string with $N = 2$ and $\alpha = \phi$, $w_1 = \log 2$. The continued fraction of ϕ is $[[1,1,1,\dots]]$; hence $q_0, q_1, \dots$ is the sequence of Fibonacci numbers $1,1,2,3,5,8,13,\dots$, and $q'_k = \phi^k$.

Numerically, we find $D \approx .7792119034$ and the following approximation to the power series $\Delta(x)$:

$$-\,.47862\,x + .08812\,x^2 + .00450\,x^3 - .00205\,x^4 - .00039\,x^5 + .00004\,x^6 + \dots.$$

For every k, we find a complex dimension close to $D + 2\pi i q_k/\log 2$. For example, $q_9 = 55$, and we find a complex dimension at $D - .00023 + 498.58i$. More generally, for numbers that can be written as a sum of Fibonacci numbers of relatively high index, like $q = 55+5$ or $q = 55-5 = 34+13+3$, we find a complex dimension close to $D + 2\pi i q/\log 2$. For $q = 60$ and $q = 50$, we find a complex dimension respectively at $D - .023561 + 543.63i$ and at $D - .033919 + 453.53i$. In both these cases, the distance to the line $\operatorname{Re} s = D$ is comparable to the distance of the complex dimension close to $D + 2\pi i \cdot (5/\log 2)$ to this line, which is located at $D - .028499 + 45.05i$. See Figure 7, where the markers indicate the Fibonacci numbers, $1,1,2,3,5,8,13,21,34,\dots$. The pattern persists for other complex dimensions as well. Indeed, every complex dimension repeats itself according to the pattern of the Fibonacci numbers. This is illustrated for one other complex dimension.

The second graph, at the bottom of Figure 7, shows the accumulative density of the real parts of the complex dimensions, compared with the predicted density (graphed for $D_0 \le x \le D$) of Theorem 6.7.

Figure 7 was obtained by approximating ϕ by $4181/2584$. Thus, the figure shows the complex dimensions of the lattice string with generator $w = (\log 2)/2584$, $k_1 = 2584$, $k_2 = 4181$. The oscillatory period of this string is $2\pi/w \approx 23423.23721$. Figure 8 gives intermediate stages of this approximation.

