UNIQUE CONTINUATION FOR THE ELASTICITY SYSTEM WITH RESIDUAL STRESS AND ITS APPLICATIONS

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Abstract. In this paper we prove the unique continuation property for the elasticity system with small residual stress. The elasticity system becomes anisotropic due to the existence of residual stress. The main technique in the proof is Carleman estimates. Having proved the unique continuation property, we will study the inverse problem of identifying the inclusion or cavity.

1. Introduction

Let $\mathcal{B}$ be an isotropic elastic body with residual stress and the reference configuration of $\mathcal{B}$ be Ω, a bounded open set in $\mathbb{R}^n$ with smooth boundary. The residual stress is modeled by a symmetric, smooth, second-rank tensor $T(x) = (t_{ij}(x))_{1\le i,j\le n}$ satisfying

$$\partial_{x_j} t_{ij} = 0 \quad \text{in } \Omega, \quad 1 \le i \le n \tag{1.1}$$

and

$$t_{ij}\nu_j = 0 \quad \text{on } \partial\Omega, \quad 1 \le i \le n, \tag{1.2}$$

where $\nu = (\nu_1, \cdots, \nu_n)$ is the unit outer normal to $\partial\Omega$. Hereafter, we adopt the summation convention. Let $u : \Omega \to \mathbb{R}^n$ be the displacement vector, then the first Piola-Kirchhoff stress is written as

$$\begin{aligned}\sigma = \; & T + (\nabla u)T + \lambda(\mathrm{tr}\epsilon)I + 2\mu\epsilon + \beta_1(\mathrm{tr}\epsilon)(\mathrm{tr}T)I + \beta_2(\mathrm{tr}T)\epsilon \\ & +\beta_3((\mathrm{tr}\epsilon)T + \mathrm{tr}(\epsilon T)I) + \beta_4(\epsilon T + T\epsilon),\end{aligned}$$

where λ, μ are the Lamé moduli and $\beta_1 \cdots \beta_4$ are material parameters and

$$\epsilon = (\epsilon_{ij}) = \frac{1}{2}(\nabla u + (\nabla u)^t)$$

is the strain tensor [16]. Moreover, we assume that the Lamé moduli satisfy the strong ellipticity condition

$$\mu(x) > \delta > 0, \; \lambda(x) + 2\mu(x) > \delta > 0 \qquad \forall x \in \Omega \tag{1.3}$$

and

$$\beta_3 = \beta_4 = 0,$$

i.e.,

$$\sigma = T + (\nabla u)T + \tilde{\lambda}(\mathrm{tr}\epsilon)I + 2\tilde{\mu}\epsilon,$$

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where

$$\tilde{\lambda} = \lambda + \beta_1(\mathrm{tr}T), \quad \tilde{\mu} = \mu + \frac{1}{2}\beta_2(\mathrm{tr}T).$$

The stationary elasticity system is expressed as

$$(Lu)_i = (\nabla \cdot \sigma)_i + \omega^2 \rho(x) u_i = \partial_j \sigma_{ij} + \omega^2 \rho(x) u_i = 0 \quad \text{in } \Omega,\ 1 \le i \le n,\ \omega \in \mathbb{R}, \tag{1.4}$$

where $\rho(x) > 0$ is the density of the medium. In another setting, if we define the elasticity tensor $\mathbb{C}$ with components

$$C_{ijkl} = \tilde{\lambda}\delta_{ij}\delta_{kl} + (\tilde{\mu}\delta_{jl} + t_{jl})\delta_{ik} + \tilde{\mu}\delta_{il}\delta_{jk} \tag{1.5}$$

and denote

$$(\mathbb{C}E)_{ij} = C_{ijkl}E_{kl} \quad \text{for any matrix} \quad E,$$

then (1.4) is equivalent to

$$(Lu)_i = (\nabla \cdot \mathbb{C}\nabla u)_i + \omega^2 \rho u_i = \partial_j (C_{ijkl}\partial_l u_k) + \omega^2 \rho u_i = 0 \quad \text{in } \Omega,\ 1 \le i \le n.$$

It is clear to see that (1.4) is an anisotropic elasticity system. In this paper, we will investigate the (weak) unique continuation property (UCP) for the system (1.4), i.e. if $u \in H^2_{\mathrm{loc}}(\Omega)$ is a solution to (1.4) in Ω and vanishes in a non-empty open subset of Ω, then u vanishes identically in Ω.

The unique continuation property for differential equations has a long history. Many deep results about scalar elliptic equations or elliptic systems have been established. We refer the reader to [3] and references therein for details. Recently, few attempts have been made at studying the unique continuation property for systems of equations in mathematical physics such as the Dirac equations and the Maxwell equations [4], [13], [17], [19], [20]. Here we mention two interesting articles [20] and [17] in which Vogelsang and Okaji, respectively, proved the strong unique continuation property for the Maxwell system with anisotropic coefficients. In this paper we pay our attention to the elasticity system. Several results of weak continuation property for the inhomogeneous isotropic elasticity have been obtained in [1], [5], (stationary) and [6], [12] (non-stationary). Moreover, a strong unique continuation property were recently proven by Alessandrini and Morassi [2]. Unlike the isotropic case, the unique continuation property for the inhomogeneous anisotropic elasticity has not been fully explored.

