THE CIRCLE TRANSFORM ON TREES

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Abstract. We consider the overdetermined problem of integral geometry on trees given by the transform that integrates functions on a tree over circles, and exhibit difference equations that describe the range. We then show how this problem modifies if we restrict the transform to some natural subcomplex of the complex of circles, proving inversion formulas and characterizing ranges.

1. Introduction

There is a strong analogy between the geometry and the harmonic analysis of trees and those of real hyperbolic spaces [BCP], [Ca]. This analogy is better understood by regarding trees as p-adic symmetric spaces of rank one [S], [M]. It is natural to interpret inversion formulas for the horospherical and geodesic Radon transforms on trees as discrete versions of corresponding formulas for hyperbolic spaces [BCCP], [CCP], and vice versa.

Overdetermined problems play an important role in integral geometry. The classical example is F. John's transform [J], which maps functions on the three-dimensional Euclidean space to their integrals over lines. Thus a function of three variables is transformed into a function of four variables. As a result, the range consists only of functions that satisfy the ultrahyperbolic equation. Correspondingly, in inversion formulas it is natural to use the values of the transformed function not on all lines, but only on those in some three-dimensional subfamily (called *complex*).

Another classical overdetermined problem is given by the *circle transform* in the plane (or, more generally, the sphere transform). This maps functions of two variables to functions of three variables, and we get again the situation described above: a description of the range by a differential equation, and inversion using two-dimensional families of circles.

Perhaps for the first time, we consider an overdetermined problem on trees, namely the discrete analogue of the circle transform: Given a function $f(v)$ on the vertices of a tree, we take $\hat{f}(v,n)$ to be its "integral" on the circle of center v and radius n. To work in greater generality, we shall in fact also allow the center to be an edge of the tree. In some sense, $\hat{f}$ depends on one extra variable with respect to f, and as a result it must satisfy a suitable equation. Indeed we obtain in Section 3 some difference equations that describe the range of the transform $f \mapsto \hat{f}$.

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To eliminate the overdetermination, as in the continuous case we restrict the transform to some families of circles that, in some sense, have codimension one; we choose those that are the analogues of some so-called admissible families of circles in the plane for which there are good inversion formulas. As a basic example, in Section 4 we restrict the circle transform to the subcomplex of circles passing through a given fixed vertex, proving an inversion and a range characterization for ℓ^1. We finally consider other subcomplexes for which, by analogy with the continuous case, we can expect good inversion formulas.

2. Preliminaries and notation

Let T be a *tree*, that is, a connected, undirected, locally finite graph without loops and self-loops. The set of vertices of T will be denoted by V, the set of edges by E, and both will be endowed with the counting measure for integration purposes. Two vertices u, v are *neighbors*, and we write $u \sim v$, if they are connected by an edge, which in this case is unique, is denoted by uv and identified with the set $\{u, v\}$. The *degree* of a vertex v is the number $q_v + 1$ of its neighbors; if this is a constant $q + 1$, then the tree is *homogeneous*. The *interior* of a subset K of V is the set of its vertices whose neighbors are all in K. Two edges uv, wz are neighbors, or $uv \sim wz$, if they share exactly one vertex. An edge uv has exactly $q_u + q_v$ neighboring edges. The *distance* between two vertices u, w is the length n of the unique injective *path* $[u, w] = (u = v_0, v_1, \ldots, v_n = w)$ from u to w for which $v_{j-1} \sim v_j$ for every $j = 1, \ldots, n$. Instead, the distance between an edge uv and a vertex w is $d(uv, w) = \min(d(u, w), d(v, w))$; it vanishes if and only if $w \in uv$. A subset K of V is *connected* (or *convex*) if it contains the path between any two of its vertices. For instance, a *geodesic*, that is, a doubly infinite path in V, is a connected subset of V.

Let $\mathbf{N}$ be the set of non-negative integers. A *circle* in V is the set

$$C_{x,n} = \{u \in V \colon d(x, u) = n\} \qquad \text{for } x \in V \cup E \text{ and } n \in \mathbf{N},$$

where x is its *center* and n its *radius*. Although all circles are subsets of V, the above definition thus encompasses those whose centers are vertices (henceforth *v-circles*) as well as those whose centers are edges (*e-circles*). The space $\mathcal{C}$ of circles is thus in one-to-one correspondence with $(V \cup E) \times \mathbf{N}$. The *circle transform* of a function f on V is the function $\hat{f}$ on $\mathcal{C}$ given by

$$\hat{f}(x, n) = \sum_{u \in C_{x,n}} f(u) \qquad \text{for } (x, n) \in (V \cup E) \times \mathbf{N}.$$

A circle $C_{x,n}$ is *externally*, respectively *internally tangent* to a subset K of V if the minimum, respectively maximum distance of x from points of K is n.

