High-Order Terms in the Asymptotic Expansions of the Steady-State Voltage Potentials in the Presence of Conductivity Inhomogeneities of Small Diameter *

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Abstract

We derive high-order terms in the asymptotic expansions of the steady-state voltage potentials in the presence of a finite number of diametrically small inhomogeneities with conductivities different from the background conductivity. Our derivation is rigorous, and based on layer potential techniques. The asymptotic expansions in this paper are valid for inhomogeneities with Lipschitz boundaries and those with extreme conductivities.

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$, with a connected Lipschitz boundary $\partial\Omega$. Let ν denote the unit outward normal to $\partial\Omega$. Suppose that Ω contains a finite

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number m of small inhomogeneities $(D_l)_{l=1}^m$, each of the form $D_l = z_l + \epsilon B_l$, where B_l, $l = 1, \ldots, m$, is a bounded Lipschitz domain in $\mathbb{R}^d$ containing the origin. We assume that the domains $(D_l)_{l=1}^m$ are separated apart from each other and apart from the boundary. More precisely, we assume that there exists a constant $c_0 > 0$ such that

$$(1.1) \qquad |z_l - z_{l'}| \geq 2c_0 > 0, \forall\, l \neq l' \quad \text{and} \quad \text{dist}(z_l, \partial\Omega) \geq 2c_0 > 0, \forall\, l,$$

and ϵ, the common order of magnitude of the diameters of the inhomogeneities, is sufficiently small, that these inhomogeneities are disjoint, and that their distance to $\mathbb{R}^d \setminus \overline{\Omega}$ is larger than c_0. We also assume that the "background" is homogeneous with conductivity 1 and the inhomogeneities D_l have conductivities k_l, $k_l \neq 1$, $1 \leq l \leq m$.

Let u_ϵ denote the steady-state voltage potential in the presence of the conductivity inhomogeneities, *i.e.*, the solution to

$$(1.2) \qquad \begin{cases} \nabla \cdot \Big(\chi(\Omega \setminus \bigcup_{l=1}^m \overline{D_l}) + \sum_{l=1}^m k_l \chi(D_l) \Big) \nabla u_\epsilon = 0 \quad \text{in } \Omega, \\ \dfrac{\partial u_\epsilon}{\partial \nu}\Big|_{\partial\Omega} = g. \end{cases}$$

Let U denote the "background" potential, that is, the solution to

$$(1.3) \qquad \begin{cases} \Delta U = 0 \quad \text{in } \Omega, \\ \dfrac{\partial U}{\partial \nu}\Big|_{\partial\Omega} = g. \end{cases}$$

The function g represents the applied boundary current; it belongs to $L_0^2(\partial\Omega) = \{g \in L^2(\partial\Omega), \int_{\partial\Omega} g = 0\}$. The potentials, u_ϵ and U, are normalized by $\int_{\partial\Omega} u_\epsilon = \int_{\partial\Omega} U = 0$.

The main achievement of this paper is a rigorous derivation, based on layer potential techniques, of high-order terms in the asymptotic expansion of $u_\epsilon|_{\partial\Omega}$ as $\epsilon \to 0$. The leading order term in this asymptotic formula has been derived by Cedio-Fengya et al. [7]; see also the prior work of Friedman and Vogelius [14] for the case of perfectly conducting or insulating inhomogeneities. The main result of this paper is the following full asymptotic expansion of the solution for the case $m = 1$:

Theorem 1.1 *Suppose that the inhomogeneity consist of single component and let u_ε be the solution of (1.2). The following pointwise asymptotic expansion on $\partial\Omega$*

holds for $d = 2, 3$:

$$
\begin{aligned}
u_\varepsilon(x) &= U(x) - \epsilon^{d-2} \sum_{|i|=1}^{n} \sum_{|j|=1}^{n-|i|+1} \frac{1}{j!} \epsilon^{|i|+|j|} \times \\
&\qquad \Big[\Big((I + \sum_{p=1}^{n+2-|i|-|j|-d} \epsilon^{d+p-1} \mathcal{Q}_p)(\partial^l U(z)) \Big)_i M_{ij} \partial_z^j N(x,z) \Big] \\
&\qquad + O(\epsilon^{d+n}),
\end{aligned} \tag{1.4}
$$

where the remainder $O(\epsilon^{d+n})$ is dominated by $C\epsilon^{d+n} \|g\|_{L^2(\partial\Omega)}$ for some C independent of $x \in \partial\Omega$. Here $N(x,z)$ is the Neumann function, M_{ij}, $i, j \in \mathbb{N}^d$, are the generalized polarization tensors defined in (3.2), and the operator $\mathcal{Q}_p$ is defined in (4.12).

We have a similar expansion for the solutions of the Dirichlet problem (Theorem 4.2.

The derivation of the asymptotic expansions for any fixed number m of well separated inhomogeneities (these are a fixed distance apart) follows by iteration of the arguments that we will present for the case $m = 1$. In other words, we may develop asymptotic formulas involving the difference between the fields u_ϵ and U on $\partial\Omega$ with l inhomogeneities and those with $l-1$ inhomogeneities, $l = m, \dots, 1$, and then at the end essentially form the sum of these m formulas (the reference fields change, but that may easily remedied). The derivation of each of the m formulas is virtually identical.

We also note that the asymptotic expansion (1.4) is valid for inhomogeneities with zero or infinity conductivity (cavity or perfect conductor). Precise definitions of generalized polarization tensors (GPT) associated with the domains B_l and the conductivities k_l will be given at the end of section 3. These GPT seem to be natural generalizations of the tensors that have been introduced by Schiffer and Szegö [23] and thoroughly studied by many other authors [22], [18], [14], [7]. (See section 3.)

The higher-order terms are essential when $\nabla U(z_l) = 0$, for then the leading order term in the asymptotic expansion of $u_\epsilon|_{\partial\Omega}$, given in [7], vanishes. We remind the reader that, for general current inputs g, ∇U vanishes at some "critical points" inside Ω.

