SUBFACTORS OF FREE PRODUCTS OF RESCALINGS OF A II_1–FACTOR

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ABSTRACT. Let Q be any II_1–factor. It is shown that any standard lattice $\mathcal{G}$ can be realized as the standard invariant of a free product of (several) rescalings of Q. In particular, if Q has fundamental group equal to the positive reals and if P is the free product of infinitely many copies of Q, then P has subfactors giving rise to all possible standard invariants. Similarly, given a II_1–subfactor $N \subset M$, it is shown there are subfactors $\hat{N} \subset \hat{M}$ having the same standard invariant as $N \subset M$ but where $\hat{M}$, respectively $\hat{N}$, is the free product of M, respectively N, with rescalings of Q.

1. INTRODUCTION

The systems of higher relative commutants of finite index subfactors of II_1–factors were classified by S. Popa [7], when he proved certain conditions on a lattice of finite dimensional commuting squares (together with certain other data such as Markov traces) to be equivalent to it arising as the higher relative commutants of a subfactor. Such an abstract lattice is called a *standard lattice*. The question of whether there is a single II_1–factor $\mathcal{M}$ whose subfactors give rise to all standard lattices was answered in the affirmative by Popa and Shlyakhtenko [8], where they proved this universal property for $\mathcal{M} = L(\mathbf{F}_\infty)$, the von Neumann algebra of the free group on infinitely many generators. Their proof has two main components. First, given a standard lattice $\mathcal{G}$, they construct a non–degenerate commuting square

$$\begin{array}{ccc} B & \subset & A \\ \cup & & \cup \\ D & \subset & C \end{array} \tag{1}$$

of type I von Neumann algebras having atomic centers, which encodes the standard lattice. Then, for any II_1–factor Q, they consider the resulting inclusion $N \subset M$ of amalgamated free product von Neumann algebras

$$\begin{aligned} \mathcal{M} &= (Q \otimes B, \tau_Q \otimes \mathrm{id}_B) *_B (A, E^A_B) \\ \mathcal{N} &= (Q \otimes D, \tau_Q \otimes \mathrm{id}_D) *_D (C, E^C_D)\ , \end{aligned} \tag{2}$$

where τ_Q is the tracial state on Q and E^A_B and E^C_D are the conditional expectations from the commuting square (1). By relating it to a construction from [7], they show

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that $\mathcal{N} \subset \mathcal{M}$ is an inculsion of II_∞ factors which is an infinite amplification of an inclusion $N \subset M$ of II_1–factors whose system of higher relative commutants is the original standard lattice $\mathcal{G}$. The second main part of their proof is to show that if Q is taken to be $L(\mathbf{F}_\infty)$, then both N and M are isomorphic to $L(\mathbf{F}_\infty)$.

In this paper, we identify N and M for general Q as a free scaled product [4] of rescalings of Q. This allows us to generalize Popa and Shlyakhtenko's results. For example, (Theorem 4.2), we show that for every standard lattice $\mathcal{G}$ of finite depth and for every II_1–factor Q there is a subfactor $N \subseteq M$ whose standard lattice is $\mathcal{G}$ and where

$$\begin{aligned} M &= Q_{s_1} * \cdots * Q_{s_m} \\ N &= Q_{t_1} * \cdots * Q_{t_n} * L(\mathbf{F}_b) \end{aligned}$$

for some $m, n \in \mathbf{N}$, positive real numbers $s_1, \ldots, s_m$ and $t_1, \ldots, t_n$ and $b > 1 - n$ depending only on $\mathcal{G}$. (Here we are using the notation of [6] for $L(\mathbf{F}_a)$ with a possibly negative). Taking Q to be the hyperfinite II_1–factor R and using the result [2] that the n–fold free product of R is $L(\mathbf{F}_n)$, this gives after rescaling if necessary that if $1 < t < \infty$ and if $\mathcal{G}$ is a standard lattice with finite depth, then $L(\mathbf{F}_t)$ has a subfactor whose standard invariant is $\mathcal{G}$; This result was first proved by Rădulescu [9].