Our study of the unique continuation property for the inhomogeneous anisotropic elasticity is motivated by its application to inverse problems. It was first recognized by Lax [15] that the Runge approximation property is a consequence of the weak unique continuation property. The Runge approximation property is shown to be a useful technique in dealing with some inverse problems, especially the inverse problem of recovering inclusions or cavities (see [11], [7], [8], [9], [10], [14] and references therein). It should be noted that the Runge approximation property with constraint for the anisotropic elasticity were proved in [9] and [10]. However, the elasticity tensor there is assumed to be either homogeneous or real-analytic. The weak unique continuation property is an obvious fact in these two situations.

To prove the unique continuation property for the general inhomogeneous anisotropic elasticity is very challenging and difficult. Here we want to consider the system (1.4) which has the simplest form of anisotropy. It turns out we are able to establish the UCP for (1.4) provided the residual stress is sufficiently small. Our main idea comes from Weck's recent article [21] where he proved the UCP for the isotropic elasticity system with zeroth or first order perturbations which contains the results previous obtained by [1], [5]. Weck actually proved something more, namely, he established the UCP for a rather general system of second order differential inequalities with the Laplacian principal part. Like many literature on the UCP, the key step in [21] is to prove appropriate Carleman estimates. Here we will adopt Weck's approach to (1.4) with small residual stress, but we have to work a little harder to derive the desired Carleman estimates because we need to deal with variable coefficients second order principal parts due to the presence of residual stress. As indicated previously, having established the UCP, we can prove the Runge approximation property for (1.4) with constraints on Dirichlet data. With this tool at hand, we can solve the inverse problem of identifying inclusions or cavities inside an elastic body with small residual stress by the localized Dirichlet-to-Neumann map using the methods in [9] and [10].

This paper is organized as follows. In Section 2, we state and prove the UCP for (1.4) with small residual stress based on suitable Carleman estimates. The derivation of these Carleman estimates is given in Section 3. In Section 4, we will discuss the application of UCP for (1.4) to the aforementioned inverse problem. In the paper, C stands for a generic constant and its value may vary from line to line.

2. Unique continuation

To begin, let us denote $v_i = u_i$ for $1 \le i \le n$ and $v_{n+1} = \partial_i u_i$. Then, it follows from (1.4) that

$$
\begin{aligned}
0 &= (Lu)_i \\
&= (\tilde\mu\Delta + t_{kj}\partial_j\partial_k)v_i + (\tilde\lambda + \tilde\mu)\partial_i v_{n+1} + (\partial_j t_{kj})\partial_k v_i + (\partial_i\tilde\lambda)v_{n+1} \\
&\quad +(\partial_j\tilde\mu)(\partial_i v_j + \partial_j v_i) + \omega^2\rho v_i \\
&= (\tilde\mu\Delta + t_{kj}\partial_j\partial_k)v_i + R_i^{(1)}(v_1,\cdots,v_n,v_{n+1}) \quad \text{in } \Omega,\ 1 \le i \le n,
\end{aligned} \tag{2.1}
$$

where $R_i^{(1)}$'s are some first order differential operators. Next, by taking the divergence of (1.4), we obtain that

$$
\begin{aligned}
0 &= \partial_i(Lu)_i \\
&= ((\tilde\lambda + 2\tilde\mu)\Delta + t_{kj}\partial_j\partial_k)v_{n+1} + 2(\partial_i\tilde\mu)\Delta v_i + (\partial_i t_{kj})\partial_j\partial_k v_i + 2\partial_i(\tilde\lambda + \tilde\mu)\partial_i v_{n+1} \\
&\quad +(\partial_j t_{kj})\partial_k v_{n+1} + (\partial_i\partial_j t_{kj})\partial_k v_i + (\Delta\tilde\lambda)v_{n+1} + (\partial_i\partial_j\tilde\mu)(\partial_i v_j + \partial_j v_i) + \omega^2(\partial_i\rho)v_i + \omega^2\rho v_{n+1} \\
&= ((\tilde\lambda + 2\tilde\mu)\Delta + t_{kj}\partial_j\partial_k)v_{n+1} + R^{(2)}(v_1,\cdots,v_n) + R_{n+1}^{(1)}(v_1,\cdots,v_{n+1}),
\end{aligned} \tag{2.2}
$$

where $R^{(2)}$ is a pure second order (containing only second derivatives) and $R_{n+1}^{(1)}$ is a first order differential operators, respectively. It should be mentioned that $R^{(2)}$ acts only on $v_1, \cdots, v_n$. In view of (1.3), we can see that if

$$\max_{kj} \|t_{kj}\|_{L^\infty(\Omega)} < \varepsilon \tag{2.3}$$

with $\varepsilon \ll 1$, then

$$\tilde{\mu} > \delta' > 0 \quad \text{and} \quad \tilde{\lambda} + 2\tilde{\mu} > \delta' > 0 \quad \forall x \in \Omega.$$