The proof of our asymptotic expansion is radically different from the ones in [14], [7], and [26]. It is based on layer potential techniques and a decomposition formula of the steady-state voltage potential into a harmonic part and a refraction part. This formula is due to Kang and Seo [15]. What makes our proof particularly original and elegant is that the rigorous derivation of high-order terms follows almost immediately. The extension of the techniques used in [14], [7], and [26] to construct higher-order terms in the expansion of $u_\epsilon|_{\partial\Omega}$ as $\epsilon \to 0$ seems to be laborious. Furthermore, the general approach developed in this paper could be carried out to obtain more precise asymptotic formulas for the full Maxwell's equations and for the equations of linear elasticity than those derived in [3] and [1]. The method of this paper also enables us to extend the asymptotic expansions to the cases of inhomogeneities with Lipschitz boundaries. Previously, the leading order term was derived under the assumption that inhomogeneities are $C^{1,\alpha}$ smooth [14] [7]. We note that our method works as well even when the inhomogeneities have extreme conductivities ($k = 0$ or $k = \infty$).

Let us now explain what makes this asymptotic formula interesting in the electrical impedance tomography (EIT). It is well-known that the ultimate objective of EIT is to recover, most efficiently and accurately, the conductivity distribution inside a body from measurements of current flows and voltages on the body's surface. The vast and growing literature reflects the many possible applications of EIT, e.g. for medical diagnosis or nondestructive evaluation of materials [6]. In its most general form EIT is severely ill-posed and nonlinear. Taking advantage of the smallness of the inhomogeneities, Cedio-Fengya et al. [7] have used the leading order term in the asymptotic expansion of $u_\epsilon|_{\partial\Omega}$ to find the locations z_l, $l = 1, \ldots, m$ of the inhomogeneities and certain properties of the domains B_l, $l = 1, \ldots, m$ (relative size, orientation). The algorithm proposed by Cedio-Fengya et al. [7] is based on a least-square algorithm. Ammari et al. [2] have also utilized this leading order term to design a variationally based direct reconstruction method. The new idea in [2] is to form the integral of the "measured boundary data" against harmonic test functions and choose the input current g so as to obtain expression involving the inverse Fourier transform of distributions supported at the locations z_l, $l = 1, \ldots, m$. Applying a direct Fourier transform to this data then pins down the locations. This approach is similar to the ideas used by Calderón [5] in his proof of uniqueness of the linearized conductivity problem and later, by Sylvester and Uhlmann in their important work [24] on uniqueness of the three-dimensional inverse conductivity problem. An other algorithm that makes use of an asymptotic expansion of the voltage potentials has been derived by Brühl et al. [4]. This algorithm is in the spirit of the linear sampling method of Colton and Kirsch [9].

In all of these algorithms, the locations z_l, $l = 1, \ldots, m$ of the inhomogeneities are found with an error $O(\epsilon)$ and little about the domains B_l can be reconstructed. Making use of higher-order terms in the asymptotic expansion of $u_\epsilon|_{\partial\Omega}$ we certainly would be able to reconstruct the small inhomogeneities with higher resolution from boundary information about specific solutions to (1.2). Perhaps, more importantly, this would allow us to identify quite general conductivity inhomogeneities without restrictions on their sizes.

The use of higher-order terms in the asymptotic expansion of $u_\epsilon|_{\partial\Omega}$ may also be decisive in dramatically improving the algorithm of Kwon et al. [19], that is based on the observation of the pattern of a simple weighted combination of an input current

g of the form $g = a \cdot \nu$ for some constant vector a and the corresponding output voltage. We also believe that the use of such higher-order terms would improve the algorithm of Mast et al. [20], that uses eigenfunctions of the scattering operator.

This paper is organized as follows. In section 2, we collect some notations and preliminary regarding layer potentials. In section 3, we introduce the generalized polarization tensors associated with the domains D_l and the conductivities k_l. In section 4, we provide a rigorous derivation of high-order terms in the asymptotic expansion of the output voltage potentials. For reasons of brevity we restrict a significant part of this derivation to the case of a single inhomogeneity ($m = 1$). The proof in the case of multiple well-separated inhomogeneities may be derived by a fairly straightforward iteration of the arguments we present, however we leave the details to the reader.

2 Layer Potentials for the Laplacian

Let us first review some well-known properties of the layer potentials for the Laplacian and prove some useful identities.

The theory of layer potentials has been developed in relation to the boundary value problems. Let D be a bounded domain in $\mathbb{R}^d, d \geq 2$. We assume that ∂D is Lipschitz. Let $\Gamma(x)$ be the fundamental solution of the Laplacian Δ:

$$\Gamma(x) = \begin{cases} \dfrac{1}{2\pi} \ln |x|, & d = 2, \\ \dfrac{1}{(2-d)\omega_d} |x|^{2-d}, & d \geq 3, \end{cases} \tag{2.1}$$

where ω_d is the area of $(d-1)$ dimensional unit sphere. The single and double layer potentials of the density function ϕ on D is defined by

$$\mathcal{S}_D \phi(x) := \int_{\partial D} \Gamma(x-y) \phi(y) d\sigma(y), \quad x \in \mathbb{R}^d, \tag{2.2}$$

$$\mathcal{D}_D \phi(x) := \int_{\partial D} \frac{\partial}{\partial \nu_y} \Gamma(x-y) \phi(y) d\sigma(y), \quad x \in \mathbb{R}^d \setminus \partial D. \tag{2.3}$$

For a function u defined on $\mathbb{R}^d \setminus \partial D$, we denote

$$\frac{\partial}{\partial \nu^{\pm}} u(x) := \lim_{t \to 0^+} \langle \nabla u(x \pm t\nu_x), \nu_x \rangle, \quad x \in \partial D$$

if the limit exists. Here ν_x is the outward unit normal to $\partial \Omega$ at x.

The proof of the following trace formula can be found in [11, 13, 21] (for Lipschitz domains, see [25]):

$$\frac{\partial}{\partial \nu^{\pm}} \mathcal{S}_D \phi(x) = (\pm \frac{1}{2} I + \mathcal{K}_D^*) \phi(x), \tag{2.4}$$

$$(\mathcal{D}\phi)|_{\pm} = (\mp \frac{1}{2} I + \mathcal{K}_D) \phi(x), \quad x \in \partial D, \tag{2.5}$$

where

$$\mathcal{K}_D \phi(x) = \frac{1}{\omega_d} \text{p.v.} \int_{\partial D} \frac{\langle x - y, \nu_y \rangle}{|x - y|^d} \phi(y) d\sigma(y)$$

and $\mathcal{K}_D^*$ is the L^2-adjoint of $\mathcal{K}_D$. When ∂D is Lipschitz, $\mathcal{K}_D$ is a singular integral operator and known to be bounded on $L^2(\partial \Omega)$ [8]. Let $L^2_0(\partial D) := \{f \in L^2(\partial D) : \int_{\partial D} f d\sigma = 0\}$. The following results are due to Verchota and Escauriaza-Fabes-Verchota.

