Hardy spaces for non-compactly causal symmetric spaces and the most continuous spectrum

Simon Gindikin*, Bernhard Krötz and Gestur Ólafsson*****

Abstract

Let G/H be a semisimple symmetric space. Then the space $L^2(G/H)$ can be decomposed into a finite sum of series representations induced from parabolic subgroups of G. The most continuous part of the spectrum of $L^2(G/H)$ is the part induced from the smallest possible parabolic subgroup. In this paper we introduce Hardy spaces canonically related to this part of the spectrum for a class of non-compactly causal symmetric spaces. The Hardy space is a reproducing Hilbert space of holomorphic functions living on a tube type bounded symmetric space, containing G/H as a boundary component. A boundary value map is constructed and we show that it induces an G-isomorphism onto a multiplicity free subspace of full spectrum in the most continuous part $L^2_{\mathrm{mc}}(G/H)$ of $L^2(G/H)$. We also relate our Hardy space with the classical Hardy space on the tube domain.

Introduction

When we transfer from harmonic analysis of Riemannian symmetric spaces to non Riemannian semisimple symmetric spaces the most important new phenomenon is the existence of different series of representations. This paper lies in the framework of a geometrical understanding of the different series of representations. The idea can be traced back to the paper [GeGi77], where a program was suggested to connect different series with a complex geometry of the complexification of symmetric spaces. The first step in this program was the realization of the holomorphic discrete series for groups of Hermitian type G in appropriate Hardy spaces in some tubes in $G_{\mathbb{C}}$ which have the group G as Shilov boundary. This was accomplished by G.I. Olshanski and R. Stanton [Ol82, St86].

It turned out that there is a natural class of symmetric spaces G/H, the *compactly causal* symmetric spaces, on which it is possible to generalize the constructions for Hermitian groups. In [ÓØ88] the holomorphic discrete series for G/H were constructed and the appropriate Hardy space was defined and investigated in [HÓØ91]. The explicit form of the Plancherel density and

* Supported in part by the NSF-grant DMS-0070816 and the MSRI
** Supported in part by the NSF-grant DMS-0097314 and the MSRI
*** Supported in part by the NSF-grant DMS-0070607 and the MSRI

the corresponding projection operators for the holomorphic discrete series were determined by the two last named authors [Kr01,KrÓl99, Ól00].

There are in principle two different classes of causal symmetric spaces, the *compactly causal* symmetric spaces and the *non-compactly causal* symmetric spaces. Let us mention that these two classes of symmetric spaces look completely different from the point of view of harmonic analysis.

For compactly causal symmetric spaces G/H one has a rich well understood complex geometry: There exist G-invariant tubes in $G_\mathbb{C}/H_\mathbb{C}$ which have G/H as Shilov boundary; secondly, most of the compactly causal symmetric spaces can be realized as an open dense orbit in the Shilov-boundary of a bounded Hermitian symmetric space of tube type [Ber96, Bet97, ÓØ99]. For non-compactly causal symmetric spaces it was believed for a long time that a similar complex geometrical picture does not exist. In [Gi98] it was conjectured that appropriate tubes (related to complex crowns of Riemannian symmetric spaces) might also exist for non-compactly causal symmetric spaces. Also it was conjectured that some Hardy spaces on these tubes are connected with the most continuous spectrum (contrary to the compactly causal symmetric spaces where Hardy spaces correspond to the holomorphic discrete series).

In the last year substantial progress was achieved on the complex geometry for non-compactly causal symmetric spaces. This gives us now the possibility to start the corresponding analytical investigations – the subject proper of this paper.

Now a few remarks on geometric results. The complex crown of a Riemannian symmetric space G/K, first studied in [AkGi90], is a certain open connected G-invariant Stein domain in $G_\mathbb{C}/K_\mathbb{C}$ which contains G/K as a totally real submanifold (cf. Section 1 for the definition). It is universal in many ways: It can be used to parametrize the compact cycles in complex flag manifolds (cf. [GiMa01]). Further it has the property that all eigenfunctions on G/K for the algebra of G-invariant differential operators holomorphically extend to Ξ [KrSt01a]. For a more detailed account on the complex crown Ξ we refer to the introduction of [GiKr01a].

By the recent work of the two first named authors [GiKr01b] we know that all non-compactly causal symmetric spaces G/H appear in the *distinguished boundary* $\partial_d\Xi \subseteq G_\mathbb{C}/K_\mathbb{C}$ of Ξ, and, moreover, the non-compactly causal spaces are the only symmetric spaces which can be realized as G-orbits in $\partial_d\Xi$. Let us emphasize that one can interpret this as a connection of the elliptic analysis of the Riemannian symmetric space G/K with the hyperbolic analysis on the non-compactly symmetric spaces G/H "through the complex crown Ξ". Such kind of connections are well known in classical mathematical physics (transformation of solutions of Laplace equations in solutions of wave equations and back etc).

In this paper we consider those complex crowns which admit a realization as a symmetric space of Hermitian type [KrSt01b]. By this we mean that there exists a Hermitian group $S \supseteq G$ with maximal compact subgroup $U \supseteq K$ such that Ξ is G-biholomorphic to S/U. For these cases the distinguished boundary $\partial_d\Xi = G(z_1) \simeq G/H$ is exactly one G-orbit and, moreover, the distinguished boundary is equal to the Shilov boundary $\partial_s\Xi$ of Ξ in $G_\mathbb{C}/K_\mathbb{C}$ [GiKr01b]. Under the additional assumption that G/H is symmetric, hence a non-compactly causal space, we arrive at the following list of our concern:

$\mathfrak{g} = \mathrm{Lie}(G)$	$\mathfrak{h} = \mathrm{Lie}(H)$	$\mathfrak{s} = \mathrm{Lie}(S)$
$\mathfrak{sp}(n, \mathbb{R})$	$\mathfrak{gl}(n, \mathbb{R})$	$\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{sp}(n, \mathbb{R})$
$\mathfrak{su}(n, n)$	$\mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{R}$	$\mathfrak{su}(n, n) \oplus \mathfrak{su}(n, n)$
$\mathfrak{so}^*(4n)$	$\mathfrak{sl}(n, \mathbb{H}) \oplus \mathbb{R}$	$\mathfrak{so}^*(4n) \oplus \mathfrak{so}^*(4n)$
$\mathfrak{so}(2, n)$	$\mathfrak{so}(1, n-1) \oplus \mathbb{R}$	$\mathfrak{so}(2, n) \oplus \mathfrak{so}(2, n)$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-26)} \oplus \mathbb{R}$	$\mathfrak{e}_{7(-25)} \oplus \mathfrak{e}_{7(-25)}$
$\mathfrak{so}(1, n)$	$\mathfrak{so}(1, n-1)$	$\mathfrak{so}(2, n)$
$\mathfrak{sp}(n, n)$	$\mathfrak{sp}(n, \mathbb{C})$	$\mathfrak{su}(2n, 2n)$
$\mathfrak{sp}(n, \mathbb{C})$	$\mathfrak{sp}(n, \mathbb{R})$	$\mathfrak{sp}(2n, \mathbb{R})$

We call the triples $(\mathfrak{s}, \mathfrak{g}, \mathfrak{h})$ from the above list *causally symmetric triples.* They can be defined axiomatically starting from a Hermitian Lie algebra $\mathfrak{s}$ of tube type with two involutions τ and σ such that:

(CST1) $\mathfrak{s}^\tau = \mathfrak{g}$ and $(\mathfrak{s}, \tau)$ is compactly causal.

(CST2) $\mathfrak{g}^\sigma = \mathfrak{h}$ and $(\mathfrak{g}, \sigma|_{\mathfrak{g}})$ is non-compactly causal.

Following É. Cartan the Hermitian symmetric space S/U admits a canonical embedding in the dual compact Hermitian symmetric space and can be realized as a bounded circled domain in $\mathbb{C}^n$ (the realization of disc type where the isotropy subgroup acts linearily). We will use Harish-Chandra's construction of the realization of S/U as a bounded symmetric domain $\mathcal{D}$. There is also the realization of S/U as an affine homogeneous tube domain $T_\Omega = \mathbb{R}^n + i\Omega$ (the realization of halfplane type). The realization Ξ we use is different from the two mentioned above– it lies inside $G_\mathbb{C}/K_\mathbb{C}$ and is connected with the non-transitive action of the group G.

With the different realizations of S/U one becomes different Shilov boundaries: A compact Shilov boundary $\partial_s\mathcal{D}$ for $\mathcal{D}$ which is a single S-orbit, and a vector space $\mathbb{R}^n$ for the Shilov boundary of the tubes T_Ω. For the Ξ-realization we have a single G-orbit for the Shilov boundary.

Generally, Hardy spaces correspond to Hilbert spaces of holomorphic functions which have L^2-boundary values on the Shilov boundary. Since we have in the three cases mentioned above different Shilov boundaries, this necessarily leads to different definitions of Hardy spaces. In the first two cases the constructions of Hardy spaces $\mathcal{H}^2(\mathcal{D})$ and $\mathcal{H}^2(T_\Omega)$ are well known; they are isomorphic as irreducible S-modules. We refer to them as *classical Hardy spaces.*

For the definition of a Hardy space on Ξ one encounters an essential geometric difficulty, namely that G does not act transitively. We use an open compression subsemigroup of $\mathcal{D}$ inside of $S_\mathbb{C}$, discovered by Olshanski. We denote by Γ^{-1} the intersection of this semigroup with $G_\mathbb{C}$. Since Ξ is biholomorphic to $\mathcal{D}$ we obtain an action of Γ on Ξ by compressions. We now can define a *Hardy space* $\mathcal{H}^2(\Xi)$ on Ξ by

$$\mathcal{H}^2(\Xi) := \{f \in \mathcal{O}(\Xi) : \|f\|^2 := \sup_{\gamma\in\Gamma} \int_{G/H} |f(\gamma^{-1}gz_1)|^2 \, d\mu_{G/H}(gH) < \infty\}.$$

It turns out that the Hardy space $\mathcal{H}^2(\Xi)$ is a Hilbert space of holomorphic functions on Ξ with a natural unitary action L of G by left translations in the arguments. In particular, $\mathcal{H}^2(\Xi)$ admits a reproducing kernel $K_\Xi(z, w)$, the so-called *Cauchy-Szegö kernel*, holomorphic in the first and antiholomorphic in the second variable.

One of our tools is to compare this space with the classical Hardy space $\mathcal{H}^2(\mathcal{D})$ which is an irreducible G-spherical unitary highest weight module for the big group S. The action of S on $\mathcal{H}^2(\mathcal{D})$ is given through a cocycle representation

$$(\pi_h(s)f)(z) = J_h(s^{-1}, z)^{-1} f(s^{-1}z) \qquad (f \in \mathcal{H}^2(\mathcal{D}), s \in S, z \in \mathcal{D}).$$

Theorem 5.7 now reads:

Theorem A. *There exists an explicit zero free holomorphic function* ψ *on* $\Xi \simeq \mathcal{D}$ *such that the mapping*

$$\Psi: (\pi_h|_G, \mathcal{H}^2(\mathcal{D})) \to (L, \mathcal{H}^2(\Xi)), \quad f \mapsto \psi f$$

is a G*-equivariant isomorphism of Hilbert spaces.*

The fact that G/K is a totally real submanifold of S/U can now be used to construct the generalized Segal-Bargmann transform, which is a unitary G-isomorphism $L^2(G/K) \to \mathcal{H}^2(\Xi)$, [DÓZ01,Ner99,Ól00,ÓØ96,Zh01]. This implies that the continuous branching of the Hardy space representation. Combining Theorem 5.5 and Theorem 5.8 we come to our main result:

Theorem B. *The boundary value mapping*

$$b: \mathcal{H}^2(\Xi) \to L^2(G/H), \quad f \mapsto \big(gH \mapsto \lim_{\substack{\gamma \to 1 \\ \gamma \in \Gamma}} f(\gamma^{-1} g z_1)\big)$$

is a G*-equivariant isometric embedding. Furthermore the image of* b *is a multiplicity free full subspace of the most continuous spectrum* $L^2(G/H)_{\mathrm{mc}}$ *(cf. Definition 4.1).*

Finally we compare the reproducing kernel K_Ξ of $\mathcal{H}^2(\Xi)$ with the Cauchy-Szegö kernel K_h from the classical Hardy space. Theorem 5.9 from below says:

Theorem C. *The Cauchy-Szegö kernels* K_Ξ *and* K_h *are related through*

$$K_\Xi(z,w) = \psi(z)\overline{\psi(w)} K_h(z,w).$$

To illustrate Theorem C consider the example where $S = G \times G$. Then Ξ is G-biholomorphic to $\mathcal{D}_G \times \mathcal{D}_G^{\mathrm{opp}}$ where $\mathcal{D}_G$ is the Harish-Chandra realization of G/K and $\mathcal{D}_G^{\mathrm{opp}}$ refers to $\mathcal{D}_G$ equipped with the opposite complex structure. Now if $K_h^G(z,w)$ denotes the Cauchy-Szegö kernel of the classical Hardy space $\mathcal{H}^2(\mathcal{D}_G)$, then Theorem C boils down to (see Example 5.10 below)

$$K_\Xi\big((z_1,w_1),(z_2,w_2)\big) = \frac{K_h^G(z_1,z_2)K_h^G(w_2,w_1)}{K_h^G(z_1,w_1)K_h^G(w_2,z_2)} \qquad (\forall (z_1,w_1),(z_2,w_2) \in \Xi \simeq \mathcal{D}_G \times \mathcal{D}_G^{\mathrm{opp}}) .$$

Note that the formula of K_Ξ is explicit, since the classical Hardy space kernels K_h^G are well known.

In the Plancherel decomposition of $L^2(G/H)$ the most continuous spectrum $L^2_{\mathrm{mc}}(G/H)$ corresponds to one series of representations, the one which is induced off from a minimal parabolic subgroup. However, the space $L^2_{\mathrm{mc}}(G/H)$ is not multiplicity free and we encounter the next problem to separate the different multiplicities in $L^2_{\mathrm{mc}}(G/H)$ geometrically. Theorem B implies an embedding

$$\mathcal{H}^2(\Xi) \oplus \mathcal{H}^2(\Xi^{\mathrm{opp}}) \hookrightarrow L^2(G/H)_{\mathrm{mc}}$$

which realizes a multiplicity *two* subspace of the most continuous spectrum of G/H. Presently it is unclear how to obtain the other multiplicities in $L^2(H/H)_{\mathrm{mc}}$ missed by $\mathcal{H}^2(\Xi) \oplus \mathcal{H}^2(\Xi^{\mathrm{opp}})$. In this context notice the striking similarity with the picture for discrete series representations. Only two out of the discrete series, the holomorphic and the antiholomorphic, can be realized as a Hardy space.

We would like to mention, as we know through several talks, that J. Faraut is working on another construction of Hardy spaces for certain non-compactly causal symmetric spaces.

Our paper is organized as follows:

It is our pleasure to thank the MSRI, Berkeley, for its hospitality during the *Integral geometry program* where this work was accomplished.

1. Non-compactly causal symmetric spaces and boundary components of complex domains

Causal symmetric Lie algebras

Let $\mathfrak{s}$ denote a semisimple real Lie algebra with complexification $\mathfrak{s}_\mathbb{C}$. Let S be a connected Lie group with Lie algebra $\mathfrak{s}$ and let $S_\mathbb{C}$ denote a complex Lie group with Lie algebra $\mathfrak{s}_\mathbb{C}$. We will assume – if not otherwise stated – that $S \subseteq S_\mathbb{C}$. Denote by θ a Cartan involution on $\mathfrak{s}$ with Cartan decomposition $\mathfrak{s} = \mathfrak{u} \oplus \mathfrak{p}_*$. Let $\tau: \mathfrak{s} \to \mathfrak{s}$ be an involution on $\mathfrak{s}$ of which we may assume to commute with θ. We will also assume that τ defines an involution on $S_\mathbb{C}$. Write $\mathfrak{s} = \mathfrak{g} \oplus \mathfrak{q}_*$ for the τ-eigenspace decomposition corresponding to the τ-eigenvalues $+1$ and -1. Then $\mathfrak{g}$ is reductive and $\theta|_\mathfrak{g}$ is a Cartan involution of $\mathfrak{g}$. Set $\mathfrak{k} = \mathfrak{g} \cap \mathfrak{u}$ and $\mathfrak{p} := \mathfrak{g} \cap \mathfrak{p}_*$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. Let G be the connected subgroup of S with Lie algebra $\mathfrak{g}$. The symmetric pair $(\mathfrak{s}, \mathfrak{g})$ is called *irreducible* if the only τ-invariant ideals in $\mathfrak{s}$ are the trivial ones, $\{0\}$ and $\mathfrak{s}$. In this case either $\mathfrak{s}$ is simple or $\mathfrak{s} \simeq \mathfrak{g} \oplus \mathfrak{g}$, with $\mathfrak{g}$ simple, and $\tau(X,Y) = (Y,X)$ the flip. In the second case we have $S/G \ni (a,b)G \mapsto ab^{-1} \in G$ defines an isomorphism of S/G with G. We say that the symmetric space S/G is irreducible if $(\mathfrak{s}, \mathfrak{g})$ is irreducible.

Definition 1.1. Let C be an open convex subset of $\mathfrak{q}_*$. Then C is called *hyperbolic*, if for all $X \in C$ the operator $\mathrm{ad}(X)$ is semisimple with real eigenvalues. We call C *elliptic*, if all operators $\mathrm{ad}(X)$, $X \in C$ are semisimple and with imaginary spectrum. ■

We recall now the definition of causal symmetric spaces (c.f. [HiÓl96], Chapter 3):

Definition 1.2. **(Causal symmetric spaces)** The symmetric space S/G is called *causal* if there exists a non-empty open G-invariant convex cone C , containing no affine lines, in $\mathfrak{q}_*$. ■

There are two different types of causal symmetric spaces, the non-compactly causal symmetric spaces (NCC) and the compactly causal symmetric spaces (CC). In addition, there is the intersection of those two classes, the Cayley type symmetric spaces (CT).