With the equations (2.1) and (2.2) in mind, motivated by Weck's paper [21], we will prove the UCP for the following system of differential inequalities

$$\begin{aligned} &|A_1(x,\partial)u^1| \le CQ(u^1,u^2)^{1/2}, \\ &|A_2(x,\partial)u^2| \le C\{\sum_{ijk}|\partial_i\partial_j u_k^1| + Q(u^1,u^2)^{1/2}\}, \end{aligned} \tag{2.4}$$

where $u^l : \Omega \to \mathbb{R}^{m_l}, m_l \in \mathbb{Z}_+$(psitive integers) and $A_l(x,\partial) = a_{ij}^l \partial_i \partial_j$ with real symmetric matrix (a_{ij}^l), $l = 1, 2$ and $Q(u^1,u^2) = \sum_{ikl}(|\partial_i u_k^l|^2 + |u_k^l|^2)$.

Theorem 2.1. *Let $a_{ij}^l \in W^{1,\infty}(\Omega)$ and $(u^1,u^2) \in H_{loc}^2(\Omega) \times H_{loc}^2(\Omega)$ satisfy (2.4). Then there exists an $\varepsilon > 0$ such that if*

$$\max_{ij} \|a_{ij}^l(x) - \delta_{ij}\|_{L^\infty(\Omega)} < \varepsilon, \tag{2.5}$$

then (u^1,u^2) vanishes identically in Ω if it vanishes in a non-empty open subset of Ω.

Theorem 2.1 immediately implies the UCP for (1.4) with small residual stress.

Corollary 2.1. *Let coefficients $\lambda, \mu, \beta_1, \beta_2, t_{kj}$ belong to $W^{2,\infty}(\Omega)$ and ρ be in $W^{1,\infty}(\Omega)$. Then there exists an $\varepsilon > 0$ such that if (2.3) is satisfied with this ε, then the system (1.4) possesses the UCP.*

The proof of Theorem 2.1 relies on the following Carleman estimates.

Proposition 2.1. *Assume that the differential operators A_1 and A_2 satisfy the assumptions in Theorem 2.1. Let $r_0 < 1$ and $U_{r_0} = \{u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) : \mathrm{supp}(u) \subset B_{r_0}\}$, where B_{r_0} is the ball centered at the origin with radius r_0. Then there exist positive constants β_0 and ε_0 such that if (2.5) is satisfied with $\varepsilon \le \varepsilon_0$, then for all $\beta \ge \beta_0$ and $u \in U_{r_0}$ we have that*

$$\int r^{-\sigma}\psi^2 \sum_{ij} |\partial_i\partial_j u|^2 dx \le C \int r^{-\sigma}\psi^2(\beta^2 r^{-2\beta-2}|\nabla u|^2 + |A_l u|^2)dx \tag{2.6}$$

and

$$\beta^2 \int r^{-\sigma-\beta-1}\psi^2(|\nabla u|^2 + |u|^2)dx \le C \int r^{-\sigma}\psi^2|A_l u|^2 dx \tag{2.7}$$

for $l = 1, 2$, where $r = |x|$, $\psi = \exp(r^{-\beta})$ and $\sigma = \sigma_0 + c\beta$ with $\sigma_0, c \in \mathbb{R}$.

The proof of Proposition 2.1 is postponed until the next section. Here we want to prove Theorem 2.1 based on this proposition.

Proof of Theorem 2.1: It suffices to prove the theorem for $m_1 = m_2 = 1$ case. Let (u^1,u^2) vanish in a neighborhood of $x_0 \in \Omega$. Without loss of generality, we assume $x_0 = 0$. We set $\tilde{r} = \min\{1/2, \mathrm{dist}(0,\partial\Omega)\}$. Now let $\chi \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function satisfying $0 \leq \chi \leq 1$, $\chi|_{B_{\tilde{r}/2}} = 1$ and $\mathrm{supp}\,(\chi) \subset B_{\tilde{r}}$. Denote $v_l = \chi u^l$, $l = 1, 2$. From (2.4) we have that

$$\begin{aligned} |A_1 v_1| &\leq C(e(v_1) + e(v_2))^{1/2} + f_1, \\ |A_2 v_2| &\leq C[\sum_{ij} |\partial_i \partial_j v_1| + (e(v_1) + e(v_2))^{1/2}] + f_2, \end{aligned} \tag{2.8}$$

where $e(v) = |\nabla v|^2 + |v|^2$ and f_l is supported in $B_{\tilde{r}} \setminus B_{\tilde{r}/2}$ for $l = 1, 2$. It follows from (2.8) that