Theorem 2.1 ([10], [25]) *$\lambda I - \mathcal{K}_D^*$ is invertible on $L_0^\infty(\partial D)$ if $|\lambda| \geq \frac{1}{2}$, and for $\lambda \in (-\infty, -\frac{1}{2}] \cup (\frac{1}{2}, \infty)$, $\lambda I - \mathcal{K}_D^*$ is invertible on $L^2(\partial D)$.*

For proofs when ∂D is smooth, see [11], [13].

The following theorem was proved in [15, 16, 17].

Theorem 2.2 *Suppose that D is a domain compactly contained in Ω with a connected Lipschitz boundary and conductivity k. Then the solution u of the problem*

$$\begin{cases} \nabla \cdot ((1 + (k-1)\chi(D)) \nabla u) = 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu}\Big|_{\partial \Omega} = g \end{cases} \tag{2.6}$$

is represented as

$$u(x) = H(x) + \mathcal{S}_D \phi(x), \quad x \in \Omega, \tag{2.7}$$

where the harmonic function H is given by

$$H(x) = -\mathcal{S}_\Omega(g)(x) + \mathcal{D}_\Omega(f)(x), \quad x \in \Omega, \quad f := u|_{\partial \Omega}, \tag{2.8}$$

and $\phi \in L^2_0(\partial D)$ satisfies the integral equation

$$(\frac{k+1}{2(k-1)} I - \mathcal{K}_D^*) \phi = \frac{\partial H}{\partial \nu}\Big|_{\partial D} \quad \text{on } \partial D. \tag{2.9}$$

Moreover, $\forall\, n \in \mathbb{N}$, there exists a constant $C_n = C(n, \Omega, dist(D, \partial\Omega))$ independent of $|D|$ and k such that

$$\|H\|_{\mathcal{C}^n(\overline{D})} \leq C_n \|g\|_{L^2(\partial \Omega)}. \tag{2.10}$$

Proof. The representation formula (2.7) was proved in [15, 17]. (2.10) was proved in [16] for $d = 2$ and it is easily seen that the same proof works for $d = 3$. We only need to check carefully whether the constant C_n in the estimate (2.10) is independent of $|D|$. Before doing this, let us point out that the harmonic function H can be computed explicitly from the boundary measurements $(\frac{\partial u}{\partial \nu}\big|_{\partial\Omega}, u\big|_{\partial\Omega})$ and the density ϕ is uniquely and explicitly determined by the domain D and the harmonic function H. The decomposition of the function u into a harmonic part H and a refraction part $\mathcal{S}_D\phi$ is unique [15, 17]. The representation formula (2.7) seems to inherit geometric properties of D.

Suppose that $\mathrm{dist}(D, \partial\Omega)) > 2c_0$ for some constant $c_0 > 0$. From the definition of H in (2.8) it is easy to see that

$$\|H\|_{\mathcal{C}^n(\overline{D})} \leq C_n\Big(\|g\|_{L^2(\partial\Omega)} + \|u|_{\partial\Omega}\|_{L^2(\partial\Omega)}\Big), \tag{2.11}$$

where C_n depends only on n, $\partial\Omega$, and c_0. Let $\vec{\alpha}$ be a vector field supported in the set $\mathrm{dist}(x, \partial\Omega) < 2c_0$ such that $\vec{\alpha}\cdot\nu(x) \geq \delta$ for some $\delta > 0$ for all $x \in \partial\Omega$. Using the Rellich identity with this $\vec{\alpha}$, we can show that

$$\|\frac{\partial u}{\partial T}\|_{L^2(\partial\Omega)} \leq C\Big(\|g\|_{L^2(\partial\Omega)} + \|\nabla u\|_{L^2(\Omega\setminus\overline{D})}\Big),$$

where C depends only on $\partial\Omega$ and c_0 and $T(x)$ is the tangent vector to $\partial\Omega$ at x. (See the proof of Lemma 2.1 of [12] for the detail of the proof.) Observe that

$$\begin{aligned}\|\nabla u\|^2_{L^2(\Omega\setminus\overline{D})} &\leq C\int_\Omega (1 + (k-1)\chi(D))\nabla u\cdot\nabla u\, dx\\ &= C\int_{\partial\Omega} gu\, d\sigma\\ &\leq C\|g\|_{L^2(\partial\Omega)}\|u|_{\partial\Omega}\|_{L^2(\partial\Omega)}.\end{aligned}$$

Since $\int_{\partial\Omega} u\, d\sigma = 0$, it follows from the Poincaré inequality that

$$\|u|_{\partial\Omega}\|_{L^2(\partial\Omega)} \leq C\|\frac{\partial u}{\partial T}\|_{L^2(\partial\Omega)}.$$

Thus we obtain

$$\|u|_{\partial\Omega}\|^2_{L^2(\partial\Omega)} \leq C\Big(\|g\|^2_{L^2(\partial\Omega)} + \|g\|_{L^2(\partial\Omega)}\|u|_{\partial\Omega}\|_{L^2(\partial\Omega)}\Big),$$

and hence

$$\|u|_{\partial\Omega}\|_{L^2(\partial\Omega)} \leq C\|g\|_{L^2(\partial\Omega)}.$$

From (2.11) we finally obtain (2.10). □

Using above representation we can derive a formula similar to (1.4) which is potentially useful in detecting the inhomogeneities (see the remark at the end of this paper). However it uses the function H which depends on D and hence on ϵ. Thus in order to derive (1.4) we will transform it to representations using only background potentials.

Let $N(x, y)$ be the Neumann function for Δ in Ω corresponding to a Dirac mass at z. That is, N is the solution to

$$\begin{cases} \Delta_x N(x,z) = -\delta_z & \text{in } \Omega, \\ \dfrac{\partial N}{\partial \nu}\Big|_{\partial\Omega} = -\dfrac{1}{|\partial\Omega|}. \end{cases}$$

In addition, we assume that

$$\int_{\partial\Omega} N(x,y)\,d\sigma(x) = 0 \quad \text{for } y \in \Omega. \tag{2.12}$$

Let us fix one more notation: For D, a subset of Ω, let

$$N_D f(x) := \int_{\partial D} N(x,y) f(y)\,d\sigma(y).$$

The following lemma relates the fundamental solution with the Neumann function.