We shall assume in Section 4 that a reference vertex $o \in V$ is fixed. The *length* of $x \in V \cup E$ is $|x| = d(x, o)$. A vertex v is *even* or *odd* if so is its length. Correspondingly, $V_{\text{even}}, V_{\text{odd}}$ are the subsets of even, respectively odd vertices of V. If $0 \leq n \leq |v|$, denote by v_n the n-th vertex of the path from o to v, so that $|v_n| = n$, and the whole path is

$$[o, v] = (o = v_0, v_1, \ldots, v_{|v|} = v).$$

The vertices $v_0, \dots, v_{|v|}$ are the *predecessors* of v. In particular, if $v \neq o$ then $v_{|v|-1}$, i.e., the unique predecessor of v that is at the same time its neighbor, is also denoted by v^-. Setting $V' = V \setminus \{o\}$, the map $\mu \colon V' \to E$ given by

$$\mu(v) = vv^- \qquad \text{for } v \in V'$$

is a bijection. The set of vertices of which u is a predecessor is

$$S_u = \{v \in V \colon |v| \geq |u|,\ v_{|u|} = u\} \qquad \text{for } u \in V,$$

and the subset of those at distance n from u is

$$S_{u,n} = S_u \cap C_{u,n} = C_{o,|u|+n} \cap C_{u,n} \qquad \text{for } u \in V \text{ and } n \in \mathbf{N}.$$

If $x \in E$ set $S_x = S_{\mu^{-1}(x)}$ and $S_{x,n} = S_{\mu^{-1}(x),n}$ for $n \in \mathbf{N}$ (all are sets of vertices). The *join* of two vertices v, w is their longest common predecessor.

In order to give a more suggestive expression of the range-characterizing equations we need some notation for finite differences. The *forward* and *backward difference operators* ∂^+, ∂^- act on a function h on $\mathbf{N}$ as

$$\begin{aligned} \partial^+ h(n) &= h(n+1) - h(n), \\ \partial^- h(n) &= h(n) - h(n-1), \end{aligned}$$

where we henceforth set $h(-1) = 0$. Also, let

$$\begin{aligned} \partial &= \frac{\partial^+ + \partial^-}{2}, \\ \partial^{(2)} &= \partial^+ - \partial^- = \partial^+\partial^- = \partial^-\partial^+ \end{aligned}$$

be the *symmetric two-step difference*, respectively *second order difference operators*. By i we shall denote the identity operator (i.e., $ih(n) = h(n)$).

The *Laplace operator* on V being given by

$$\Delta f(v) = \frac{1}{q_v + 1} \sum_{u \sim v} f(u) - f(v),$$

then for any real $c > 0$ the *discrete-time wave operator*, acting on functions on $V \times \mathbf{N}$, can be defined as

$$\Box_c = \Delta - c\partial^{(2)}.$$

The same expression provides a wave operator on functions on $E \times \mathbf{N}$ in terms of the Laplace operator on E, which, if T is homogeneous of degree $q+1$, is given by

$$\Delta f(uv) = \frac{1}{2q} \sum_{wz \sim uv} f(wz) - f(uv)$$

(every edge has $2q$ neighboring edges).

The *backward difference operator* D^- on V takes a function f to the function Df given by

$$D^- f(v) = f(v) - f(v^-) \qquad \text{for every } v \in V,$$

where we set $f(o^-)=0$. The inverse of D is given by

$$f(v)=\sum_{w\in[o,v]} D^-f(w) \qquad \text{for every } v\in V. \tag{2.1}$$

Instead, the *forward difference operator* on V can be defined by

$$D^+f(v)=\sum_{u^-=v} f(u)-f(v) \qquad \text{for every } v\in V.$$

Lemma 2.1. *The operator D^+ maps $\ell^1(V)$ into itself with operator norm not greater than 2, and it can be inverted, for $f\in\ell^1(V)$, by*

$$f(v)=-\sum_{u\in S_v} D^+f(u) \qquad \textit{for every } v\in V,$$

where the series converges absolutely.