We also show, (Theorem 4.4), that for every standard lattice $\mathcal{G}$ and for every II_1–factor Q, there is a subfactor $N \subset M$ whose standard lattice is $\mathcal{G}$ and such that

$$\begin{aligned} M &= \mathop{*}_{i=1}^{\infty} Q_{s_i} \\ N &= \mathop{*}_{i=1}^{\infty} Q_{t_i} \end{aligned}$$

for some positive numbers $s_1, s_2, \ldots$ and $t_1, \ldots, t_2$ depending only on $\mathcal{G}$. As a consequence of these two results, we obtain (Theorem 4.5) that if Q is a II_1–factor whose fundamental group is equal to the set of all positive reals then the free product $Q * Q * \cdots$ of infinitely many copies of Q has the universal property with respect to subfactors, namely, its subfactors give rise to all standard lattices as their standard invariants.

Popa and Shlyakhtenko [8] also proved that if $N \subset M$ is any subfactor of a II_1–factor then there is a subfactor pair $\hat{N} \subseteq \hat{M}$ having the same standard lattice as $N \subseteq M$ and such that $\hat{N} \cong N * L(\mathbf{F}_\infty)$ and $\hat{M} \cong M * L(\mathbf{F}_\infty)$. Their proofs involved amalgamated free products of the form

$$(Q \otimes B, \tau_Q \otimes \mathrm{id}_B) *_B (M \otimes B(H), E)\,, \tag{3}$$

where E is a conditional expectation onto a copy of the type I algebra B embedded in the II_∞–factor $M \otimes B(H)$, and where they use $Q = L(\mathbf{F}_\infty)$.

We will identify the free product (3) for an arbitrary II_1–factor Q in terms of free subproducts, and thereby prove generalizations of the above mentioned result where $L(\mathbf{F}_\infty)$ in the formula for $\hat{N}$ and $\hat{M}$ is replaced by free products of rescalings of Q. (See Theorems 4.6, 4.7 and 4.8.)

The organization of the rest of the paper is as follows: in §2 and, respectively, §3, isomorphism results are proved for amalgamated free products as in (3) and, respectively (2). (However, the simplifying assumption is made that the algebra B is commutative — an easy trick reduces the general case to this. Moreeover, because it entails no extra difficulty and in fact makes the proof more transparent, we consider free products where the algebra $Q \otimes B$ of (3) and (2) is replaced by a direct sum $\bigoplus_{i\in I} Q(i)$ of II_1–factors whose center is B.) In §4 we prove the existence of subfactors, including those described above.

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2. Amalgamated free products, I

In this section, we consider amalgamated free products related to those of the form (3) when B is taken to be commutative, and we show the result can be expressed in terms of free scaled products of von Neumann algebras. See [4] for the definition of and results about free scaled products.

Take an index set I equal to $\{1, 2, \ldots, m\}$ for some integer $m \geq 2$ or to the natural numbers (in which case we write $m = \infty$) and let $\mathcal{Q} = \bigoplus_{i\in I} Q(i)$, where each $Q(i)$ is a II_1–factor. let B be the center of $\mathcal{Q}$ and let p_i denote the minimal projection of B that is the support of the ith summand $Q(i)$ of $\mathcal{Q}$. Let $E_B^{\mathcal{Q}} : \mathcal{Q} \to B$ be the conditional expectation that is the center–valued trace on $\mathcal{Q}$. Let M be a type II_1 or II_∞ factor with fixed faithful (finite or semifinite) trace Tr. Let N denote the II_1–factor obtained as eMe if M is type II_∞, where $e \in M$ is a projection of trace 1, and as $M_{\mathrm{Tr}(1)^{-1}}$ if M is type II_1; (thus if M is type II_1 and $\mathrm{Tr}(1) \geq 1$ then $N \cong eMe$ where e is as above). Suppose B is embedded in M as a unital subalgebra in such a way that $\mathrm{Tr}(p_i) < \infty$ for all $i \in I$ and let $E_B^M : M \to B$ be the Tr–preserving conditional expectation. Finally, let $\beta(i) = \mathrm{Tr}(p_i)$.

