Stability and Reconstruction in Gel'fand Inverse Boundary Spectral Problem

Atsushi Katsuda*, Yaroslav Kurylev†, and Matti Lassas‡

January 30, 2002

Abstract. We consider stability and approximate reconstruction of Riemannian manifold when the finite number of eigenvalues of the Laplace-Beltrami operator and the boundary values of the corresponding eigenfunctions are given. The reconstruction can be done in stable way when manifold is a priori known to satisfy natural geometrical conditions related to curvature and other invariant quantities.

I. Introduction In this paper we consider the questions of stability and approximate reconstruction in the inverse boundary spectral problem which is also called the generalized Gelfand inverse problem [13] for Riemannian manifolds. To formulate the problem and the main results, we need to introduce some basic notations. We will denote by (M, g) an unknown, m-dimensional, compact connected Riemannian manifold with a (smooth) metric g and non-empty boundary ∂M. Our goal is to find (M, g) from boundary data. The boundary ∂M is itself an $(m-1)$-dimensional compact differentiable manifold. We note that we do not assume the knowledge of the metric $i^*(g)$ generated on ∂M by the metric g , where $i : \partial M \to M$ is an embedding or even the corresponding area element dS_g.

Because in the inverse problems the boundary ∂M is usually known, we will consider a class $\mathbf{M} = \mathbf{M}_{\partial M}$ of compact, connected Riemannian manifolds which have the same, i.e. diffeomorphic, boundary, ∂M.

Let $-\Delta_g$ be the Laplace operator on (M, g) with Neumann boundary condition. Denote by $\{\lambda_j, \varphi_j; j = 1, 2, \ldots\}$ the complete set of eigen-

*Department of Mathematics, Okayama University, Tsushima-naka, Okayama, 700-8530, Japan

†Department of Mathematical Sciences, Loughborough University, Loughborough, LE11 3TU, UK

‡Rolf Nevanlinna Institute, University of Helsinki Helsinki, P.O.Box 4, FIN-00014, Finland

values and corresponding orthonormal eigenfunctions of $-\Delta_g$, $0 = \lambda_1 < \lambda_2, \dots$, $\varphi_1 = \mathrm{vol}^{-1/2}(M, g)$, where vol stands for the volume of (M, g).

Denote by $\mathbf{B}$ the set of sequences $\{\mu_j,\, \psi_j;\ j = 1, 2, \dots\}$ where $\mu_j \in \mathbf{R}$ and $\psi_j \in L^2(\partial M)$ and by $\mathcal{D} : \mathbf{M} \to \mathbf{B}$, the map

$$\mathcal{D}\,(M, g) = \{\lambda_j,\, \varphi_j|_{\partial M},\, j = 1, 2, \dots\}\,.$$

Definition 1 *The collection* $\{\lambda_j,\, \varphi_j|_{\partial M};\ j = 1, 2, \dots\}$ *is called the boundary spectral data of* (M, g) *and the map* $\mathcal{D}$ *- the boundary spectral map.*

It was shown by Belishev-Kurylev [7] who used the boundary control method [5] and the unique continuation result of Tataru [35] that

Theorem 1 *The map* $\mathcal{D} : \mathbf{M} \to \mathbf{B}$ *is injective.*

(For earlier uniqueness results in the Gelfand inverse problem for isotropic operators obtained by the boundary control method see, e.g. [6]). The complex geometric optics method [33] was applied to this problem in [26], [27].)

In this paper, we will first analyse the question of stability of the inverse problem, i.e. the question of continuity of $\mathcal{D}^{-1}$. Later, we will also describe a procedure of an approximate reconstruction of (M, g).

Alessandrini [1],[2] considered an inverse boundary problem for a Schrödinger operator, $-\Delta + q$ in a bounded domain of $\mathbf{R}^m$ and obtained a log-type stability estimate in the Gel'fand inverse problem (see also [23]).

Alessandrini and Sylvester [3] and Stefanov and Uhlmann [31] considered inverse boundary value problems for the wave equation which are closely related to the Gel'fand inverse problem. Let

$$u_{tt} + a(x, D)u = 0, \quad \text{in} \quad \Omega \times [0, T], \quad u|_{t<0} < 0, \quad u|_{\partial\Omega \times [0,T]} = f,$$

where $a(x, D) = -\Delta + q$ in [3] and $a(x, D) = -\partial_i a^{ij} \partial_j$ in [31] and the inverse data is given as a non-stationary Dirichlet-Neumann map. For sufficiently large T and, in [31] a^{ij} close to δ^{ij}, they showed Hölder-type stability. These results raise the question of an optimal stability estimates in the Gel'fand inverse problem and inverse boundary value problem for the wave equation. Although exact stability estimates lie outside the scope of this paper which is devoted to the analysis of geometric conditions upon (M, g) to provide continuity of $\mathcal{D}^{-1}$, we believe that Hölder-type estimates could be valid only under rather strong requirements upon (M, g), like strong geodesic property (see e.g. [10], [30]). Further results on stability for different types of inverse boundary value problems could be found in [16]. For counterexamples see e.g. [24].

Clearly, to discuss the continuity of $\mathcal{D}^{-1}$ it is necessary to define appropriate topologies in $\mathbf{M}$ and $\mathbf{B}$. However, it is well-known (see e.g. [11]) that inverse problems are, in general, ill-posed. Therefore, to gain stability we need to know *a priori* that the unknown object, which we are going to reconstruct, belongs to some compact class $\mathbf{K}$ (see e.g. [16]). To obtain stability results, the topologies of $\mathbf{M}$ and $\mathbf{B}$ have to satisfy two properties. First, the map

$$\mathcal{D} : \mathbf{K} \to \mathbf{B}$$

has to be continuous. Second, $\mathcal{D}$ has to be injective. When these two properties are satisfied and $\mathbf{K} \subset \mathbf{M}$ is a compact set then the continuity of the map

$$\mathcal{D}^{-1} : \mathcal{D}(\mathbf{K}) \to \mathbf{K}$$

follows from basic results of general topology. In the first part of the paper we carry out the constructions of the appropriate topologies. We would like to note that a proper topology on $\mathbf{M}$ is the Gromov-Hausdorff topology and the (pre)-compact subsets we actually use are the manifolds of bounded geometry (see e.g. [9], [14], [28], [29]). The topology on $\mathbf{B}$ is defined from the boundary spectral convergence.