Proof. The assertion about the norm follows from the uniqueness of the immediate predecessor u^{-1} of each given $u\neq o$. The series converges absolutely because $D^+f\in\ell^1(V)$. The inversion formula is obtained by summing for $n\in\mathbf{N}$ the relation

$$\sum_{u\in S_{v,n}} D^+f(u)=\sum_{u\in S_{v,n+1}} f(u)-\sum_{u\in S_{v,n}} f(u) \qquad \text{for every } v\in V \text{ and } n\in\mathbf{N}. \quad \square$$

$\square$

We shall need in Section 4 to extend D^+,D^- to $V\cup E$ in a somewhat hybrid fashion as

$$\begin{aligned} D^+f(vv^-)&=f(v)-f(vv^-) &&\text{for every } v\in V',\\ D^-f(vv^-)&=f(vv^-)-f(v^-) &&\text{for every } v\in V'. \end{aligned}$$

3. The full circle transform

We can now characterize the range of the circle transform on the space $\mathcal{C}$ of all circles.

Theorem 3.1. *A function ϕ on $(V\cup E)\times\mathbf{N}$ is in the range of the circle transform if and only if the following two equations hold:*

$$\sum_{u\sim v}\phi(u,n)=\phi(v,n+1)+\phi(v,n-1)\cdot\begin{cases}\text{anything} & \textit{if } v\in V \textit{ and } n=0,\\ (q_v+1) & \textit{if } v\in V \textit{ and } n=1,\\ q_v & \textit{if } v\in V \textit{ and } n>1,\end{cases} \tag{3.1}$$

$$\phi(u,n)+\phi(v,n)=\phi(uv,n)+\phi(uv,n-1) \qquad \textit{for } uv\in E \textit{ and } n\in\mathbf{N} \tag{3.2}$$

(with $\phi(\,\cdot\,,-1)\equiv 0$). Equation (3.1) *alone characterizes the range on v-circles, and can be rewritten as $Z_V\phi=0$ where*

$$Z_V=\square_{1/2}+\frac{q_v-1}{q_v+1}\partial+\frac{\partial^- - i}{q_v+1}\cdot\begin{cases}\text{anything} & \textit{on } V\times\{0\},\\ 1 & \textit{on } V\times\{1\},\\ 0 & \textit{on } V\times(\mathbf{N}\setminus\{0,1\}).\end{cases}$$

If T is a homogeneous tree of degree $q+1$, equation (3.2) can be replaced by

$$(3.2') \quad \begin{cases} \phi(u,0)+\phi(v,0)=\phi(uv,0) \quad \text{(this is (3.2) for } n=0\text{)}, \\ \displaystyle\sum_{wz\sim uv}\phi(wz,n)=\phi(uv,n+1)+q\phi(uv,n-1) \\ \qquad\qquad\qquad\qquad + \phi(uv,n)\cdot\begin{cases} q & \text{if } n=0, \\ (q-1) & \text{if } n>0, \end{cases} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for every } uv\in E. \end{cases}$$

The second equation of (3.2′), which only involves values at e-circles, can be rewritten as $Z_E\phi=0$, where

$$Z_E=\square_{(q+1)/4q}+\frac{q-1}{2q}\partial-\begin{cases} \dfrac{i}{2q} & \text{on } E\times\{0\}, \\ 0 & \text{on } E\times(\mathbf{N}\setminus\{0\}); \end{cases}$$

the first two summands of this right-hand side can also be written as

$$\square_{1/2}+\frac{q-1}{2q}\partial^{+}.$$

Proof. Let f be a function on V, and let $\phi=\hat f$. Then (3.1) can be inferred from the equivalent [Co, Theorem 1], or shown as follows. Let $n>1$. For each neighbor u of v, every vertex $w\in C_{u,n}$ is at distance $n+1$ or $n-1$ from v, according if it is closer to u or to v. In the former case there is exactly one neighbor u of v for which $w\in C_{u,n}$, and this yields the first summand of the right-hand side. In the latter case there are q_v such neighbors u of v, namely all but the one that lies on the path from v to w; this accounts for the other summand. If $n=1$, in this latter case w equals v, so it belongs to $C_{u,1}$ for all the q_v+1 neighbors u of v. Finally, if $n=0$ the equality is immediate.