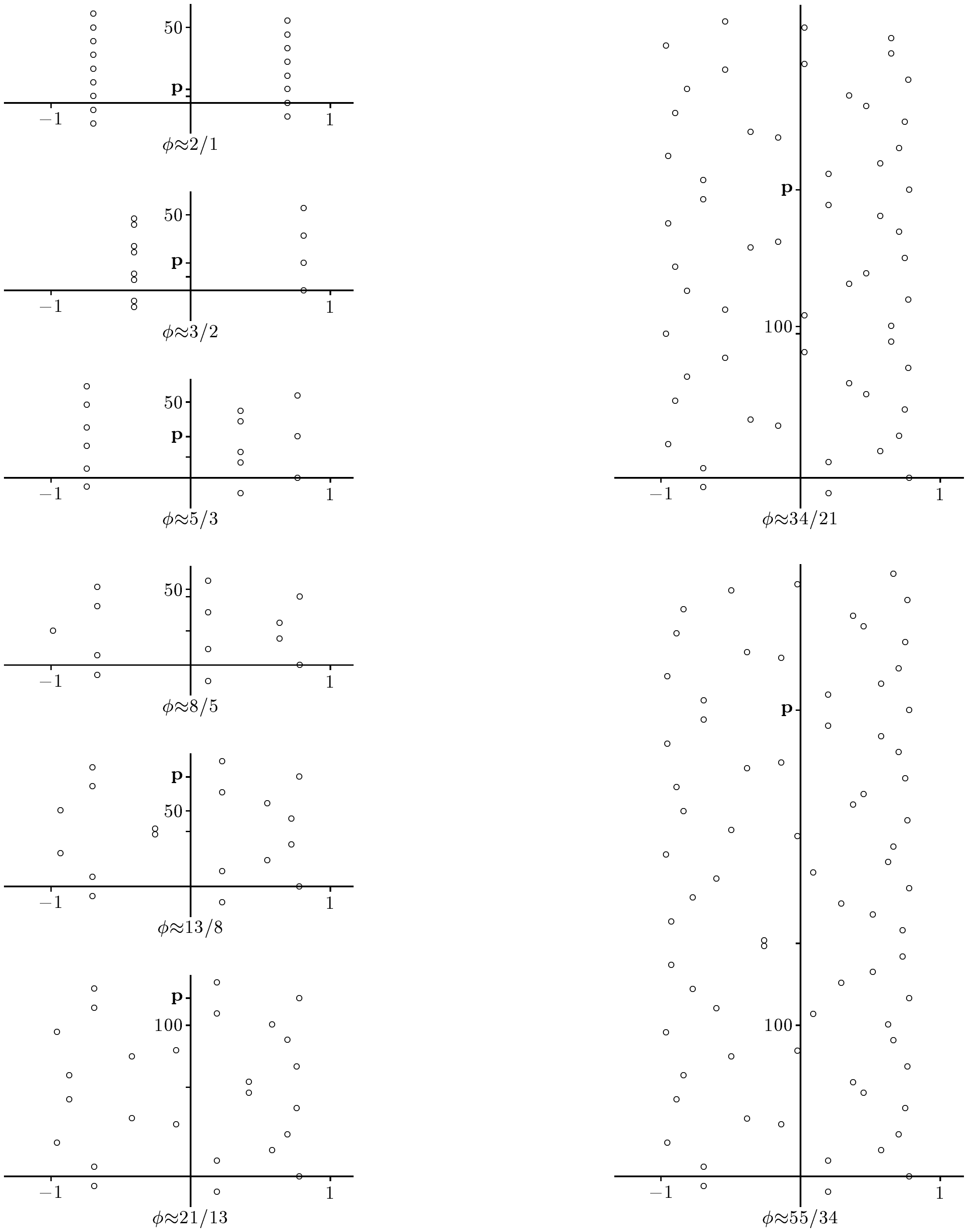

FIGURE 8. Approximations to the complex dimensions of the Golden string. Emergence of the almost periodic pattern.

Example 7.4. Figure 9 gives the complex dimensions and the density of their real parts of the string with weights $w_1 = \log 2$, $w_2 = \alpha \log 2$, where α is the positive real number with continued fraction $[[1, 2, 3, 4, \ldots]]$. One can compute that

$$\alpha = \frac{\sum_{n=0}^{\infty} \frac{1}{(n!)^2}}{\sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}} = \frac{I_0(2)}{I_1(2)}, \tag{7.3}$$

where $I_k(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{2n+k}}{n!(n+k)!}$ is the modified Bessel function. Indeed, by [WW, pp. 359, 373], we have the recursion relation

$$\frac{I_{n-1}(z)}{I_n(z)} = \frac{2n}{z} + \frac{1}{\frac{I_n(z)}{I_{n+1}(z)}},$$

for $n = 1, 2, \ldots$. Thus $I_0(z)/I_1(z) = [[2/z, 4/z, 6/z, \ldots]]$ and so α is given by Equation (7.3), as claimed. (See also [W].) For this reason, we propose to call this string the *Bessel string*. Even though α is better approximable by rationals than numbers with bounded partial quotients, qualitatively there does not seem to be a significant difference with the Golden string.

Figure 9 is obtained by approximating α by $1393/972$. The markers illustrate again the different periodic patterns. The repetitions occur at denominators of approximations of α, which are the numbers $1, 2, 7, 30, 157, \ldots$, and combinations of these (in the sense of the α-adic expansion of Equation (6.6)).

Example 7.5. The Two-Three-Five string is an example with $N > 2$. It has $N = 3$ scaling ratios $r_1 = 1/2, r_2 = 1/3, r_3 = 1/5$. See Figure 10 for a diagram of the complex dimensions and of the density of their real parts. Figure 10 is obtained using the approximation $\log_2 3 \approx 2826/1783, \log_2 5 \approx 4140/1783$.