In this and the next section, when X_1 and X_2 are subsets of an algebra, we will use the symbols $\Lambda^{\mathrm{o}}(X_1, X_2)$ to denote the set of all alternating words in X_1 and X_2, i.e. all words of the form $a_1 a_2 \cdots a_n$, where $a_j \in X_{i(j)}$ and $i(j) \neq i(j+1)$. In fact, we will frequently blur the distinction between the set of words *per se* and the set of elements in the algebra that are equal to such words.

Proposition 2.1. *Consider the amalgamated free product of von Neumann algebras*

$$(\mathcal{M}, E) = (M, E_B^M) *_B (\mathcal{Q}, E_B^{\mathcal{Q}}) . \tag{4}$$

If $\sum_{i=1}^m \beta(i) < \infty$ then $\mathcal{M}$ is a II_1–factor, while if $\sum_{i=1}^m \beta(i) = \infty$ then $\mathcal{M}$ is a II_∞–factor. In either case, we have

$$p_1 \mathcal{M} p_1 \cong N_{\beta(1)} \mathop{*}_{i\in I} \left[\tfrac{\beta(i)}{\beta(1)}, Q(i)\right] . \tag{5}$$

Proof. We denote also by Tr the semifinite trace $\mathrm{Tr}\restriction_B \circ E$ on $\mathcal{M}$. The method of [3] shows that E is faithful, hence Tr is faithful on $\mathcal{M}$; we will show the isomorphism (5), from which it will follow that $\mathcal{M}$ is a factor with faithful trace Tr, which is finite or semifinite depending on $\sum_{i=1}^m \beta(i)$.

Let $p_{[1,n]}=p_1+p_2+\cdots+p_n$ and let $\beta[1,n]=\mathrm{Tr}(p_{[1,n]})$. For every $n\in I$, let

$$\mathcal{M}(n)=W^*(p_{[1,n]}Mp_{[1,n]}\cup p_{[1,n]}\mathcal{Q})\ .$$

Then $\mathcal{M}(1)$ is generated by $p_1Mp_1\cong N_{\beta(1)}$ and $p_1\mathcal{Q}\cong Q(1)$, which are free, so

$$\mathcal{M}(1)\cong N_{\beta(1)}*Q(1)\ .$$

Now for $n\in I$, $n\geq 2$, we will find $\mathcal{M}(n)$ in terms of $\mathcal{M}(n-1)$. Let $K\in\mathbf{N}$ be such that $\beta[1,n-1]\geq\beta(n)/K$ and let $(e_{ij})_{1\leq i,j\leq K}$ be a system of matrix units in M such that $\sum_1^K e_{ii}=p_n$. Let $A=\mathrm{span}\,\{e_{ij}\mid 1\leq i,j\leq K\}$ and let $\mathcal{P}=W^*(A\cup p_n\mathcal{Q})$. Clearly A and $p_n\mathcal{Q}$ are free with respect to $\beta(n)^{-1}\mathrm{Tr}\restriction_{p_n\mathcal{M}p_n}$, so

$$\mathcal{P}\cong\mathcal{Q}(n)*M_K(\mathbf{C})\cong\mathcal{Q}(n)*L(\mathbf{F}_{1-K^{-2}}).$$

Let $v\in M$ be such that $v^*v=e_{11}$ and $q:=vv^*\leq p_{[1,n-1]}$. Then

$$\mathcal{M}(n)=W^*(\mathcal{M}(n-1)\cup\mathcal{P}\cup\{v\})\ ,\qquad q\mathcal{M}(n)q=W^*(q\mathcal{M}(n-1)q\cup v\mathcal{P}v^*)\ .$$