Equation (3.2) follows from the identity

$$C_{u,n}\cup C_{v,n}=C_{uv,n}\cup C_{uv,n-1} \qquad \text{for } uv\in E \text{ and } n\in\mathbf{N}$$

(without the last term if $n=0$) and the fact that these are disjoint unions. Indeed, a vertex $w\in C_{u,n}$ is, as before, at distance $n+1$ or $n-1$ from the neighbor v of u. In the former case it belongs to $C_{uv,n}$, in the latter to $C_{uv,n-1}$.

Finally, (3.2′) is verified as follows. Let $n>0$. If $u\neq w\sim v$, then every vertex $y\in C_{wv,n}$ is at distance $n+1$, n, or $n-1$ from uv, according if it is closest to w, v, or u. In the first case there is exactly one w as above, so we get the first summand in the bottom right-hand side. In the second case there are q_v-1 such neighbors w of v, namely all except u and the one that lies on the path from v to y. In the third case there are q_v such neighbors w of v, namely all but u. If $n=0$, in the second case $y=v$, so it belongs to $C_{wv,0}$ for all the q_v neighbors w of v that are different from u.

Conversely, let ϕ satisfy (3.1), and set $f(v)=\phi(v,0)$ for each $v\in V$. We want to prove by induction on n that $\hat f(\,\cdot\,,n)=\phi(\,\cdot\,,n)$ on V. For $n=0$ this holds by definition, so assume it proved up to n. Equation (3.1) has been proved to hold for $\hat f$ in place of ϕ, and it expresses $\phi(\,\cdot\,,n+1)$ in terms of $\phi(\,\cdot\,,n)$ and $\phi(\,\cdot\,,n-1)$. Since

$\phi(\,\cdot\,,k) = \hat{f}(\,\cdot\,,k)$ for $k = n, n-1$, we conclude that this is also true for $k = n+1$. If (3.2) also holds then we see that $\hat{f}(\,\cdot\,,n) = \phi(\,\cdot\,,n)$ on E, again by induction in a similar fashion. Same if (3.2′) replaces (3.2).

The verifications concerning Z_V, Z_E are straightforward. □

4. Circles through a fixed point

4.1. **Inversion.** The overdetermination described in Theorem 3.1 can be compensated by restricting the circle transform to a suitable subcomplex of the space $\mathcal{C}$ of all circles of V. One possibility is to take the circles that pass through a given reference vertex o. Since these can be parametrized by their centers, the resulting subcomplex is therefore in one-to-one correspondence with $V \cup E$ via $x \mapsto C_x = C_{x,|x|}$. For functions $f \in \ell^1(V)$, the corresponding restriction R of the circle transform is given by

$$Rf(x) = \hat{f}(x, |x|) = \sum_{u \in C_x} f(u) \qquad \text{for every } x \in V \cup E.$$

The circles through o are not enough to separate all vertices of V. We shall describe a partition of V with the property that, if a vertex belongs to a circle, then so do all the elements of the same coset. One can therefore recover from Rf at most the sum of values of f at the vertices in each of these cosets. We shall actually prove that this is possible, and give the inversion formula.

On each given circle of positive radius through o, all vertices, except o itself, share the unit-length predecessor v_1 of the center. Therefore it is enough to restrict attention to the subtree spanned by $\{o\} \cup S_{o'}$ for each fixed neighbor o' of o. Thus in the sequel we shall assume without loss of generality that $q_o = 0$, that is, o has only one neighbor o'. In order to avoid exceptions in forthcoming formulas, set $o_1 = o'$ (although o' is not a predecessor of o) and $S_{o',-1} = \{o\}$.

First consider the restriction R_V of R to v-circles (through o). Any such circle contains only even vertices, so $R_V f$ depends only on the restriction of f to V_{even}. The cosets of V_{even} will be described as inverse images of single vertices by a sort of contraction map β, in terms of which R_V can be explicitly factored through an invertible transform.

We first describe v-circles more explicitly.