Definition 1.3. **(NCC)** Assume that S/G is an irreducible causal symmetric space. Then the following two conditions are equivalent:

(a) There exists a non-empty G-invariant open hyperbolic cone $C \subseteq \mathfrak{q}_*$ which contains no affine lines;

(b) There exists an element $T_0 \in \mathfrak{q}_* \cap \mathfrak{p}_*$, $T_0 \neq 0$, which is fixed by K.
If one of those equivalent conditions are satisfied, then S/G is called *non-compactly causal*. ■

Definition 1.4. **(CC)** Assume that S/G is an irreducible causal symmetric space. Then the following two conditions are equivalent:

(a) There exists a non-empty G-invariant open elliptic cone $C \subseteq \mathfrak{q}_*$ which contains no affine lines;

(b) There exists an element $X_0 \in \mathfrak{q}_* \cap \mathfrak{u}$, $X_0 \neq 0$, which is fixed by K.
If one of those equivalent conditions are satisfied, then S/G is called *compactly causal*. ■

Definition 1.5. **(CT)** Assume that S/G is an irreducible causal symmetric space. Then S/G is called a *symmetric space of Cayley type*, if it is both non-compactly causal and compactly causal. ■

Remark 1.6. (a) The elements T_0 and X_0 in Definition 1.3 and Definition 1.4 are unique up to multiplication by scalar. If S/G is NCC then we normalize T_0 such that $\mathrm{ad}(T_0)$ has spectrum $\{0, 1, -1\}$. The eigenspace corresponding to 0 is exactly $\mathfrak{s}^{\theta\tau} = \mathfrak{k} \oplus (\mathfrak{p}_* \cap \mathfrak{q}_*)$. If S/G is CC, then we normalize X_0 such that the spectrum of $\mathrm{ad}(iX_0)$ is $\{0, 1, -1\}$. In this case the zero eigenspace is exactly $\mathfrak{u}$.
(b) If S/G is compactly causal, then $\mathfrak{u}$ has a non trivial center $\mathfrak{z}(\mathfrak{u})$ and $\mathfrak{z}(\mathfrak{u}) \cap \mathfrak{q}_* = \mathbb{R}X_0$. If S is simple then $\mathfrak{z}(\mathfrak{u}) = \mathbb{R}X_0$. If $\mathfrak{s} \simeq \mathfrak{g} \oplus \mathfrak{g}$. Then $X_0 = (X_0', -X_0')$ with X_0' central in $\mathfrak{k}$ and $\mathfrak{z}(\mathfrak{u}) = \mathbb{R}X_0 \oplus \mathbb{R}(X_0', X_0')$. ■

Denote the complex linear extension of τ to $\mathfrak{s}_{\mathbb{C}}$ also by τ. The c-dual $(\mathfrak{s}^c, \tau^c)$ of $(\mathfrak{s}, \tau)$ is defined by $\mathfrak{s}^c = \mathfrak{g} \oplus i\mathfrak{q}_* \subseteq \mathfrak{s}_{\mathbb{C}}$ with involution $\tau|_{\mathfrak{s}^c}$. Notice that the c-dual of $(\mathfrak{s}^c, \tau^c)$ is $(\mathfrak{s}, \tau)$ back again. Let S^c denote the analytic subgroup of $S_{\mathbb{C}}$ with Lie algebra $\mathfrak{s}^c$. Then $G = (S \cap S^c)_0$, where the subscript $_0$ denotes the connected component containing $\mathbf{1}$.

We recall the following fact:

Proposition 1.7. *Assume that S/G is an irreducible symmetric space. Then the following holds:*

(i) *S/G is non-compactly causal if and only if S^c/G is compactly causal.*

(ii) *S/G is of Cayley type if and only if $S/G \simeq S^c/G$.*

(iii) *Suppose that S/G is irreducible and causal. Then S/G is of Cayley type if and only if G is not simple. In that case $\mathfrak{z}(\mathfrak{g})$ is one dimensional and contained in $\mathfrak{p}$.* ■

Example 1.8. **(The group case)** Let G be a connected semisimple Lie group, $S = G \times G$, and $\tau(a, b) = (b, a)$. Then S/G is compactly causal if and only if G/K is a Hermitian symmetric space. In this case $S_{\mathbb{C}} = G_{\mathbb{C}} \times G_{\mathbb{C}}$ and $S^c/G \simeq G_{\mathbb{C}}/G$ (cf. [HiÓl96], Example 1.2.2). ■

The following three tables list the causal symmetric pairs (cf. [HiÓl96, Th. 3.2.8]).

Table I
The group case

$\mathfrak{s}$ CC	$\mathfrak{s}^c$ NCC	$\mathfrak{g}$
$\mathfrak{su}(p,q)\oplus\mathfrak{su}(p,q)$	$\mathfrak{sl}(p+q,\mathbb{C})$	$\mathfrak{su}(p,q)$
$\mathfrak{so}^*(2n)\oplus\mathfrak{so}^*(2n)$	$\mathfrak{so}(2n,\mathbb{C})$	$\mathfrak{so}^*(2n)$
$\mathfrak{so}(2,n)\oplus\mathfrak{so}(2,n)$	$\mathfrak{so}(n+2,\mathbb{C})$	$\mathfrak{so}(2,n)$
$\mathfrak{sp}(n,\mathbb{R})\oplus\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{sp}(n,\mathbb{R})$
$\mathfrak{e}_{6(-14)}\oplus\mathfrak{e}_{6(-14)}$	$\mathfrak{e}_6$	$\mathfrak{e}_{6(-14)}$
$\mathfrak{e}_{7(-25)}\oplus\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_7$	$\mathfrak{e}_{7(-25)}$

Table II
$\mathfrak{s}$ simple and not CT

$\mathfrak{s}$ CC	$\mathfrak{s}^c$ NCC	$\mathfrak{g}$
$\mathfrak{su}(p,q)$	$\mathfrak{sl}(p+q,\mathbb{R})$	$\mathfrak{so}(p,q)$
$\mathfrak{su}^*(2p,2q)$	$\mathfrak{su}^*(2(p+q)$	$\mathfrak{sp}(p,q)$
$\mathfrak{so}^*(2n)$	$\mathfrak{so}(n,n)$	$\mathfrak{so}(n,\mathbb{C})$
$\mathfrak{so}(2,n)$	$\mathfrak{so}(1,n+1)$	$\mathfrak{so}(1,n)$
$\mathfrak{so}(2,p+q)$	$\mathfrak{so}(p+1,q+1)$	$\mathfrak{so}(1,p)\oplus\mathfrak{so}(1,q)\,,\ p,q\geq 2$
$\mathfrak{sp}(2n,\mathbb{R})$	$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{sp}(n,\mathbb{C})$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{e}_{6(6)}$	$\mathfrak{sp}(2,2)$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{e}_{6(-26)}$	$\mathfrak{f}_{4(-20)}$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{7(7)}$	$\mathfrak{su}^*(8)$

Table III
$\mathfrak{s}$ simple and CT

$\mathfrak{s}$	$\mathfrak{s}^c\simeq\mathfrak{s}$	$\mathfrak{g}$
$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{gl}(n,\mathbb{R})$
$\mathfrak{su}(n,n)$	$\mathfrak{su}(n,n)$	$\mathfrak{sl}(n,\mathbb{C})\oplus\mathbb{R}$
$\mathfrak{so}^*(4n)$	$\mathfrak{so}^*(4n)$	$\mathfrak{su}^*(2n)\oplus\mathbb{R}$
$\mathfrak{so}(2,n)$	$\mathfrak{so}(2,n)$	$\mathfrak{so}(1,n-1)\oplus\mathbb{R}$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-26)}\oplus\mathbb{R}$

The domain Ξ – the complex crown of G/K

According to [GiKr01a] all non-compactly causal symmetric spaces G/H can be realized as a boundary components of a certain complex domain wich we discuss now.

Let $\mathfrak{g}$ be a reductive Lie algebra with complexification $\mathfrak{g}_\mathbb{C}$. We let $G_\mathbb{C}$ be a connected complex Lie group with Lie algebra $\mathfrak{g}_\mathbb{C}$ and take G as the analytic subgroup of $G_\mathbb{C}$ corresponding to $\mathfrak{g}$. Let $K<G$ be a maximal compact subgroup and $\mathfrak{k}$ its Lie algebra. Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ the Cartan decomposition attached to $\mathfrak{k}$. Let $\mathfrak{a}\subseteq\mathfrak{p}$ a maximal abelian subspace, set $\mathfrak{m}=\mathfrak{z}_\mathfrak{k}(\mathfrak{a})$ and for $\alpha\in\mathfrak{a}^*$ define

$$\mathfrak{g}^\alpha=\{X\in\mathfrak{g}\colon(\forall H\in\mathfrak{a})[H,X]=\alpha(H)X\}\,.$$

Let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ be the corresponding set of (restricted) roots, i.e., $\Sigma = \{\alpha \in \mathfrak{a}^* \setminus \{0\} : \mathfrak{g}^\alpha \neq \{0\}\}$. Then

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}^\alpha .$$

We write A for the analytic subgroup of G corresponding to $\mathfrak{a}$ and $\mathfrak{n}$. We also need the *Weyl group* of $\Sigma(\mathfrak{a}, \mathfrak{g})$, $\mathcal{W} := N_K(\mathfrak{a})/Z_K(\mathfrak{a})$.

Let

$$\Omega = \{X \in \mathfrak{a} : (\forall\alpha \in \Sigma)\ |\alpha(X)| < \frac{\pi}{2}\}.$$

Then the *complex crown* of the Riemannian symmetric space G/K is defined by

$$\Xi := G \exp(i\Omega) K_{\mathbb{C}}/K_{\mathbb{C}} .$$

Note that Ξ is an open domain in $G_{\mathbb{C}}$ and that G acts properly on it (cf. [AkGi90]). Write $\partial\Xi$ for the topological boundary of Ξ and notice that $G \exp(i\partial\Omega) K_{\mathbb{C}}/K_{\mathbb{C}} \subseteq \partial\Xi$ (cf. [AkGi90]).

The distinguished boundary of Ξ

Write $\overline{\Xi}$ for the closure of Ξ in $G_{\mathbb{C}}/K_{\mathbb{C}}$, and note that $\overline{\Xi} = \Xi \amalg \partial\Xi$. Write $\partial_e\Omega$ for the extreme points of the compact convex set $\overline{\Omega}$. Recall from [GiKr01a] that $\partial_e\Omega$ is a finite union of $\mathcal{W}$-orbits:

$$\partial_e\Omega = \mathcal{W}(X_1) \amalg \ldots \amalg \mathcal{W}(X_n).$$

We define the *distinguished boundary of* Ξ *in* $G_{\mathbb{C}}/K_{\mathbb{C}}$ by

$$\partial_d\Xi := G \exp(i\partial_e\Omega) K_{\mathbb{C}}/K_{\mathbb{C}}.$$

It was shown in [GiKr01a, Th. 2.3] that $\partial_d\Xi$ is a subset of $\partial\Xi$ which satisfies the property

$$\sup_{z\in\Xi} |f(z)| = \sup_{z\in\partial_d\Xi} |f(z)| \tag{1.1}$$

for all bounded holomorphic functions f on Ξ which continuously extend to $\overline{\Xi}$. Moreover, if $\partial_d\Xi$ is connected, i.e., if $\partial_e\Omega = \mathcal{W}(X_1)$ is a single orbit, then $\partial_d\Xi$ is minimal with respect to the property (1.1). In this case we call $\partial_d\Xi$ the *Shilov boundary* of Ξ and denote it by $\partial_s\Xi$.

Write $z_j := \exp(iY_j)K_{\mathbb{C}} \in \partial_d\Xi$ and write H_j for the isotropy subgroup of G in z_j. Then we have the following (cf. [GiKr01a]):

Theorem 1.9. *Assume that $G_{\mathbb{C}}$ is simply connected. Then, with the notation introduced above, the following assertions hold:*

(i) $\partial_d\Xi \simeq G/H_1 \amalg \ldots \amalg G/H_n$.

(ii) *If one of the boundary components G/H_j is a symmetric space, then it is a non-compactly causal symmetric space. Moreover every non-compactly causal symmetric space G/H attached to G appears G-locally in the distinguished boundary of $\partial_d\Xi$ of Ξ.* ■

Realization of Ξ as tube domains

We consider now the case where Ξ has a realization as a tube domain. The material in this section is taken from [KrSt01b]. Throughout this section $(\mathfrak{s}, \tau)$ denotes a compactly causal symmetric Lie algebra. Let $\mathfrak{s}_\mathbb{C}$ denote the complexification of $\mathfrak{s}$ and let $S_\mathbb{C}$ denote the *simply connected* complex Lie group with Lie algebra $\mathfrak{s}_\mathbb{C}$. Let $S \subseteq S_\mathbb{C}$ be the connected subgroup corresponding to $\mathfrak{s}$. Let G be the analytic subgroup of S corresponding to $\mathfrak{g}$. Write U and K for the maximal compact subgroups in S and G. Note that the embedding

$$G/K \hookrightarrow S/U, \quad gK \mapsto gU$$

realizes G/K as a totally real submanifold of the Hermitian symmetric space S/U.

Recall that $\mathfrak{z}(\mathfrak{u}) \cap \mathfrak{q}_* = \mathbb{R}X_0$ is one dimensional and that we can normalize X_0 such that $\mathrm{Spec}(\mathrm{ad}\, iX_0) = \{0, 1, -1\}$. We let $\mathfrak{t} \subseteq \mathfrak{u}$ be a compact τ-stable Cartan algebra of $\mathfrak{s}$ and note that $\mathfrak{z}(\mathfrak{u}) \subseteq \mathfrak{t}$. Write $\Delta = \Delta(\mathfrak{s}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ for the root system with respect to $\mathfrak{t}_\mathbb{C}$. Call a root α *compact* if $\alpha\,|_{\mathfrak{z}(\mathfrak{u})} = 0$ and *non-compact* otherwise. Thus α is non-compact if and only if $\mathfrak{g}_\mathbb{C}^\alpha \subset \mathfrak{p}_\mathbb{C}$. Denote by Δ_c the set of compact roots and by Δ_n the set of non-compact roots. Let $\Delta_n^+ := \{\alpha \in \Delta \colon \alpha(iX_0) = 1\}$ and fix a positive system Δ^+ such that $\Delta_n^+ \subseteq \Delta^+$. Write $\mathfrak{p}^\pm = \oplus_{\alpha\in\Delta_n^+} \mathfrak{g}_\mathbb{C}^{\pm\alpha} \subseteq \mathfrak{p}_\mathbb{C}$. Then we have the triangular decomposition of $\mathfrak{s}_\mathbb{C}$:

$$\mathfrak{s}_\mathbb{C} = \mathfrak{p}^+ \oplus \mathfrak{u}_\mathbb{C} \oplus \mathfrak{p}^-.$$

Let $P^\pm = \exp(\mathfrak{p}^\pm)$. Let $U_\mathbb{C} \subseteq S_\mathbb{C}$ be the complexification of U. Then it is well known that $S \subseteq P^+ U_\mathbb{C} P^-$ and we have the Borel embedding

$$S/U \hookrightarrow S_\mathbb{C}/U_\mathbb{C} P^-.$$

The map $P^+ \times U_\mathbb{C} \times P^- \ni (p^+, u, p^-) \mapsto p^+ u p^- \in P^- U_\mathbb{C} P^+$ is a diffeomorphism onto the open dense subset $P^+ U_\mathbb{C} P^- \subseteq S_\mathbb{C}$. We write $s = p^+(s)\kappa(s)p^-(s)$ for the unique decomposition of an element $s \in P^+ U_\mathbb{C} P^-$. Then we have the Harish Chandra realization of S/U as a bounded symmetric domain $\mathcal{D} \subseteq \mathfrak{p}^+$ given by

$$S/U \ni sU \mapsto \log(p^+(s)) \in \mathfrak{p}^+$$

where $\log = (\exp|_{\mathfrak{p}^+})^{-1} : P^+ \to \mathfrak{p}^+$.

Write $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ for the restricted root system with respect to $\mathfrak{a}$ and $\widehat{\Sigma} = \Sigma(\mathfrak{s}, \mathfrak{a})$ for the double restricted root system. If Σ is an abstract root system on $\mathfrak{a}$, then we also write $\Omega = \Omega(\Sigma)$ to indicate the dependance of Ω on Σ. Define

$$\Xi_0 := G \exp(i\Omega(\widehat{\Sigma}))K_\mathbb{C}/K_\mathbb{C},$$

and notice that $\Xi_0 \subseteq \Xi$ since $\Sigma \subseteq \widehat{\Sigma}$.

Theorem 1.10. *(cf.* [KrSt01b]*) The embedding $G/K \hookrightarrow S/U$ extends to a G-equivariant biholomorphism*

$$\Phi \colon \Xi_0 \to S/U \subseteq S_\mathbb{C}/U_\mathbb{C}P^-.$$

Further the following statements are equivalent:

(1) $\Xi = \Xi_0$.

(2) $\Omega(\widehat{\Sigma}) = \Omega(\Sigma)$.