In the second part of the paper our attention will be focused on the problem of an approximate reconstruction of (M, g) when we do not know all boundary spectral data. Rather, we know only the first eigenvalues, $\lambda_j < \delta^{-1}$ with some small $\delta > 0$ and boundary values of the corresponding eigenfunctions and, furthermore, we know them with some error. Namely, we have a finite collection of pairs

$$\{\mu_j, \psi_j;\, j = 1, \dots, n(\delta^{-1})\}, \quad \mu_j \in \mathbf{R},\, \psi_j \in L^2(\partial M)$$

and know that they are close to $\{\lambda_j,\, \varphi_j|_{\partial M};\, j = 1, \dots, n(\delta^{-1})\}$, where $n(\delta^{-1})$ is the number of eigenvalues that are smaller than δ^{-1}. Our definition of topology on $\mathbf{B}$ is adjusted to make the intuitive notion of closedness of $\{\mu_j, \psi_j\}$ to $\{\lambda_j,\, \varphi_j|_{\partial M};\, j \le n(\delta^{-1})\}$, rigorous.

In the second part of the paper we describe the stable reconstruction procedure but omit the proof showing that this procedure is stable. These proof are to be published elsewhere. We note that to carry out the reconstruction procedure we will need to work under stronger requirements on (M, g) than those that are needed for stability. Similarly, in the first part we will only formulate necessary results omitting all proofs. The proofs and detailed description of the reconstruction procedure will be given in a forthcoming paper.

II. Main Results. Stability. We start with a proper topology on $\mathbf{M}$.

Definition 2 *(Gromov-Hausdorff topology). Let $\varepsilon > 0$. The Riemannian manifolds $(M^i, g^i) \in \mathbf{M}$, $i = 1, 2$ are ε-close in the Gromov-Hausdorff topology,*

$$d_{GH}((M^1, g^1),\, (M^2, g^2)) \leq \varepsilon,$$

if there are ε-nets $\{x_j^i;\, j = 1, \dots, J(\varepsilon)\} \subset M^i$, $i = 1, 2$, such that

$$|d_{g_1}(x_j^1, x_k^1) - d_{g_2}(x_j^2, x_k^2)| \leq \varepsilon, \quad j, k = 1, \dots, J(\varepsilon),$$

where $d_{g^i}(\cdot, \cdot)$ stands for the distance on (M^i, g^i).

We remind that $\{x_j;\, j = 1, \dots, J(\varepsilon)\} \subset M$ is an ε-net if for every $x \in M$ there is x_j such that $d_g(x_j, x) \leq \varepsilon$.

Remark 1. In the future we will identify diffeomorphic manifolds assuming implicitly that desired statements are valid after an automorphism of M.

Definition 3 *(Riemannian manifolds of bounded geometry). For any Λ, D, $i_0 > 0$, $\mathbf{M}(\Lambda, D, i_0) \subset \mathbf{M}$ consists of Riemannian manifolds $(M, g) \in \mathbf{M}$ such that*
i) $\|\nabla Rm(M, g)\| + \|Rm(M, g)\| \leq \Lambda$, *iii) $diam(M, g) \leq D$,*
ii) $\|\nabla S(M, g)\| + \|S(M, g)\| \leq \Lambda$, *iv) $inj(M, g) \geq i_0$.*

Here Rm is the Riemannian curvature tensor on (M, g) considered as a 4-linear form on TM, ∇ is the covariant derivative, and S is the second fundamental form with respect to the inner product, i.e. a quadratic form on $T\partial M$. Furthermore, diam is the diameter of (M, g) and inj is the injectivity radius of (M, g). Condition iv) consists of three different conditions:

(α) Let $x \in \partial M$. Denote, as usual, by $B_{\partial M}(x, r)$ the open metric ball (on ∂M) of the radius r with center in x. Then, for any $r < i_0$, $B_{\partial M}(x, r)$ is a domain of normal coordinates on ∂M centered at x.

(β) Let $x \in M$, $d(x, \partial M) \geq i_0$. Then, for any $r < i_0$, $B_M(x, r)$ is a domain of normal coordinates on M centered at x.

(γ) Let $x \in M$, $d(x, \partial M) < i_0$. Then, for any $r < i_0$, the cylinder $C(x, r)$ is a domain of boundary normal coordinates. Here $C(x, r)$ consists of all points $y \in M$ such that $d(y, \partial M) < r$ and the unique boundary point $z(y) \in \partial M$, nearest to y is in the ball $B_{\partial M}(z(x), r)$.

Next we introduce topology on the set $\mathbf{B}$.

Definition 4 *(Boundary spectral topology.) Let $\delta > 0$. Collections*

$$\{\mu_j^i,\, \psi_j^i;\, j = 1, 2, \dots\} \in \mathbf{B}, \quad i = 1, 2$$

are δ-close if there are disjoint intervals

$$I_p = (a_p, b_p) \subset (-\delta^{-1} - \delta,\ \delta^{-1} + \delta), \quad p = 1, \dots, P,$$

such that

i) $b_p - a_p < \delta$,

ii) For any μ_j^i, $i = 1, 2$ *with* $\mu_j^i < \delta^{-1}$ *there is* p *such that* $\mu_j^i \in I_p$.

iii) For any p*, the total number* n_p^i *of* μ_j^i *inside* I_p *coincide, i.e.*

$$n_p^1 = n_p^2\,(= n_p),$$

iv) For any p *there is a unitary matrix* $U_p = [u_p^{kl}]_{k,l=1}^{n_p}$ *such that*

$$\|U_p \Psi_p^1 - \Psi_p^2\|_{L^2(\partial M)^{n_p}} \leq \delta.$$

Here, Ψ_p^i *is the vector-function* $(\psi_{j(1)}^i \cdots \psi_{j(n_p)}^i)$, $j(1) < \cdots < j(n_p)$, *with* $\mu_{j(k)} \in I_p$ *for* $k = 1, \dots, n_p$.

Clearly, condition iv) depends on the choice of the boundary measure used in the definition of $L^2(\partial M)$. However, due to the compactness of ∂M different smooth measures on ∂M determine equivalent L^2-norms.

We note that such topology was introduced by Alessandrini [1], [2] who studied stability in the Gel'fand inverse problem for a Schrödinger operator.

We are now able to formulate the principal stability result for inverse boundary spectral problem.

Theorem 2 *For any* $\Lambda, D, i_0 > 0$*, the map* $\mathcal{D}^{-1}$ *exists and is continuous on* $\mathbf{M}(\Lambda, D, i_0)$*. That is, there is* $\delta > 0$ *such that if* $(M^i, g^i) \in \mathbf{M}(\Lambda, D, i_0)$, $i = 1, 2$, *and their boundary spectral data,*

$$\left\{\lambda_j^1,\, \varphi_j^1|_{\partial M};\, j = 1, 2, \dots\right\} \quad \text{and} \qquad \left\{\lambda_j^2,\, \varphi_j^2|_{\partial M};\, j = 1, 2, \dots\right\},$$

are δ-close then M^1 *and* M^2 *are diffeomorphic. Moreover, for any* $\alpha \in (0, 1)$, $g^1 \to g^2$ *in* $C^{1,\alpha}$ *-topology when* $\delta \to 0$.

Remark 2. The convergence of the metrics in the $C^{1,\alpha}$-topology means that in a proper coordinate system on $M = M_i$, $i = 1, 2$, the metric tensors g_{ij} converge in the $C^{1,\alpha}$-sense, see e.g. [28].