Lemma 4.1. *If $o \neq v \in C_x$ for some $x \in V$, then v is even and $x \in S_{v_{|v|/2}} \setminus S_{v_{|v|/2+1}}$. Therefore, if a vertex u is such that*

$$u_{|u|/2+1} = v_{|v|/2+1}$$

(in particular, $|u| = |v|$), then $u \in C_x$.

Proof. It is enough to realize that $|x| \geq |v|/2$, and that the join of v, x is $v_{|v|/2}$. □

Now we can give a natural definition of the contraction map $\beta\colon V_{\text{even}} \to V'$ by

$$\beta(v) = v_{|v|/2+1} \qquad \text{for every } v \in V_{\text{even}}$$

(with the special case $\beta(o) = o'$). Since every level set of β is finite, there is an induced operator B that maps a function f on V_{even} (or V) to the function Bf on V' given by

$$Bf(w) = \sum_{v \in \beta^{-1}(w)} f(v) \qquad \text{for every } w \in V'.$$

The map β is easily seen to be onto, then so is the operator B. Moreover B maps $\ell^1(V_{\text{even}})$ (or $\ell^1(V)$) into $\ell^1(V')$ with unit operator norm, its kernel is the set of summable functions on V_{even} whose sum vanishes on each level set of β, namely on

$$\beta^{-1}(w) = S_{w,|w|-2} \qquad \text{for } w \in V',$$

and its range is the set of summable functions on V'. A right inverse of B is the operator B^{-1} that maps a function g on V' to the function $B^{-1}g$ on V_{even} given by

$$B^{-1}g(v) = \frac{g(\beta(v))}{\#\beta^{-1}(\beta(v))} \qquad \text{for every } v \in V_{\text{even}}.$$

Such a right inverse also has unit ℓ^1 operator norm, and its range is the space $\mathcal{B}$ of summable functions on V_{even} that are constant on each level set of β.

Let us describe the images of circles through the contraction.

Lemma 4.2. *If $v \in V$, the image $\beta(C_v)$ of the circle C_v under β is*

$$C'_v = \{o'\} \cup \{w \in V' \colon w \notin [o, v],\ w^- \in [o, v]\}$$

(with the special case $C'_o = \{o'\}$). Indeed $\beta^{-1}(C'_v) = C_v$.

Proof. If $o \neq u \in C_v$, let $v_n = u_n$ be the join of v, u. Then $|v| = d(v, u) = d(v, v_n) + d(v_n, u) = (|v| - n) + (|u| - n)$, so $n = |u|/2$. If $w = \beta(u)$, a predecessor of u, then $w^- = u_n \in [o, v]$, while $w \notin [o, v]$.

Conversely, if $o' \neq w \in C'_v$ and if $u \in \beta^{-1}(w)$ then w^- is the join of v, w, so $d(v, u) = d(v, w^-) + d(w^-, w) + d(w, u) = (|v| - |w| + 1) + 1 + (|w| - 2) = |v|$. $\square$

By Lemma 4.1, each circle through o is a union of level sets of β, hence we can factor $R_V = R'B$, where R' is the Radon transform that integrates summable functions on V' along the sets C'_v. Namely, if $g \in \ell^1(V')$, then

$$R'g(v) = \sum_{w \in C'_v} g(w) \qquad \text{for every } v \in V.$$

Notice that $|R'g| \leq \|g\|_{V'}$, the ℓ^1 norm of g. (Both transforms R_V, R' take values among functions on V.) Certainly R_V is not injective, since $\ker R_V$ contains $\ker B$. We shall prove that this is the only obstruction to invertibility—that is, R' is injective, and the inverse of R_V is only determined up to averages on level sets of β, each of which is finite. In other words, we shall show how to invert R_V on $\mathcal{B}$.

Proposition 4.3. *If g is a function on V' then*

$$D^- R'g(w) = D^+ g(w) + \begin{cases} g(w) & \text{if } w = o, o', \\ 0 & \text{if } o, o' \neq w \in V. \end{cases}$$

Therefore D^-R' maps $\ell^1(V')$ into $\ell^1(V)$ with operator norm not greater than 2, and any function $g \in \ell^1(V')$ can be expressed in terms of $\phi = R'g$ by

$$g(v) = \begin{cases} D^-\phi(o) & \text{if } v = o', \\ -\sum_{w\in S_v} D^-\phi(w) & \text{if } o' \neq v \in V', \end{cases} \tag{4.1}$$

where the series converges absolutely. In particular, D^-R', R' are one-to-one. Moreover ϕ satisfies the relation

$$\sum_{w\neq a} D^-\phi(w) = 0, \tag{4.2}$$

again with absolute convergence.