In the following corollary, we use the expression $w_1, \ldots, w_N$ ($N \geq 2$) is 'a-approximable' if $a\colon [1, \infty) \to \mathbb{R}^+$ is an increasing function such that for every $q \geq 1$, $j = 1, \ldots, N$ and integers p_j,

$$|qw_j - p_j w_1| \geq \frac{w_1}{a(q)} q^{-1/(N-1)}.$$

From Theorem 6.12, we deduce

Corollary 7.6. *Let $N \geq 2$. The best dimension free region that $\mathcal{L}$ can have is of size*

$$\left\{\sigma + it\colon \sigma \geq D - O\left(t^{-2/(N-1)}\right)\right\}. \tag{7.4}$$

The implied constant depends only on $w_1, \ldots, w_N$.

If $w_1, \ldots, w_N$ is 'a-approximable', then the dimension free region has the form

$$\left\{\sigma + it\colon \sigma \geq D - O\left(a^{-2}(w_1 t/(2\pi)) t^{-2/(N-1)}\right)\right\}. \tag{7.5}$$

Remark 7.7. If $a(q)$ grows more than polynomially, we obtain a bound of the form $x^D/a^{\mathrm{inv}}(\log x)$ for the error in the explicit formula for $V(\varepsilon)$; see (2.11). Here, a^{inv} is the inverse function of a.