Remark 3. It follows from definition 4 that to apply Theorem 2 it is sufficient to know only the parts of the boundary spectral data of (M^1, g^1) and (M^2, g^2) that correspond to the eigenvalues $\lambda_j^i < \delta^{-1}$.

Theorem 2 and the fact that in the class $\mathbf{M}(\Lambda, D, i_0)$ the Gromov-Hausdorff topology is equivalent to the Lipschitz topology (see ([21]) implies the following corollary that describes the Lipschitz convergence of Riemannian manifolds.

Corollary 1 *Let* $(M^i, g^i) \in \mathbf{M}(\Lambda, D, i_0)$, $i = 1, 2$. *Then, for any* $\varepsilon > 0$ *there is* $\delta > 0$ *such that if the boundary spectral data of these manifolds are* δ*-close then* M^1 *and* M^2 *are diffeomorphic and*

$$(1+\varepsilon)^{-1} \leq \frac{d_{g^1}(x,y)}{d_{g^2}(x,y)} \leq (1+\varepsilon), \quad x, y \in M.$$

Remark 4. It should be noted that $\mathbf{M}(\Lambda, D, i_0)$ is not closed in $\mathbf{M}$ in the Gromov-Hausdorff topology. Therefore, to obtain the desired stability result we should first extend Theorem 1 onto the closure $\overline{\mathbf{M}(\Lambda, D, i_0)}$. The set $\mathbf{M}(\Lambda, D, i_0)$ consists of some class of $C^{1,\alpha}$-smooth Riemannian manifolds. Although the boundary control method that was instrumental to prove Theorem 1 is, in general, not applicable to $C^{1,\alpha}$ Riemannian manifolds it was possible to show that

$$\mathcal{D} : \overline{\mathbf{M}(\Lambda, D, i_0)} \to \mathbf{B}$$

is injective (see section IV).

In Theorem 2, the condition iii) in $\mathbf{M}(\Lambda, D, i_0)$ is automatically satisfied from an estimate of λ_2 in [25]. The other conditions are natural for the stability of the inverse problem because, otherwise, the convergence of the boundary spectral data does not imply the convergence of the corresponding Riemannian manifolds. We illustrate this thesis with the following examples.

Example 1. Let (M, g) be a two-dimensional manifold obtained by the following surgeries of the two-sphere S^2. Add a handle H near the North pole and cut-off a small piece B to make a boundary near the South pole. If the metric size of the handle tend to 0 uniformly then, in the boundary spectral topology, the boundary spectral data of (M, g) tend to the boundary spectral data of the sphere with a hole but without a handle (see e.g. [34]). This shows the necessities of i) and iv) in $\mathbf{M}(\Lambda, D, i_0)$.

Example 2. Let (M, g) be cylinder in $\mathbf{R}^3$ with the boundary made of two components, $\partial M = \partial M_1 \cap \partial M_2$ (see Fig.1). We pinch the cylinder in the middle so that the diameter tends to 0. Therefore, the manifold eventually "splits" into two disjoint semispheres. Meanwhile, in the boundary spectral topology on ∂M, the boundary spectral data of (M, g) tend to the

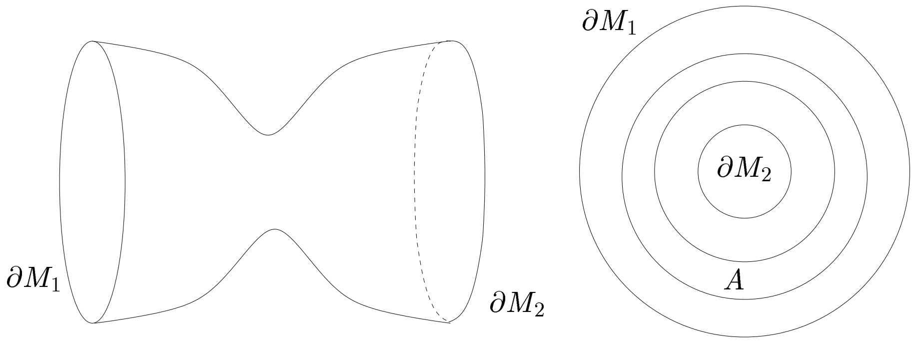

Figure 1: Two representations of the same manifold: Cylinder in $\mathbf{R}^3$ pinched in the middle and its representation as an annulus in $\mathbf{R}^2$ with variable metric. When $\varepsilon \to 0$ the arclength of circles in the region A tend to 0 so that the manifold splits into two components.

boundary spectral data of this disconnected manifold. Again, conclusion of the theorem is no more valid.

This second example may be also described in terms of a perturbation of the metric in an annulus $A(1,4) = \{(r,\theta) : 1 \le r \le 4\}$, where $(r,\theta))$ are the polar coordinates in $\mathbf{R}^2$. We define the length element by

$$dl_\varepsilon^2 = dr^2 + \frac{r^2}{1+\varepsilon^{-1}\chi(r)}\, d\theta^2, \quad \chi(r) \in C_0^\infty,$$

where

$$\chi = 0 \quad \text{for} \quad r < 2 \quad \text{or} \quad r > 3, \quad \chi > 0 \quad \text{for} \quad r \in (2,3).$$

Then, for $\varepsilon \to 0$ each circle $r \in (2,3)$ shrinks to a point so that the internal part near $r = 1$ of A becomes separated from the external part near $r = 4$.

We would note that although this example does not precisely satisfies the definition of $\mathbf{M}_{\partial M}$ because the limiting manifold is no longer connected. However, this definition can be extended onto the case of, probably disconnected Riemannian manifolds with the same boundary ∂M which have no components without the boundary if we change the condition of boundedness of $\mathrm{diam}(M)$ to the boundedness of $\mathrm{vol}(M)$.

Example 3 Take X and Y to be isospectral manifolds without boundary of different topological types (see e.g. [32]) and Z be a Riemannian manifold with boundary ∂Z. Let M_ε be a Riemannian manifold obtained by connecting Z with X by a thin tube of radius ε^2 of length ε. Similarly, let N_ε be a Riemannian manifold obtained by connecting Z with Y by a thin tube of radius ε^2 of length ε. When $\varepsilon \to 0$, the boundary spectral data of M_ε tend to the union of the boundary spectral data of Z and $\{\mu_j, 0\}$, where μ_j is the spectrum of X. Clearly, the boundary spectral data of N_ε tend to the same limit. Thus, for any $\delta > 0$, there exists $\varepsilon_0 > 0$ such that the boundary spectral data of M_ε and N_ε are δ-close when $\varepsilon \le \varepsilon_0$. However, M_ε and N_ε are of different topological types.

In these examples $\|Rm\| \to \infty$ and inj $\to 0$ when $\varepsilon \to 0$ which show the importance of conditions i) and iv) of the theorem. Similar examples can be given to illustrate the importance of condition ii).