Lemma 2.3 *For $z \in \Omega$ and $x \in \partial\Omega$, let $\Gamma_z(x) := \Gamma(x - z)$ and $N_z(x) := N(x, z)$. Then*

$$(-\frac{1}{2}I + \mathcal{K}_\Omega)(N_z)(x) = \Gamma_z(x) \quad \textit{modulo constants}, \quad x \in \partial\Omega, \tag{2.13}$$

or to be more precise, for any simply connected Lipschitz domain D compactly contained in Ω and for any $g \in L^2_0(\partial D)$, we have

$$\int_{\partial D} (-\frac{1}{2}I + \mathcal{K}_\Omega)(N_z)(x) g(z)\,d\sigma(z) = \int_{\partial D} \Gamma_z(x) g(z)\,d\sigma(z), \quad \forall x \in \partial\Omega. \tag{2.14}$$

Proof. Let $f \in L^2_0(\partial\Omega)$ and define

$$u(z) := \langle (-\frac{1}{2}I + \mathcal{K}_\Omega)(N_z), f\rangle_{\partial\Omega} \quad z \in \Omega.$$

Then

$$u(z) = \int_{\partial\Omega} N(x,z)(-\frac{1}{2}I + \mathcal{K}^*_\Omega) f(z)\,d\sigma(z).$$

Therefore, $\Delta u = 0$ in Ω and $\frac{\partial u}{\partial \nu}|_{\partial\Omega} = (-\frac{1}{2}I + \mathcal{K}^*_\Omega) f$. Thus by (2.4) we have

$$u(z) - \mathcal{S}_\Omega f(z) = \text{constant}, \quad z \in \Omega.$$

Thus if $g \in L^2_0(\partial\Omega)$, then we obtain

$$\begin{aligned}
&\int_{\partial\Omega}\int_{\partial D}(-\frac{1}{2}I + \mathcal{K}_\Omega)(N_z)(x)g(z)d\sigma(z)f(x)d\sigma(x) \\
&= \int_{\partial\Omega}\int_{\partial D}\Gamma_z(x)g(z)d\sigma(z)f(x)d\sigma(x).
\end{aligned}$$

Since f is arbitrary, we have (2.13) or equivalently, (2.14). This completes the proof. $\square$

Let $g \in L^2_0(\partial\Omega)$. Let $U(y) := \int_{\partial\Omega} N(x,y)g(x)d\sigma(x)$. Then U satisfies

$$\begin{cases} \Delta U = 0 \quad \text{in } \Omega, \\ \dfrac{\partial U}{\partial \nu}|_{\partial\Omega} = g \in L^2_0(\partial\Omega), \\ \displaystyle\int_{\partial\Omega} U(x)d\sigma(x) = 0. \end{cases} \tag{2.15}$$

Theorem 2.4 *The solution u of (2.6) can be represented as*

$$u(x) = U(x) - N_D\phi(x), \quad x \in \partial\Omega \tag{2.16}$$

where ϕ is defined in (2.9).

Proof. By substituting (2.7) into the equation (2.8), we obtain

$$H(x) = -\mathcal{S}_\Omega(g)(x) + \mathcal{D}_\Omega(H|_{\partial\Omega} + (\mathcal{S}_D\phi)|_{\partial\Omega})(x), \quad x \in \Omega.$$

It then follows from (2.5) that

$$(\frac{1}{2}I - \mathcal{K}_\Omega)(H|_{\partial\Omega}) = -(\mathcal{S}_\Omega g)|_{\partial\Omega} + (\frac{1}{2}I + \mathcal{K}_\Omega)((\mathcal{S}_D\phi)|_{\partial\Omega}), \quad \text{on } \partial\Omega. \tag{2.17}$$

Since $U = -\mathcal{S}_\Omega(g) + \mathcal{D}_\Omega(U|_{\partial\Omega})$ in Ω, we have

$$(\frac{1}{2}I - \mathcal{K}_\Omega)(U|_{\partial\Omega}) = -(\mathcal{S}_\Omega g)|_{\partial\Omega}. \tag{2.18}$$

Since $\phi \in L^2_0(\partial D)$, it follows from (2.13) that

$$-(\frac{1}{2}I - \mathcal{K}_\Omega)((N_D\phi)|_{\partial\Omega}) = (\mathcal{S}_D\phi)|_{\partial\Omega}. \tag{2.19}$$

From (2.17), (2.18), and (2.19), we conclude that

$$(\frac{1}{2}I - \mathcal{K}_\Omega)(H|_{\partial\Omega} - U|_{\partial\Omega} + (\frac{1}{2}I + \mathcal{K}_\Omega)((N_D\phi)|_{\partial\Omega})) = 0.$$

Therefore, we have

$$H|_{\partial\Omega} - U|_{\partial\Omega} + (\frac{1}{2}I + \mathcal{K}_\Omega)((N_D\phi)|_{\partial\Omega}) = C \text{ (constant)}. \tag{2.20}$$

Note that $(\frac{1}{2}I + \mathcal{K}_\Omega)((N_D\phi)|_{\partial\Omega}) = (N_D\phi)|_{\partial\Omega} + (\mathcal{S}_D\phi)|_{\partial\Omega}$. Thus we get from (2.7) and (2.20)

$$u|_{\partial\Omega} = U|_{\partial\Omega} - (N_D\phi)|_{\partial\Omega} + C. \tag{2.21}$$

Since all the functions entering in (2.21) belong to $L^2_0(\partial\Omega)$, we conclude that $C = 0$, and the theorem is proved. □

We have a similar representation for solutions of the Dirichlet problem. Let $G(x,y)$ be the Green function for the Dirichlet problem, i.e., the function V defined by $V(x) := \int_{\partial\Omega} \frac{\partial G}{\partial\nu(y)}(x,y)f(y)d\sigma(y)$ is the solution of the problem $\Delta V = 0$ in Ω and $V|_{\partial\Omega} = f$ for any $f \in L^2(\partial\Omega)$. Then we have the following representation theorem.