(3) Σ *is of type* C_n *or* BC_n *for* $n \geq 2$ *or* $(\mathfrak{g}, \mathfrak{s})$ *is one of* $(\mathfrak{su}(1,1), \mathfrak{su}(1,1) \oplus \mathfrak{su}(1,1))$ *or* $(\mathfrak{so}(1,n), \mathfrak{so}(2,n))$. ■

The non-compactly causal symmetric spaces G/H which are singled out by the additional requirement that $\Xi = \Xi_0 \simeq S/U$ are the ones that are most important for the rest of this article. We will therefore discuss them in more details now. The non-compactly causal symmetric spaces which satisfy in addition $\Xi = \Xi_0$ are exactly those where Σ is of type C_r, say

$$\Sigma = \{\frac{1}{2}(\pm\gamma_i \pm \gamma_j): 1 \leq i,j \leq r\}\backslash\{0\}.$$

Define $Y_j \in \mathfrak{a}$ by $\gamma_i(Y_j) = 2\delta_{ij}$ and $Y_0 = \frac{1}{2}(Y_1 + \ldots + Y_r)$. In this case $n = 1$, $X_1 = \frac{\pi}{2}Y_0$ and

$$\partial_e\Omega = \mathcal{W}(\frac{\pi}{2}Y_0)\,.$$

Hence with $z_1 := \exp(i\frac{\pi}{2}Y_0)K_{\mathbb{C}}$ we have $\partial_s\Xi = G(z_1)$ is a connected G-space. Write H for the isotropy subgroup of G in z_1 and $\mathfrak{h}$ for the Lie algebra of H. Sometimes we also write $\mathbf{c}$ for z_1 since $\mathrm{Ad}(\mathbf{c})$ is a partial Cayley transform. It follows from [GiKr01b] that

$$\sigma: G \to G, \quad g \mapsto (\mathrm{Ad}(\mathbf{c}^2) \circ \theta)(g)$$

is an involution on G such that $H = G^\sigma$. Comparing Theorem 1.10 with the list in the tables I, II and III, we arrive at the following:

Table IV
$\Xi \simeq S/U$ and $\partial_s\Xi \simeq G/H$

$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{s}$
$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{gl}(n,\mathbb{R})$	$\mathfrak{sp}(n,\mathbb{R}) \oplus \mathfrak{sp}(n,\mathbb{R})$
$\mathfrak{su}(n,n)$	$\mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{R}$	$\mathfrak{su}(n,n) \oplus \mathfrak{su}(n,n)$
$\mathfrak{so}^*(4n)$	$\mathfrak{sl}(n,\mathbb{H}) \oplus \mathbb{R}$	$\mathfrak{so}^*(4n) \oplus \mathfrak{so}^*(4n)$
$\mathfrak{so}(2,n)$	$\mathfrak{so}(1,n-1) \oplus \mathbb{R}$	$\mathfrak{so}(2,n) \oplus \mathfrak{so}(2,n)$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-26)} \oplus \mathbb{R}$	$\mathfrak{e}_{7(-25)} \oplus \mathfrak{e}_{7(-25)}$
$\mathfrak{so}(1,n)$	$\mathfrak{so}(1,n-1)$	$\mathfrak{so}(2,n)$
$\mathfrak{sp}(n,n)$	$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{su}(2n,2n)$
$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{sp}(2n,\mathbb{R})$

Note that the first five pairs $(\mathfrak{g},\mathfrak{h})$ in this list are of Cayley type. We call the triple $(\mathfrak{s},\mathfrak{g},\mathfrak{h})$ from table IV a *causally symmetric triple.* Note that we always have

$$\mathrm{rank}_{\mathbb{R}}(\mathfrak{h}) = \mathrm{rank}_{\mathbb{R}}(\mathfrak{g}) = \frac{1}{2}\,\mathrm{rank}_{\mathbb{R}}\,\mathfrak{s}.$$

We write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ for the σ-eigenspace decomposition of $\mathfrak{g}$. Note that σ extends to an involution on $\mathfrak{s}$ since $\mathrm{Spec}\,\mathrm{ad}_{\mathfrak{s}}(Y_0) = \{-1,0,1\}$. Our choice of the maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$ is then such that $\mathfrak{a} \subseteq \mathfrak{q} \cap \mathfrak{p}$.

Causally symmetric triples can also be defined axiomatically:

Definition 1.11. **(Causally symmetric triples)** A triple of Lie algebras $(\mathfrak{s},\mathfrak{g},\mathfrak{h})$ with $\mathfrak{h} \subseteq \mathfrak{g} \subseteq \mathfrak{s}$ is called a *causally symmetric triple* if the following axioms are satisfied:

(CST1) $\mathfrak{s}$ is Hermitian and of tube type.

(CST2) There exists two involutions τ and σ on $\mathfrak{s}$ such that:

(1) $(\mathfrak{s},\tau)$ is compactly causal and $\mathfrak{g} = \mathfrak{s}^\tau$.

(2) $(\mathfrak{g},\sigma|_{\mathfrak{g}})$ is non-compactly causal and $\mathfrak{h} = \mathfrak{g}^\sigma$. ■

If we only require that G/K embeds into S/U and that $\partial_d\Xi$ has at least one boundary G/H_j component (cf. Theorem 1.9) which is a non-compactly causal symmetric space, then we arrive at the following more general situation (cf. [KrSt01b] and [GiKr01b]).

Theorem 1.12. *Suppose that G/K is a totally real submanifold of S/U and that there exists a symmetric subgroup H of G such that G/H is a non-compactly causal symmetric space which is a G-orbit in $\partial_d\Xi$. With $\partial_d\Xi_0 = G\exp(i\partial_e\Omega(\widehat{\Sigma}))K_{\mathbb{C}}/K_{\mathbb{C}}$ the distinguished boundary of Ξ_0 we arrive at the following four possibilities:*

(i) $(\mathfrak{s},\mathfrak{g},\mathfrak{h})$ *is a causally symmetric triple.*

(ii) G *is the structure group of an Euclidian Jordan algebra, i.e.,* $\mathfrak{g}$ *is isomorphic to one of the following*

$$\mathfrak{gl}(n,\mathbb{R}) \qquad \mathfrak{sl}(n,\mathbb{C})\oplus\mathbb{R} \qquad \mathfrak{sl}(n,\mathbb{H})\oplus\mathbb{R} \qquad \mathfrak{so}(1,n)\oplus\mathbb{R} \qquad \mathfrak{f}_{4(-20)}\oplus\mathbb{R}\ ,$$

and $\partial_d\Xi_0 = \partial_d\Xi \amalg G/K \amalg G/K$.

(iii) $G = \mathrm{SO}(n,n)$, $\partial_d\Xi_0 = G/\mathrm{SO}(n,\mathbb{C})$, *and*

$$\partial_d\Xi = \partial_d\Xi_0 \amalg G/\big(\mathrm{SO}(1,n-1)\times\mathrm{SO}(1,n-1)\big)\,.$$

(iv) $G = \mathrm{SO}(2n,\mathbb{C})$, $\partial_d\Xi_0 = G/\mathrm{SO}^*(2n)$, *and* $\partial_d\Xi = \partial_d\Xi_0 \amalg G/\mathrm{SO}(2,2n-2)$. ∎

The compactification of the Shilov boundary $\partial_s\Xi$

For the remainder of this section we assume that $(\mathfrak{s},\mathfrak{g},\mathfrak{h})$ is a causally symmetric triple.

Proposition 1.13. *Assume that $(\mathfrak{s},\mathfrak{g},\mathfrak{h})$ is a causally symmetric triple. Then the mapping*

$$\iota\colon G_{\mathbb{C}}/K_{\mathbb{C}} \to S_{\mathbb{C}}/U_{\mathbb{C}}P^-, \quad gK_{\mathbb{C}} \mapsto gU_{\mathbb{C}}P^-$$

is an open $G_{\mathbb{C}}$-equivariant embedding.

Proof. Since $\mathfrak{s}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}} + \mathfrak{u}_{\mathbb{C}} + \mathfrak{p}^-$ it is clear that the image of ι is open. It remains to show that ι is injective, which means that $G_{\mathbb{C}} \cap U_{\mathbb{C}}P^- = K_{\mathbb{C}}$. Also denote by τ and θ the holomorphic extension of τ and θ to an involution on $S_{\mathbb{C}}$. Note that $S_{\mathbb{C}}^{\tau} = G_{\mathbb{C}}$ and $S_{\mathbb{C}}^{\theta} = U_{\mathbb{C}}$ since $S_{\mathbb{C}}$ was assumed to be simply connected.

We have $\tau(P^-) = P^+$. Thus we obtain

$$G_{\mathbb{C}} \cap U_{\mathbb{C}}P^- = G_{\mathbb{C}} \cap U_{\mathbb{C}}P^- \cap \tau(U_{\mathbb{C}}P^-) = G_{\mathbb{C}} \cap \big(U_{\mathbb{C}}P^- \cap U_{\mathbb{C}}P^+\big).$$

Now the well known fact $U_{\mathbb{C}}P^- \cap U_{\mathbb{C}}P^+ = U_{\mathbb{C}}$ implies that

$$G_{\mathbb{C}} \cap U_{\mathbb{C}}P^- = G_{\mathbb{C}} \cap U_{\mathbb{C}}.$$

It remains to show that $G_{\mathbb{C}} \cap U_{\mathbb{C}} = K_{\mathbb{C}}$. Now we are going to use the fact that the pair $(\mathfrak{s},\mathfrak{g},\mathfrak{h})$ is from table IV. Quick inspection shows that $G_{\mathbb{C}}$ is simply connected. Hence $G_{\mathbb{C}}^{\theta} = K_{\mathbb{C}}$ is connected and so $G_{\mathbb{C}} \cap U_{\mathbb{C}} = G_{\mathbb{C}} \cap S_{\mathbb{C}}^{\theta} = K_{\mathbb{C}}$, concluding the proof of the proposition. ∎

Write $\overline{\mathcal{D}}$ for the closure of the bounded symmetric domain in $\mathfrak{p}^+$. Recall that this is also the closure from S/U in $S_{\mathbb{C}}/U_{\mathbb{C}}P^-$. Hence Proposition 1.13 implies that the isomorphism $\Phi\colon\Xi\to\mathcal{D}$ from Theorem 1.10 extends to a G-equivariant embedding

$$\overline{\Phi}\colon\overline{\Xi}\to\overline{\mathcal{D}}.$$

Thus $\overline{\mathcal{D}}$ is the natural G-invariant compactification of Ξ. Let $z_1{:=}\exp(i\frac{\pi}{2}Y_0)K_{\mathbb{C}} \in \partial_d\Xi$ as before. It is known that the stabilizer of z_1 in S is a maximal parabolic subgroup P of S. Note that P is given by

$$P = \mathbf{c}U_{\mathbb{C}}P^-\mathbf{c}^{-1} \cap S.$$

Furthermore the Shilov boundary $\partial_s\mathcal{D}$ of $\mathcal{D}$ is the S-orbit through z_1 and

$$\partial_s\mathcal{D} = S(x_1) = U(x_1) \simeq U/K.$$

From the G-equivariance of $\overline{\Phi}$ it hence follows that $\overline{\Phi}(\partial_s\Xi) \subseteq \partial_s\mathcal{D}$.

Theorem 1.14. *The image of the Shilov boundary $\partial_s\Xi$ under the embedding $\overline{\Phi}$ is open and dense in $\partial_s\mathcal{D}$. In particular, $\partial_s\mathcal{D}$ is a compactification of $\partial_s\Xi$.*

Proof. First note that $G/H \hookrightarrow S/P$ is an open G-invariant embedding. Hence $\overline{\Phi}(\partial_s\Xi)$ is open and in $\partial_s\mathcal{D}$.

Let us now show that $\overline{\Phi}(\partial_s\Xi)$ is dense. For that we realize $\overline{\Xi}$ in $\overline{\mathcal{D}}$ via the embedding $\overline{\Phi}$. If $Y := \overline{\Phi}(\partial_s\Xi)$ is not dense, then we find a continuous function f on $\overline{\mathcal{D}}$, holomorphic on $\mathcal{D} = \Xi$ which does not attain its maximum on $\overline{Y}$. But this contradicts the fact that $\Xi = \mathcal{D}$ and that $\partial_s\Xi$ is the Shilov boundary of Ξ. ∎

2. Continuous branching of spherical highest weight representations

Throughout this section $(\mathfrak{s}, \tau)$ denotes a compactly causal symmetric Lie algebra with τ-eigenspace decomposition $\mathfrak{s} = \mathfrak{g} \oplus \mathfrak{q}_*$. We let S be a simply connected Lie group with Lie algebra $\mathfrak{s}$ and write $G := S^\tau$ for the fixed point group. Observe that G is connected since S was supposed to be simply connected.

In this section we discuss the *generalized Segal-Bargmann transform* introduced in [ÓØ85], see also [DÓZ01,Ner99,Ól00,Zh01]. This allows us to give a sharp criterion for a G-spherical unitary highest weight representation $(\pi, \mathcal{H})$ of S for continuous branching of $\pi|_G$.

As before we write U for the analytic subgroup of S associated to $\mathfrak{u}$ and $U_\mathbb{C}$ for its complexification. We denote by $P^\pm$ the analytic subgroup of $S_\mathbb{C}$ with Lie algebras $\mathfrak{p}^\pm$. Assume for a moment that S is not simply connected and that $S \subseteq S_\mathbb{C}$. By the Harish Chandra decomposition of S we have $S \subseteq P^+U_\mathbb{C}P^-$. For $s \in S$ write $s = p^+(s)\kappa(s)p^-(s)$ with the obvious notation. Recall that the map $S \ni s \mapsto \log p^+(s) \in \mathfrak{p}^+$ gives the realization of S/U as a bounded symmetric domain in $\mathfrak{p}^+$. Further we have the middle projection $\kappa : S \to U_\mathbb{C}$. By the usual lifting argument we obtain such a map κ also for a simply connected group S. This lifting is unique by the requirement of $\kappa(\mathbf{1}) = \mathbf{1}$. We denote by $X \mapsto \overline{X}$ the conjugation in $\mathfrak{s}_\mathbb{C}$ with respect to the real form $\mathfrak{s}$. Then $\overline{\mathfrak{p}^+} = \mathfrak{p}^-$.

Realization of unitary highest weight representations.

Our source of reference for the facts collected below is [Ne99, Ch. XII]. Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be a unitary highest weight representation of S with highest weight $\lambda \in i\mathfrak{t}^*$ and with respect to the positive system Δ^+. Write $F(\lambda)$ for the finite dimensional lowest U-type in $\mathcal{H}_\lambda$, i.e., the space which is generated by applying U to a highest weight vector v_λ of $\mathcal{H}_\lambda$. We now briefly recall the realization of $\mathcal{H}_\lambda$ in $\mathcal{O}(\mathcal{D}, F(\lambda))$. For that we write σ_λ for the representation of $U_\mathbb{C}$ on $F(\lambda)$. Define

$$\mathcal{K}: \mathcal{D} \times \mathcal{D} \to U_\mathbb{C}, \quad (z, w) \mapsto \kappa(\exp(-\overline{w})\exp(z))^{-1}$$

$$J: S \times \mathcal{D} \to U_\mathbb{C}, \quad (s, z) \mapsto \kappa(g\exp(z)).$$

Set

$$K_\lambda = \sigma_\lambda \circ \mathcal{K} \quad \text{and} \quad J_\lambda = \sigma_\lambda \circ J\,. \tag{2.1}$$

Note that K_λ is holomorphic in the first variable, antiholomorphic in the second and satisfies the cocyle identity

$$K_\lambda(s(z), s(w)) = J_\lambda(s, z)K_\lambda(z, w)J_\lambda(s, w)^* \tag{2.2}$$

for all $s \in S$, $z, w \in \mathcal{D}$. Here A^* denotes the adjoint of an operator $A \in B(F(\lambda)) = \mathrm{End}(F(\lambda))$.

The relation with $(\pi_\lambda, \mathcal{H}_\lambda)$ is now as follows: π_λ is unitary if and only if K_λ is positive definite. If this is the case, then $(\pi_\lambda, \mathcal{H}_\lambda)$ can be realized in $\mathcal{O}(\mathcal{D}, F(\lambda))$ as the reproducing kernel Hilbert space corresponding to K_λ. So henceforth we will assume that $\mathcal{H}_\lambda \subseteq \mathcal{O}(\mathcal{D}, F(\lambda))$. The representation π_λ is then given by

$$(\pi_\lambda(s)f)(z) = J_\lambda(s^{-1}, z)^{-1} f(s^{-1}(z)) \tag{2.3}$$

for $f \in \mathcal{H}$, $s \in S$ and $z \in \mathcal{D}$.

G-spherical unitary highest weight representation

A unitary highest weight representation $(\pi_\lambda, \mathcal{H}_\lambda)$ is called *G-spherical* if there exists a non-trivial G-fixed distribution vector ν in $\mathcal{H}_\lambda^{-\infty}$. Here $\mathcal{H}_\lambda^{-\infty}$ denotes the module of distribution vectors of $(\pi_\lambda, \mathcal{H}_\lambda)$. If $(\pi_\lambda, \mathcal{H}_\lambda)$ is a unitary highest weight representation of S, then we call λ *regular* if $\mathcal{H}_\lambda$ is dense in $\mathcal{O}(\mathcal{D}, F(\lambda))$, or, equivalently if $\lambda|_{\mathfrak{z}(\mathfrak{u})}$ lies on the continuous line of the Wallach set (this is made precise in Appendix A). We say that $(\pi_\lambda, \mathcal{H}_\lambda)$ if $F(\lambda) = \mathbb{C}$. This is equivalent to $\lambda \in i\mathfrak{z}(\mathfrak{u})^*$.

We now recall some facts concernig G-spherical unitary highest weight representation. For more information and an almost complete classification see [KrNe00].

- If $(\pi_\lambda, \mathcal{H}_\lambda)$ is G-spherical, then $\nu_0 := \nu|_{F(\lambda)}$ is non-zero and K-fixed. In particular, $F(\lambda)$ is a K-spherical module for U.
- If λ is regular, then $(\pi_\lambda, \mathcal{H}_\lambda)$ is G-spherical if and only if $F(\lambda)$ is K-spherical.
- If $\lambda \in i\mathfrak{z}(\mathfrak{u})^*$ is regular, then $(\pi_\lambda, \mathcal{H}_\lambda)$ is G-spherical if and only if $\lambda \in i(\mathfrak{z}(\mathfrak{u}) \cap \mathfrak{q}_*)^*$. In particular, if S is simple, then all regular $\lambda \in i\mathfrak{z}(\mathfrak{u})^*$ correspond to G-spherical representations.
- If we are in the group case $S = G \times G$, then the G-spherical unitary highest weight representations of S are exactly the representations $\pi_\lambda \otimes \pi_\lambda^*$ with π_λ a unitary highest weight representation of G and π_λ^* its dual representation.