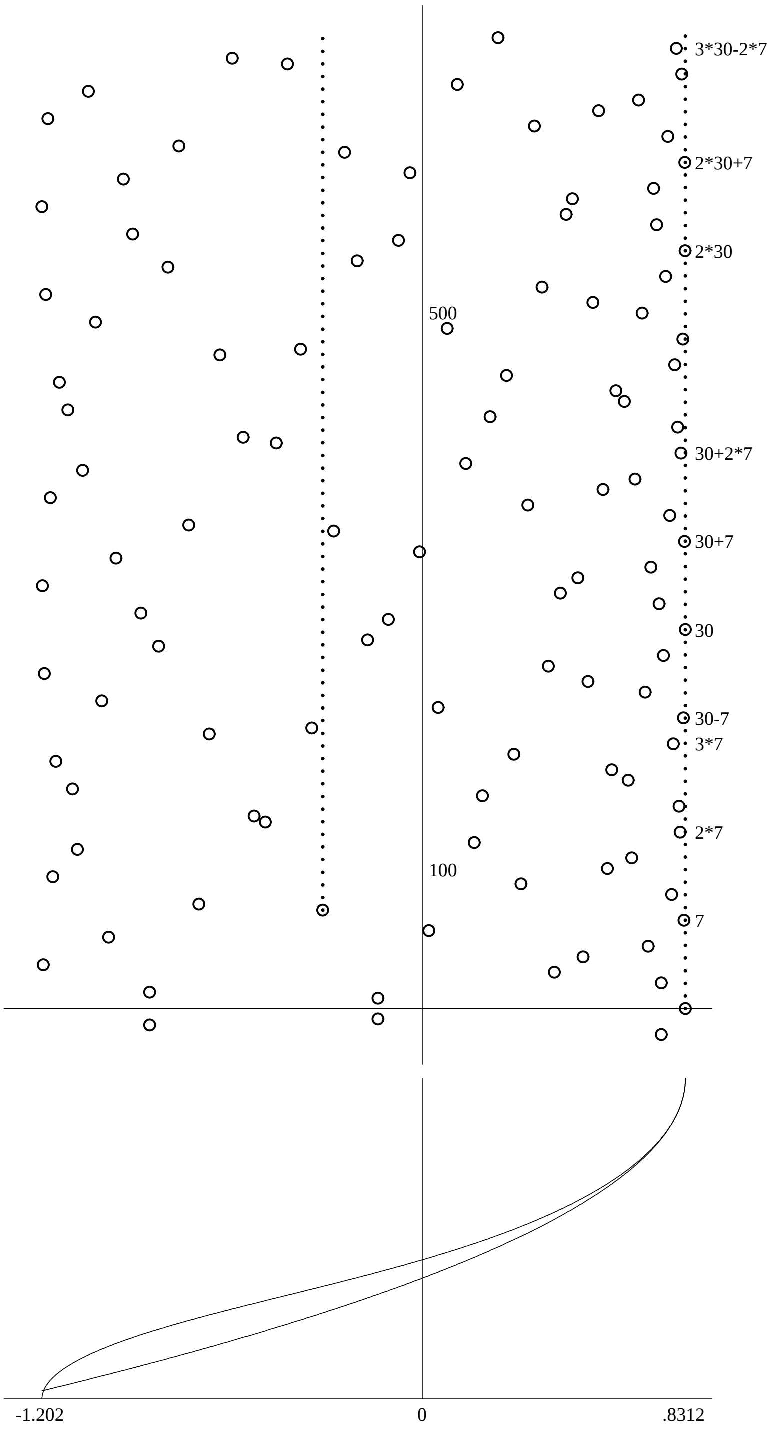

FIGURE 9. The complex dimensions of the Bessel string; the accumulative density of their real parts, compared with the theoretical prediction of Theorem 6.7.

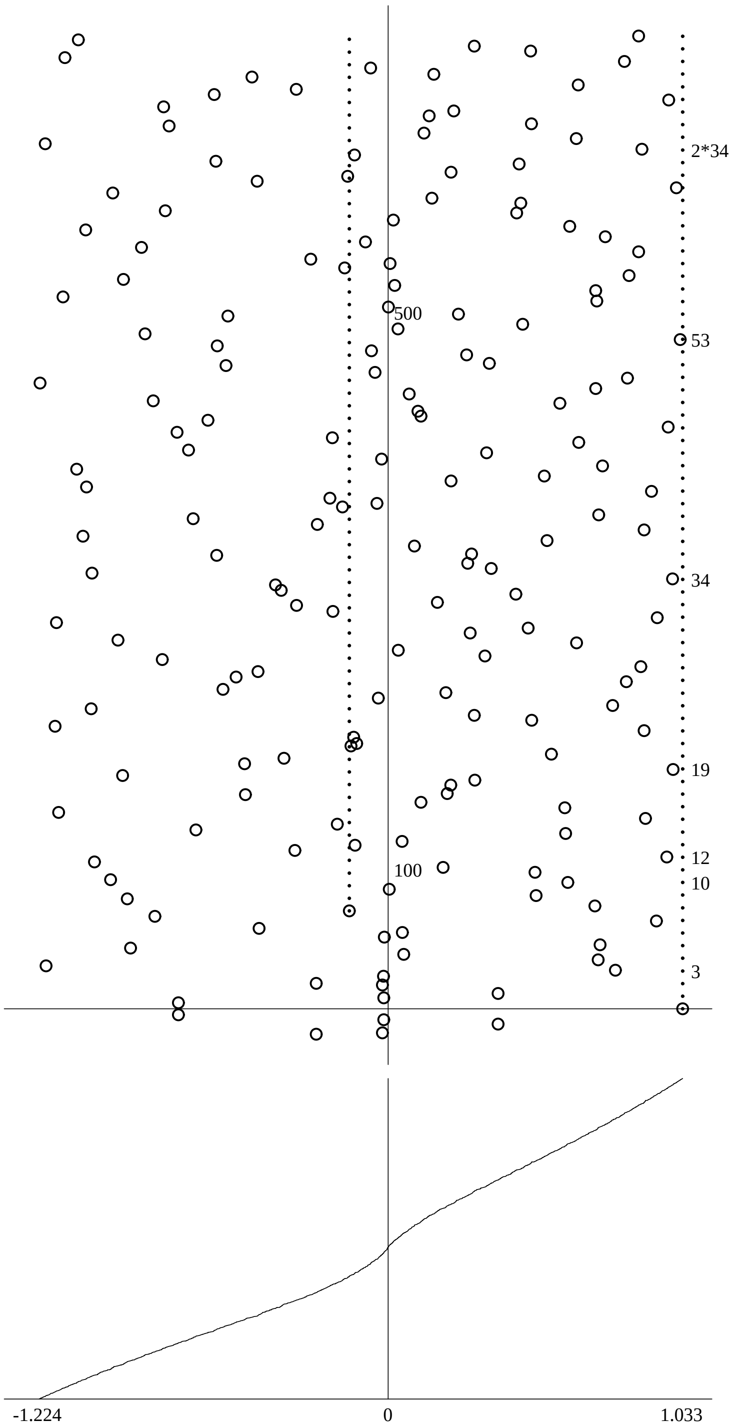

FIGURE 10. The almost periodic behavior of the complex dimensions of the Two-Three-Five string; the accumulative density of the real parts of the complex dimensions.