Theorem 2.5

$$(\frac{1}{2}I + \mathcal{K}^*_\Omega)^{-1}(\frac{\partial\Gamma_z(y)}{\partial\nu(y)})(x) = \frac{\partial G_z}{\partial\nu(x)}(x), \quad x \in \partial\Omega, \ z \in \Omega. \tag{2.22}$$

Let u be the solution of (2.6) with the Neumann condition replaced by the Dirichlet condition $u|_{\partial\Omega} = f$. Then u can be represented as

$$\frac{\partial u}{\partial\nu}(x) = \frac{\partial V}{\partial\nu}(x) - G_D\phi(x), \quad x \in \partial\Omega \tag{2.23}$$

where ϕ is defined in (2.9) and $G_D\phi(x) := \displaystyle\int_{\partial D} \frac{\partial G}{\partial\nu(y)}(x,y)\phi(y)d\sigma(y)$.

Theorem 2.5 can be proved in the same way as Theorem 2.4. In fact, it is simpler because of the solvability of the Dirichlet problem, or equivalently, the invertibility of $(\frac{1}{2}I + \mathcal{K}^*_\Omega)$. So we omit the proof.

3 Generalized Polarization Tensors

In this section we introduce the generalized polarization tensors (GPT) associated with a domain B and a conductivity k. These GPT are the basic building block for the asymptotic expansions in this paper.

Let B be a Lipschitz bounded domain in $\mathbb{R}^d$ and the conductivity of B be k $(k \neq 1)$. The polarization tensor is $M = (m_{ij})$, $1 \le i,j \le d$, is defined by

$$m_{ij} := (k-1)\left[\delta_{ij}|B| + \int_{\partial B} y_i \frac{\partial}{\partial\nu^+}\psi_j(y)d\sigma(y)\right],$$

where ψ_j is the unique solution of the following transmission problem:

$$\begin{cases} \Delta\psi_j(x) = 0, \quad x \in B \cup \mathbb{R}^d \setminus \overline{B}, \\ \psi_j|_+ - \psi_j|_- = 0 \quad \text{on } \partial B, \\ \dfrac{\partial}{\partial\nu^+}\psi_j - k\dfrac{\partial}{\partial\nu^-}\psi_j = \nu_j \quad \text{on } \partial B, \\ \psi_j(x) \to 0 \text{ as } |x| \to \infty. \end{cases}$$

See [23], [7], and [14]. One can easily check using (2.4) that

$$\psi_j = \mathcal{S}_B(\lambda I - \mathcal{K}_B^*)^{-1}(\nu_j).$$

Using (2.4) again, we have

$$\begin{aligned} &\int_{\partial B} y_i \frac{\partial}{\partial\nu^+}\psi_j(y)d\sigma(y) \\ &\quad = -\int_{\partial B} y_i(\frac{1}{2}I - \mathcal{K}_B^*)(\lambda I - \mathcal{K}_B^*)^{-1}(\nu_j)(y)d\sigma(y) \\ &\quad = -\int_{\partial B} y_i\nu_j d\sigma(y) + (\lambda - \frac{1}{2})\int_{\partial B} y_i(\lambda I - \mathcal{K}_B^*)^{-1}(\nu_j)(y)d\sigma(y) \\ &\quad = -\delta_{ij}|B| + \frac{1}{k-1}\int_{\partial B} y_i(\lambda I - \mathcal{K}_B^*)^{-1}(\nu_j)(y)d\sigma(y). \end{aligned}$$

Therefore we prove that the polarization tensor M associated with B and k is given by

$$m_{ij} = \int_{\partial B} y_i(\lambda I - \mathcal{K}_B^*)^{-1}(\nu_j)(y)d\sigma(y). \tag{3.1}$$

Recall $\lambda := \frac{k+1}{2(k-1)}$.

For a multi-index $i = (i_1, \ldots, i_d) \in \mathbb{N}^d$, let $\partial^i f = \partial_1^{i_1} \ldots \partial_d^{i_d} f$ and $x^i := x_1^{i_1} \cdots x_d^{i_d}$. For $i, j \in \mathbb{N}^d$, we define the *generalized polarization tensor* M_{ij} by

$$M_{ij} := \int_{\partial B} y^j \phi_i(y)d\sigma(y), \tag{3.2}$$

where ϕ_i is defined by

$$\phi_i(x) := (\lambda I - \mathcal{K}_B^*)^{-1}(\frac{1}{i!}\nu_y \cdot \nabla y^i)(x), \quad x \in \partial B.$$

4 Derivation of the Full Asymptotic Formula

In this section we derive our asymptotic formula (1.4). As stated in the introduction, we restrict our derivation to the case of a single inhomogeneity ($m = 1$). We only give the details when considering the difference between the fields corresponding to one and zero inhomogeneities. In order to further simplify notation we assume that the single inhomogeneity D has the form $D = \epsilon B + z$, where $z \in \Omega$ and B is a bounded Lipschitz domain in $\mathbb{R}^d$ containing the origin. Suppose that the conductivity of D is k. Let $\lambda := \frac{k+1}{2(k-1)}$. Then by (2.7) and (2.9), the solution u of (2.6) takes the form

$$u(x) = U(x) - N_D(\lambda I - \mathcal{K}_D^*)^{-1}(\frac{\partial H}{\partial \nu}|_{\partial D})(x), \quad x \in \partial\Omega,$$

where U is the background potential given in (2.15).

Define

$$H_n(x) := \sum_{|i|=0}^{n} \frac{1}{i!}(\partial^i H)(z)(x-z)^i.$$

Then we have from (2.10) that

$$\begin{aligned}\left\| \frac{\partial H}{\partial \nu} - \frac{\partial H_n}{\partial \nu} \right\|_{L^2(\partial D)} &\leq \sup_{x \in \partial D} |\nabla H(x) - \nabla H_n(x)||\partial D|^{1/2} \\ &\leq \|H\|_{\mathcal{C}^{n+1}(\overline{D})} |x-z|^n |\partial D|^{1/2} \\ &\leq C\|g\|_{L^2(\partial\Omega)} \epsilon^n |\partial D|^{1/2}.\end{aligned}$$

Note that

$$\text{if} \int_{\partial D} h d\sigma = 0, \text{ then} \int_{\partial D} (\lambda I - \mathcal{K}_D^*)^{-1} h d\sigma = 0. \tag{4.1}$$