(2.9)
$$I := \gamma \int r^{-\beta} \psi^2 |A_1 v_1|^2 dx + \int r \psi^2 |A_2 v_2|^2 dx \leq C(F + G + \int r \psi^2 \sum_{ij} |\partial_i \partial_j v_1|^2 dx),$$

where

$$\begin{aligned} F &= \gamma \int r^{-\beta} \psi^2 f_1^2 dx + \int r \psi^2 f_2^2 dx, \\ G &= \int (r + \gamma r^{-\beta}) \psi^2 (e(v_1) + e(v_2)) dx. \end{aligned}$$

Here γ is a large positive parameter which will be chosen later on. By the standard approximation argument, we can see that v_1 and v_2 satisfy estimates (2.6) and (2.7). Taking $\sigma = -1$ in the estimate (2.6) for $l = 1$ and substituting it into (2.9) yield

$$I \leq C(F + G + \int r \psi^2 |A_1 v_1|^2 dx + \beta^2 \int r^{-2\beta - 1} \psi^2 |\nabla v_1|^2 dx). \tag{2.10}$$

Replacing the last term of (2.10) with the help of (2.7) for $\sigma = \beta$ and $l = 1$, we obtain that

$$I \leq C(F + G + \int r^{-\beta} \psi^2 |A_1 v_1|^2 dx). \tag{2.11}$$

Now taking γ sufficiently large, we can absorb the last term of (2.11) and get

$$I \leq C(F + G). \tag{2.12}$$

From now on we fix the parameter γ.

Next using $\sigma = \beta$ in (2.7) for $l = 1$ and $\sigma = -1$ in (2.7) for $l = 2$, we find that

$$\begin{aligned} H : &= \beta^2 \int r^{-2\beta - 1} \psi^2 e(v_1) dx + \beta^2 \int r^{-\beta} \psi^2 e(v_2) dx \\ &\leq C(\int r^{-\beta} \psi^2 |A_1 v_1|^2 dx + \int r \psi^2 |A_2 v_2|^2 dx). \end{aligned} \tag{2.13}$$

Combining (2.12) and (2.13) gives

$$H \leq C(F + G) \leq C(F + \int (r + \gamma r^{-\beta}) \psi^2 (e(v_1) + e(v_2)) dx). \tag{2.14}$$

Now observing that $r < r^{-\beta} < \beta r^{-\beta} < \beta r^{-2\beta - 1}$ when $r \leq \tilde{r}$ and $\beta > 1$, we obtain from (2.14) that

$$H \leq C(F + \beta \int r^{-2\beta - 1} \psi^2 e(v_1) dx + \beta \int r^{-\beta} \psi^2 e(v_2) dx). \tag{2.15}$$

Taking β sufficiently large in (2.15), we get that

$$H \leq CF,$$

i.e.

$$\beta^2\int r^{-2\beta-1}\psi^2 e(v_1)dx+\beta^2\int r^{-\beta}\psi^2 e(v_2)dx\le C(\int r^{-\beta}\psi^2 f_1^2dx+\int r\psi^2 f_2^2dx)$$

from which we immediately have

$$\beta^2\int_{B_{\tilde r/2}} r^{-\beta}\psi^2(v_1^2+v_2^2)dx\le C\int_{B_{\tilde r}\setminus B_{\tilde r/2}} r^{-\beta}\psi^2(f_1^2+f_2^2)dx. \tag{2.16}$$

Since $r^{-\beta}\psi^2$ is a strictly decreasing function, (2.16) implies that

$$\beta^2\int_{B_{\tilde r/2}}(v_1^2+v_2^2)dx\le C\int_{B_{\tilde r}\setminus B_{\tilde r/2}}(f_1^2+f_2^2)dx$$

and therefore $(v_1,v_2)=0$ on $B_{\tilde r/2}$ if we choose β sufficiently large. Clearly, (u^1,u^2) must be zero throughout Ω. $\square$

3. Proof of Carleman estimates

This section is devoted to the proof of Proposition 2.1. It suffices to prove (2.6) and (2.7) for A_1. Therefore, we denote $a^1_{ij}=a_{ij}$ and $A_1=A$. To prove (2.6), we first recall the following estimate in [21]

$$\int r^{-\sigma}\psi^2\sum_{ij}|\partial_i\partial_j u|^2dx\le C\int r^{-\sigma}\psi^2(\beta^2r^{-2\beta-2}|\nabla u|^2+|\Delta u|^2)dx$$

(see [21, Lemma 2]) from which we have that

$$\begin{aligned}\int r^{-\sigma}\psi^2\sum_{ij}|\partial_i\partial_j u|^2dx &\le C\int r^{-\sigma}\psi^2(\beta^2r^{-2\beta-2}|\nabla u|^2+|Au|^2+|\Delta u-Au|^2)dx\\ &\le C\int r^{-\sigma}\psi^2(\beta^2r^{-2\beta-2}|\nabla u|^2+|Au|^2+\varepsilon^2\sum_{ij}|\partial_i\partial_j u|^2)dx.\end{aligned}$$

Thus, choosing ε small enough immediately implies the estimate (2.6).