Remark 7.8. The best dimension free region is obtained when $w_1, \dots, w_N$ is badly approximable by rationals; i.e., when the weights are only a-approximable for a constant function $a = a(q)$. This corresponds to the case of two generators when α has bounded partial quotients in its continued fraction (Corollary 7.2). When $w_1, \dots, w_N$ is better approximable by rationals, the dimension free region is smaller and there are complex dimensions closer to the line $\operatorname{Re} s = D$. Such nonlattice strings behave more like lattice strings (compare Theorem 4.1).

Remark 7.9. Regarding the dependence on N, we see from comparing Corollary 7.2 and 7.6 that in general, depending of course on the properties of Diophantine approximation of the weights, nonlattice string with a larger number of scaling ratios have a wider (better) dimension free region.

Remark 7.10. By formulas (1.9) and (1.10) of the introduction, we deduce that for a nonlattice self-similar string, $V(\varepsilon)$ is approximated by its leading term up to an error of order $|\log \varepsilon|^{-(N-1)/2}$, or worse if the weights are well approximable.

8. The Real Parts of the Complex Dimensions of a Nonlattice String

In this section, we obtain a rigorous result and make several conjectures regarding the real parts of the complex dimensions of nonlattice strings. It is noteworthy that both the theorem and the conjectures were themselves suggested by computer experimentation guided by our theoretical investigation.

Recall from Remark 6.11 that the analogue of formula (6.16) holds for any complex dimension ω besides D.

Theorem 8.1. *The set of real parts of the complex dimensions of a nonlattice string has no isolated points.*

Proof. By formula (6.16), applied for ω, every complex dimension gives rise to a sequence of complex dimensions close to the points $\omega + 2\pi i q/w_1$, for integers q. Since the corrective terms are not all purely imaginary, we find complex dimensions with real parts close to $\operatorname{Re}\omega$. When q increases through a sequence of integers such that for each $j = 1, \dots, N$, $x_j = 2\pi i(qw_j/w_1 - p_j) \to 0$, we find a sequence of complex dimensions whose real parts approach $\operatorname{Re}\omega$. □

Remark 8.2. Recall from [LvF2, §10.2 and §10.3] that a string is said to be fractal in dimension α if it has a complex dimension with real part α. Each complex dimension gives rise to oscillations in the geometry of the fractal string. The frequencies of these oscillations are determined by the imaginary part of the complex dimension, and the real part of the complex dimension determines the amplitude of the oscillations. We define the set of *dimensions of fractality* of a fractal string as the closure of the set of real parts of its complex dimensions. Thus, Theorem 8.1 can be interpreted as saying that a nonlattice string is fractal in a perfect set of fractal dimensions.

8.1. **The density of the real parts.**

The density of the real parts of the complex dimensions of four different nonlattice strings are plotted in Figures 7, 9, 10 and 11, respectively. These figures show the graph of the function given by formula (6.12)