The generalized Segal-Bargmann transform

In this subsection $(\pi_\lambda, \mathcal{H}_\lambda)$ denotes a G-spherical unitary highest weight representation of S. Note that this means in particular that there exists a non-trivial K-fixed vector $\nu_0 \in F(\lambda)^*$. Write $\nu_0 = \langle \cdot, v_0 \rangle$ where $0 \neq v_0 \in F(\lambda)$ is a K-fixed vector. Define

$$D_\lambda : G \to B(F(\lambda)), \quad g \mapsto J_\lambda(g, 0)^{-1} = \sigma_\lambda(\kappa(g))^{-1}$$

and

$$\|D_\lambda\| : G/K \to \mathbb{C}, \quad gK \to \|D_\lambda(g)\|$$

where $\|\cdot\|$ denotes the operator norm on $B(F(\lambda))$. Write $\mathcal{A}(G/K)$ for the space of analytic functions on G/K. Define a restriction map

$$R_\lambda : \mathcal{H}_\lambda \to \mathcal{A}(G/K), \quad R_\lambda(f)(gK) := \langle D_\lambda(g) f(gK), v_0 \rangle.$$

Note that this map is well defined, since v_0 is K-fixed. We write L for the left regular representation of G on functions on G/K, i.e., $(L(g)f)(xK) = f(g^{-1}xK)$ for $g, x \in G$ and f a function on G/K. We have then the following Lemma, see [Ól00], Lemma 3.4:

Lemma 2.1. *Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be a G-spherical unitary highest weight representation of S.*

(i) *The map $R_\lambda: \mathcal{H}_\lambda \to \mathcal{A}(G/K)$ intertwines $\pi_\lambda|_G$ with L. Moreover, if $\lambda \in i\mathfrak{z}(\mathfrak{u})^*$, then R_λ is injective.*

(ii) *If λ is regular and $\|D_\lambda\|: G/K \to \mathbb{C}$ is square integrable, then* $\mathrm{Im} R_\lambda \subseteq L^2(G/K)$ *is dense and $R_\lambda: \mathcal{H}_\lambda \to L^2(G/K)$ is continuous.* ■

Assume now that $\|D_\lambda\| \in L^2(G/K)$ and that λ is regular. Then $R_\lambda: \mathcal{H}_\lambda \to L^2(G/K)$ is continuous with dense image. Thus we can polarize the continuous operator R_λ as $R_\lambda = U_\lambda P_\lambda$ where $P_\lambda = (R_\lambda^* R_\lambda)^{\frac{1}{2}}$ and U_λ is a partial isometry. If $\lambda \in i\mathfrak{z}(\mathfrak{u})^*$, then U_λ is an isometry and the unitary G-equivariant isomorphism

$$U_\lambda^* : L^2(G/K) \to \mathcal{H}_\lambda$$

is called the *generalized Segal-Bargmann transform* (cf. [Ól00]).

We are going now to specify the parameters for which $\|D_\lambda\|$ is square integrable. We will distinguish the two cases of S simple and $S = G \times G$. The group case is much simpler, since then the square integrability of D_λ can be easily reduced to known integrals. Although we will present a general proof for a an exact criterion on λ for the square integrability of D_λ we will treat the group case separately since the general proof requires some knowledge of advanced structure theory the general reader might not be so familiar with.

The square integrability of $\|D_\lambda\|$: group case

In this section we consider the case where $S = G \times G$. We let $(\pi_\lambda, \mathcal{H}_\lambda)$ be a unitary representation of G. Then the S-representation we consider is $\widehat{\pi}_\lambda = \pi_\lambda \otimes \pi_\lambda^*$. Recall that these are precisely the G-spherical unitary highest weight representations of S. As explained before we can realize $\mathcal{H}_\lambda$ in $\mathcal{O}(G/K, F(\lambda))$. As before we write K_λ for its reproducing kernel and J_λ for the corresponding cocycle. We recall now some well known facts on representations of the holomorphic discrete series on G.

Write Z for the center of G. We say $(\pi_\lambda, \mathcal{H}_\lambda)$ belongs to the *relative discrete series* if all matrix coefficients of $(\pi_\lambda, \mathcal{H}_\lambda)$ are square integrable modulo Z.

By abuse of notation we will denote here by $\mathfrak{t}$ a compact Cartan algebra of $\mathfrak{g}$ and Δ for the corresponding root system. We write β for the highest root in Δ^+ and write $\rho := \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$.

Lemma 2.2. *Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be a unitary highest weight representation of the simply connected Hermitian Lie group G. Then the following assertions are equivalent:*

(1) *$(\pi_\lambda, \mathcal{H}_\lambda)$ belongs to the relative discrete series.*

(2) *The highest weight $\lambda \in i\mathfrak{t}^*$ satisfies the Harish Chandra condition*

$$\langle \lambda + \rho, \beta \rangle < 0.$$

(3) *The integral*

$$\int_{G/K} \|K_\lambda(z,z)\|^{-1} \, d\mu_{G/K}(z) < \infty$$

is finite. Here $\|\cdot\|$ denotes the operator norm on $B(F(\lambda))$ and $\mu_{G/K}$ the invariant measure on G/K.

(4) *The integral*

$$\int_{G/Z} \|J_\lambda(g,0)^{-1}(J_\lambda(g,0)^{-1})^*\| \, d\mu_{G/Z}(gZ) < \infty$$

is finite.

Proof. (1) $\iff$ (2) is the criterion of Harish Chandra (cf. [Ne99, Th. XII.5.12]. For the equivalence of (2) and (3) see [Ne99, Th. XII.5.6]. Finally the equivalence of (3) and (4) is immediate from (2.1). ∎

As before we realize $(\widehat{\pi}_\lambda, \mathcal{H}_\lambda \widehat{\otimes} \mathcal{H}_\lambda^*)$ in $\mathcal{O}(G/K \times \overline{G/K}, B(F(\lambda)))$. Here we used the identification $B(F(\lambda)) = F(\lambda) \otimes F(\lambda)^*$. The $U = K \times K$-spherical vector in $B(F(\lambda))$ is the trace in this case. The function $\|D_\lambda\|$ in this case is given by

$$\|D_\lambda\|: G/K \to \mathbb{R}^+, \quad gK \mapsto \operatorname{tr}(J_\lambda(g,0)^{-1}(J_\lambda(g,0)^{-1})^*).$$

If G is a locally compact group, then we write μ_G for a left invariant Haar measure on G.

Lemma 2.3. *Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be a unitary highest weight representation of G. Then D_λ is square integrable if and only if*

$$\langle 2\lambda + \rho, \beta \rangle < 0$$

for β the highest root in Δ^+.

Proof. That D_λ is square integrable means that

$$\int_{G/Z} \|J_\lambda(g,0)^{-1}(J_\lambda(g,0)^{-1})^*)\|^2 \, d\mu_{G/Z}(gZ) < \infty. \tag{2.4}$$

Assume for a moment that $\lambda \in i\mathfrak{z}(\mathfrak{k})^*$, i.e., $F(\lambda) \simeq \mathbb{C}$. Then $J_\lambda(g,0)^2 = J_{2\lambda}(g,0)$ and (2.2) becomes equivalent to

$$\int_{G/Z} \|J_{2\lambda}(g^{-1},0) J_{2\lambda}(g^{-1},0)^*\| \, d\mu_{G/Z}(gZ) < \infty. \tag{2.5}$$

From Lemma 2.1 (2) $\iff$ (4) we know obtain that (2.5) is equivalent to

$$\langle 2\lambda + \rho, \beta \rangle < 0$$

for β the highest root in Δ^+. Finally the technique to estimate the integrals in Lemma 2.2 (4) as presented in [Wa79, Lemma 3.1] shows that the assumption that $\lambda \in i\mathfrak{z}(\mathfrak{k})^*$ is no restriction to the general case. ∎

Lemma 2.4. *If the highest weight λ of $(\pi_\lambda, \mathcal{H}_\lambda)$ satisfies the condition of $\langle 2\lambda + \rho, \beta \rangle < 0$, then λ is regular.*

Proof. This follows from the classification of unitary highest weight modules (cf. [EHW83]). In the scalar case this is proved in Lemma A.2 (iv) in Appendix A. ∎

Assume now that $\langle 2\lambda + \rho, \beta \rangle < 0$ holds. In view of Lemma 2.3 and Lemma 2.4, this implies in particular that D_λ is square integrable. In view of Lemma 2.1 we have thus proved:

Theorem 2.5. *Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be a unitary highest weight representation with highest weight λ satisfying the condition $\langle 2\lambda + \rho, \beta \rangle < 0$ with β the highest root. Then we have an onto G-equivariant partial isometry*

$$U_\lambda: \mathcal{H}_\lambda \widehat{\otimes} \mathcal{H}_\lambda^* \to L^2(G/K)$$

which is an isomorphism when $\lambda \in i\mathfrak{z}(\mathfrak{k})^*$. In particular, for $\lambda \in i\mathfrak{z}(\mathfrak{k})^*$ the branching of the irreducible unitary representation $\pi_\lambda \otimes \pi_\lambda^*$ of $G \times G$ to the diagonal subgroup G is completely continuous and multiplicity free. ∎

The square integrability of $\|D_\lambda\|$: general case

In this subection we will prove a theorem which will give us the exact range of parameters for the square integrability of D_λ. For that recall some results from [ÓØ88] applied to the symmetric Lie algebra $(\mathfrak{s}, \theta\tau)$, which is also compactly causal. The proof of the statements uses simple $\mathfrak{su}(1,1)$-reduction (cf. Appendix B). Choose the Cartan subspace $\mathfrak{t}$ such that $\mathfrak{e} := \mathfrak{t} \cap \mathfrak{q}_*$ is maximal abelian in $\mathfrak{q}_*$. Write $\Delta_\mathfrak{e} = \Delta(\mathfrak{s}_\mathbb{C}, \mathfrak{e}_\mathbb{C})$ for the restricted root system with respect to $\mathfrak{e}_\mathbb{C}$. Note that $\Delta_\mathfrak{e}$ is always of type C_n or BC_n. Define a positive system $\Delta_\mathfrak{e}^+$ by $\Delta_\mathfrak{e}^+ := \Delta^+|_\mathfrak{e} \backslash \{0\}$. Further set $\Delta_{\mathfrak{e},n}^+ := \Delta_n^+|_\mathfrak{e}$. Note that $\gamma|_\mathfrak{e} \neq 0$ for all $\gamma \in \Delta_n$.

Let $\gamma_1, \ldots, \gamma_r$ be a maximal set of long strongly orthogonal roots in $\Delta_{\mathfrak{e},n}^+$. Then we can choose $E_{\pm j} \in \mathfrak{s}_\mathbb{C}^{\pm\gamma_j}$ such that with $X_j = -i(E_j - E_{-j})$ the space $\mathfrak{a} = \bigoplus_{j=1}^r \mathbb{R}X_j$ is maximal abelian in $\mathfrak{p}$. Let $H_j \in i\mathfrak{e}$ be such that $\gamma_i(H_j) = 2\delta_{ij}$ then

$$\kappa(\exp(\sum_{j=1}^r t_j X_j)) = \exp\left(-\frac{1}{2}\sum_{j=1}^r \log(\cosh(2t_j))H_j\right).$$

Hence

$$\|D_\lambda(\exp(\sum_{j=1}^r t_j X_j))\| = \exp\left(\frac{1}{2}\langle\lambda, \sum_{j=1}^r \log(\cosh(2t_j))H_j\rangle\right).$$

Thus with $a = \exp(\sum_{j=1}^r t_j X_j)$ and $k_1, k_2 \in K$:

$$\|D(k_1 a k_2)\|^2 = \prod_{j=1}^r \cosh(2t_j)^{\langle\lambda, H_j\rangle}. \tag{2.6}$$

The root vectors E_j-determine a Cayley transform c such that $c(\sum_{j=1}^r \mathbb{R}H_j) = \mathfrak{a}$. Recall that $\Sigma = \Delta(\mathfrak{g}, \mathfrak{a})$ is the set of restricted roots of $\mathfrak{a}$ in $\mathfrak{g}$. Choose Σ^+ such that $\Sigma^+ \subseteq \Delta^+ \circ c^{-1}$. Let ρ_G be the half sum of positive roots counted with multiplicities and let $\rho_G^c = \rho_G \circ c^{-1} \in i\mathfrak{e}^*$.

Theorem 2.6. *We have $D_\lambda \in L^2(G/K)$ if and only if*

$$\langle\lambda + \rho_G^c, \beta\rangle < 0$$

for β the highest root in Δ^+.

Proof. First note that $\langle\lambda + \rho_G^c, \beta\rangle < 0$ is equivalent to

$$\langle\lambda + \rho_G^c, \alpha\rangle < 0 \quad (\forall\alpha \in \Delta_n^+).$$

Further the fact that $\lambda, \rho_G^c \in i\mathfrak{e}^*$ implies that this is equivalent to

$$\langle\lambda + \rho_G^c, \alpha\rangle < 0 \quad (\forall\alpha \in \Delta_{\mathfrak{e},n}^+).$$

Let

$$\Delta_G(k_1 a k_2) := \prod_{\alpha\in\Sigma^+} \sinh(\alpha(\log a))^{m_\alpha} \qquad k_1, k_2 \in K, a \in A$$

where $m_\alpha = \dim \mathfrak{g}^\alpha$. Then for $f \in L^1(G)$:

$$\int_G f(g)\, dg = \int_K \int_{A^+} \int_K f(k_1 a k_2)\ \Delta_G(a)\ d\mu_K(k_1)\ d\mu_A(a)\ d\mu_K(k_2)$$

There exists a non-zero polynomial function $p(X)$ on $\mathfrak{a}$ such that for all $a \in A^+$ and some costants $c_1, c_2 > 0$ one has

$$c_1 |p(\log a)| a^{2\rho_G^c} \leq \Delta(a) \leq c_2 a^{2\rho_G^c}\,.$$

From this and (2.6) we get for $a = \exp(\sum_{j=1}^r t_j X_j) \in A^+$ some constants $C_1, C_2 > 0$:

$$C_1 |p(\log a)| \prod_{j=1}^r \exp(2t_j \langle \lambda + \rho_G^c, H_j\rangle) \leq \|D(k_1 a k_2)\|^2 \Delta(a) \leq C_2 \prod_{j=1}^r \exp(2t_j \langle \lambda + \rho_G^c, H_j\rangle)\,.$$

The claim now follows as $t_j > 0$ for $a \in A^+$. ∎

From Theorem 2.6 and Lemma 2.1 we now obtain the following result:

Theorem 2.7. *Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be a G-spherical unitary highest weight representation of S. Suppose the highest weight λ is regular and satisfies the condition*

$$\langle \lambda + \rho_G^c, \beta\rangle < 0$$

for β the highest root in Δ^+. Then we have an onto G-equivariant partial isometry

$$U_\lambda : \mathcal{H}_\lambda \to L^2(G/K)$$

which is an isomorphism when $\lambda \in i(\mathfrak{z}(\mathfrak{u}) \cap \mathfrak{q}_*)^*$. In particular, for $\lambda \in i(\mathfrak{z}(\mathfrak{u}) \cap \mathfrak{q}_*)^*$ the representation $\pi_\lambda|_G$ of G is multiplicity free with completely continuous spectrum. ∎

Remark 2.8. (a) The condition in Theorem 2.7 is the same as the one in Lemma 2.3 in the group case. The condition on the regularity of λ in Theorem 2.7 is probably superfluous, but we did not check it.
(b) The results in [GrKo01] for $\mathfrak{s} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ show that the condition $\langle \lambda + \rho_G^c, \beta\rangle < 0$ on λ in Theorem 2.7 is also necessary for continuous branching. It would be interesting to know if this is generally true. ∎

Example 2.9. Let us consider $S = \widetilde{\mathrm{Sl}}(2,\mathbb{R})$ with $G = \mathrm{SO}(1,1)$. In this case $\rho_G^c = 0$ and this means that all unitary highest weight representations π_λ have continuous branching for $\pi_\lambda|_G$.

On the other hand consider the group case $S = \widetilde{\mathrm{Sl}}(2,\mathbb{R}) \times \widetilde{\mathrm{Sl}}(2,\mathbb{R})$. Here the condition $\langle \lambda + \rho_G^c, \beta\rangle < 0$ gives a real restriction. Further by the results in [GrKo01] we do not have continuous branching for $\pi_\lambda \otimes \pi_\lambda^*$ to G on the full half line. ∎

Applications to the classical Hardy space

Write $\partial_s \mathcal{D}$ for the Shilov boundary of $\mathcal{D} \simeq S/U$ and recall that $\partial_s \mathcal{D}$ is the U-orbit through a certain $E \in \partial_s \mathcal{D}$. Note that $\mathcal{D}$ is a circular domain. In particular we have $r\partial_s \mathcal{D} \subseteq \mathcal{D}$ for all $0 \leq r < 1$.

The *Hardy space parameter* $\lambda_h \in i(\mathfrak{z}(\mathfrak{u}) \cap \mathfrak{q}_*)^*$ is defined by

$$\lambda_h = -\rho_n := -\frac{1}{2} \sum_{\alpha \in \Delta_n^+} \alpha.$$

The corresponding Hilbert space $\mathcal{H}_{\lambda_h}$ is the classical Hardy space on $\mathcal{D}$:

$$\mathcal{H}^2(\mathcal{D}) := \{f \in \mathcal{O}(\mathcal{D}) : \sup_{0 \le r < 1} \int_U |f(ru(E))|^2 \, d\mu_U(u) < \infty\}.$$

Here we identified the Shilov boundary of $\partial_s \mathcal{D}$ with $U(E)$ as explained before. We write $\pi_h := \pi_{\lambda_h}$ for the unitary irredducible representation of S on $\mathcal{H}^2(\mathcal{D})$.