III. Main results. Approximate reconstruction. Definition 2 of the Gromov-Hausdorff topology in $\mathbf{M}$ can be easily extended to the set of all compact metric spaces (X, d). Thus, two Riemannian manifolds (M^i, g^i) are ε-close in the Gromov-Hausdorff metric if they possess ε-nets X^i which, being considered as finite metric spaces with distance inherited from (M^i, g^i), are ε-close in the Gromov-Hausdorff topology. Therefore, to approximately reconstruct the Riemannian manifold (M, g) we should construct a metric space $(Y, d), Y = \{y_1, \dots, y_J\}$ which is ε-close to (M, g), i.e.

$$d_{GH}((M, g),\, (Y, d)) \le \varepsilon.$$

To construct a metric space approximation to (M, g) we will assume further regularity properties of the class of manifolds.

Theorem 3 *Let $\Lambda, D, i_0 > 0$. Then for any $\varepsilon > 0$ there is $\delta > 0$ such that if $\{\mu_j,\, \psi_j|_{\partial M};\, j = 1, 2, \dots\} \in \mathbf{B}$ is δ-close to the boundary spectral data $\{\lambda_j,\, \varphi_j|_{\partial M};\, j = 1, 2, \dots\}$ of a Riemannian manifold, $(M, g) \in \overline{\mathbf{M}(\Lambda, D, i_0)}$, then there is a finite metric space (Y, d) such that*

$$d_{GH}((M, g),\, (Y, d)) \le \varepsilon$$

Moreover, there is a constructive algorithm to find (Y, d) from $\{\mu^i_j,\, \psi^i_j;\, j = 1, 2, \dots\} \in \mathbf{B}$.

The algorithm to construct (Y, d) is described in section V.

Remark 5. By definition 4 the δ-closedness does not involve $\{\mu^i_j,\, \psi^i_j\}$ with $|\mu_j| > \delta^{-1}$. Similarly, the construction of (Y, d) does not use these data.

IV. Geometric convergence and stability. The proof of Theorem 2 is based upon the following result.

Proposition 1 *For any $\Lambda, D, i_0 > 0$, the set $\mathbf{M}(\Lambda, D, i_0)$ is pre-compact in the Gromov-Hausdorff topology. Its closure, $\overline{\mathbf{M}(\Lambda, D, i_0)}$ consists of differentiable manifolds with $C^{1,\alpha}$-smooth metric, where $\alpha \in (0,1)$ is arbitrary.*

If a sequence of Riemannian manifolds (M^n, g^n) converges to (M, g) in the Gromov-Hausdorff topology on $\overline{\mathbf{M}(\Lambda, D, i_0)}$, there is $n_0 \geq 1$ such that

i) For $n \geq n_0$, there is a diffeormorphism $F_n : M \to M^n$, i.e. M^n and M are diffeomorphic.

ii) for any $\alpha \in (0,1)$, $F_n^*(g^n)$ converge to g in the $C^{1,\alpha}$-topology.

Remark 6. Proposition 1 is obtained by applying the ideas developed by M. Anderson ([4], see also [15]) in the interior of the manifold and the fundamental equation of Riemannian geometry (see [28]) near the boundary. The latter equations is also known as the Riccati equation for the second fundamental form along normal geodesics. This equation gives the desired regularity estimates for the metric tensor in the boundary normal coordinates.

The geometric convergence described by Proposition 1 yields the continuity of the direct problem.

Proposition 2 *$\mathcal{D} : \overline{\mathbf{M}(\Lambda, D, i_0)} \to \mathbf{B}$ is continuous.*

Remark 7. Proposition 1 was proven by Kodani [21] in the case of weaker regularity. Namely, Kodani showed that the manifolds satisfying

i) $\|Rm(M,g)\| \leq \Lambda$,	iii) $\mathrm{diam}(M,g) \leq D$,
ii) $\|S(M,g)\| \leq \Lambda$,	iv) $\mathrm{inj}(M,g) \geq i_0$.

are pre-compact in the Lipschitz topology rather than the $C^{1,\alpha}$-topology. This result is sufficient for continuity of $\mathcal{D}$. This can be obtained by means of the perturbation theory of quadratic form (see e.g. [12] in the case of manifolds without boundary). Therefore, the result of Kodani would have been sufficient to prove the continuity of $\mathcal{D}^{-1} : \mathcal{D}(\overline{\mathbf{M}(\Lambda, D, i_0)}) \to \overline{\mathbf{M}(\Lambda, D, i_0)}$ if it were possible to show the injectivity of $\mathcal{D}$ on $\overline{\mathbf{M}(\Lambda, D, i_0)}$. Unfortunately, the method used to prove Theorem 1 fails in this case in its parts related to the approximate controllability result by Tataru [35] and also to the injectivity of the boundary distance map (see the next section for the definition of this map).

Proposition 1, on the contrary, guarantees a stronger regularity of the Riemannian manifolds in $\overline{\mathbf{M}(\Lambda, D, i_0)}$. It was used to extend Theorem 1 on the class

$$\widehat{\mathbf{M}} = \bigcup_{\Lambda, D, i_0} \overline{\mathbf{M}(\Lambda, D, i_0)}.$$

Theorem 4 *The map* $\mathcal{D} : \widehat{\mathbf{M}} \to \mathbf{B}$ *is injective.*

V. Construction of a finite ε-net. The procedure of constructing an approximation (Y, d) consists of several steps which we will explain separately. We will also point out smoothness requirements which are necessary to carry out different steps of the procedure.

a. Construction of Fourier coefficients of waves. For simplicity, we assume here that the metric on the boundary, $i^*(g)$, $i : \partial M \to M$ is given. Consider an initial-boundary value problem associated with the Neumann Laplace operator,

$$u^f_{tt} - \Delta_g u^f = 0, \quad \text{in} \quad M \times [0, D],$$

$$\partial_\nu u^f|_{\partial M \times [0,D]} = f \in C^1_p([0, D],\, L^2(\partial M)),$$

$$u^f|_{t=0} = u^f_t|_{t=0} = 0, \tag{1}$$

where $C^1_p([0, D],\, L^2(\partial M))$ consists of piecewise C^1-functions of t. We introduce the wave operator W^t given by

$$W^t(f) = u^f(\cdot, t) \in C^1([0, D],\, L^2(M))$$

and, if we denote by $u^f_k(t)$ the k-th Fourier coefficient of u^f,

$$u^f_k(t) = (u^f(t),\, \varphi_k) = \int_0^t \int_{\partial M} s^t_k(y, t') f(y, t')\, dt' dS_g(y), \tag{2}$$

where dS_g is the boundary area element and

$$s^t_k(y, t') = \frac{\sin(\sqrt{\lambda_k}(t - t'))}{\sqrt{\lambda_k}}\, \varphi_k(y).$$