If $\int_{\partial D} h d\sigma = 0$, then we have for $x \in \partial\Omega$ that

$$\begin{aligned}\left|N_D(\lambda I - \mathcal{K}_D^*)^{-1} h(x)\right| &= \left| \int_{\partial D} [N(x-y) - N(x-z)](\lambda I - \mathcal{K}_D^*)^{-1} h(y) d\sigma(y) \right| \\ &\leq C\epsilon |\partial D|^{1/2} \|h\|_{L^2(\partial D)} .\end{aligned}$$

It then follows that

$$\begin{aligned}\left| N_D(\lambda I - \mathcal{K}_D^*)^{-1}(\frac{\partial H}{\partial \nu}|_{\partial D} - \frac{\partial H_n}{\partial \nu}|_{\partial D})(x) \right| &\leq C\epsilon |\partial D|^{1/2} \left\| \frac{\partial H}{\partial \nu} - \frac{\partial H_n}{\partial \nu} \right\|_{L^2(\partial D)} \\ &\leq C\|g\|_{L^2(\partial\Omega)} \epsilon^{d+n}.\end{aligned}$$

Therefore, we have

$$u(x) = U(x) - N_D(\lambda I - \mathcal{K}_D^*)^{-1}(\frac{\partial H_n}{\partial \nu}|_{\partial D})(x) + O(\epsilon^{d+n}), \quad x \in \partial\Omega \tag{4.2}$$

where $O(\epsilon^{d+n})$ term is dominated by $C\|g\|_{L^2(\partial\Omega)}\epsilon^{d+n}$ for some C depending only on c_0. Note that

$$(\lambda I - \mathcal{K}_D^*)^{-1}(\frac{\partial H_n}{\partial \nu}|_{\partial D})(x) = \sum_{|i|=1}^{n} (\partial^i H)(z)(\lambda I - \mathcal{K}_D^*)^{-1}(\frac{1}{i!}\nu_x \cdot \nabla (x-z)^i)(x).$$

Since $D = \epsilon B + z$, one can prove by using the change of variables $y = \frac{x-z}{\epsilon}$ that

$$(\lambda I - \mathcal{K}_D^*)^{-1}(\frac{1}{i!}\nu_x \cdot \nabla (x-z)^i)(x) = \epsilon^{|i|-1}(\lambda I - \mathcal{K}_B^*)^{-1}(\frac{1}{i!}\nu_y \cdot \nabla y^i)(\frac{1}{\epsilon}(x-z)).$$

Put

$$\phi_i(x) := (\lambda I - \mathcal{K}_B^*)^{-1}(\frac{1}{i!}\nu_y \cdot \nabla y^i)(x), \quad x \in \partial B. \tag{4.3}$$

Then we get

$$\begin{aligned} & N_D(\lambda I - \mathcal{K}_D^*)^{-1}(\frac{\partial H_n}{\partial \nu}|_{\partial D})(x) \\ &= \sum_{|i|=1}^{n} (\partial^i H)(z)\epsilon^{|i|-1}\int_{\partial D} N(x,y)\phi_i(\epsilon^{-1}(y-z))d\sigma(y) \\ &= \sum_{|i|=1}^{n} (\partial^i H)(z)\epsilon^{|i|+d-2}\int_{\partial B} N(x,\epsilon y + z)\phi_i(y)d\sigma(y). \end{aligned} \tag{4.4}$$

We now expand $N(x,\epsilon y+z)$ asymptotically as $\epsilon \to 0$. By (2.13) we have the following relation:

$$(-\frac{1}{2}I + \mathcal{K}_\Omega)[N(\cdot,\epsilon y + z)](x) = \Gamma(x - z - \epsilon y) \quad \text{modulo constants}, \quad x \in \partial\Omega.$$

Since

$$\Gamma(x - \epsilon y) = \sum_{|j|=0}^{+\infty} \frac{1}{j!}\epsilon^{|j|}\partial^j(\Gamma(x))y^j,$$

we obtain

$$\begin{aligned} (-\frac{1}{2}I + \mathcal{K}_\Omega)[N(\cdot,\epsilon y + z)](x) &= \sum_{|j|=0}^{+\infty} \frac{1}{j!}\epsilon^{|j|}\partial^j(\Gamma(x-z))y^j \\ &= (-\frac{1}{2}I + \mathcal{K}_\Omega)\left[\sum_{|j|=0}^{+\infty} \frac{1}{j!}\epsilon^{|j|}\partial_z^j N(\cdot,z)y^j\right](x). \end{aligned}$$

Since $\int_{\partial\Omega} N(x,w)d\sigma(x) = 0$ for all $w \in \Omega$, we have the following asymptotic expansion of the Neumann function which is of independent interest.

Lemma 4.1 *For $x \in \partial\Omega$, $z \in \Omega$, and $y \in \partial B$, and $\epsilon \to 0$,*

$$N(x, \epsilon y + z) = \sum_{|j|=0}^{+\infty} \frac{1}{j!} \epsilon^{|j|} \partial_z^j N(x,z) y^j. \tag{4.5}$$

We now have from (4.4)

$$\begin{aligned} & N_D(\lambda I - \mathcal{K}_D^*)^{-1}(\frac{\partial H_n}{\partial \nu}|_{\partial D})(x) \\ & = \sum_{|i|=1}^{n} (\partial^i H)(z) \epsilon^{|i|+d-2} \sum_{|j|=0}^{+\infty} \frac{1}{j!} \epsilon^{|j|} \partial_z^j N(x,z) \int_{\partial B} y^j \phi_i(y) d\sigma(y). \end{aligned}$$

Observe that

$$\sum_{|i|=l} \frac{1}{i!} (\partial^i H)(z) \Delta(y^i) = \Delta_y \left(\sum_{|i|=l} \frac{1}{i!} (\partial^i H)(z) y^i \right) = 0,$$

and therefore, by the Green's theorem, it follows that

$$\int_{\partial B} \sum_{|i|=l} \frac{1}{i!} (\partial^i H)(z) \nabla (y^i) \cdot \nu(y) \; d\sigma(y) = 0.$$

Thus, in view of (4.3), the following identity holds by using observation (4.1)

$$\sum_{|i|=l} (\partial^i H)(z) \int_{\partial B} \phi_i(y) d\sigma(y) = 0, \quad \forall \, l \geq 1. \tag{4.6}$$

In fact, this follows immediately from (4.1). Recall now that $M_{ij} = \displaystyle\int_{\partial B} y^j \phi_i(y) d\sigma(y)$ is the generalized polarization tensor associated with the domain B and the conductivity k to obtain the following pointwise asymptotic formula: for $x \in \partial\Omega$,

$$u(x) = U(x) - \epsilon^{d-2} \sum_{|i|=1}^{n} \sum_{|j|=1}^{n-|i|+1} \frac{1}{j!} \epsilon^{|i|+|j|} (\partial^i H)(z) M_{ij} \partial_z^j N(x,z) + O(\epsilon^{d+n}). \tag{4.7}$$

Observing that the formula (4.7) still contains $\partial^i H$ factors, the remaining task is to convert (4.7) to a formula given solely by U and its derivatives.