The proof of (2.7) is lengthy. Here we will adopt some techniques from [18], [21] and [22]. Let $\phi=\psi^{-1}$ and $u=r^{\tau/2}\phi z$, then

$$\begin{aligned}r^{-\sigma/2}\psi Au &= r^{-\sigma/2}\psi A(r^{\tau/2}\phi z)\\ &= r^{-\sigma/2}\psi[r^{\tau/2}\phi Az+2a_{ij}\partial_i z\partial_j(r^{\tau/2}\phi)+zA(r^{\tau/2}\phi)].\end{aligned}$$

By virtue of the inequality $(a+b+c)^2\ge 2ab+2bc$, we have that

$$\begin{aligned}\int r^{-\sigma}\psi^2|Au|^2dx &\ge 4\int r^{-\sigma}\psi^2a_{ij}\partial_i z\partial_j(r^{\tau/2}\phi)r^{\tau/2}\phi Azdx\\ &\quad +4\int r^{-\sigma}\psi^2a_{ij}\partial_i z\partial_j(r^{\tau/2}\phi)zA(r^{\tau/2}\phi)dx.\end{aligned} \tag{3.1}$$

With the choice of $\tau=\sigma+\beta+2$, we can compute

$$\begin{aligned}I &:= \int r^{-\sigma}\psi^2a_{ij}\partial_i z\partial_j(r^{\tau/2}\phi)r^{\tau/2}\phi Azdx\\ &= \beta\int a_{ij}\partial_i zx_jAzdx+\tau/2\int r^{\beta}a_{ij}\partial_i zx_jAzdx.\end{aligned}$$

It is readily seen that the leading term (for large β) of I is $\beta\int a_{ij}\partial_i z x_j Az dx$. Repeated integration by parts shows that

$$\begin{aligned}
2\int a_{ij}\partial_i z x_j Az dx &= 2\int a_{ij}\partial_i z x_j a_{kl}\partial_k\partial_l z dx\\
&= -\int \partial_i z\partial_l(a_{kl}a_{ij}x_j)\partial_k z dx + \int \partial_k z\partial_i(a_{kl}a_{ij}x_j)\partial_l z dx\\
&\quad -\int \partial_l z\partial_k(a_{kl}a_{ij}x_j)\partial_l z dx.
\end{aligned} \tag{3.2}$$

Using (3.2) we obtain that

$$\begin{aligned}
&|I| \le C\beta|\int \partial_i z\partial_l(a_{kl}a_{ij}x_j)\partial_k z dx|\\
&\le C\beta\|\nabla z\|^2\\
&\le C\beta(\|\nabla(r^{-\tau/2}\psi)u\|^2 + \|r^{-\tau/2}\psi\nabla u\|^2)\\
&\le C(\beta^3\int r^{-\sigma-3\beta-4}\psi^2|u|^2dx + \beta\int r^{-\sigma-\beta-2}\psi^2|\nabla u|^2 dx).
\end{aligned} \tag{3.3}$$

Next we observe that

$$\begin{aligned}
J &:= \int r^{-\sigma}\psi^2 a_{ij}\partial_i z\partial_j(r^{\tau/2}\phi) z A(r^{\tau/2}\phi)dx\\
&= \beta\int r^{-\sigma+\tau/2-\beta-2}\psi a_{ij}\partial_i z x_j z A(r^{\tau/2}\phi)dx\\
&\quad +\tau/2\int r^{-\sigma+\tau/2-2}\psi a_{ij}\partial_i z x_j z A(r^{\tau/2}\phi)dx.
\end{aligned}$$

Straightforward calculations show that

$$\partial_i\partial_j\phi = (\beta^2 x_i x_j r^{-2\beta-4} + \beta\delta_{ij}r^{-\beta-2} - \beta(\beta+2)x_i x_j r^{-\beta-4})\phi$$

and

$$\partial_i\partial_j r^{\tau/2} = (\tau/2)(\tau/2-2)r^{\tau/2-4}x_i x_j + (\tau/2)r^{\tau/2-2}\delta_{ij}.$$