This formula makes possible to find approximately the first Fourier coefficients of the waves u^f if we know $\{\mu_j, \psi_j;\, j = 1, \ldots, n(\delta^{-1})\}$.

b. Construction of domains of influence. Let $\Gamma \subset \partial M$ be open. Consider the waves $u^f(\tau)$ with $f \in C^\infty_0(\Gamma \times [0, \tau])$, $\tau > 0$. Clearly,

$$\text{supp}[u^f(\tau)] \subset M(\Gamma, \tau) = \{x \in M : d_g(x, \Gamma) \le \tau\}.$$

By the fundamental result of Tataru ([35], see also [18], Ch.2.5), we have

Theorem 5 *Assume that the metric tensor g is C^1-smooth. For any $\tau > 0$ and $\Gamma \subset \partial M$,*

$$cl_{L^2(M)} \left\{ u^f(\tau) : f \in C_0^\infty(\Gamma \times [0,\tau]) \right\} = L^2(M(\Gamma,\tau)),$$

where $L^2(\Omega) \subset L^2(M)$ consists of functions with support in $\overline{\Omega}$.

Let $\eta > 0$, $D > \mathrm{diam}(M)$ as in part iii) of Definition 4 and $\Gamma_l, \quad l = 1, \ldots, L$, be open subsets of ∂M satisfying

$$\mathrm{diam}(\Gamma_l) < \eta, \quad \Gamma_l \cap \Gamma_k = \emptyset, \quad \partial M = \overline{\cup \Gamma_l}.$$

Let

$$\alpha = (\alpha_1, \cdots, \alpha_L), \quad \alpha_l \le D/\eta, \tag{3}$$

be a multi-index. Denote by $\Sigma_\alpha \subset \partial M \times [0, D]$ the set

$$\Sigma_\alpha = \bigcup_l (\Gamma_l \times [D - \alpha_l \eta,\, D]) \cap (\partial M \times [0, D])$$

and by $M_\alpha \subset M$ the subdomain

$$M_\alpha = \bigcup_l M(\Gamma_l,\, \alpha_l \eta). \tag{4}$$

By Theorem 5, for any $\sigma > 0$ there is $f = f_\alpha \in C_0^\infty(\Sigma_\alpha)$ such that

$$\|u^f(D) - \chi_{M_\alpha} \varphi_1\|_{L^2(M)} \le \sigma/2,$$

where χ_Ω is the characteristic function of a set Ω.

It then follows from the general results of control theory that there is a finite linear combination $\widehat{f}$,

$$\widehat{f}(x,t) = \sum_{j=1}^{J} a_j s_j^D(x,t) \chi_{\Sigma_\alpha}(x,t), \quad x \in \partial M, \tag{5}$$

such that

$$\|u^{\widehat{f}}(D) - \chi_{M_\alpha} \varphi_1\|_{L^2(M)} \le \sigma. \tag{6}$$

The coefficients a_j of the function (5) which satisfies equation (6) can be found using the boundary spectral data $\{\lambda_k,\, \varphi_k|_{\partial M};\, k = 1, \ldots, K\}$, $K > J$ by means of some variational procedure.

To define this procedure rigorously, let $H_N(\Sigma_\alpha)\subset C^1_p([0,D],\,L^2(\partial M))$ be an N-dimensional subspace spanned by the functions $\{s^D_n(y,t)\,\chi_{\Sigma_\alpha}(y,t);\ n=1,\dots,N\}$.

Let $A_{N,I,K}(f)$ be a functional on $H_N(\Sigma_\alpha)$,

$$A_{N,I,K}(f)= \tag{7}$$

$$=\sup\left\{\left\|(P_KW^Df-\chi_{M_\alpha}\varphi_1,W^Dh)\right\|:\ h\in H_I(\Sigma_\alpha),\quad \|P_KW^Dh\|\le 1\right\}.$$

Here P_N is the orthoprojector in $L^2(M)$,

$$P_Nu=\sum_{n\le N}(u,\varphi_n)\,\varphi_n.$$

Clearly, due to (2) the functional $A(f)$ can be evaluated in terms of $\{\lambda_j,\,\varphi_j|_{\partial M};\ j=1,\dots,n(\delta^{-1})\}$ if $n(\delta^{-1})\ge\max(N,I,K)$. As we have $\{\mu_j,\,\psi_j;\,j=1,\dots,n(\delta^{-1})\}$ rather then $\{\lambda_j,\,\varphi_j|_{\partial M};\,j=1,\dots,n(\delta^{-1})\}$, the functional $A(f)$ can be evaluated only approximately.

Theorem 6 *For any $\sigma>0$ there are parameters $C,N,I,K,\delta,\ N\le I\le K\le n(\delta^{-1})$, such that if $\{\mu_j,\,\psi_j\}$ is δ-close to the boundary spectral data $\{\lambda_j,\,\varphi_j|_{\partial M}\}$ of a Riemannian manifold (M,g), then a minimizer $f^*\in H_N$ of the functional $A_{N,I,K}$ such that*

$$\|f^*\|_{L^2(\partial M\times[0,D])}\le C,$$

satisfies the equation

$$\left\|W^Df^*-\chi_{M_\alpha}\varphi_1\right\|\le\sigma.$$

By Proposition 2 the boundary spectral data depend continuously on the Gromov-Hausdorff distance restricted on $\overline{\mathbf{M}(\Lambda,D,i_0)}$. Consider

$$u^f(\cdot,t)=u^f_{(M,g)}(\cdot,t)$$

as a function of $(M,g)\in\overline{\mathbf{M}(\Lambda,D,i_0)}$. By Proposition 1, the perturbation theory [19] shows that the wave $u^f_{(M,g)}(\cdot,t)$ depends continuously (in the $C^1([0,D],L^2(M))$-topology) on the Gromov-Hausdorff distance. As the set $\overline{\mathbf{M}(\Lambda,D,i_0)}$ is compact, the parameters C,N,I,K and δ used in the above construction are uniform on $\overline{\mathbf{M}(\Lambda,D,i_0)}$ and depend only on σ.

c. Construction of the boundary distance map. As

$$\|\chi_{M_\alpha}\varphi_1\|^2=\frac{\mathrm{vol}(M_\alpha)}{\mathrm{vol}(M)},\quad \mathrm{vol}(M)=\varphi_1(z)^{-2},\ z\in\partial M,$$

Proposition 1 makes it possible to evaluate an approximate volume, $\mathrm{vol}^a(M_\alpha)$. We will use this observation to construct a finite approximation to the set of the boundary distance functions associated with (M, g). Namely, for any $x \in M$, let $r_x \in L^\infty(\partial M)$ be given by

$$r_x(z) = d_g(x, z), \quad \text{for any} \quad z \in \partial M.$$

The boundary distance functions determine the boundary distance map $R : (M, g) \to L^\infty(\partial M)$,

$$R(x) = r_x.$$

It turns out that the map R or, more precisely, its image $R(M, g)$ is sufficient to reconstruct (M, g) (see [22] for the procedure of the reconstruction of (M, g) from $R(M)$). We will use an approximation R^* to $R(M, g)$ to construct the desired metric space (Y, d) in Theorem 3. Note that the metric d is different to the induced metric as a (metric) subspace of $L^\infty(\partial M)$ (cf. Step 1 in subsection **d**.