As a simpliest case, let us now take $n = 1$ to find the leading order term in the asymptotic expansion of $u|_{\partial\Omega}$ as $\epsilon \to 0$. From (2.7) and (2.16), we get

$$\|H - U\|_{L^\infty(\partial\Omega)} \leq C\epsilon^d \|\phi\|_{L^2(\partial D)} \leq C\epsilon^d \|g\|_{L^2(\partial\Omega)}$$

for some C depending only on Ω and c_0. It then follows from the maximum principle that

$$\|H - U\|_{L^\infty(\Omega)} \leq C\epsilon^d \|g\|_{L^2(\partial\Omega)}.$$

Then, from the mean value property of harmonic functions, we obtain

$$|\nabla H(z) - \nabla U(z)| \leq C\epsilon^d \|g\|_{L^2(\partial\Omega)}.$$

It thus follows from (4.7) that

$$u(x) = U(x) - \epsilon^d \sum_{|i|=1,|j|=1} (\partial^i U)(z) M_{ij} \partial^j N(x,z) + O(\epsilon^{d+1}), \quad x \in \partial\Omega, \tag{4.8}$$

which is in view of (3.1) exactly the formula derived in [14] and [7] when D has $C^{1,\alpha}$ boundary.

We now return to (4.7). Substitution of (4.7) into (2.8) yields that, for any $x \in \Omega$,

(4.9)

$$H(x) = U(x) - \epsilon^{d-2} \sum_{|i|=1}^{n} \sum_{|j|=1}^{n-|i|+1} \frac{1}{j!} \epsilon^{|i|+|j|} (\partial^i H)(z) M_{ij} \mathcal{D}_\Omega(\partial_z^j N(\cdot, z))(x) + O(\epsilon^{d+n}).$$

In (4.9) the remainder $O(\epsilon^{d+n})$ is uniform in the $\mathcal{C}^n$ norm on any compact subset of Ω for any n and therefore

$$(\partial^l H)(z) + \sum_{|i|=1}^{n} \epsilon^{d-2} \sum_{|j|=1}^{n-|i|+1} \epsilon^{|i|+|j|} (\partial^i H)(z) P_{ijl} = (\partial^l U)(z) + O(\epsilon^{d+n}), \tag{4.10}$$

for all $l \in \mathbb{N}^d$ with $|l| \leq n$ where

$$P_{ijl} = \frac{1}{j!} M_{ij} \partial_x^l \mathcal{D}_\Omega(\partial_z^j N(\cdot, z))|_{x=z}. \tag{4.11}$$

Define the operator

$$\mathcal{P}_\epsilon : (v_l)_{l \in \mathbb{N}^d, |l| \leq n} \mapsto \Big(v_l + \epsilon^{d-2} \sum_{|i|=1}^{n} \sum_{|j|=1}^{n-|i|+1} \epsilon^{|i|+|j|} v_i P_{ijl} \Big)_{l \in \mathbb{N}^d, |l| \leq n}.$$

Observe that

$$\mathcal{P}_\epsilon = I + \epsilon^d \mathcal{P}_1 + \ldots + \epsilon^{n+d-1} \mathcal{P}_{n-1}.$$

Defining $\mathcal{Q}_p, p = 1, \ldots, n-1$, by

$$(I + \epsilon^d \mathcal{P}_1 + \ldots + \epsilon^{n+d-1} \mathcal{P}_{n-1})^{-1} = I + \epsilon^d \mathcal{Q}_1 + \ldots + \epsilon^{n+d-1} \mathcal{Q}_{n-1} + O(\epsilon^{n+d}), \tag{4.12}$$

we finally obtain that

$$(4.13)\qquad ((\partial^i H)(z))_{i\in\mathbb{N}^d,|i|\le n} = (I+\sum_{p=1}^{n}\epsilon^{d+p-1}\mathcal{Q}_p)((\partial^i U)(z))_{i\in\mathbb{N}^d,|i|\le n} + O(\epsilon^{d+n})$$

which yields the main result of this paper stated in Theorem 1.1.

We also have a complete asymptotic expansion of the solutions of the Dirichlet problem:

Theorem 4.2 *Suppose that the inhomogeneity consist of single component and let u be the solution of (1.2) with the Neumann condition replaced by the Dirichlet condition $u|_{\partial\Omega} = f$. Let V be the solution of $\Delta V = 0$ in Ω with $V|_{\partial\Omega} = f$. The following pointwise asymptotic expansion on $\partial\Omega$ holds for $d = 2,3$:*

$$(4.14)\qquad \begin{aligned}\frac{\partial u}{\partial \nu}(x) &= \frac{\partial V}{\partial \nu}(x) - \epsilon^{d-2}\sum_{|i|=1}^{n}\sum_{|j|=1}^{n-|i|+1}\frac{1}{j!}\epsilon^{|i|+|j|}\times \\ &\qquad \Big[\Big((I+\sum_{p=1}^{n+2-|i|-|j|-d}\epsilon^{d+p-1}\mathcal{Q}_p)(\partial^l U(z))\Big)_i M_{ij}\partial_z^j\frac{\partial}{\partial\nu_x}G(x,z)\Big] \\ &\quad + O(\epsilon^{d+n}),\end{aligned}$$

where the remainders $O(\epsilon^{d+n})$ are is dominated by $C\epsilon^{d+n}\|g\|_{L^2(\partial\Omega)}$ for some C independent of $x\in\partial\Omega$. Here $G(x,z)$ is the Dirichlet Green function, M_{ij}, $i,j\in\mathbb{N}^d$, are the generalized polarization tensors, and $\mathcal{Q}_p$ is the operator defined in (4.12) where $\mathcal{P}_{ijk}$ is defined, in this case, by

$$(4.15)\qquad P_{ijl} = \frac{1}{j!}M_{ij}\partial_x^l\mathcal{S}_\Omega(\partial_z^j(\frac{\partial}{\partial\nu_x}G)(\cdot,z))|_{x=z}.$$

Theorem 4.2 can be proved in the exactly same manner as Theorem 1.1. We begin with Lemma 2.5. Then the same arguments give us

$$u(x) = U(x) - \epsilon^{d-2}\sum_{|i|=1}^{n}\sum_{|j|=1}^{n-|i|+1}\frac{1}{j!}\epsilon^{|i|+|j|}(\partial^i H)(z)M_{ij}\partial_z^j G(x,z) + O(\epsilon^{d+n}).$$

From this we can get (4.14) as before.