Proof. The first equality holds in the case $w = o$ because $C'_o = \{o'\}$, whereas, if $w \neq o$, because

$$C'_w \setminus C'_{w^-} = S_{w,1},$$
$$C'_{w^-} \setminus C'_w = \begin{cases} \emptyset & \text{if } w = o', \\ \{w\} & \text{if } w \neq o'. \end{cases}$$

The assertion about the norm is a consequence of the definition of D^+ and, as in Lemma 2.1, of the uniqueness of the immediate predecessor u^{-1} of each given $u \neq o$. As to (4.1), the case $v = o'$ is again immediate, whereas for $o' \neq v \in V'$ it follows from Lemma 2.1, as does relation (4.2). □

Remark 4.4. The subtraction due to the difference operator D^- in the series in (4.1) cannot in general be distributed, namely as $\sum_{w\in S_v} \phi(w) - \sum_{w\in S_v} \phi(w^-)$, to obtain cancelations.

An inversion for R_V can only work on a space, such as $\mathcal{B}$, that has trivial intersection with its kernel. Such partial inversion can now be obtained by simply applying B^{-1} to (4.1).

Having inverted the restriction R_V of R to the set of v-circles (through o), let us now turn our attention to R itself. Lemma 4.1 carries on word by word if we replace $x \in V$ with each of the endpoint vertices of $x \in E$. Therefore the partition on V_{even} described above, also in terms of the map β, is not properly refined by e-circles; in other words, these do not separate a pair of even vertices unless also v-circles do. Nonetheless, e-circles also contain odd vertices, which in turn divide into cosets. Each odd vertex v belongs to exactly one circle, namely the e-circle centered at the edge $v_{(|v|+1)/2}v_{(|v|-1)/2}$. Therefore V_{odd} gets partitioned into the cosets $S_{w,|w|-1}$ for $w \in V'$. In order to get simple unified expressions, we define a map $\theta \colon V \to V' \cup E$ so that the inverse images of vertices are these cosets, and that θ restricts to $\mu \circ \beta$ on V_{even}. Indeed

$$\theta(v) = \begin{cases} v_{|v|/2+1}v_{|v|/2} & \text{if } v \in V_{\text{even}}, \\ v_{(|v|+1)/2} & \text{if } v \in V_{\text{odd}}. \end{cases}$$

As with β, there is an induced operator Θ that maps a function f on V to the function Θf on $V' \cup E$ given by

$$\Theta f(x) = \sum_{v \in \theta^{-1}(x)} f(v) \qquad \text{for every } x \in V' \cup E.$$

Since θ is onto, then so is Θ, which maps $\ell^1(V)$ into $\ell^1(V' \cup E)$ with unit operator norm, its kernel is the set of summable functions on V whose sum vanishes on each level set of θ, namely on

$$\theta^{-1}(x) = S_{x,|x|-1} \qquad \text{for every } x \in V' \cup E,$$

and its range is the set of summable functions on $V' \cup E$. A right inverse of Θ is the operator Θ^{-1} that maps a function h on $V' \cup E$ to the function $\Theta^{-1}h$ on V given by

$$\Theta^{-1}h(v) = \frac{h(\theta(v))}{\#\theta^{-1}(\theta(v))} \qquad \text{for every } v \in V.$$

Such a right inverse also has unit ℓ^1 operator norm, and its range is the space of summable functions on V that are constant on each level set of θ.

The function Θf contains as much information on f as can be recovered from the knowledge of Rf. For $f \in \ell^1(V)$ and $\phi = Rf$, we have shown in (4.1) that

$$\Theta f(x) = \begin{cases} D^-\phi(o) & \text{if } x = o'o, \\ -\sum_{w \in S_x} D^-\phi(w) & \text{if } o'o \neq x \in E. \end{cases}$$

The sum of f on each coset in V_{odd} can also be recovered from Rf:

Proposition 4.5. *If f is a function on V then*

$$D^-Rf(x) = D^+\Theta f(x) + \begin{cases} \Theta f(x) & \text{if } x = o'o, \\ 0 & \text{if } o'o \neq x \in E. \end{cases}$$

Therefore D^-R maps $\ell^1(V)$ into $\ell^1(V \cup E)$ with operator norm not greater than 3. For any $f \in \ell^1(V)$ one can recover Θf on V' in terms of $\phi = Rf$ by

$$\Theta f(v) = D^-\phi(vv^-) - \sum_{w \in S_v} D^-\phi(w) \qquad \textit{for } v \in V', \tag{4.3}$$

where the series converges absolutely.