To find R^* consider subdomains $M^*_\beta \subset M$,

$$M^*_\beta = \{x \in M : d_g(x, \Gamma_l) \in ((\beta_l - 2)\eta, (\beta_l + 2)\eta), \quad l = 1, \dots, L\}. \tag{8}$$

For any β the subdomain M^*_β can be obtained as a finite number of unions, intersections and compliments of the subdomains M_α of form (4). For any $\Omega, \Omega' \subset M$,

$$\mathrm{vol}(\Omega^c) = \mathrm{vol}(M) - \mathrm{vol}(\Omega),$$

where $\Omega^c = M \setminus \Omega$, and

$$\mathrm{vol}(\Omega \cap \Omega') = \mathrm{vol}(\Omega) + \mathrm{vol}(\Omega') - \mathrm{vol}(\Omega \cup \Omega').$$

Moreover, for any α, β,

$$\mathrm{vol}(M_\alpha \cup M_\beta) = \mathrm{vol}(M_\gamma), \quad \text{where} \quad \gamma_l = \max(\alpha_l, \beta_l).$$

Therefore, it is possible to evaluate an approximate volume $\mathrm{vol}^a(M^*_\alpha)$, if we know $\{\mu_j, \psi_j\}$. Clearly, $|\mathrm{vol}^a(M^*_\beta) - \mathrm{vol}(M^*_\beta)| < \sigma$ when $\{\mu_j, \psi_j\}$ is δ-close to $\{\lambda_j, \varphi_j|_{\partial M}\}$ with sufficiently small δ.

When $\mathrm{vol}^a(M^*_\beta) \geq \sigma$ we associate with this β a function $r_\beta \in L^\infty(\partial M)$,

$$r_\beta(z) = \beta_l \eta \quad \text{for} \quad z \in \Gamma_l.$$

Thus, if $\mathrm{vol}(M^*_\beta) \geq 2\sigma$ so that $\mathrm{vol}^a(M^*_\beta) \geq \sigma$, there is an approximate function r_β such that for any $x \in M^*_\beta$,

$$\|r_\beta - r_x\| \leq 2\eta.$$

In the construction of the approximate volume of a set $M^*_\beta \neq \emptyset$ such that $\mathrm{vol}(M^*_\beta) < 2\sigma$ it may happen that $\mathrm{vol}^a(M^*_\beta) < \sigma$. Thus, according to the procedure, R^* do not contain r_β corresponding to this multi-index β. Consider next a point $x \in M^*_\beta$ for such β. By condition (3) there is an upper bound $N(\eta)$ for the number of all multi-indexes β. Moreover, due to condition iv) of definition 3, $\mathrm{vol}(B_M(x,\eta)) \geq c\eta^m$. Therefore, if $\sigma \leq C'\eta^m N(\eta)^{-1}$, there is a multiindex γ, $\gamma \neq \beta$ such that

i. $\mathrm{vol}(M^*_\gamma) \geq 2\sigma$, so that $\mathrm{vol}^a(M^*_\gamma) \geq \sigma$,

ii. $d(x, M^*_\gamma) \leq \eta$.

Thus

$$\|r_\gamma - r_x\| \leq 4\eta.$$

Summarizing the previous considerations we obtain the following result.

Lemma 1 *For any $(M,g) \in \overline{\mathbf{M}}$ and any $\eta > 0$ there is $\delta > 0$ such that if $\{\mu_j,\, \psi_j\}$ is δ-close to the boundary spectral data $\{\lambda_j,\, \varphi_j|_{\partial M}\}$ then the set R^* constructed from $\{\mu_j,\, \psi_j;\, j = 1, \dots, n(\delta^{-1})\}$ is η-close to $R(M,g)$, i.e.*

$$d_H(R^*, R(M,g)) \leq \eta.$$

Here d_H is the Hausdorff distance between subsets in the metric space $L^\infty(\partial M)$. Moreover, δ can be chosen uniformly on any $\overline{\mathbf{M}(\Lambda, D, i_0)}$.

d. Construction of an ε-net The set R^* has a metric structure inherited from $L^\infty(\partial M)$,

$$d_\infty(r_1, r_2) = \|r_1 - r_2\|_{L^\infty(\partial M)}.$$

If (M,g) is strongly geodesic (e.g. [10], [30]), i.e. all geodesic inside M are minimal, then

$$d_g(x,y) = \|r_x - r_y\|_{L^\infty(\partial M)}, \tag{9}$$

for any $x, y \in M$. Thus, the metric space (R^*, d_∞) provides a 2η-net for (M,g),

$$d_{GH}((R^*, d_\infty),\, (M,g)) \leq 2\eta.$$

However, condition (9) is not valid for general manifolds. Therefore, d_∞-metric is inappropriate to construct an approximation to (M,g) in the Gromov-Hausdorff topology. In this section we will describe how to equip the finite metric space R^* with another metric d^* so that $(Y,d) = (R^*, d^*)$ becomes close to (M,g) in the Gromov-Hausdorff metric. In our description we will assume that the whole $R(M)$ rather then R^* is known. We will show how to find an approximation to $d_g(x,y)$ from $R(M)$. This procedure is valid also in the case when we know only R^*. The corresponding result is given at the end of this section.

1. Assume first that $x_1, x_2 \in M^{int}$ can be connected by a unique shortest geodesic, $\gamma(x_1,x_2)$. Assume in addition that this geodesic can be continued beyond x_2 to a boundary point $z \in \partial M$ so that $\gamma(x_1,z)$ is a shortest geodesic between x_1 and z. Then,

$$d_g(x_1,x_2) = d_g(x_1,z) - d_g(x_2,z) = \|r_{x_1} - r_{x_2}\|,$$

can be found from $R(M,g)$.

It remains to identify those $r_1, r_2 \in R(M)$ which correspond to x_1, x_2 with this property. As i^*g on ∂M is known, it is possible to extend (M,g) to a larger Riemannian manifold $(\widetilde{M}, \widetilde{g})$ which has a continuous, piecewise C^1 metric $\widetilde{g}$. We can then extend each function $r_x \in C(\partial M)$ to the function $\widetilde{r}_x \in C(\widetilde{M} \setminus M)$, $\widetilde{r}_x(y) = d_{\widetilde{M}}(x,y)$. Take a point $z \in \partial M$ and consider minimal geodesics $\gamma(x_i,z)$, $i=1,2$. Assume that $\gamma(x_i,z) \setminus \{z\} \subset M^{int}$, z is not a cut point along any of $\gamma(x_i,z)$ and $\gamma(x_i,z)$ are transversal to ∂M. Then these geodesics can be extended as minimal geodesics beyond z upto some points $y_i \in \widetilde{M} \setminus M$. Denote by $\gamma_i(z,y_i) \subset \widetilde{M} \setminus M$ these extensions. Then the geodesic $\gamma(x_1,x_2)$ connecting x_1 and x_2 extends to a minimal geodesic to $z \in \partial M$ if and only if $\gamma_1(z,y_1) = \gamma_2(z,y_2)$.