We claim that $q\mathcal{M}(n-1)q$ and $v\mathcal{P}v^*$ are free with respect to $K\beta(n)^{-1}\mathrm{Tr}\restriction_{q\mathcal{M}q}$. This is equivalent to freeness of $v^*\mathcal{M}(n-1)v$ and $e_{11}\mathcal{P}e_{11}$ with respect to $K\beta(n)^{-1}\mathrm{Tr}\restriction_{e_{11}\mathcal{M}e_{11}}$, which is what we will show. Using the trace–preserving conditional expectation $\mathcal{P}\to A$ along with the facts that e_{11} is a minimal projection in A and that $\{p_n\}\cup\Lambda^{\mathrm{o}}(A\cap\ker\mathrm{Tr},p_n\mathcal{Q}\cap\ker\mathrm{Tr})$ spans a dense subspace of $\mathcal{P}$, we find that a dense subspace of $e_{11}\mathcal{P}e_{11}\cap\ker\mathrm{Tr}$ is spanned by

$$\widetilde{\Phi}:=\bigcup_{1\leq i,j\leq K}e_{1i}\Phi e_{j1}\ ,$$

where Φ is the set of all words belonging to $\Lambda^{\mathrm{o}}(A\cap\ker\mathrm{Tr},p_n\mathcal{Q}\cap\ker\mathrm{Tr})$ whose first and last letters come from $p_n\mathcal{Q}\cap\ker\mathrm{Tr}$. On the other hand, since

$$p_{[1,n-1]}B\cup\Lambda^{\mathrm{o}}(p_{[1,n-1]}Mp_{[1,n-1]}\cap\ker E,p_{[1,n-1]}\mathcal{Q}\cap\ker E)$$

spans a dense subspace of $\mathcal{M}(n-1)$, we see that $v^*\mathcal{M}(n-1)v\cap\ker\mathrm{Tr}$ is contained in the closed linear span of

$$\widetilde{\Psi}:=(e_{11}Me_{11}\cap\ker\mathrm{Tr})\cup\Psi\ ,$$

where Ψ is the set of all words belonging to $\Lambda^{\mathrm{o}}(M\cap\ker E,p_{[1,n-1]}\mathcal{Q}\cap\ker E)$ with first letter from $e_{11}Mp_{[1,n-1]}$ and with last letter from $p_{[1,n-1]}Me_{11}$. Now to show freeness of $q\mathcal{M}(n-1)q$ and $v\mathcal{P}v^*$, it will suffice to show

$$\Lambda^{\mathrm{o}}(\widetilde{\Phi},\widetilde{\Psi})\subseteq\ker\mathrm{Tr}\ . \tag{6}$$

Let $x\in\Lambda^{\mathrm{o}}(\widetilde{\Phi},\widetilde{\Psi})$. If the first letter in x comes from $e_{11}\Phi e_{j1}\subseteq\widetilde{\Phi}$ then substitute $e_{11}=(e_{11}-K^{-1}p_n)+K^{-1}p_n$ for this first e_{11}, while if the last letter in x comes from $e_{1i}\Phi e_{11}\subseteq\widetilde{\Phi}$ then substitute this value for this last e_{11} and then distribute. Using $e_{j1}e_{11}Mp_{[1,n-1]}\subseteq\ker E$ and $e_{j1}(e_{11}Me_{11}\cap\ker\mathrm{Tr})e_{i1}\subseteq\ker E$ for all $i,j\in\{1,\ldots,K\}$ and regrouping, we see that x is equal to a linear combination of at most four words from $\Lambda^{\mathrm{o}}(M\cap\ker E,\mathcal{Q}\cap\ker E)$, and hence $\mathrm{Tr}(x)=0$ by freeness. We have shown (6) and thereby freeness of $q\mathcal{M}(n-1)q$ and $v\mathcal{P}v^*$.

What we have shown above implies there is an isomorphism

$$p_{[1,n-1]}\mathcal{M}(n)p_{[1,n-1]}\overset{\sim}{\longrightarrow}\mathcal{M}(n-1)*\left[\tfrac{\beta(n)}{K\beta[1,n-1]},v\mathcal{P}v^*\right] \tag{7}$$

intertwining the inclusion $\mathcal{M}(n-1) \hookrightarrow p_{[1,n-1]}\mathcal{M}(n)p_{[1,n-1]}$ and the canonical embedding of $\mathcal{M}(n-1)$ in the RHS of (7). By results of [6],