Lemma 2.10. *Assume that $(\mathfrak{s}, \mathfrak{g}, \mathfrak{h})$ is a causally symmetric triple. Then λ_h is regular and satisfies the condition $\langle \lambda_h + \rho_G^c, \beta \rangle < 0$.*

Proof. See Proposition A.3 in the appendix. ■

From this we obtain the following important corollary to Theorem 2.8:

Corollary 2.11. *Let $(\mathfrak{s}, \mathfrak{g}, \mathfrak{h})$ be a causally symmetric triple and $\mathcal{H}^2(\mathcal{D})$ be the classical Hardy space on $\mathcal{D} \simeq S/U$. Then the inverse of the Segal-Bargmann transform*

$$U : (\pi_h|_G, \mathcal{H}^2(\mathcal{D})) \to (L, L^2(G/K))$$

is a G-equivariant unitary isomorphism. In particular, the branching $\pi_h \mid_G$ is completely continuous and multiplicity free. ■

3. Analytical and geometrical constructions on Ξ

From now on we will assume that $(\mathfrak{s}, \mathfrak{g}, \mathfrak{h})$ is a causally symmetric triple. We let $S_{\mathbb{C}}$ be a simply connected Lie group with Lie algebra $\mathfrak{s}_{\mathbb{C}}$ and let S, U, G, K, H the analytic subgroups of $S_{\mathbb{C}}$ with Lie algebras $\mathfrak{s}$, $\mathfrak{u}$, $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{h}$. The objective of this section is to establish certain geometrical and analytical facts which will be needed later on.

The function ψ

We will construct a function ψ which will be used for comparing the classical Hardy space with our Hardy space on Ξ in Section 6.

Denote by (π_m, V_m) the irreducible representation of $U_{\mathbb{C}}$ with lowest weight $-m\rho_n$. If such representation exists it is $K_{\mathbb{C}}$-spherical. Note that $V_m \simeq \mathbb{C}$ is one dimenisonal since $\rho_n \in i(\mathfrak{z}(\mathfrak{u}) \cap \mathfrak{q}_*)^*$. Also notice that the representation $\det \mathrm{Ad}|_{\mathfrak{p}^-}$ of U has lowest weight $-2\rho_n$. Let

$$\zeta = \frac{1}{2}(\gamma_1 + \ldots + \gamma_r) \in i(\mathfrak{z}(\mathfrak{u}) \cap \mathfrak{q}_*)^*$$

where $\gamma_1, \ldots, \gamma_r$ is a maximal set of long strongly orthogonal roots in Δ_n^+. Let

$$d = \dim \mathfrak{s}_{\mathbb{C}}^{\frac{1}{2}(\gamma_i + \gamma_j)} = \dim \mathfrak{s}_{\mathbb{C}}^{\frac{1}{2}(\gamma_i - \gamma_j)}$$

(cf. Appendix A for the table for d). Then

$$\rho_n = (1 + \frac{d(r-1)}{2})\zeta \tag{3.1}$$

(cf. Appendix A).

Proposition 3.1. *Let the notation be as above. Let $m \in \mathbb{N}$ be minimal such that (π_m, V_m) exists. Then the following holds:*

(i) *If $(\mathfrak{g}, \mathfrak{h})$ is of Cayley type and $\mathfrak{g} \neq \mathfrak{so}(2, 2k+1), \mathfrak{sp}(2n, \mathbb{R})$ then $m = 1$.*

(ii) *If $(\mathfrak{g}, \mathfrak{h})$ is of Cayley type and $\mathfrak{g} = \mathfrak{so}(2, 2k+1), \mathfrak{sp}(2n, \mathbb{R})$ then $m = 2$.*

(iii) *If $(\mathfrak{s}, \mathfrak{g}) = (\mathfrak{su}(2n, 2n), \mathfrak{sp}(n, n))$ then $m = 1$.*

(iv) *If $(\mathfrak{s}, \mathfrak{g}) = (\mathfrak{sp}(2n, \mathbb{R}), \mathfrak{sp}(n, \mathbb{C}))$ then $m = 2$.*

(v) *If $(\mathfrak{s}, \mathfrak{g}) = (\mathfrak{so}(2, n), \mathfrak{so}(1, n))$ for $n > 2$ then $m = 1$ if n is even and $m = 2$ for n odd.*

Proof. Using the table for d in Appendix A, this follows from (3.1) by straightforward computation. ■

Lemma 3.2. *The set is $G_{\mathbb{C}} U_{\mathbb{C}} P^-$ open and dense in $S_{\mathbb{C}}$. Furthermore $S \subseteq G_{\mathbb{C}} U_{\mathbb{C}} P^-$.*

Proof. This is Theorem 2.4 in [ÓØ88]. ■

Assume now that (π_m, V_m) exists. Then $V_m = \mathbb{C} v_m$ and v_m is both a lowest weight vector and a $K_{\mathbb{C}}$-spherical vector. Normalize v_m such that $(v_m, v_m) = 1$. Let

$$\mathcal{U} := \{ z \in \mathfrak{p}^+ : \exp(z) \in G_{\mathbb{C}} U_{\mathbb{C}} P^- \cap P^+ U_{\mathbb{C}} P^- \}.$$

Since $P^+ U_{\mathbb{C}} P^-$ is open an dense in $S_{\mathbb{C}}$ it follows from Lemma 3.2 that $\mathcal{U}$ is an open dense, hence locally $G_{\mathbb{C}}$ invariant subset of $\mathfrak{p}^+$. Moreover we have $\mathcal{D} \subseteq \mathcal{U}$.

For $s \in \mathcal{U}$ define $K_{\mathbb{C}} u_G(s) \in K_{\mathbb{C}} \backslash U_{\mathbb{C}}$ by

$$s \in G_{\mathbb{C}} u_G(s) P^- . \tag{3.2}$$

Define a function $\psi_m : \mathcal{U} \to \mathbb{C}$ by

$$\psi_m(z) := (\pi_m(u_G(\exp(z))) v_m, v_m) . \tag{3.3}$$

Proposition 3.3. *Let ψ_m be as above. Then the following assertions hold:*

(i) *Let $g \in G_{\mathbb{C}}$ and $z \in \mathcal{U}$ such that $gz \in \mathcal{U}$. Then*

$$\psi_m(gz) = J_{m\rho_n}(g, z) \psi_m(z) .$$

(ii) *ψ_m is holomorphic and has no zeros on $\mathcal{D}$.*

Proof. (i) We have $g \exp(z) = \exp(gz) J(g, z) p$ with $p \in P^-$. Hence

$$\exp(gz) = g \exp(z) J(g, z)^{-1} p'$$

for some $p' \in P^-$. It follows that

$$\psi_m(gz) = (\pi_m(\exp(z)) \pi_m(J(g, z)^{-1}) v_m, v_m)) = J_{m\rho_n}(g, z) \psi_m(z)$$

because v_m is a weight vector with weight $-m\rho_n$.

(ii) The holomorphicity of ψ_m is clear by construction. To see that $\psi_m(z) \neq 0$ for $z \in \mathcal{D}$ write $z = g(0)$ for some $g \in G \exp(i\Omega)$. Since $\mathcal{D} \subseteq \mathcal{U}$, we obtain from (i) that

$$\psi_m(z) = \psi_m(g(0)) = J_{m\rho_n}(g, 0) \psi_m(0).$$

Now $\psi_m(0) = (v_m, v_m) = 1 \neq 0$ and $J_{m\rho_n}(g, 0) \neq 0$ by construction. ■

From now on we consider ψ_m as a function on $\mathcal{D}$ only. Note that ψ_2 always exists by Proposition 3.1. Since $\mathcal{D}$ is simply connected and ψ_m is zero-free it follows from Proposition 3.3(ii) that we can define a holomorphic square root $\psi(z) := \sqrt{\psi_2(z)}$ which becomes unique under the requirement $\psi(0) = 1$. Note that $\psi = \psi_1$ in case (π_1, V_1) exists.

Write $\widetilde{S}$ for the universal covering of S and G_1 for the analytic subgroup of $\widetilde{S}$ with Lie algebra $\mathfrak{g}$. Note that $J_{\rho_n}(s, z)$ exists for all $s \in \widetilde{S}$ and $z \in \mathcal{D}$ (cf. Section 2). From Proposition 3.3(i) we thus obtain that $\psi(gz) = J_{\rho_n}(g, z) \psi(z)$ for all $g \in G_1$ and $z \in \mathcal{D}$. In particular, $J_{\rho_n}(g, z) = \frac{\psi(z)}{\psi(gz)}$ and so $J_{\rho_n}(g, z)$, initially defined only on $G_1 \times \mathcal{D}$, factors to a function on $G \times \mathcal{D}$, which we denote by J_h^{-1}. Summarizing our discussions we have proved:

Proposition 3.4. *There exist unique holomorphic function $\psi: \mathcal{D} \to \mathbb{C}^*$ and a unique analytic function $J_h: G \times \mathcal{D} \to \mathbb{C}^*$ with the following properties:*

(i) $\psi(gz) = J_h^{-1}(g,z)\psi(z)$ *for all* $g \in G$, $z \in \mathcal{D}$.

(ii) $\psi^2 = \psi_2$ *and* $\psi(0) = 1$.

(iii) J_h *satisfies the cocycle property* $J_h(g_1g_2, z) = J_h(g_1, g_2z)J_h(g_2, z)$ *for* $g_1, g_2 \in G$, $z \in \mathcal{D}$ *as well as* $J_h^{-2} = J_{2\rho_n}$ *and* $J_h(\mathbf{1}, z) = 1$ *for all* $z \in \mathcal{D}$. ∎

Example 3.5. Let $S = G \times G$, and G the diagonal group. Write $G \subset P_G^+ K_{\mathbb{C}} P_G^-$ for the triangular decomposition for $G_{\mathbb{C}}$ and let $k_1(g) \in K_{\mathbb{C}}$ be the corresponding Harish-Chandra projection. If $\mathcal{D}_G \subseteq \mathfrak{p}_G^+$ is the bounded realization of G/K, then $\mathcal{D} = \mathcal{D}_G \times \mathcal{D}_G^{\mathrm{opp}}$ where $\mathcal{D}_G^{\mathrm{opp}}$ denotes $\mathcal{D}_G$ equipped with the opposite complex structure. Note that $P^+ = P_G^+ \times P_G^-$ and $P^- = P_G^- \times P_G^+$. In particular we are realizing S/U inside $(G_{\mathbb{C}}/K_{\mathbb{C}}P^-) \times (G_{\mathbb{C}}/K_{\mathbb{C}}P^+)$. Recall that the conjugation $g \mapsto \overline{g}$ with respect to G satisfies $\overline{P^+} = P^-$.

Let $(s,t) \in G \times G$. Then

$$(s,t) = (t,t)(t^{-1}s, \mathbf{1}) = (t,t)(p_1^+ k_1(t^{-1}s)p_1^-, \mathbf{1}) = (tp_1^+, tp_1^+)(k_1(t^{-1}s)p_1^-, (p_1^+)^{-1}).$$

Hence

$$G_{\mathbb{C}} u_G(s,t) = G_{\mathbb{C}}(k_1(t^{-1}s), \mathbf{1}).$$

As $U_{\mathbb{C}} = K_{\mathbb{C}} \times K_{\mathbb{C}}$ it follows that any $K_{\mathbb{C}}$-spherical representation π_U of $U_{\mathbb{C}}$ is of the form $\pi_U = \pi \otimes \pi^*$ where π is an irreducible representation of $K_{\mathbb{C}}$. Hence

$$\psi(z,w) = \pi_1(k_1(\exp(-\bar{w})\exp(z))) = K_{-\rho_n^G}(z,w)^{-1} \qquad (z,w \in \mathcal{D}_G) \tag{3.4}$$

where ρ_n^G is the ρ_n for the Hermitian group G. Notice that $K_{-\rho_n^G}$ is nothing else but the reproducing kernel of the classical Hardy space on $\mathcal{D}_G$. ∎

SU(1,1)-reduction

In this subsection we will discuss the case $S = \mathrm{SU}(1,1)$ even if it is not in our list of causally symmetric triples. The reason is, that many calculations can be reduced to this situation using τ-equivariant embedding of $\mathrm{SU}(1,1)$ into S. Let

$$S = \mathrm{SU}(1,1) = \left\{\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C},\ |\alpha|^2 - |\beta|^2 = 1\right\}$$

Define $\tau : S \to S$ by conjugation with $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. On $S_{\mathbb{C}} = SL(2,\mathbb{C})$ the involution τ is given by

$$\tau\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

In particular

$$G = \left\{\begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} : t \in \mathbb{R}\right\}.$$

In this situation we have

$$Y_0 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$X_0 = -i\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$Z_0 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

For the general situation of a causally symmetric triple recall the $\mathfrak{su}(1,1)$-triple $\{X_j, Y_j, Z_j\}$ from Appendix B. From the relations (B.1) and (B.2) in Appendix B it now follows:

Lemma 3.6. *Let the notation be as above. Then the map* $X_j \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $Y_j \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *and* $Z_j \mapsto \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ *defines a Lie algebra homomorphism into* $\mathfrak{su}(1,1)$ *intertwining the involution* τ *on* $\mathfrak{s}$ *and the above involution on* $\mathfrak{su}(1,1)$. *Furthermore it also intertwines the Cartan involution* θ *on* $\mathfrak{s}$ *and the Cartan involution* $X \mapsto -X^*$ *on* $\mathfrak{su}(1,1)$. ■

On the group level we have:

$$U_{\mathbb{C}} = \{\begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{pmatrix} : \gamma \in \mathbb{C}^*\},$$

$$P^+ = \{\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C}\},$$

and

$$P^- = \{\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} : w \in \mathbb{C}\}.$$

Further

$$\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} = \begin{pmatrix} \gamma + zw\gamma^{-1} & z\gamma^{-1} \\ \gamma^{-1}w & \gamma^{-1} \end{pmatrix}.$$

Thus, if $d \neq 0$ we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & b/d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{d} & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ c/d & 1 \end{pmatrix}.$$

For the $G_{\mathbb{C}}U_{\mathbb{C}}P^-$-decomposition we notice that

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} = \begin{pmatrix} a\gamma + bw\gamma^{-1} & b\gamma^{-1} \\ b\gamma + a\gamma^{-1}w & a\gamma^{-1} \end{pmatrix}.$$

Hence, using that $a^2 - b^2 = 1$, we obtain that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{d}{\sqrt{d^2-b^2}} & \frac{b}{\sqrt{d^2-b^2}} \\ \frac{b}{\sqrt{d^2-b^2}} & \frac{d}{\sqrt{d^2-b^2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{d^2-b^2}} & 0 \\ 0 & \sqrt{d^2-b^2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{cd-ab}{d^2-b^2} & 1 \end{pmatrix}.$$

Thus the open sets $G_{\mathbb{C}}K_{\mathbb{C}}P^-$ and $P^+K_{\mathbb{C}}P^-$ in $S_{\mathbb{C}}$ can now be described as:

$$G_{\mathbb{C}}U_{\mathbb{C}}P^- = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : b^2 - d^2 \neq 0\}$$

$$P^+U_{\mathbb{C}}P^- = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid d \neq 0\}$$

Furthermore we notice that the $P^+, U_{\mathbb{C}}$, and P^- components, whenever defined, are given by

$$s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto u_G(s) = \begin{pmatrix} \frac{1}{\sqrt{d^2-b^2}} & 0 \\ 0 & \sqrt{d^2-b^2} \end{pmatrix}$$

and

$$s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 1 & b/d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c/d & 1 \end{pmatrix} .$$

Notice that a double covering is needed in general for u_G, but as an element of the coset $K_{\mathbb{C}} \backslash U_{\mathbb{C}}$, with $K_{\mathbb{C}} = \{\pm 1\}$ it is well defined. In the following we will identify P^+ with $\mathbb{C}$ by $z \mapsto \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ and $\mathbb{C}^*$ with $U_{\mathbb{C}}$ by $\gamma \mapsto \begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{pmatrix}$. In particular we get the following lemma:

Lemma 3.7. *Indentify $\mathfrak{p}^+, \mathfrak{p}^-$ with $\mathbb{C}$ and $U_{\mathbb{C}}$ with $\mathbb{C}^*$ in the way explained above. Let $z, w \in \mathbb{C}$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then the following assertions hold:*

(i) *If $z\bar{w} \neq 1$ then $\mathcal{K}(z, w) \in U_{\mathbb{C}}$ is defined and given by*

$$\mathcal{K}(z, w) = \frac{1}{1 - z\bar{w}} .$$

(ii) *If $az + d \neq 0$ then $J(g, z)$ is defined and*

$$J(g, z) = cz + d.$$

(iii) *Assume that $|z| < 1$. Then $u_G(z)$ is defined and*

$$u_G(z) = \frac{1}{\sqrt{1 - z^2}} .$$

■

Description of $\partial_s \Xi$ in $\partial_s \mathcal{D}$

In this subsection we will use the $\mathfrak{su}(1,1)$-reduction to identify $\partial_s \Xi$ in $\partial_s \mathcal{D}$ as the non-vanishing locus of the function ψ.