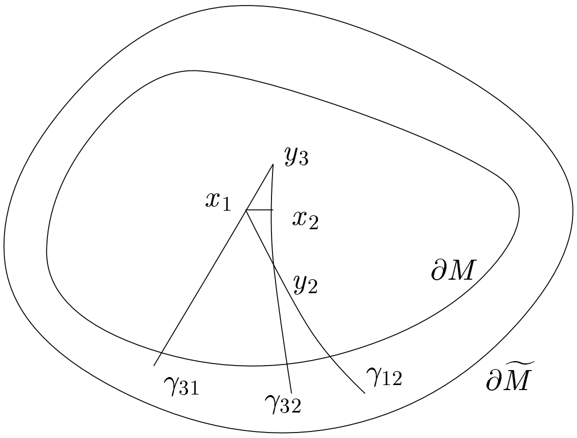

Figure 2: Example of a good triangle used to construct $d(x_1,x_2)$. In this case $x_1 = y_1$ and x_2 lies on the geodesic $\gamma(y_2,y_3)$.

2. Let now $x_1, x_2 \in M^{int}$ lie on the sides $\gamma(y_3, y_1)$ and $\gamma(y_3, y_2)$ of a good geodesic triangle $\Delta(y_1, y_2, y_3)$. A geodesic triangle Δ is called good if all three geodesics $\gamma(y_i, y_j)$ can be continued to ∂M as the shortest geodesics (see Fig. 2). Then, by part **1** we can find $d_g(y_i, y_j)$ and also $d_g(x_i, y_i)$. Let $\widetilde{\Delta}(\widetilde{y}_1, \widetilde{y}_2, \widetilde{y}_3)$ be a triangle in $\mathbf{R}^2$ with $d_e(\widetilde{y}_i, \widetilde{y}_j) = d_g(y_i, y_j)$, where d_e stands for the Euclidean distance. Let $\widetilde{x}_i, i = 1, 2$ be the points on the sides $[\widetilde{y}_3, , \widetilde{y}_i]$ of $\widetilde{\Delta}$ with $d_e(\widetilde{x}_i, \widetilde{y}_i) = d_g(x_i, y_i)$. Then, by the Alexandrov lemma,

$$|d_g(x_1, x_2) - d_e(\widetilde{x}_1, \widetilde{x}_2)| \leq c\sigma^2,$$

where $\sigma = \max d_g(y_i, y_j)$ and c is uniform on $\overline{\mathbf{M}(\Lambda, D, i_0)}$.

Therefore, if for some given x_1, x_2, we can find a proper geodesic triangle $\Delta(y_1, y_2, y_3)$ of a small size, we can find $d_g(x_1, x_2)$ with an error of order σ^2.

3. The existence of a good geodesic triangle for any sufficiently close $x_1, x_2 \in M^{int}$ is based on the following result.

Lemma 2 *Let $\Lambda, D, i_0 > 0$ and $(M, g) \in \overline{\mathbf{M}(\Lambda, D, i_0)}$. Let $x \in M$. Then there is a constant $\rho > 0$ such that for any $(M, g) \in \overline{\mathbf{M}(\Lambda, D, i_0)}$ and $x \in (M, g)$ with $d_g(x, \partial M) \geq C\rho$ there is a vector $u \in T_x M$ $|u| = 1$ with the following property:*
Let $(y, v) \in B_\rho(x, u)$, $|v| = 1$. Then the geodesic $\gamma_y(tv)$, $t \geq 0$, can be extended as a shortest geodesic to the boundary.

Here $B_\rho(x, u)$ is the ball of radius ρ with center at (x, u) in the Sasakian metric on TM. Combining steps **1-3** we can find an approximation to $d_g(x_1, x_2)$ when x_1, x_2 are sufficiently close to each other and not too close to ∂M. (For the definition of the Sasakian metric, see e.g. [29].)

4. Let

$$d_g(x_i, \partial M) = \min_{z \in \partial M} r_i(z) \leq \eta < i_0, \quad r_i = r_{x_i}.$$

There are unique $z_i \in \partial M$ with $d_g(x_i, z_i) = r_i(z_i)$. Denote

$$\widetilde{d} = \widetilde{d}(x_1, x_2) = \left[d^2_{\partial M}(z_1, z_2) + |r_1(z_1) - r_2(z_2)|^2\right]^{1/2},$$

where $d_{\partial M}(\cdot, \cdot)$ is the distance along ∂M. Then,

$$|d_g(x_1, x_2) - \widetilde{d}(x_1, x_2)| \leq C(\eta\widetilde{d} + \widetilde{d}^2).$$

Therefore, $\widetilde{d}$ is a good approximation to d when x_i are close to ∂M.

5. Steps **1-4** define approximate distance $\widetilde{d}(x_1, x_2)$ for sufficiently close x_1, x_2. Using a standard procedure we can extend $\widetilde{d}(x_1, x_2)$ onto arbitrary $x_i \in M$.

When we know the metric space (R^*, d_∞) rather then $R(M,g)$ we can carry out the above constructions approximately. This gives rise to the following result.

Lemma 3 *Let $Y \subset L^\infty(\partial M)$ be a set described in Lemma 1. It is possible to equip Y with a metric structure $d(r_1, r_2)$ so that*

$$d_{GH}((Y,d),(M,g)) \leq \varepsilon(\eta),$$

where $\varepsilon(\eta) \to 0$ when $\eta \to 0$. The function $\varepsilon(\eta)$ can be found uniformly for $\overline{\mathbf{M}(\Lambda, D, i_0)}$.

Combining steps **a.-d.** we obtain a procedure to construct a finite ε-net for (M,g), $\varepsilon = \varepsilon(\delta)$.

VII. Concluding remarks.

Remark 8. The above considerations lack quantitative estimate for the function $\varepsilon(\delta)$. Combining our construction with the Carleman estimates (see, e.g. [35], [36], [18]) we will obtain quantitative estimates for $\varepsilon = \varepsilon(\delta)$. This will be described in a forthcoming paper.

Remark 9. All considerations remain valid for the Dirichlet Laplacian. We need just to take into account that $\varphi_1 \neq 0$ in M^{int}.