$$v\mathcal{P}v^* \cong \mathcal{P}_{1/K} \cong \mathcal{Q}(n)_{1/K} * L(\mathbf{F}_{K^2-1}) ,$$

so using the technology of free scaled products [4], we get isomorphisms

$$\begin{aligned}
&\mathcal{M}(n-1) * \left[\tfrac{\beta(n)}{K\beta[1,n-1]}, \mathcal{Q}(n)_{1/K} * L(\mathbf{F}_{K^2-1})\right] \xrightarrow{\sim} \\
&\quad\xrightarrow{\sim} \mathcal{M}(n-1) * \left[\tfrac{\beta(n)}{K\beta[1,n-1]}, \mathcal{Q}(n)_{1/K}\right] * \left[\tfrac{\beta(n)}{K\beta[1,n-1]}, L(\mathbf{F}_{K^2-1})\right] \\
&\quad\xrightarrow{\sim} \mathcal{M}(n-1) * L(\mathbf{F}_{(\beta(n)/\beta[1,n-1])^2(1-K^{-2})}) * \left[\tfrac{\beta(n)}{K\beta[1,n-1]}, \mathcal{Q}(n)_{1/K}\right] \\
&\quad\xrightarrow{\sim} \mathcal{M}(n-1) * \left[\tfrac{\beta(n)}{\beta[1,n-1]}, \mathcal{Q}(n)\right] ,
\end{aligned}$$

all intertwining the canonical embeddings of $\mathcal{M}(n-1)$ into the free scaled products; (the last isomorphism above is obtained from Theorem 5.5 of [4], and the others follow routinely from the definition of free scaled product). Now compressing by p_1 and appealing to Theorem 4.9 of [4], we get an isomorphism

$$p_1\mathcal{M}(n)p_1 \xrightarrow{\sim} p_1\mathcal{M}(n-1)p_1 * \left[\tfrac{\beta(n)}{\beta(1)}, \mathcal{Q}(n)\right] \tag{8}$$

intertwining the inclusion $p_1\mathcal{M}(n-1)p_1 \hookrightarrow p_1\mathcal{M}(n)p_1$ and the canonical embedding of $p_1\mathcal{M}(n-1)p_1$ in the RHS of (8).

Taking the inductive limit as n increases gives the desired isomorphism (5). □

The following corollary is a direct consequence of Proposition 2.1 and results from [6] and [4].

Corollary 2.2. *Let*

$$(\mathcal{M}, E) = (M, E_B^M) *_B (\mathcal{Q}, E_B^{\mathcal{Q}})$$

with accompanying notation be as in Proposition 2.1.

(A) *If $I = \{1, \ldots, m\}$ is finite then*

$$\mathcal{M} \cong M * Q(1)_{\frac{1}{\beta(1)}} * Q(2)_{\frac{1}{\beta(2)}} * \cdots * Q(m)_{\frac{1}{\beta(m)}} * L(\mathbf{F}_t)$$

where

$$t = -m + \sum_{i=1}^{m} \beta(i)^2 .$$

(B) *If $I = \mathbf{N}$ is infinite and if one or more of the following conditions hold:*

(i) *$Q(i) \cong Q(i) * L(\mathbf{F}_\infty)$ for some $i \in \mathbf{N}$,*
(ii) *$N \cong N * L(\mathbf{F}_\infty)$,*
(iii) $\sum_{i=1}^{\infty} \beta(i)^2 = \infty$,

then

$$p_1\mathcal{M}p_1 \cong N_{\beta(1)} * \left(\mathop{*}_{i=1}^{\infty} Q(i)_{\frac{\beta(1)}{\beta(i)}} \right) .$$

3. Amalgamated free products, II

In this section, we will describe amalgamated free product von Neumann algebra related to those in (2), but with B commutative, in terms of free scaled products.