Proof. If $v \in V'$ then

$$C_{vv^-} \setminus C_{v^-} = S_{v,|v|-1} = \theta^{-1}(v),$$

$$C_{v^-} \setminus C_{vv^-} = \begin{cases} \emptyset & \text{if } v = o', \\ S_{v,|v|-2} = \theta^{-1}(vv^-) & \text{if } v \neq o'. \end{cases}$$

Therefore

$$D^-\phi(vv^-) - \Theta f(v) = \begin{cases} 0 & \text{if } v = o', \\ -\Theta f(vv^-) & \text{if } v \neq o', \end{cases}$$

that equals $\sum_{w\in S_v} D^-\phi(w)$ in both cases, by (4.2) in the former, while applying (4.1) to $g = Bf$ in the latter. As to norms, we have

$$\begin{aligned}\|D^-Rf\|_{V\cup E} &= \|D^-Rf\|_V + \|D^-Rf\|_E \\ &\leq 2\|f\|_{V_{\text{even}}} + \big(\|f\|_{V_{\text{odd}}} + \|f\|_{V_{\text{even}}}\big). \quad \square\end{aligned}$$

□

This completes the description of the inversion of R.

4.2. **Range.** For every $f \in \ell^1(V)$, relation (4.2) is the only non-trivial one satisfied by Rf:

Proposition 4.6. *The range of R (respectively R_V) on $\ell^1(V)$ is the space of functions ϕ on $V \cup E$ (respectively V) such that $D^-\phi$ is in ℓ^1 of $V \cup E$ (respectively V), relation* (4.2) *holds, and the function g given by* (4.1) *is in $\ell^1(V')$.*

Proof. We first show that $R'g = \phi$ on V. Indeed, $R'g(o) = g(o') = D^-\phi(o) = \phi(o)$ because $C'_o = \{o'\}$, while if $u \neq o$ we have

$$R'g(u) = D^-\phi(o) - \sum_{\substack{v\in C'_u\setminus\{o'\}\\ w\in S_v}} D^-\phi(w) = \sum_{w\in[o,u]} D^-\phi(w) = \phi(u)$$

by (4.2), since V is the disjoint union of $[o,u]$ and of S_v for $v \in C'_u \setminus \{o'\}$, and by (2.1). Then $f = B^{-1}g \in \ell^1(V_{\text{even}})$ and $Rf = R'g = \phi$ on V.

Let h be the function on V' given by

$$h(v) = D^-\phi(vv^-) - \begin{cases} 0 & \text{if } v = o', \\ -g(v) & \text{if } v \neq o', \end{cases}$$

and extend to V_{odd} the f determined before by $f = \Theta^{-1}h$. Then $\|f\|_{V_{\text{odd}}} = \|h\| \leq \|D^-\phi\| + \|g\|$, while $Rf = \phi$ on E is obtained by comparing the definition of h with (4.3), in view of (4.1), of the definition of D^-, and the relation $Rf = \phi$ already proved on V. □

5. Some other subcomplexes

We conclude by observing that the restriction of the full circle transform to some other natural subcomplexes, inspired by the classical setting in the plane, is essentially equivalent to the situation described in Section 4.

One possible subcomplex is that of circles tangent to a geodesic K of V, that are those that intersect K in exactly one vertex each. This tangency point is also the closest vertex of K to the center as well as to each of the vertices of the circle. Therefore, one can restrict to the subtree of T spanned by the vertices of V whose closest vertex of K is a given vertex $o \in K$, and realize that on such subtree we have the situation of the preceding section. There follows a similar description of the lack of injectivity of the transform, a partial inversion, and a range characterization.

The same considerations carry on verbatim if we take circles externally tangent to any given connected subset K of V—if K is infinite, the adverb "externally" can be omitted. In order for every point of K to belong to some such circles one has to add the requirement that the interior of K be empty.

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