Theorem 3.8. *We have:*

$$\partial_s \Xi = \{z \in \partial_s \mathcal{D} : \psi(z) \neq 0\} .$$

Proof. "$\subseteq$": Recall the element $z_1 \in \partial_s \Xi$ with isotropy subgroup H, i.e., $\partial_s \Xi = G(z_1) \simeq G/H$. We first show that $\psi(z_1) \neq 0$ by using $\mathrm{SU}(1,1)$-reduction. For $S = \mathrm{SU}(1,1)$ we have $\mathcal{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $z_1 = i$. In particular $\psi(z_1) \neq 0$ by Lemma 3.7(iii). Now $\psi(z_1) \neq 0$ in the general case follows from simple $\mathrm{SU}(1,1)$-reduction (cf. Lemma 3.6 and our structural results in Appendix B). From the covariance property of ψ (cf. Proposition 3.4) we hence get that $\psi(gz_1) \neq 0$ for all $g \in G$. In particular ψ has no zeros on $G(z_1) = \partial_s \Xi$.
"$\supseteq$": By Theorem 1.14 we know that $\partial_s \Xi$ is open and dense in $\partial_s \mathcal{D}$. Let Y_j and $\mathfrak{a} = \sum_{j=1}^r \mathbb{R} Y_j$ be as in Appendix B. Let $A = \exp(\mathfrak{a})$. Then $G = KAH$. Let g_n be a sequence in G such that

$g_n(z_1) \to \partial_s\mathcal{D}\backslash\partial_s\Xi$. Write $g_n = k_n a_n h_n$ with $k_n \in K$, $h_n \in H$ and $a_n = \exp(\sum_{j=1}^r t_{r,n} Y_j) \in A$. Note that $g_n(z_1) \to \partial_s\mathcal{D}\backslash\partial_s\Xi$ means precisely that $a_n \to \infty$.

We have $g_n(z_1) = k_n a_n(z_1)$. We also have that $|J_h(k,z)| = 1$ for all $k \in K$. Hence it follows from Proposition 3.4 and $c := |\psi(z_1)| > 0$ that

$$|\psi(k_n a_n(z_1))| = |J_h^{-1}(k_n, a_n(z_1))| \cdot |\psi(a_n(z_1))| = |\psi(a_n, z_1)| = c|J_h^{-1}(a_n, z_1)|\,.$$

By $\mathrm{SU}(1,1)$-reduction (cf. Lemma 3.6 and Lemma 3.7(ii)) we get

$$J_h^{-1}(a_n, z_1) = \prod_{j=1}^r (i\sinh(t_{n,j}) + \cosh(t_{n,j}))^{-\rho_n(H_j)}\,.$$

As $a_n \to \infty$ there exists a j and subsequence $s_n = t_{n,j}$ such that $|s_n| \to \infty$ and therefore

$$\prod_{j=1}^r (i\sinh(t_{n,j}) + \cosh(t_{n,j}))^{-\rho_n(H_j)} \to 0$$

because $\rho_n(H_j) > 0$. ∎

L^2-isomorphism on the Shilov boundaries

Let $\mu_{\partial_s\mathcal{D}}$ be the unique (up to constant) U-invariant measure on $\partial_s\mathcal{D}$. If $\lambda_h = -\rho_n$ is analytically integral, then S acts unitarily on $L^2(\partial_s\mathcal{D}, \mu_{\partial_s\mathcal{D}})$ by

$$(\pi_h(s)f)(z) = J_h(s^{-1}, z)^{-1} f(s^{-1}z) \qquad (s \in S,\ z \in \mathcal{D},\ f \in \mathcal{H}^2(\mathcal{D})) \tag{3.5}$$

with $J_h = J_{-\rho_n}$. In the case where $m = 2$ (see Proposition 3.1) we need to go to a double covering of S. But notice, that $\pi_h(s)$ is always defined for $s \in G$ according to Proposition 3.4. One of the consequences of this observation is the following identification of L^2-spaces:

Lemma 3.9. *The following assertions hold:*

(i) *The G-invariant measure on G/H as a subset of $\partial_s\mathcal{D}$ is given by*

$$f \mapsto \int_{\partial_s\mathcal{D}} f(z)|\psi(z)|^{-2}\, d\mu_{\partial_s\mathcal{D}}(z)$$

where $d\mu_{\partial_s\mathcal{D}}$ is the unique (up to constant) U-invariant measure on $\partial_s\mathcal{D}$.

(ii) *The mapping $f \mapsto \frac{1}{\psi} f$ is a G-equivariant isomorphism of $L^2(G/H)$ onto $L^2(\partial_s\mathcal{D}, \mu_{\partial_s\mathcal{D}})$.*

Proof. This follows from the covariance of ψ in Proposition 3.4 (cf. [BÓ01], Theorem 5.1). ∎

4. The classical Hardy space inside $L^2(G/H)_{\mathrm{mc}}$

In this section we will show how to realize the classical Hardy space $\mathcal{H}^2(\mathcal{D})$ in the most continuous spectrum of $L^2(G/H)$.

The most continuous spectrum $L^2(G/H)_{\rm mc}$

Recall the involution σ on $\mathfrak{g}$ which is associated to $\mathfrak{h}$. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be the σ-eigenspace decomposition. Note that we can choose $\mathfrak{a} \subseteq \mathfrak{q} \cap \mathfrak{p}$ since $(\mathfrak{g}, \sigma)$ is non-compactly causal (cf. [HiÓl96]). In particular, the minimal parabolic subgroup $P_{\min} = MAN$ is also a minimal $\theta\sigma$-*stable* parabolic subgroup of G. Here, as usual, $M = Z_K(\mathfrak{a})$. If $\mathfrak{m}$ is the Lie algebra of M, then note that $\mathfrak{m} \subseteq \mathfrak{h}$ since $\mathfrak{a} \subseteq \mathfrak{q} \cap \mathfrak{p}$. Moreover we have

$$M = Z_H(\mathfrak{a}) \subseteq H$$

as $G_{\mathbb{C}}$ is simply connected (cf. [HiÓl96, Lemma 3.1.22]).

Write $\pi_\delta = \mathrm{Ind}_{MAN}^G(\delta)$ for the K-spherical unitary principal series of parameter $\delta \in i\mathfrak{a}^*$. Notice that $\pi_\delta \simeq \pi_{w\delta}$ for $w \in \mathcal{W}$. The *most continuous part* $L^2(G/H)_{\rm mc}$ in $L^2(G/H)$ is by definition the G-invariant subspace in $L^2(G/H)$ which corresponds in the Plancherel formula to to all principal series induced off from a minimal $\theta\sigma$-stable parabolic subgroup. Denote by $\mathcal{W}_0 = N_{K\cap H}(\mathfrak{a})/Z_{K\cap H}(\mathfrak{a})$ the very little Weyl group. Since $M = M \cap H$ we now obtain from [vdBS97] or [De98] that

$$(L, L^2(G/H)_{\rm mc}) \simeq \Big(\int_{i\mathfrak{a}^*/\mathcal{W}}^{\oplus} \pi_\delta \otimes \mathrm{id} \ d\mu(\delta), \int_{i\mathfrak{a}^*/\mathcal{W}}^{\oplus} \mathcal{H}_\delta \otimes \mathbb{C}^{|\mathcal{W}/\mathcal{W}_0|} \ d\mu(\delta) \Big)$$

where μ is a Borel measure which is completely continuous with respect to the Lebesgue measure on $i\mathfrak{a}^*$. In particular for a measure dense subset the multiplicity of π_δ in $L^2(G/H)_{\rm mc}$ is $|\mathcal{W}/\mathcal{W}_0|$. Let $\omega \subseteq i\mathfrak{a}^*/\mathcal{W}$ be an open subset. Then we write

$$P_\omega : L^2(G/H)_{\rm mc} \to \int_\omega \mathcal{H}_\delta \otimes \mathbb{C}^{|\mathcal{W}/\mathcal{W}_0|} \ d\mu(\delta)$$

for the orthogonal G-invariant projection associated to ω.

Definition 4.1. We call a closed G-invariant subspace of $W \subseteq L^2(G/H)_{\rm mc}$ a subspace with *full spectrum* if $P_\omega(W) \neq \{0\}$ for all non-empty open subets $\omega \subseteq i\mathfrak{a}^*/\mathcal{W}$. ∎

It is interesting to compare the most-continuous spectrum $L^2(G/H)_{\rm mc}$ with the spectrum of $L^2(G/K)$. The Plancherel Theorem for $L^2(G/K)$ has the form

$$(L, L^2(G/K)) \simeq \int_{i\mathfrak{a}^*/\mathcal{W}} \pi_\delta \ d\mu'(\delta)$$

with μ' a Borel measure completely continuous with respect to $d\delta$. Hence we see that the spectrum of $L^2(G/K)$ is the same as the most continuous part of the spectrum of $L^2(G/H)$, but the difference is that the spectrum of $L^2(G/K)$ is multiplicity free.

Embedding of $\mathcal{H}^2(\mathcal{D})$ in the most continuous spectrum

Recall the U-invariant measure $\mu_{\partial_s\mathcal{D}}$ on the Shilov boundary of $\mathcal{D}$ and the boundary value map of the classical Hardy space:

$$b : (\pi_h, \mathcal{H}^2(\mathcal{D})) \to (\pi_h, L^2(\partial_s\mathcal{D}, \mu_{\partial_s\mathcal{D}})), \quad f \mapsto (z \mapsto \lim_{\substack{r\to 1 \\ r<1}} f(rz))$$

which is an S-equivariant isometric embedding. On the other hand we have the G-equivariant isomorphism from Lemma 3.9:

$$\Psi: (\pi_h, L^2(\partial_s \mathcal{D})) \to (L, L^2(G/H)), \quad f \mapsto \psi f.$$

Theorem 4.2. *The mapping*

$$\Psi \circ b: (\pi_h|_G, \mathcal{H}^2(\mathcal{D})) \to (L, L^2_{\mathrm{mc}}(G/H))$$

is a G-equivariant isometric embedding. Moreover the image $\mathrm{im}\,\Psi \circ b$ *is a mulitiplicity free subspace of full spectrum in* $L^2_{\mathrm{mc}}(G/H)$.

Proof. From Lemma 3.9 it is clear that $\Psi \circ b$ is a G-equivariant map from $\mathcal{H}^2(\mathcal{D})$ into $L^2(G/H)$. It remains to specify the image. From Corollary 2.11 we obtain that $\pi_h|_G \simeq L^2(G/K)$. Thus the remaining assertions of the theorem follow from our discussion of the most continuous spectrum in the previous subsection. ∎

5. The Hardy space on Ξ

In order to define the Hardy space on Ξ we first have to recall some facts on semigroups compressing $\mathcal{D}$, resp. Ξ.

Recall the space $\mathfrak{e} = \mathfrak{t} \cap \mathfrak{q}_*$ and associated restricted root system $\Delta_\mathfrak{e}$ from Section 2. For $\alpha \in \Delta_\mathfrak{e}$ we denote by $\check{\alpha} \in i\mathfrak{e}$ the coroot of α. Define

$$C_{\min} := \mathrm{cone}(\{\check{\alpha} : \alpha \in \Delta^+_{\mathfrak{e},n}\}),$$

where $\mathrm{cone}(\cdot)$ refers to the convex cone generated by $(\cdot)$. Then

$$W_{\min} := \mathrm{Ad}(G) C_{\min}$$

is a minimal open convex cone in $i\mathfrak{q}_*$ and

$$\Gamma := G \exp(W_{\min})$$

is an open G-biinvariant subsemigroup of $S_\mathbb{C}$ (cf. [HiÓl96, Ch. 4]). The closure of Γ in $S_\mathbb{C}$ is given by $\overline{\Gamma} = G \exp(\overline{W_{\min}})$. The polar mapping

$$G \times \overline{W_{\min}} \to \overline{\Gamma}, \quad (g, X) \mapsto g \exp(X)$$

is a homeomorphism (cf. [Ne99, Th. XI.1.7]). Moreover, $\overline{\Gamma}$ is an involutive semigroup with involution

$$\overline{\Gamma} \to \overline{\Gamma}, \quad \gamma = g \exp(X) \mapsto \gamma^* = \exp(X) g^{-1}.$$

Note that Γ^{-1} compresses $\mathcal{D}$:

$$(\forall \gamma \in \Gamma^{-1}) \qquad \gamma(\overline{\mathcal{D}}) \subseteq \mathcal{D}$$

(cf. [Ne99, Th. XII.3.3]). From the realization of $\overline{\Xi}$ in $\overline{\mathcal{D}}$ we therefore obtain an action of Γ^{-1} on $\overline{\Xi}$ with the property

$$(\forall \gamma \in \Gamma^{-1}) \qquad \gamma(\overline{\Xi}) \subseteq \Xi. \tag{5.1}$$

Definition 5.1. **(Hardy space on Ξ)** The Hardy space on Ξ is defined by

$$\mathcal{H}^2(\Xi) = \{f \in \mathcal{O}(\Xi) : \|f\|^2 := \sup_{\gamma \in \Gamma} \int_{G/H} |f(\gamma^{-1} g z_1)|^2 \, d\mu_{G/H}(gH) < \infty\}.$$ ∎

From the definition of the Hardy space it is not clear yet that $\mathcal{H}^2(\Xi)$ is a Hilbert space. This will follow from the following geometric fact and its proof:

Lemma 5.2. *The semigroup orbit $\Gamma^{-1}(z_1) \subseteq \Xi$ is open and measure dense.*

Proof. Recall the elements $H_j \in i\mathfrak{c}$ and note that $\sum_{j=1}^n t_j H_j \in C_{\min}$ whenever $t_j \geq 0$ and not all $t_j = 0$. By simple $\mathrm{SU}(1,1)$-reduction we obtain for all j that

$$\exp(-t_j H_j)(z_1) = \exp(i \arctan(e^{-t_j}) Y_j)(0). \tag{5.2}$$

Write $D := \{z \in \mathbb{C} : |z| < 1\}$ for the unit disc and identify $A \exp(i\Omega)(0)$ with D^n in the obvious way through $\mathrm{SU}(1,1)$-reduction. With $D^\times = D \backslash \{0\}$. Equation (5.2) then gives

$$A\mathcal{W} \exp(-C_{\min})(z_1) = (D^\times)^n. \tag{5.3}$$

In particular $A\mathcal{W} \exp(-C_{\min})(z_1)$ is open and Zariski dense in $A\exp(i\Omega)(0) = D^n$. Sweeping out Ξ by G-orbits through $\exp(i\Omega)(0)$ proves the lemma. ■

If M is a complex manifold, then we write $\mathcal{O}(M)$ for the Fréchet space of holomorphic functions on M.

Corollary 5.3. *The Hardy space $\mathcal{H}^2(\Xi)$ is a Hilbert space of holomorphic functions. Moreover, the inclusion mapping $\mathcal{H}^2(\Xi) \hookrightarrow \mathcal{O}(\Xi)$ is continuous.*

Proof. If we know that $\mathcal{H}^2(\Xi)$ is complete, then the standard semigroup techniques imply that $\mathcal{H}^2(\Xi)$ is a Hilbert space (cf. [HÓØ91] or [Ne99, Ch. XIV]). The completeness and the continuity of the embedding $\mathcal{H}^2(\Xi) \hookrightarrow \mathcal{O}(\Xi)$ is easily obtained from (5.3) and Cauchy's Theorem (use $\Gamma = G\Gamma G$ and choose local coordinates – recall the definition of the Hardy space). ■

Knowing that $\mathcal{H}^2(\Xi)$ is a Hilbert space, it is straightforward from the definition of $\mathcal{H}^2(\Xi)$ that the left regular representation L of G on $\mathcal{H}^2(\Xi)$ is unitary. In the terminology of [FaTh99] or [Kr99] this means that $\mathcal{H}^2(\Xi)$ is a *G-invariant Hilbert space of holomorphic functions on Ξ*. Those Hilbert spaces feature the following properties:

Remark 5.4. (a) The inclusion mapping $\mathcal{H}^2(\Xi) \hookrightarrow \mathcal{O}(\Xi)$ is continuous and so all point evaluations

$$\mathrm{ev}_z : \mathcal{H}^2(\Xi) \to \mathbb{C}, \quad f \mapsto f(z) \qquad (z \in \Xi)$$

are continuous. In particular for every $z \in \Xi$ there exists an element $K_z \in \mathcal{H}^2(\Xi)$ such that $f(z) = \langle f, K_z \rangle$.

(b) The Hardy space admits a reproducing kernel

$$K_\Xi : \Xi \times \Xi \to \mathbb{C}, \quad (z,w) \mapsto \langle K_w, K_z \rangle.$$

Note that K_Ξ is holomorphic in the first and antiholomorphic in the second variable. We call K_Ξ the *Cauchy-Szegö kernel* of Ξ.

(c) By construction the representation $(L, \mathcal{H}^2(\Xi))$ of G is unitary. This can also be phrased as that K_Ξ is G-invariant:

$$K_\Xi(gz, gw) = K_\Xi(z,w)$$

for all $g \in G$, $z, w \in \Xi$. ■

The boundary value map

Using the standard semigroup techniques for Hardy spaces (cf. [HÓØ91] or [Ne99, Ch. XIV]) one easily shows that the prescription

$$L:\overline{\Gamma} \to B(\mathcal{H}^2(\Xi)), \quad (L(\gamma)f)(z) := f(\gamma^{-1}z)$$

defines a strongly continuous involutive contractive representation of $\overline{\Gamma}$ whose restriction to Γ is a holomorphic mapping. Here *strongly continuous* means that the mapping $\overline{\Gamma} \to \mathcal{H}^2(\Xi)$, $\gamma \mapsto L(\gamma)f$ is continuous for all $f \in \mathcal{H}^2(\Xi)$; *involutive* means $L(\gamma^*) = L(\gamma)^*$ for all $\gamma \in \overline{\Gamma}$, and *contractive* means that $\|L(\gamma)\| \leq 1$ for all $\gamma \in \overline{\Gamma}$. Note that L is involutive implies in particular that $L|_G$ is unitary.