So the leading term of J is

$$\beta^3\int r^{-2\beta-4}a_{ij}\partial_i z x_j a_{kl}x_k x_l z dx.$$

Note that we have chosen $\tau = \sigma+\beta+2$. Performing the integration by parts, we can see that

$$\begin{aligned}
&\beta^3\int r^{-2\beta-4}a_{ij}\partial_i z x_j a_{kl}x_k x_l z dx\\
&= -2\beta^3\int z\partial_i(r^{-2\beta-4}a_{ij}x_j a_{kl}x_k x_l)z dx\\
&\ge (4-o(\beta))\beta^4\int r^{-2\beta-6}a_{ij}x_i x_j a_{kl}x_k x_l|z|^2 dx\\
&\ge (4-o(\beta))\beta^4(1-O(\varepsilon))\int r^{-2\beta-2}|z|^2 dx\\
&\ge (4-o(\beta))\beta^4(1-O(\varepsilon))\int r^{-\sigma-3\beta-4}\psi^2|u|^2 dx,
\end{aligned}$$

where $o(\beta)\to 0$ as $\beta\to\infty$ and $O(\varepsilon)$ is a constant bounded by $C\varepsilon$. In other words, we have that

$$J \ge (4-o(\beta))\beta^4(1-O(\varepsilon))\int r^{-\sigma-3\beta-4}\psi^2|u|^2dx. \tag{3.4}$$

Notice that we need to keep track the leading constant here in order to obtain the desired estimate. Combining (3.1), (3.3) and (3.4) gives

$$\begin{aligned}
&\int r^{-\sigma}\psi^2|Au|^2dx + C(\beta^3\int r^{-\sigma-3\beta-4}\psi^2|u|^2dx + \beta\int r^{-\sigma-\beta-2}\psi^2|\nabla u|^2dx)\\
&\ge 4(4-o(\beta))\beta^4(1-O(\varepsilon))\int r^{-\sigma-3\beta-4}\psi^2|u|^2dx
\end{aligned}$$

from which we can derive that

$$\int r^{-\sigma}\psi^2|Au|^2dx + C\beta\int r^{-\sigma-\beta-2}\psi^2|\nabla u|^2dx \ge 4(4-o(\beta))\beta^4(1-O(\varepsilon))\int r^{-\sigma-3\beta-4}\psi^2|u|^2dx. \tag{3.5}$$

By the ellipticity condition and performing the integration by parts, we can get that

$$
\begin{aligned}
&(1-O(\varepsilon))\int r^{-\sigma-\beta-2}\psi^2|\nabla u|^2dx\\
&\leq \int r^{-\sigma-\beta-2}\psi^2 a_{ij}\partial_i u\partial_j udx\\
&\leq |\int u\partial_i(r^{-\sigma-\beta-2}\psi^2)a_{ij}\partial_j udx|+|\int r^{-\sigma-\beta-2}u\partial_i(a_{ij})\partial_j udx|\\
&\quad +|\int r^{-\sigma-\beta-2}ua_{ij}\partial_i\partial_j udx|\\
&:= K_1+K_2+K_3.
\end{aligned}
\tag{3.6}
$$

Using the relation $|ab|\leq (a^2+b^2)/2$, we can estimate

$$
\begin{aligned}
K_1 &= |\int u\partial_i(r^{-\sigma-\beta-2}\psi^2)a_{ij}\partial_j udx|\\
&\leq \int 2\beta r^{-\sigma-2\beta-4}\psi^2|ua_{ij}x_i\partial_j u|dx\\
&\leq (2+o(\beta))\beta^2(1+O(\varepsilon))\int r^{-\sigma-3\beta-4}\psi^2|u|^2dx+(1+O(\varepsilon))/2\int r^{-\sigma-\beta-2}\psi^2|\nabla u|^2dx.
\end{aligned}
\tag{3.7}
$$

Likewise, for K_1 and K_2, we have that

$$
K_2\leq C(r_0^\beta\int r^{-\sigma-3\beta-4}\psi^2|u|^2dx+r_0^{\beta+2}\int r^{-\sigma-\beta-2}\psi^2|\nabla u|^2dx)
\tag{3.8}
$$

and

$$
K_3\leq C(r_0^\beta\beta^2\int r^{-\sigma-3\beta-4}\psi^2|u|^2dx+\beta^{-2}\int r^{-\sigma}\psi^2|Au|^2dx).
\tag{3.9}
$$

Plugging (3.7), (3.8) and (3.9) into (3.6) and multiplying the new inequality by β^2, we obtain that

$$
\begin{aligned}
&\beta^2(1-O(\varepsilon))\int r^{-\sigma-\beta-2}\psi^2|\nabla u|^2dx\\
&\leq (2+o(\beta))\beta^4(1+O(\varepsilon))\int r^{-\sigma-3\beta-4}\psi^2|u|^2dx+\beta^2(1+O(\varepsilon))/2\int r^{-\sigma-\beta-2}\psi^2|\nabla u|^2dx\\
&\quad +C(r_0^\beta\beta^2\int r^{-\sigma-3\beta-4}\psi^2|u|^2dx+r_0^{\beta+2}\beta^2\int r^{-\sigma-\beta-2}\psi^2|\nabla u|^2dx)\\
&\quad +C(r_0^\beta\beta^4\int r^{-\sigma-3\beta-4}\psi^2|u|^2dx+\int r^{-\sigma}\psi^2|Au|^2dx).
\end{aligned}
\tag{3.10}
$$

Adding (3.10) to (3.5) and taking β sufficiently large and ε small enough, we conclude that

$$
\beta^4\int r^{-\sigma-3\beta-4}\psi^2|u|^2dx+\beta^2\int r^{-\sigma-\beta-2}\psi^2|\nabla u|^2dx\leq C\int r^{-\sigma}\psi^2|Au|^2dx
$$

which immediately implies (2.7).