We begin by letting A be a separable, type I von Neumann algebra with atomic center $Z(A)$ and let B be a unital commutative subalgebra of A with the property that $B \cap Z(A) = \mathbf{C}1$. We assume that there is a normal, faithful, semifinite trace Tr_A on A such that $\mathrm{Tr}_A(p) < \infty$ for every minimal projection p in B. Let $E_B^A : A \to B$ be the Tr_A–preserving conditional expectation onto B. Let $(p_i)_{i\in I}$ be the minimal projections in B, where we take either $I = \{1, \ldots, n\}$, some $n \in \mathbf{N}$, or $I = \mathbf{N}$, in which case we set $n = \infty$. Let $Q(i)$ be a II_1–factor ($i \in I$) and let $\mathcal{Q} = \bigoplus_{i\in I} Q(i)$, with the center of $\mathcal{Q}$ identified with B in such a way that p_i is the support projection of the ith summand $Q(i)$. Let $E_B^{\mathcal{Q}} : \mathcal{Q} \to B$ be the conditional expectation that is the center valued trace on $\mathcal{Q}$. Our goal in this section is to write the amalgamated free product von Neumann algebra

$$(\mathcal{P}, E) = (A, E_B^A) *_B (\mathcal{Q}, E_B^{\mathcal{Q}})$$

as a free scaled product of II_1–factors.

We have implicitly choosen an ordering of the minimal projections in B. Let $(q_j)_{j\in J}$ be the minimal projections in $Z(A)$. Define J_m recursively ($m \in I$) by

$$\begin{aligned} J_1 &= \{j \in J \mid q_j p_1 \neq 0\} \\ J_m &= \{j \in J \mid q_j p_m \neq 0\} \backslash (J_1 \cup \cdots \cup J_{m-1}) \qquad (m \in I \backslash \{1\}) \,. \end{aligned}$$

Let $\beta(i) = \mathrm{Tr}_A(p_i)$, ($i \in I$) and if f_j is a minimal projection of A that lies under q_j, then let $\alpha(j) = \mathrm{Tr}_A(f_j)$ ($j \in J$). For every $m \in I\backslash\{1\}$, let e_m be a minimal projection in A such that $e_m \leq p_1 + \cdots + p_{m-1}$ and e_m is equivalent to a subprojection of p_m. (Such a projection must exist by the condition $Z(A) \cap B = \mathbf{C}$.) Let $j(m) \in J$ be such that $e_m \leq q_{j(m)}$ and let $\gamma(m) = \alpha(j(m)) = \mathrm{Tr}_A(e_m)$.

Proposition 3.1. *Let*

$$(\mathcal{P}, E) = (A, E_B^A) *_B (\mathcal{Q}, E_B^{\mathcal{Q}})$$

be the free product of von Neumann algebras with amalgamation over B. If $\sum_{i=1}^n \beta(i) < \infty$ then $\mathcal{P}$ is a II_1–factor, while if $\sum_{i=1}^n \beta(i) = \infty$ then $\mathcal{P}$ is a II_∞–factor. In either case,

$$p_1 \mathcal{P} p_1 \cong (Q(1) * L(\mathbf{F}_r)) \mathop{*}_{i=2}^{n} \left[\tfrac{\gamma(i)}{\beta(1)}, Q(i)_{\frac{\gamma(i)}{\beta(i)}} \right], \tag{9}$$

where

$$r = \frac{1}{\beta(1)^2} \Big(\beta(1)^2 - \sum_{j\in J_1} \alpha(j)^2 \Big) + \frac{1}{\beta(1)^2} \sum_{m=2}^{n} \Big(\beta(m)^2 - \gamma(m)^2 - \sum_{j\in J_m} \alpha(j)^2 \Big) \,.$$

Proof. The condition $Z(A) \cap B = \mathbf{C}$ ensures that $\mathcal{P}$ is a factor. We have the normal, faithful, semifinite trace $\mathrm{Tr} = \mathrm{Tr}_A \circ E$ on $\mathcal{P}$, which is finite if and only if $\sum_{i=1}^n \beta(i) < \infty$.