In order to define the boundary value map we first have to exhibit a good G-invariant subspace of $\mathcal{H}^2(\Xi)$ whose elements extend continuously up to the boundary. A good choice herefore is the space of analytic vectors $\mathcal{H}^2(\Xi)^\omega$ of the representation $(L, \mathcal{H}^2(\Xi))$ of G. According to [KNÓ97, App.] we have

$$\mathcal{H}^2(\Xi)^\omega = \bigcup_{\gamma\in\Gamma} L(\gamma)\mathcal{H}^2(\Xi).$$

In particular it follows from the compression property (5.1) that all functions in $\mathcal{H}^2(\Xi)^\omega$ holomorphically extend over the closure $\overline{\Xi}$ of Ξ. As a consequence we obtain a well defined boundary value mapping

$$b^\omega: \mathcal{H}^2(\Xi)^\omega \to L^2(G/H), \quad f \mapsto \big(gH \mapsto \lim_{\substack{\gamma\to 1\\ \gamma\in\Gamma}} f(\gamma^{-1}gz_1)\big)$$

which is G-equivariant and continuous.

Theorem 5.5. *The boundary value mapping*

$$b^\omega: \mathcal{H}^2(\Xi)^\omega \to L^2(G/H), \quad f \mapsto \big(gH \mapsto \lim_{\substack{\gamma\to 1\\ \gamma\in\Gamma}} f(\gamma^{-1}gz_1)\big)$$

extends to a G-equivariant isometric embedding

$$b: \mathcal{H}^2(\Xi) \to L^2(G/H).$$

Proof. This follows from our discussion above and the standard semigroup techniques (cf. [HÓO91] or [Ne99, Ch. XIV]). ■

The isomorphism between $\mathcal{H}^2(\Xi)$ and $\mathcal{H}^2(\mathcal{D})$

In the sequel we will identify Ξ with $\mathcal{D}$ under the biholomorphism $\Phi: \Xi \to \mathcal{D}$ from Section 2. We let G act on $\mathcal{O}(\Xi)$ via the left regular representation L and we let S (or its double covering if necessary) act on $\mathcal{O}(\mathcal{D})$ via the cocycle action π_h

$$(\pi_h(s)f)(z) = J_h(s^{-1}, z)^{-1} f(s^{-1}z)$$

for all $s \in S$, $z \in \mathcal{D}$ and $f \in \mathcal{O}(\mathcal{D})$. Then we have an isomorphism of topological vector spaces

$$\Psi: \mathcal{O}(\mathcal{D}) \to \mathcal{O}(\Xi), \quad F \mapsto \psi F$$

with inverse mapping

$$\Psi^{-1}: \mathcal{O}(\Xi) \to \mathcal{O}(\mathcal{D}), \quad f \mapsto \frac{1}{\psi} f$$

Note that Ψ and Ψ^{-1} are defined since ψ is a zero free holomorphic function according to Proposition 3.4.

Lemma 5.6. *The maps Ψ and Ψ^{-1} are G-equivariant.*

Proof. Since $J_h = J_{-\rho_n}$ this follows from Proposition 3.4. ■

From Lemma 5.6 we obtain immediately that the mapping

$$\Psi\colon (\pi_h|_G, \mathcal{H}^2(\mathcal{D})) \to (L, \mathcal{O}(\Xi)), \quad f \mapsto \psi f$$

is a continuous G-equivariant embedding. More precisely we have:

Theorem 5.7. *The mapping*

$$\Psi\colon (\pi_h|_G, \mathcal{H}^2(\mathcal{D})) \to (L, \mathcal{H}^2(\Xi)), \quad f \mapsto \psi f$$

is a G-equivariant isomorphism of Hilbert spaces.

Proof. Recall the element $X_0 \in \mathfrak{z}(\mathfrak{u}) \cap \mathfrak{q}_*$. For $s \in]-\infty, 0[$ define

$$\gamma_s := \exp(isX_0)$$

and note that $\gamma_s \in Z(U_{\mathbb{C}}) \cap \Gamma$. Furthermore we have $\gamma_s^{-1}(z_1) = z_{e^s}$ which follows by simple $\mathfrak{su}(1,1)$-reduction using the results in Appendix B. We first show that $\frac{1}{\psi} f \in \mathcal{H}^2(\mathcal{D})$ for all $f \in \mathcal{H}^2(\Xi)$ and that $\|f\| = \|\frac{1}{\psi} f\|$. For every $s < 0$ set $f_s(gH) := f(\gamma_s^{-1} g z_1)$ for $g \in G$. Now it follows from the definition of $\mathcal{H}^2(\Xi)$ and Lemma 3.9 that

$$\begin{aligned}
\|f\|^2 &= \lim_{\substack{\gamma\to\mathbf{1}\\ \gamma\in\Gamma}} \int_{G/H} |f(\gamma^{-1} g z_1)|^2 \, d\mu_{G/H}(gH) \\
&= \sup\nolimits_{s<0} \int_{G/H} |f(\gamma_s^{-1} g z_1)|^2 \, d\mu_{G/H}(gH) \\
&= \sup\nolimits_{s<0} \int_{G/H} |f_s(g z_1)|^2 \, d\mu_{G/H}(gH) \\
&= \sup\nolimits_{s<0} \int_{\partial_s \mathcal{D}} |f_s(z)|^2 |\psi(z)|^{-2} \, d\mu_{\partial_s\mathcal{D}}(z) \\
&= \sup\nolimits_{0\le r<1} \int_{\partial_s \mathcal{D}} \Big|\frac{f(rz)}{\psi(z)}\Big|^2 \, d\mu_{\partial_s\mathcal{D}}(z).
\end{aligned} \tag{5.4}$$

Now consider the function

$$F\colon [0,1] \times \partial_s\mathcal{D}, \quad (r,z) \mapsto \Big|\frac{\psi(z)}{\psi(rz)}\Big|.$$

Note that F is continuous, hence bounded since defined on a compact set. In particular we find a constant $C > 0$ such that $\frac{1}{|\psi(z)|} \geq C \frac{1}{|\psi(rz)|}$ for all r, z. With this information we obtain from (5.4) that

$$\|f\|^2 \geq C \sup\nolimits_{0\le r<1} \int_{\partial_s \mathcal{D}} \Big|\frac{f(rz)}{\psi(rz)}\Big|^2 \, d\mu_{\partial_s\mathcal{D}}(z), \tag{5.5}$$

i.e., $\frac{1}{\psi} f \in \mathcal{H}^2(\mathcal{D})$. Moreover from $F(1,z) = 1$ for all $z \in \partial_s\mathcal{D}$ we now obtain from (5.4) and (5.5) that

$$\begin{aligned}
\|f\|^2 &= \lim_{r\to 1} \int_{\partial_s \mathcal{D}} \Big|\frac{f(rz)}{\psi(z)}\Big|^2 \, d\mu_{\partial_s\mathcal{D}}(z) \\
&= \lim_{r\to 1} \int_{\partial_s \mathcal{D}} |F(r,z)|^2 \Big|\frac{f(rz)}{\psi(rz)}\Big|^2 \, d\mu_{\partial_s\mathcal{D}}(z).
\end{aligned}$$

Thus we have shown that Ψ^{-1} maps $\mathcal{H}^2(\Xi)$ isometrically into $\mathcal{H}^2(\mathcal{D})$. Finally, reversing the arguments from above, also yields that Ψ maps $\mathcal{H}^2(\mathcal{D})$ isometrically into $\mathcal{H}^2(\Xi)$, concluding the proof of the theorem. ∎

An immediate conclusion from Theorem 5.7 in conjunction with Theorem 5.5 and Theorem 4.2 now is:

Theorem 5.8. *The image of the boundary value map $b : \mathcal{H}^2(\Xi) \to L^2(G/H)$ is a mulitiplicity free subspace of $L^2(G/H)_{\mathrm{mc}}$ of full spectrum.* ∎

The Cauchy-Szegö kernel

We conclude this section with the derivation of the Cauchy-Szegö kernel for $\mathcal{H}^2(\Xi)$.

Theorem 5.9. *If K_h is the reproducing kernel of $\mathcal{H}^2(\mathcal{D})$ then the Cauchy-Szegö kernel K_Ξ is given by*

$$K_\Xi(z,w) = \psi(z)\overline{\psi(w)}K_h(z,w).$$

Proof. In this proof denote by $(\cdot|\cdot)$ the scalar product on $\mathcal{H}^2(\Xi)$ and by $\langle\cdot,\cdot\rangle$ the scalar product on $\mathcal{H}^2(\mathcal{D})$. Fix $z \in \Xi$ and let $f \in \mathcal{H}^2(\Xi)$. Consider the function $w \mapsto \overline{\psi(z)}\psi(w)K_h(w,z)$ and note that this function lies in $\mathcal{H}^2(\Xi)$. We now have

$$\begin{aligned}(f|\overline{\psi(z)}\psi(\cdot)K_h(\cdot,z)) &= \psi(z)(f|\psi(\cdot)K_h(\cdot,z))\\ &= \psi(z)\langle\frac{1}{\psi(\cdot)}f, K_h(\cdot,z)\rangle\\ &= \psi(z)\frac{1}{\psi(z)}f(z) = f(z).\end{aligned}$$

This concludes the proof of the theorem. ∎

Example 5.10. Let us consider the case of $S = G\times G$. We keep the notation of Example 3.5 and identify $\Xi = \mathcal{D}$ with $\mathcal{D}_G \times \mathcal{D}_G^{\mathrm{opp}}$. Write $K_h^G(z,w)$ for the kernel of the classical Hardy space on $\mathcal{D}_G$. Then the kernel K_h of the classical Hardy space on $\mathcal{D}$ is given by

$$K_h\big((z_1,w_1),(z_2,w_2)\big) = K_h^G(z_1,z_2)\overline{K_h^G(w_1,w_2)}$$

for all (z_1,w_1), (z_2,w_2) in $\mathcal{D}$. From Theorem 5.8 and 3.4 we thus obtain that

$$K_\Xi\big((z_1,w_1),(z_2,w_2)\big) = \frac{K_h^G(z_1,z_2)\overline{K_h^G(w_1,w_2)}}{K_h^G(z_1,w_1)\overline{K_h^G(z_2,w_2)}}. \tag{5.6}$$

In particular for $G = \mathrm{SU}(1,1)$ where $\mathcal{D}_G = D$ is the unit disc we have $K_h(z,w) = \frac{1}{1-z\overline{w}}$ and so

$$K_\Xi\big((z_1,w_1),(z_2,w_2)\big) = \frac{(1-z_1\overline{w_1})(1-\overline{z_2}w_2)}{(1-z_1\overline{z_2})(1-\overline{w_1}w_2)}.$$ ∎

Final remarks

It is possible to extend the results in this paper to the cases in Theorem 1.12 (ii), (iii) and (iv). Let us explain this for groups G which are structure groups of Euclidean Jordan algebras (the case in Theorem 1.12(ii)). Here one has

$$\partial_d \Xi_0 = \underbrace{\coprod_{j=1}^{n} G/H_j}_{\partial_s \Xi} \amalg G/K \amalg G/K$$

with each G/H_j non-compactly causal (for more precise information see [GiKr01b, Sect. 4]). For every boundary component G/H_j of $\partial_d \Xi_0$ one can define a Hardy space

$$\mathcal{H}_j^2(\Xi_0) = \{f \in \mathcal{O}(\Xi_0)\colon \sup_{\gamma\in\Gamma} \int_{G/H_j} |f(\gamma^{-1} g z_j)|^2 \; d\mu_{G/H_j}(gH_j) < \infty\}.$$

We expect a boundary value map

$$b_j\colon \mathcal{H}_j^2(\Xi_0) \hookrightarrow L^2(G/H_j)$$

with image a full subspace in $L^2(G/H_j)_{\mathrm{mc}}$. Also we expect for each boundary component G/H_j a G-invariant isometric isomorphism

$$\mathcal{H}_j^2(\Xi_0) \hookrightarrow \mathcal{H}^2(\mathcal{D})$$

with the classical Hardy space.

Appendix A: parameter calculations

Unless not otherwise specified $\mathfrak{s}$ denots a simple Hermitian Lie algebra with compact Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{u}$. Write $\gamma_1, \ldots, \gamma_n$ for a maximal system of strongly orthogonal roots. Define

$$\zeta := \frac{1}{2}(\gamma_1 + \ldots + \gamma_r)$$

and note that $\zeta \in i\mathfrak{z}(\mathfrak{u})^*$. Note that we can choose the positive system Δ^+ such that $\beta = \gamma_1$ is the highest root. We write $\check{\gamma}_j \in i\mathfrak{t}$ for the coroot of γ_j. Denote by $\mathfrak{e}$ the subspace of $\mathfrak{t}$ which is spanned by the $i\check{\gamma}_j$. Write $\Sigma := \Delta\,|_{\mathfrak{e}} \backslash \{0\}$ for the restricted root system. Define $\Sigma_n := \Delta_n\,|_{\mathfrak{e}}$, $\Sigma_c := \Delta_c\,|_{\mathfrak{e}} \backslash \{0\}$. Similarly define Σ^+, Σ_n^+, Σ_c^+. By results of Harish-Chandra and Moore we have:

$$\Sigma_n^+ = \{\frac{1}{2}(\gamma_i + \gamma_j)\colon 1 \le i, j \le r\}$$

and

$$\Sigma_c^+ = \{\frac{1}{2}(\gamma_i - \gamma_j)\colon 1 \le i < j \le r\} \cup \{\frac{1}{2}\gamma_i\colon 1 \le i \le r\}.$$

The second term in Σ_c^+ only appears if $\mathfrak{s}$ is not of tube type.

Define the number $d := \dim_{\mathbb{C}} \mathfrak{s}_{\mathbb{C}}^{\frac{1}{2}(\gamma_i - \gamma_j)} = \dim_{\mathbb{C}} \mathfrak{s}_{\mathbb{C}}^{\frac{1}{2}(\gamma_i + \gamma_j)}$ for $i \neq j$. This number does not depend on i and j, only on $\mathfrak{s}$. The table is as follows:

$\mathfrak{s}$	d
$\mathfrak{sp}(n, \mathbb{R})$	1
$\mathfrak{su}(p, q)$	2
$\mathfrak{so}(2, n)$	$n - 2 \quad (n > 2)$
$\mathfrak{so}^*(2n)$	2
$\mathfrak{e}_{6(-14)}$	3
$\mathfrak{e}_{(7(-25)}$	4

Theorem A.1. (Wallach) (cf. [Wa79]) *Let $\lambda_z := z\zeta$ for $z \in \mathbb{R}$. Then the set of scalar unitary highest weights $\lambda_z \in i\mathfrak{z}(\mathfrak{u})^*$ is parametrized by*

$$z \in \mathbb{W} \quad \textit{with} \quad \mathbb{W} =]-\infty, -\frac{d(r-1)}{2}[\cup \{-\frac{d(r-1)}{2}, -\frac{d(r-2)}{2}, \ldots, 0\}. \qquad \blacksquare$$

The set $\mathbb{W}$ in Theorem A.1 is called the *Wallach set.* Note that λ_z is regular means precisely that $z < -\frac{d(r-1)}{2}$.

Lemma A.2. *If $\mathfrak{s}$ is of tube type, i.e, if Σ is of type C_r, then the following assertions hold:*

(i) $\rho_n = (1 + \frac{d(r-1)}{2})\zeta$.

(ii) $\langle \rho, \beta \rangle = \frac{1}{2}(1 + d(r-1))$.

(iii) *If $\lambda \in i\mathfrak{z}(\mathfrak{u})^*$ is such that $\langle 2\lambda + \rho, \beta \rangle < 0$, then λ is regular.*

(iv) *The Hardy space weight $\lambda_h = -\rho_n$ satisfies the condition $\langle 2\lambda_h + \rho, \beta \rangle < 0$.*

Proof. (i) [ÓØ99, Lemma 2.5].

(ii) Note that $\rho = \rho_n + \rho_c$. Thus (ii) is immediate from (i) and the structure of Σ_c^+.

(iii) Write $\lambda = \lambda_z$. Then by (ii) the condition $\langle 2\lambda + \rho, \beta \rangle < 0$ means that

$$z < -\frac{1}{2}(1 + d(r-1))$$

and Wallach's Theorem A.1 implies the assertion.

(iv) We obtain from (i) that

$$\langle 2\lambda_h + \rho, \beta \rangle = \langle -\rho_n + \rho_c, \gamma_1 \rangle = -\frac{1}{2}(1 + \frac{d(r-1)}{2}) + \langle \rho_c, \gamma_1 \rangle.$$

Now $\langle \rho_c, \gamma_1 \rangle = \frac{d(r-1)}{4}$. Thus

$$\langle 2\lambda_h + \rho, \beta \rangle = -\frac{1}{2}(1 + \frac{d(r-1)}{2}) + \frac{d(r-1)}{4} = -\frac{1}{2} < 0,$$

as was to be shown. $\blacksquare$

Proposition A.3. *Let $(\mathfrak{s}, \mathfrak{g}, \mathfrak{h})$ be a causally symmetric triple. Then the following assertions hold:*

(i) *The Hardy space parameter $\lambda_h = -\rho_n$ is regular.*

(ii) *The Hardy space parameter $\lambda_h = -\rho_n$ satisfies the condition $\langle \lambda_h + \rho_G^c, \beta \rangle < 0$.*

Proof. If $\mathfrak{s} = \mathfrak{g} \oplus \mathfrak{g}$ is the group case, then $\mathfrak{g}$ is of tube type and $\rho_G^c = \frac{1}{2}\rho$. Thus (i) and (ii) follow in this case from Lemma A.2. Hence we may assume that $\mathfrak{s}$ is simple. Since $\mathfrak{s}$ is of tube type, (i) follows from Lemma A.2(i) and Theorem A.1.