4. Applications to inverse problems

In this section we will discuss the application of the UCP for (1.4) to the inverse problem of identifying inclusions or cavities by boundary measurements. To begin, assume that D is an open subset of Ω with Lipschitz boundary such that $\Omega\setminus\bar{D}$ is connected. The domain D stands for region of the inclusion or cavity embedded in Ω. Let the reference elasticity tensor $\mathbb{C}(x)$ with components $C_{ijkl}(x)$ be defined by (1.5), i.e.

$$
C_{ijkl}=\tilde{\lambda}\delta_{ij}\delta_{kl}+(\tilde{\mu}\delta_{jl}+t_{jl})\delta_{ik}+\tilde{\mu}\delta_{il}\delta_{jk},
$$

where $\tilde{\lambda} = \lambda + \beta_1(\mathrm{tr}T)$ and $\tilde{\mu} = \mu + (1/2)\beta_2(\mathrm{tr}T)$. Here we require that the Lamé moduli satisfy the strong convexity condition

$$\mu(x) > \delta > 0 \quad \text{and} \quad n\lambda(x) + 2\mu(x) > \delta > 0 \quad \forall\, x \in \Omega \tag{4.1}$$

which is equivalent to

$$\mathbb{C}(x)E \cdot E \geq \kappa E_{ij}E_{ij} = \kappa |E|^2, \quad \kappa > 0,\ \forall\, x \in \Omega \tag{4.2}$$

for all matrix E provided that ε in (2.3) is sufficiently small. It is obvious that (4.1) implies (1.3). Next we assume that $\tilde{\mathbb{C}}$ is some fourth-rank tensor such that $\mathbb{C} + \chi_D\tilde{\mathbb{C}}$ satisfies the strong convexity condition (4.2), where χ_D denotes the characteristic function of D. Moreover, suppose that $\tilde{\mathbb{C}}$ satisfies the following jump condition

(4.3)

$$\forall\, x \in \partial D,\ \exists\, C_x > 0,\ \exists\, \delta_x > 0 \text{ such that } \tilde{\mathbb{C}}(y)E \cdot E \geq C_x|E|^2 \text{ or } \tilde{\mathbb{C}}(y)E \cdot E \leq -C_x|E|^2$$

for almost all $y \in B_{\delta_x}(x) \cap D$ and all real matrices E. Let all components of $\mathbb{C}(x)$ and $\tilde{\mathbb{C}}(x)$ be in $L^\infty(\Omega)$, then it is easy to show that there exists a unique solution $u \in H^1(\Omega)$ to

$$\begin{cases} \nabla \cdot ((\mathbb{C} + \chi_D\tilde{\mathbb{C}})\nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases}$$

for any $f \in H^{1/2}(\partial\Omega)$. In this case, the domain D is an inclusion. So we can define the Dirichlet-to-Neumann (displacement-to-traction) map $\Lambda_I : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ by

$$\Lambda_I(f) = (\mathbb{C}\nabla u)\nu|_{\partial\Omega}.$$

We are interested in the following inverse problem

IP.A Reconstruct the inclusion D from the knowledge of $\Lambda_I(f)_{\Gamma_0}$ for infinitely many $f \in H^{1/2}(\partial\Omega)$ with $\mathrm{supp}\,(f) \subset \Gamma_0$, where Γ_0 is a non-empty subset of $\partial\Omega$.

Likewise, in the extreme case, if the tensor $\tilde{\mathbb{C}}$ becomes $-\mathbb{C}$, then the domain D corresponds to a cavity. In the same way, we can prove that there exists a unique solution $u \in H^1(\Omega \setminus \bar{D})$ to the following boundary value problem

$$\begin{cases} \nabla \cdot (\mathbb{C}\nabla u) = 0 & \text{in } \Omega \setminus \bar{D}, \\ (\mathbb{C}\nabla u)\nu = 0 \quad \text{on } \partial D, & (\mathbb{C}\nabla u)\nu = g \quad \text{on } \partial\Omega \end{cases}$$

for any $g \in H^{1/2}(\partial\Omega)$. Therefore, we can define the Dirichlet-to-Neumann map $\Lambda_C : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ by

$$\Lambda_C(g) = (\mathbb{C}\nabla u)\nu|_{\partial\Omega}.$$

Similarly, we will consider the inverse problem

IP.B Reconstruct the cavity D from the knowledge of $\Lambda_C(g)|_{\Gamma_0}$ for infinitely many g with $\mathrm{supp}\,(g) \subset \Gamma_0$.