It remains to find the isomorphism class of $p_1\mathcal{P}p_1$. For $m \in I$, let

$$\begin{aligned} p_{[1,m]} &= p_1 + \cdots + p_m \\ \beta[1,m] &= \beta(1) + \cdots + \beta(m) = \mathrm{Tr}(p_{[1,m]}) \end{aligned}$$

and let

$$\begin{aligned} \mathcal{N}(m) &= W^*(p_m\mathcal{Q} \cup p_mAp_m) \\ \mathcal{P}(m) &= W^*(p_{[1,m]}\mathcal{Q} \cup p_{[1,m]}Ap_{[1,m]}) \ . \end{aligned}$$

Then

$$\mathcal{N}(m) \cong Q(m) * L(\mathbf{F}_{s(m)})$$

where $s(m)$ is the free dimension[1] of p_mAp_m with respect to $\beta(m)^{-1}\mathrm{Tr}\restriction_{p_mAp_m}$. We have $\mathcal{P}(1) = \mathcal{N}(1)$ and

$$s(1) = \frac{1}{\beta(1)^2}\left(\beta(1)^2 - \sum_{j\in J_1}\alpha(j)^2\right) \ .$$

Note that $\mathcal{P}$ is the increasing union of $\mathcal{P}(1) \subseteq \mathcal{P}(2) \subseteq \cdots$ and thus $p_1\mathcal{P}p_1$ is the increasing union of $p_1\mathcal{P}(1)p_1 \subseteq p_1\mathcal{P}(2)p_1 \subseteq \cdots$. We will show that each $p_1\mathcal{P}(m)p_1$ is isomorphic to a free scaled product of II_1–factors in such a way that the inclusions $p_1\mathcal{P}(m)p_1 \subseteq p_1\mathcal{P}(m+1)p_1$ become canonical embeddings. Taking the inductive limit will allow us to express $p_1\mathcal{P}p_1$ as a free scaled product of II_1–factors.

Fix m such that $m, m+1 \in I$. Let

$$\begin{aligned} K_m &= \bigcup_{i=1}^m J_i, \\ K_m^{(1)} &= \{k \in K_m \mid q_kp_{m+1} \neq 0\} \\ K_m^{(0)} &= K_m \backslash K_m^{(1)} \ . \end{aligned}$$

The hypothesis $Z(A) \cap B = \mathbf{C}$ ensures that $K_m^{(1)}$ is nonempty; in fact, $j(m+1) \in K_m^{(1)}$. Make an ordering $K_m^{(1)} = \{k(i) \mid i \in L\}$ where $L = \{0, 1, \ldots, \ell\}$ or $L = \{0\} \cup \mathbf{N}$, such that $k(0) = j(m+1)$. For every $k \in K_m^{(1)}$, let $v_k \in A$ be such that $v_k^*v_k$ is a minimal projection in A, $v_k^*v_k \le p_{[1,m]}$ and $v_kv_k^* \le p_{m+1}$. Note that $\mathrm{Tr}(v_k^*v_k) = \alpha(k)$; we assume without loss of generality that $v_{k(0)}^*v_{k(0)} = e_{m+1}$.

Let

$$\mathcal{R}(0) = W^*(\mathcal{P}(m) \cup \mathcal{N}(m+1) \cup \{v_{k(0)}\}) \ .$$

Then

$$p_{[1,m]}\mathcal{R}(0)p_{[1,m]} = W^*(\mathcal{P}(m) \cup v_{k(0)}^*\mathcal{N}(m+1)v_{k(0)}) \ .$$

We will show that the pair

$$e_{m+1}\mathcal{P}(m)e_{m+1}, \qquad v_{k(0)}^*\mathcal{N}(m+1)v_{k(0)} \tag{10}$$

[1]We are using the "free dimension" that was introduced in [1] — see [4] for further discusion. Since each p_mAp_m is finite dimensional, its free dimension is well defined, and we may use the term "free dimension" without inhibition.