We conclude this paper by making a remark. The following formula is not exactly an asymptotic formula. However, since the formula is simple and has some potential applicability in solving the inverse conductivity problem, we make a record of it as a theorem.

Theorem 4.3 *We have*

$$u(x) = H(x) + \epsilon^{d-2} \sum_{|i|=1}^{n} \sum_{|j|=1}^{n-|i|+1} \frac{1}{j!} \epsilon^{|i|+|j|} \partial^i H(z) M_{ij} \partial^j \Gamma(x-z) + O(\epsilon^{d+n}), \tag{4.16}$$

where $x \in \Omega_0$ *and* $O(\epsilon^{d+n})$ *term is dominated by* $C\|g\|_{L^2(\partial\Omega)}\epsilon^{d+n}$ *for some* C *depending only on* c_0*, and* H *is given in (2.8).*

References

[1] C. Alves and H. Ammari, Boundary integral formulae for the reconstruction of imperfections of small diameter in an elastic medium, SIAM J. Appl. Math. 62 (2002), 94-106.

[2] H. Ammari, S. Moskow, and M. Vogelius, Boundary integral formulas for the reconstruction of electromagnetic imperfections of small diameter, to appear in ESAIM: Cont. Opt. Calc. Var.

[3] H. Ammari, M. Vogelius, and D. Volkov, Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of imperfections of small diameter II.The full Maxwell equations, J. Math. Pures Appl. 80 (2001), 769-814.

[4] M. Brühl, M. Hanke, and M. Vogelius, A direct impedance tomography algorithm for locating small inhomogeneities, submitted.

[5] A. P. Calderón, On an inverse boundary value problem, Seminar on Numerical Analysis and its Applications to Continuum Physics, Soc. Brasileira de Matemática, Rio de Janeiro, 1980, 65-73.

[6] M. Cheney, D. Isaacson, and J.C. Newell, Electrical impedance tomography, SIAM Rev. 41 (1999), 85-101.

[7] D.J. Cedio-Fengya, S. Moskow, and M. Vogelius, Identification of conductivity imperpections of small diameter by boundary measurements: Continuous dependence and computational reconstruction, Inverse Problems 14 (1998), 553-595.

[8] R.R. Coifman, A. McIntosh, Y. Meyer, L'integrale de Cauchy definit un opérateur bournée sur L^2 pour courbes lipschitziennes, Ann. of Math., 116 (1982), 361–387.

[9] D. Colton and A. Kirsch, A simple method for solving inverse scattering problems in the resonance region, Inverse Problems 12 (1996), 383-393.

[10] L. Escauriaza, E.B. Fabes, and G. Verchota, On a regularity theorem for weak solutions to transmission problems with internal Lipschitz boundaries, Proc. A.M.S 115 (1992), 1069–1076.

[11] E.B. Fabes, M. Jodeit, and N.M. Riviére, Potential techniques for boundary value problems on $\mathcal{C}^1$ domains, Acta Math., 141 (1978), 165-186.

[12] E.B. Fabes, H. Kang, and J.-K. Seo, Inverse conductivity problem: error estimates and approximate identification for perturbed disks, SIAM Jour. of Math. Anal. 30 (4) (1999), 699–720.

[13] G.B. Folland, *Introduction to partial differential equations*, Princeton Univ. Press, Princeton, 1976.

[14] A. Friedman and M. Vogelius, Identification of small inhomogeneities of extreme conductivity by boundary measurements: a theorem on continuous dependence, Arch. Rat. Mech. Anal. 105 (1989), 299–326.

[15] H. Kang and J.K. Seo, Layer potential technique for the inverse conductivity problem, Inverse Problems 12 (1996), 267–278.

[16] ———, Identification of domains with near-extreme conductivity: global stability and error estimates, Inverse Problems 15 (1999), 851–867.

[17] ———, Recent progress in the inverse conductivity problem with single measurement, in *Inverse Problems and Related Fields*, CRC Press, 2000, 69–80.

[18] R.E. Kleinman and T.B.A. Senior, Rayleigh scattering in *Low and High Frequency Asymptotics,* edited by V.K. Varadan and V.V. Varadan, North-Holland, 1986, 1-70.

[19] O. Kwon, J.K. Seo, and J.R. Yoon, A real-time algorithm for the location search of discontinuous conductivities with one measurement, Commun. Pure Appl. Math. LV (2002), 1-29.

[20] T.D. Mast, A. Nachman, and R.C. Waag, Focusing and imagining using eigenfunctions of the scattering operator, J. Acoust. Soc. Am. 102 (1997), 715-725.

[21] A. Nachmann, Reconstructions from boundary measurements, Ann. Math. 128 (1988), 531-587.

[22] G. Pólya and G. Szegö, Isoperimetric Inequalities in Mathematical Physics, Annals of Mathematical Studies Number 27, Princeton University Press, Princeton 1951.

[23] M. Schiffer and G. Szegö, Virtual mass and polarization, Trans. AMS 67 (1949), 130-205.

[24] J. Sylvester and G. Uhlmann, A global uniqueness theorem for an inverse boundary value problem, Ann. Math. 125 (1987), 153-169.

[25] G.C. Verchota, Layer potentials and boundary value problems for Laplace's equation in Lipschitz domains, J. of Functional Analysis 59 (1984), 572–611.

[26] M. Vogelius and D. Volkov, Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities, Math. Model. Numer. Anal. 34 (2000), 723-748.