It remains to check that (ii) is satisfied for the pairs

$$(\mathfrak{s},\mathfrak{g})=(\mathfrak{sp}(2n,\mathbb{R}),\ \mathfrak{sp}(n,\mathbb{R})),\ (\mathfrak{su}(2n,2n),\ \mathfrak{sp}(n,\mathbb{C})),\quad \text{and}\quad (\mathfrak{so}(2,n),\mathfrak{so}(1,n)).$$

Let us start with $(\mathfrak{sp}(2n,\mathbb{R}),\mathfrak{sp}(n,\mathbb{R}))$. In this case it is easy to see that $\langle\rho_G^c,\beta\rangle=\frac{1}{2}\langle\rho,\beta\rangle$. Thus we obtain

$$\langle\lambda_h+\rho_G^c,\beta\rangle=\frac{1}{2}\langle 2\lambda_h+\rho,\beta\rangle<0$$

by Lemma A.2 (iv).

Next consider the case $(\mathfrak{su}(2n,2n),\mathfrak{sp}(n,\mathbb{C}))$. Note that the restricted root system of $\mathfrak{sp}(n,\mathbb{C})$ is of type C_n with with root space dimension 2 for all roots. Thus we obtain from Lemma A.2(ii) that

$$\langle\rho_G^c,\beta\rangle=\frac{1}{2}\Big(\frac{1}{2}(1+2(n-1))\Big)=-\frac{1}{4}+\frac{n}{2}.$$

On the other hand we have

$$\langle\lambda_h,\beta\rangle=-\frac{1}{2}(1+1\cdot\frac{2n-1}{2})=-(\frac{1}{4}+\frac{n}{2}).$$

Thus $\langle\lambda_h+\rho_G^c,\beta\rangle=-\frac{1}{2}<0$.

Finally consider the pair $(\mathfrak{so}(2,n),\mathfrak{so}(1,n))$. Here one has

$$\langle\lambda_h+\rho_g^c,\beta\rangle=\langle\lambda_h,\beta\rangle+\langle\rho_g^c,\beta\rangle=-\frac{n}{4}+\frac{n-1}{4}=-\frac{1}{4}<0,$$

concluding the proof of the proposition. ■

Appendix B: Structure theory for totally symmetric triples

In this subsection we will develop some structure theory for causally symmetric triples $(\mathfrak{s},\mathfrak{g},\mathfrak{h})$. The results will be useful for certain $\mathfrak{su}(1,1)$-reductions used throughout the paper. Also we will give a structural description of the maximal parabolic subgroup P of S. We will state all results without proof since they entirely consist of minor variations of results already in the literature (cf. [HiÓl96, Ch. 1, Ch. 4, App. A]).

As in Section 2 let $\mathfrak{t}\subseteq\mathfrak{u}$ denote a compact τ-stable Cartan subalgebra of $\mathfrak{s}$. Set $\mathfrak{e}:=\mathfrak{t}\cap\mathfrak{q}_*$ and write $\Delta_{\mathfrak{e}}=\Delta(\mathfrak{s}_{\mathbb{C}},\mathfrak{e}_{\mathbb{C}})$ for the restricted root system with respect to $\mathfrak{e}_{\mathbb{C}}$. We can choose Δ^+ (without changing Δ_n^+) such that $\Delta_{\mathfrak{e}}^+:=\Delta^+|_{\mathfrak{e}}\backslash\{0\}$ is a positive system for $\Delta_{\mathfrak{e}}$. Further we set $\Delta_{\mathfrak{e},n}^+:=\Delta_n^+|_{\mathfrak{e}}$. Note that $\Delta_{\mathfrak{e}}$ is a root system of type C_r. Write $\gamma_1,\ldots,\gamma_r$ for a maximal set of long positive strongly orthogonal roots in $\Delta_{\mathfrak{e}}$. Define $H_j\in i\mathfrak{e}$ by $\gamma_k(H_j)=2\delta_{kj}$. Note that

$$X_0=-i\frac{1}{2}(H_1+\ldots+H_r).$$

We can choose $E_{\pm j}\in\mathfrak{s}_{\mathbb{C}}^{\pm\gamma_j}$ such that

$$(B.1)\qquad \overline{E_j}=E_{-j},\quad \tau(E_j)=E_{-j},\quad \theta(E_j)=-E_{-j}\quad\text{and}\quad [E_j,E_{-j}]=H_j.$$

Note that the elements $\{H_j,E_j,E_{-j}\}$ form an $\mathfrak{su}(1,1)$-triple in $\mathfrak{s}_{\mathbb{C}}$. Set

$$Z_j:=-i(E_j-E_{-j})\qquad\text{and}\qquad Y_j:=E_j+E_{-j}$$

and note that $Z_j \in \mathfrak{q}_* \cap \mathfrak{p}_*$ and $Y_j \in \mathfrak{g} \cap \mathfrak{p}_* = \mathfrak{p}$.

If we set $X_j := -iH_j$ then the elements $\{X_j, Y_j, Z_j\}$ form an $\mathfrak{su}(1,1)$-triple in $\mathfrak{s}$ with relations

$$(B.2) \qquad [X_j, Y_j] = Z_j, \quad [X_j, Z_j] = -Y_j, \quad [Y_j, Z_j] = X_j.$$

Our choice of the maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$ then is

$$(B.3) \qquad \mathfrak{a} := \bigoplus_{j=1}^{r} \mathbb{R} Y_j .$$

Further

$$(B.4) \qquad \mathfrak{b} := \bigoplus_{j=1}^{r} \mathbb{R} Z_j$$

defines a maximal abelian subspace in $\mathfrak{q}_* \cap \mathfrak{p}_*$. Define elements

$$Y_0 := \frac{1}{2}(Y_1 + \ldots + Y_r) \quad \text{and} \quad Z_0 := \frac{1}{2}(Z_1 + \ldots + Z_r).$$

Note that $\mathrm{Spec}(\mathrm{ad}\, Y_0) = \mathrm{Spec}(\mathrm{ad}\, Z_0) = \{-1, 0, 1\}$. Next we define a Cayley transform

$$\mathbf{C}\colon \mathfrak{s}_{\mathbb{C}} \to \mathfrak{s}_{\mathbb{C}}, \quad X \mapsto e^{i\frac{\pi}{2}\,\mathrm{ad}\, Y_0}(X).$$

Notice the relations

$$\mathbf{C}(H_j) = Z_j, \quad \mathbf{C}(Z_j) = -H_j, \quad \mathbf{C}(Y_j) = Y_j.$$

Since $\mathrm{Spec}(\mathrm{ad}\, Y_0) = \{-1, 0, 1\}$, the prescription

$$\sigma(X) = (e^{i\pi\,\mathrm{ad}\, Y_0} \circ \theta)(X) \qquad (X \in \mathfrak{s})$$

defines an involution on $\mathfrak{s}$. Note that σ commutes with τ by construction. Write $\mathfrak{s} = \mathfrak{h}^\star \oplus \mathfrak{q}^\star$ for the σ-eigenspace decomposition of $\mathfrak{s}$ and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ for the σ-eigenspace decomposition of $\mathfrak{g}$, i.e., $\mathfrak{h} = \mathfrak{h}^\star \cap \mathfrak{g}$ and $\mathfrak{q} = \mathfrak{g} \cap \mathfrak{q}^\star$. Then

$$(B.5) \qquad \mathfrak{a} \subseteq \mathfrak{p} \cap \mathfrak{q} \qquad \text{and} \qquad \mathfrak{b} \subseteq \mathfrak{q}_* \cap \mathfrak{q}^\star \cap \mathfrak{p}_*.$$

As in Section 2 we let $S_{\mathbb{C}}$ be a simply connected Lie group with Lie algebra $\mathfrak{s}_{\mathbb{C}}$ and S, G, U, K the analytic subgroups of $S_{\mathbb{C}}$ corresponding to $\mathfrak{s}$, $\mathfrak{g}$, $\mathfrak{u}$ and $\mathfrak{k}$. Our final goal is to describe the maximal parabolic subgroup $P = \mathbf{C}(U_{\mathbb{C}} P^-) \cap S$ of S. For that write $\Sigma_{\mathfrak{b}} = \Sigma(\mathfrak{s}, \mathfrak{b})$ for the restricted root system of $\mathfrak{s}$ with respect to $\mathfrak{b}$. Note that $\Sigma_{\mathfrak{b}} = \mathbf{C}^t(\Delta_{\mathfrak{c}})$ and so $\Sigma_{\mathfrak{b}}$ is of type C_r. Define $\Sigma_{\mathfrak{b},n}^+ := C^t(\Delta_{\mathfrak{c},n}^+)$ and write

$$(B.6) \qquad \mathfrak{n}_{\mathfrak{b}}^{\pm} := \bigoplus_{\alpha \in \pm\Sigma_{\mathfrak{b},n}^+} \mathfrak{s}^{\alpha}.$$

Note that $\mathfrak{n}_{\mathfrak{b}}^{\pm}$ are both abelian and real forms of $\mathbf{C}(\mathfrak{p}^{\pm})$. Write $N_{\mathfrak{b}}^{\pm}$ for the analytic subgroups of S corresponding to $\mathfrak{n}_{\mathfrak{b}}^{\pm}$. Write $L := \mathbf{C}(U_{\mathbb{C}}) \cap S$. Then L is the Levi factor of P and we have

$$P = L \rtimes N_{\mathfrak{b}}^{-}. \tag{B.7}$$

We will show that $L = H^{\star} := S^{\sigma}$. First we describe $\mathfrak{l} := \mathrm{Lie}(\mathfrak{l})$. For that define subalgebras

$$\mathfrak{s}(0) := \mathfrak{z}_{\mathfrak{s}}(Y_0) \quad \text{and} \quad \mathfrak{g}(0) = \mathfrak{z}_{\mathfrak{g}}(Y_0).$$

Then $\mathfrak{s}(0) = \mathfrak{s}^{\sigma\theta}$ and $\mathfrak{g}(0) = \mathfrak{g}^{\sigma\theta}$ and therefore

$$\mathfrak{g}(0) = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{p} \cap \mathfrak{q}) \quad \text{and} \quad \mathfrak{s}(0) = (\mathfrak{h}^{\star} \cap \mathfrak{u}) \oplus (\mathfrak{q}^{\star} \cap \mathfrak{p}_{*}). \tag{B.8}$$

Thus we obtain

$$\begin{aligned} \mathfrak{l} = \mathbf{C}(\mathfrak{u}_{\mathbb{C}}) \cap \mathfrak{s} &= (\mathfrak{u}_{\mathbb{C}} \cap \mathfrak{s}(0)) \oplus \big(\mathbf{C}(\mathfrak{u}_{\mathbb{C}}) \cap (\mathfrak{h}^{\star} \cap \mathfrak{p}_{*} \oplus \mathfrak{q}^{\star} \cap \mathfrak{u})\big) \\ &= (\mathfrak{h}^{\star} \cap \mathfrak{u}) \oplus (\mathfrak{h}^{\star} \cap \mathfrak{p}_{*}) = \mathfrak{h}^{\star}. \end{aligned}$$

Hence $L = H^{\star}$. Summarizing our discussion we have shown:

Lemma B.1. *Let $P = \mathbf{C}(U_{\mathbb{C}} P^{-}) \cap S$ be the stabilizer of $x_1 \in \partial_s \mathcal{D}$ in S. Then the Levi-decomposition of the maximal parabolic subgroup P is given by*

$$P = H^{\star} \rtimes N_{\mathfrak{b}}^{-}.$$

■

References

[AkGi90] Akhiezer, D. N., and S. G. Gindikin, *On Stein extensions of real symmetric spaces*, Math. Ann. **286**, 1–12, 1990.

[vdBS97] van den Ban, E., and H. Schlichtkrull, *The most continuous part of the Plancherel decomposition for a reductive symmetric space*, Ann. of Math. **(2) 145** (1997), no. **2**, 267–364.

[Ber96] Bertram, W., *On some causal and conformal groups*, J. Lie Theory **6** (1996), 215–247.

[Bet97] Betten, F., *Causal compactification of compactly causal spaces*, TAMS, to appear.

[BÓ01] Betten, F., and G. Ólafsson, *Causal compactification and Hardy spaces for spaces of Hermitian type*, Pacific, J. of Math. Vol. **200 (2)** (2001), 273–312.

[DÓZ01] Davidson, M., G. Ólafsson, G. Zhang, *Segal-Bargmann transform on Hermitian symmetric spaces and orthogonal polynomials.* Preprint, 2001.

[De98] Delorme, P., *Formule de Plancherel pour les espaces symétriques réductifs*, Ann. of Math. **(2) 147** (1998), no. **2**, 417–452.

[EHW83] Enright, T., R. Howe, and N. Wallach, *A classification of unitary highest weight modules*, Representation theory of reductive groups (Park City, Utah, 1982), 97–143, Progr. Math., **40**, Birkhäuser Boston, Boston, MA, 1983.

[FaTh99] Faraut, J., and E. G. F. Thomas, *Invariant Hilbert spaces of holomorphic functions*, J. Lie Theory **9** (1999), no. **2**, 383–402.

[GeGi77] Gel'fand, M., and S.G. Gindikin, *Complex Manifolds whose Skeletons are real Lie Groups, and Analytic Discrete Series of Representations*, Funct. Anal. and Appl. **11**(1977), 19–27.

[Gi98] Gindikin, S., *Tube domains in Stein symmetric spaces*, Positivity in Lie theory: open problems, 81–97, de Gruyter Exp. Math., **26**, de Gruyter, Berlin, 1998.

[GiKr01a] Gindikin, S., and B. Krötz, *Complex crowns of Riemannian symmetric spaces and non-compactly causal symmetric spaces*, MSRI preprint 2001-038.

[GiKr01b] —, *Invariant Stein domains in Stein symmetric spaces and a non-linear complex convexity theorem*, MSRI preprint 2001-032.

[GiMa01] Gindikin, S., and T. Matsuki, *Stein Extensions of Riemann Symmetric Spaces and Dualities of Orbits on Flag Manifolds*, MSRI preprint 2001-028.

[GrKo01] Groenevelt, W., and E. Koelnik, *Meixner functions and polynomials related to Lie algebra representations*, preprint.

[Hel78] Helgason, S., "Differential Geometry, Lie Groups, and Symmetric Spaces", Acad. Press, London, 1978.

[HiÓl96] Hilgert, J. and G. Ólafsson, "Causal Symmetric Spaces, Geometry and Harmonic Analysis," Acad. Press, 1996.

[HÓØ91] Hilgert, J., G. 'Olafsson, and B. Ørsted, *Hardy spaces on affine symmetric spaces*, J. Reine Angew. Math. **415** (1991), 189–218.

[Kr99] Krötz, B., *The Plancherel theorem for biinvariant Hilbert spaces*, Publ. Res. Inst. Math. Sci. **35** (1999), no. **1**, 91–122.

[Kr01] —, *Formal dimension for semisimple symmetric spaces*, Compositio Math. **125** (2001), no. **2**, 155–191.

[KrNe00] Krötz, B., and K.–H. Neeb, *Unitary spherical highest weight representations*, TAMS, to appear.

[KNÓ97] Krötz, B., K.–H. Neeb, and G. Ólafsson, *Spherical representations and mixed symmetric spaces*, Representation Theory **1** (1997), 424–461.

[KrÓl99] Krötz, B., and G. Ólafsson, *The c-function for a non-compactly causal symmetric space*, submitted.

[KrSt01a] Krötz, B., and R.J. Stanton, *Holomorphic extensions of representations: (I) automorphic functions*, Ohio State Mathematical Research Institute preprint 01-29.

[KrSt01b] Krötz, B., and R.J. Stanton, *Holomorphic extensions of representations: (II) geometry and harmonic analysis*, preprint.

[Ne99] Neeb, K.–H., "Holomorphy and Convexity in Lie Theory," Expositions in Mathematics, de Gruyter **28**, 1999.

[Ner99] Neretin, Y. A., *Matrix analogs of the B-function and Plancherel formula for Berezin Kernel representation*. Math. Sb. **191** (2000), 57–100. Translation in Sb. Math. **191** (2000), 683–715.

[Ól00] Ólafsson, G., *Analytic continuation in representation theory and harmonic analysis*, Global analysis and harmonic analysis (Marseille-Luminy, 1999), 201–233, Sémin. Congr., **4**, Soc. Math. France, Paris, 2000.

[ÓØ88] Ólafsson, G., and B. Ørsted, *The holomorphic discrete series for affine symmetric spaces. I*, J. Funct. Anal. **81** (1988), no. **1**, 126–159.

[ÓØ96] —, *Generalizations of the Bargmann transfrom*, Lie theory and its applications in physics (Clausthal, 1995) (H.-D. Doebner, V.K. Dobrev, and J. Hilgert eds.) World Scientific, River Edge, New Jersey, 1996.

[ÓØ99] —, *Causal compactification and Hardy spaces*, Trans. Amer. Math. Soc. **351** (1999), no. **9**, 3771–3792.

[Ol82] Olshanski, G. I., *Invariant cones in Lie algebras, Lie semigroups, and the holomorphic discrete series*, Funct. Anal. and Appl. **15**, 275–285 (1982).

[St86] Stanton, R. J., *Analytic Extension of the holomorphic discrete series*, Amer. J. Math. **108**(1986), 1411–1424.

[Wa79] Wallach, N., *The analytic continuation of the discrete series.I, II*, Trans. Amer. Math. Soc. **251** (1979), 1–17, 19–37.

[Zh01] Zhang, G., *Berezin transform on real bounded symmetric domains*, Trans. Amer. Math. Soc., **353** (2001), 3769–3787.

Simon Gindikin
Department of Mathematics
Rutgers University
New Brunswick, NJ 08903
USA
gindikin@math.rutgers.edu

Bernhard Krötz
The Ohio State University
Department of Mathematics
231 West 18th Avenue
Columbus, OH 43210–1174
USA
kroetz@math.ohio-state.edu

Gestur Ólafsson
Louisiana State University
Department of Mathematics
Baton Rouge, LA 70803
USA
olafsson@math.lsu.edu