Remark 10. It is possible to use the convergence of the heat kernels restricted to ∂M instead of the boundary spectral convergence. Besides, it is possible to modify the procedure to find an approximation to the heat kernel associated with $-\Delta_g$ rather then R^*. This puts our results within the framework of the spectral convergence of Riemannian manifolds (see e.g. [8], [17]). This will be described in the forthcoming paper.

Remark 11. Requirements of C^1-regularity of Rm and S are of technical nature. We intend to improve reconstruction procedure to be valid for weaker regularity assumptions for Rm and S.

Part of our results also briefly announced in [20].

Acknowledgements The authors would like to express their gratitude to Y. D. Burago, D. Y. Burago, M. Gromov, I. M. Gel'fand, A. Katchalov, T. Sakai, E. Somersalo, J. Takahashi for helpful discussions and friendly support. This work was partly supported by EPSRC (UK) grant EPSRC (UK) grants GR/M14463 and 36595, RiP Program, Oberwolfach (Germany), Grant-in Aid for Scientific Research 09640109 and 12640073 (Japan), IHES (France), Finnish Academy project 42013 and 172434, MSRI NSF grant DMS-9810361 (USA), and TEKES (Finland). We are grateful to all these organizations.

References

[1] Alessandrini G. *Stable determination of conductivity by boundary measurements*, Appl. Anal., **27** (1988), 153–172.

[2] Alessandrini G., *Determining conductivity by boundary measurements, the stability issue*, in R. Spigler, Appl. and Ind. Math., Kluwer 1991.

[3] Alessandrini G., Sylvester J. *Stability for a multidimensional inverse spectral theorem.* Comm. Part. Diff. Eq. **15** (1990), 711–736.

[4] Anderson M.T. *Convergence and rigidity of manifolds under Ricci curvature bounds*, Invent. Math., **102** (1990), 429-445.

[5] Belishev, M.I. *An approach to multidimensional inverse problems for the wave equation.* (Russian) Dokl. Akad. Nauk SSSR, **297** (1987), 524–527

[6] Belishev, M.I., Kurylev, Y.V. *A nonstationary inverse problem for the multidimensional wave equation "in the large".* (Russian) Zap. Nauchn. Sem. LOMI, **165** (1987), 21–30.

[7] Belishev M. and Kurylev Y. *To the reconstruction of a Riemannian manifold via its spectral data (BC-method)*, Comm. Part. Diff. Eq., **17** (1992), 767-804.

[8] Berard P., Besson G. and Gallot S. *Embedding Riemannian manifolds by their heat kernel*, Geom. Funct. Anal., **4** (1994), 373-398.

[9] Burago D., Burago Y. and Ivanov S. A Course in Metric Geometry. AMS, Providence (2001)

[10] Croke, C. *Rigidity and the distance between boundary points.* J. Diff. Geom., **33** (1991), 445–464.

[11] Colton D. and Kress R. Inverse Acoustic and Electromagnetic Scattering Theory, Springer, Berlin (1998).

[12] Fukaya, K. *Collapsing of Riemannian manifolds and eigenvalues of Laplace operator.* Invent. Math. **87** (1987), 517–547.

[13] Gelfand I.M., *Some aspects of functional analysis and algebra.* 1957 Proceed. the Intern. Congr. Mathem., Amsterdam, 1954, **1**, 253–276.

[14] Gromov M. with appendices by Katz M., Pansu P. and Semmes S., Metric Structures for Riemannian and Non-Riemanian Spaces, based on " Structures metriques pour les varietes riemanniennes" (LaFontaine J. and Pansu P.eds), English translation by M.Bates, Birkhauser (1999).

[15] Hebey E. and Herzlich M. *Harmonic coordinates, harmonic radius and convergence of Riemannian manifolds*, Rend. di Matem., ser.VII, **17**, Roma (1997), 569-605.

[16] Isakov V. Inverse Problems for Partial Differential Equations, Springer, New York (1998).

[17] Kasue A. and Kumura H. *Spectral convergence of Riemannian manifolds*, Tohoku Math. J., **46** (1994), 147-179.

[18] Katchalov A.P., Kurylev Y. and Lassas M. Inverse Boundary Spectral Problems, Chapman/CRC, Boca Raton (2001).

[19] Kato T. Perturbation Theory for Linear Operators, Springer, Berlin (1995).

[20] Katsuda A. BC-method and stability of Gel'fand inverse spectral problem, RIMS Kokyuroku **1208** (Spectral and Scattering Theory and Related Topics) (2001) 24-35.

[21] Kodani S. *Convergence theory for Riemannian manifolds with boundary*, Compos. Math., **75** (1990), 171-192.

[22] Kurylev Y. *Multidimensional Gel'fand inverse problem and boundary distance map*, in: Inv. Probl. related with Geom., Ibaraki Univ. (1997), 1-16.

[23] Kurylev, Y.V. Starkov, A. *Directional moments in the acoustic inverse problem.* Inv. Probl. in Wave Propag. (Minneapolis, MN, 1995), 295–323, IMA Vol. Math. Appl., **90**.

[24] Mandache N. *Exponential instability in an inverse problem for Schrodinger equation*, to appear in Inverse Problems

[25] Meyer, D. *Minoration de la premiere valeur propre non nulle du probleme de Neumann sur les varietes,* Ann. Inst. Fourier (Grenoble) **36** (1986), 113–125.

[26] Nachman, A. Sylvester, J., Uhlmann, G. *An n-dimensional Borg-Levinson theorem.* Comm. Math. Phys. **115** (1988), 595–605

[27] Novikov, R.G. *A multidimensional inverse spectral problem for the equation* $-\Delta\psi + (v(x) - Eu(x))\psi = 0$. (Russian) Funk. Anal. i Prilozhen. **22** (1988), 11–22.

[28] Petersen P. Riemannian Geometry, Springer, New York (1998).

[29] Sakai T. Riemannian Geometry. Trans. Math. Monogr., **149**, AMS (1995).

[30] Sharafutdinov, V. A. Integral Geometry of Tensor Fields. Inverse and Ill-posed Problems Series. VSP, Utrecht, 1994.

[31] Stefanov, P., Uhlmann, G. *Stability estimates for the hyperbolic Dirichlet to Neumann map in anisotropic media.* J. Funct. Anal. **154** (1998), 330–358.

[32] Sunada T. *Riemannian coverings and isospectral manifolds.* Ann. of Math. (2) **121** (1985), 169–186.

[33] Sylvester J., Uhlmann G. *A global uniqueness theorem for an inverse boundary value problem.* Ann. of Math. (2) **125** (1987), 153–169.

[34] Takahashi J., *Collasping of connected sums and the eigenvalues of the Laplacian,* to appear in J. Geom. Phys. (2001).

[35] Tataru D. *Unique continuation for solutions to PDE's: between Hormander's theorem and Holmgren's theorem*, Comm. Part. Diff. Eq., **20** (1995), 855-884.

[36] Tataru D. Carleman estimates, unique continuation and applications, preprint.