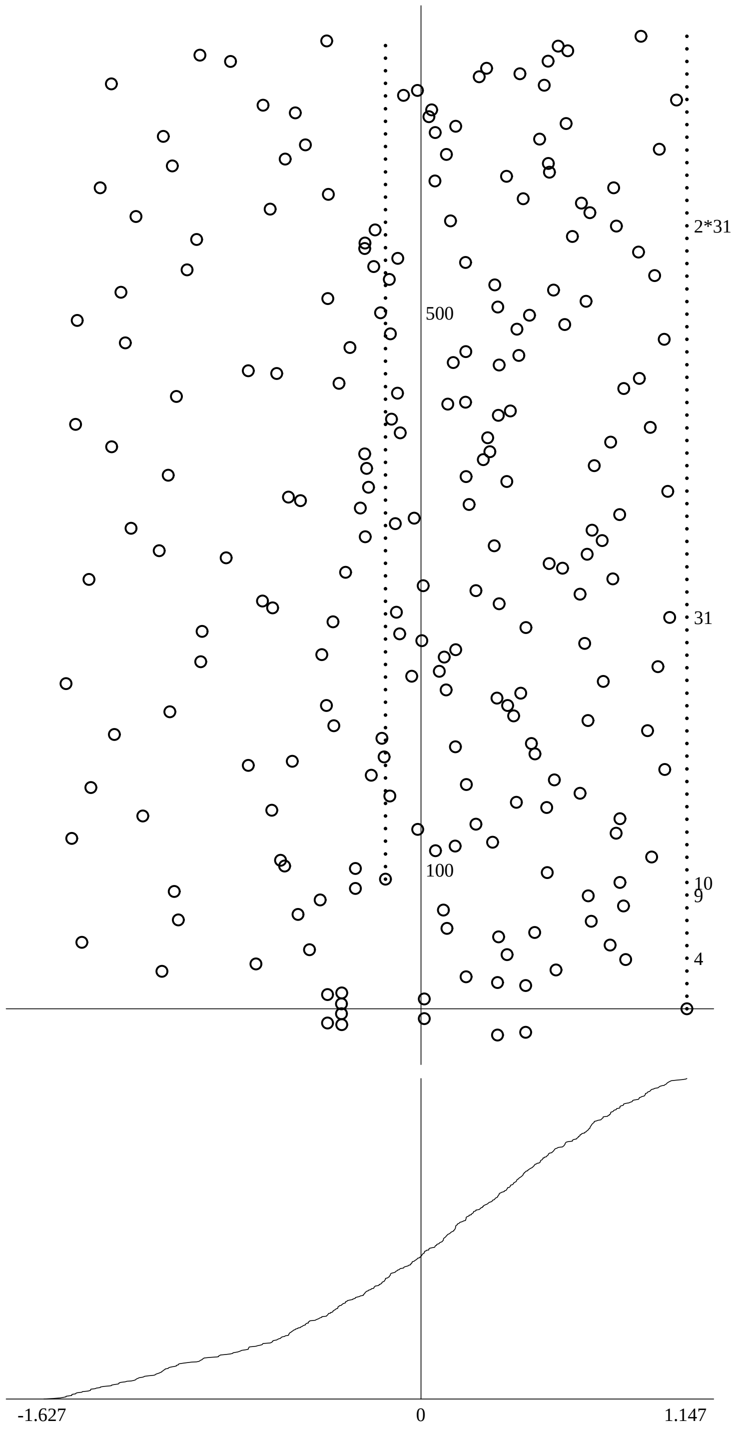

FIGURE 11. The complex dimensions of the nonlattice string with $N = 4$ and $r_1 = 1/2$, $r_2 = 1/3$, $r_3 = 1/5$, $r_4 = 1/7$; the accumulative density of their real parts.

for some large value of T. Since there are no horizontal pieces in these graphs, we conjecture that the real parts are dense in $[D_0, D]$. More generally, we expect the same to be true for the real parts of the complex dimensions of any nonlattice string.

The graphs of densities are qualitatively different for $N = 2$ and $N = 3$: the complex dimensions of the Two-Three-Five string show a 'phase transition' between negative and positive real part. (See Figure 10, and compare with Figures 7 and 9 where $N = 2$.) In the case of the Two-Three-Five string of Example 7.5 (see Figure 10), the density for negative real parts is approximately .77 (i.e., below average density), for positive real parts it is 1.08, slightly above average density. The density around vanishing real part becomes as large as 2.1.

We close this section with an example of the complex dimensions and the density of the real parts of the string with $r_1 = 1/2$, $r_2 = 1/3$, $r_3 = 1/5$ and $r_4 = 1/7$. See Figure 11. As pointed out in Remark 7.9, the complex dimensions tend to be more concentrated in the middle, away from $\operatorname{Re} s = D$ and $\operatorname{Re} s = D_0$. We have used the approximation $\log_2 3 \approx 699/441$, $\log_2 5 \approx 1024/441$ and $\log_2 7 \approx 1238/441$.

Clearly, more mathematical experimentation—guided by our theoretical investigations—is needed to determine the generality of this phenomenon and to formulate suitable additional conjectures regarding the fine structure of the complex dimensions of nonlattice strings.

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