Note that explicit reconstruction algorithms for recovering the inclusion or cavity embedded in an elastic body have been developed in [9] and [10] where the reference medium is assumed to be either inhomogeneous isotropic or anisotropic with homogeneous or analytic elasticity tensors. Here we want to extend their results to the elasticity system with residual stress (1.4). To this end, we will need the Runge approximation property with constraint for (1.4) which is a consequence of the UCP (see Corollary 2.1). Its proof can be found in [10].

Proposition 4.1. *Assume that all coefficients of* $\mathbb{C}$ *are in* $W^{2,\infty}(\Omega)$ *and the residual stress satisfies* (2.3) *with* ε *given in Corollary 2.1. Let* U *and* Ω *be two open bounded domains with Lipschitz and* C^2 *boundaries, respectively, such that* $\bar{U} \subset \Omega$. *Denote* Γ_0 *a subset of the boundary* $\partial\Omega$. *Let* $u \in H^1(U)$ *satisfy*

$$\nabla \cdot (\mathbb{C}\nabla u) = 0 \quad in \quad U.$$

Then for any compact subset $K \subset U$ *such that* $\Omega \setminus K$ *is connected and any* $\tilde{\varepsilon} > 0$ *there exists* $w \in H^1(\Omega)$ *satisfying*

$$\nabla \cdot (\mathbb{C}\nabla w) = 0 \quad in \quad \Omega$$

with $\operatorname{supp}(w|_{\partial\Omega}) \subset \Gamma_0$ *such that*

$$\|w - u\|_{H^1(K)} < \tilde{\varepsilon}.$$

Remark: The reason for using C^2 boundary on Ω is that we want to extend all coefficients of $\mathbb{C}$ into a larger domain $\tilde{\Omega}$ and the newly extended coefficients have the same regularity $W^{2,\infty}$ in $\tilde{\Omega}$.

Having the Runge approximation property Proposition 4.1 at hand, we now can apply the methods in [9] and [10] to solve **IP.A** and **IP.B**. It should be pointed out that the reference elasticity tensor in [9] and [10] satisfies the full symmetry properties, i.e.

$$C_{ijkl} = C_{klij} = C_{jikl}.$$

Nevertheless, it is not hard to check that the proofs in [9] and [10] are still valid if we only assume $C_{ijkl} = C_{klij}$ which is the case for the elasticity system with residual stress (1.4). Precisely, we can prove that

Theorem 4.1 (Identification of inclusion)**.** *Let the domain* Ω *have* C^2 *boundary. Assume that the elasticity tensor* $\mathbb{C}$ *given by* (1.5) *possesses* $W^{2,\infty}(\Omega)$ *coefficients satisfying* (4.1). *Furthermore, the residual stress tensor* T *in* $\mathbb{C}$ *satisfies the smallness condition described in Corollary 2.1. Let* $(D_1, \tilde{\mathbb{C}}_1)$ *and* $(D_2, \tilde{\mathbb{C}}_2)$ *be two inclusions such that* $\mathbb{C} + \chi_D \tilde{\mathbb{C}}_i$ *and* $\tilde{\mathbb{C}}_i$ *satisfy* (4.2) *and* (4.3), *respectively, and* $\Omega \setminus \bar{D}_i$ *is connected,* $i = 1, 2$. *If*

$$\Lambda_{I_1}(f) = \Lambda_{I_2}(f) \quad on \quad \Gamma_0$$

for all $f \in H^{1/2}(\partial\Omega)$ *with* $\operatorname{supp}(f) \subset \Gamma_0$, *then*

$$D_1 = D_2.$$

Theorem 4.2 (Identification of cavity)**.** *Let the assumptions in Theorem 4.1 on* Ω *and* $\mathbb{C}$ *hold. Assume that* D_1 *and* D_2 *are two cavities and* $\Omega \setminus \bar{D}_1$ *and* $\Omega \setminus \bar{D}_2$ *are connected. Let*

$$\Lambda_{C_1}(f) = \Lambda_{C_2}(f) \quad on \quad \Gamma_0$$

for all $f \in H^{1/2}(\partial\Omega)$ *with* $\operatorname{supp}(f) \subset \Gamma_0$, *then* $D_1 = D_2$.

Remarks:
1. We do not assume $\tilde{\mathbb{C}}_1 = \tilde{\mathbb{C}}_2$ in Theorem 4.1. Also, the regularity of the medium inside of the inclusions is only assumed to be essentially bounded.
2. In [10], a reconstruction algorithm for recovering the cavity D is given. The same algorithm can be applied to Theorem 4.2 as well.

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