is free with respect to $\gamma(m+1)^{-1}\mathrm{Tr}\restriction_{e_m \mathcal{P} e_m}$. Using C^{o} to denote $C \cap \ker E$ for any subalgebra C of $\mathcal{P}$, since e_m is a minimal projection in A, $\mathcal{P}(m)^{\mathrm{o}}$ is densely spanned by

$$\Theta := e_m\Big(\Lambda^{\mathrm{o}}((\mathcal{Q}p_{[1,m]})^{\mathrm{o}}, (p_{[1,m]}Ap_{[1,m]})^{\mathrm{o}})\backslash(p_{[1,m]}Ap_{[1,m]})^{\mathrm{o}}\Big)e_m$$

and $\big(v_{k(0)}^*\mathcal{N}(m+1)v_{k(0)}\big)^{\mathrm{o}}$ is densely spanned by

$$\Omega := v_{k(0)}^*\Big(\Lambda^{\mathrm{o}}((\mathcal{Q}p_{m+1})^{\mathrm{o}}, (p_{m+1}Ap_{m+1})^{\mathrm{o}})\backslash(p_{m+1}Ap_{m+1})^{\mathrm{o}}\Big)v_{k(0)}\ .$$

To show freeness of the pair (10), it will suffice to show $\Lambda^{\mathrm{o}}(\Theta,\Omega) \subseteq \ker E$. However, since

$$(p_{[1,m]}Ap_{[1,m]})v_{k(0)}^*(p_{m+1}Ap_{m+1}) \subseteq \ker E_B^A\ ,$$

given $x \in \Lambda^{\mathrm{o}}(\Theta,\Omega)$, by regrouping we see that x is equal to some $x' \in \Lambda^{\mathrm{o}}(\mathcal{Q}^{\mathrm{o}}, A^{\mathrm{o}})$, which is contained in $\ker E$ by freeness. This shows that the pair (10) is free. Therefore, there is an isomorphism

$$p_{[1,m]}\mathcal{R}(0)p_{[1,m]} \xrightarrow{\sim} \mathcal{P}(m) * \left[\tfrac{\gamma(m+1)}{\beta[1,m]}\ ,\ \mathcal{N}(m+1)_{\frac{\gamma(m+1)}{\beta(m+1)}}\right] \tag{11}$$

intertwining the inclusion $\mathcal{P}(m) \hookrightarrow p_{[1,m]}\mathcal{R}(0)p_{[1,m]}$ and the canonical embedding of $\mathcal{P}(m)$ in the RHS of (11). By [4], $\mathcal{R}(0)$ is a factor.

Define $\mathcal{R}(i)$ recursively for every $i \in L\backslash\{0\}$ by

$$\mathcal{R}(i) = W^*(\mathcal{R}(i-1) \cup \{v_{k(i)}\}).$$

We will show that every $\mathcal{R}(i)$ is a factor that can be written as a free subproduct involving $\mathcal{R}(i-1)$. Fixing i, suppose that $\mathcal{R}(i-1)$ is a factor. Let $w_i \in \mathcal{R}(i-1)$ be such that $w_i^*w_i = v_{k(i)}^*v_{k(i)}$ and $w_iw_i^* = v_{k(i)}v_{k(i)}^*$. Let $f = w_i^*w_i$ and $g = w_iw_i^*$; note that $f \leq p_{[1,m]}q_{k(i)}$ and $g \leq p_{m+1}q_{k(i)}$. We will show that in $f\mathcal{P}f$ and with respect to $\alpha(k(i))^{-1}\mathrm{Tr}\restriction_{f\mathcal{P}f}$, $w_i^*v_{k(i)}$ is a Haar unitary and is $*$–free from $f\mathcal{R}(i-1)f$. Let

$$C = W^*(p_{[1,m]}Ap_{[1,m]} \cup p_{m+1}Ap_{m+1} \cup \{v_{k(0)},\dots,v_{k(i-1)}\})\ .$$

Then $fCg = \{0\}$ and $\mathcal{R}(i-1)$ is densely spanned by $\Lambda^{\mathrm{o}}(C^{\mathrm{o}}, (p_{[1,m+1]}\mathcal{Q})^{\mathrm{o}})$. In order to show that $w_i^*v_{k(i)}$ is a Haar unitary and is $*$–free from $f\mathcal{R}(i-1)f$, it will suffice to show

$$\Lambda^{\mathrm{o}}\big((f\mathcal{R}(i-1)f)^{\mathrm{o}} \cup f\mathcal{R}(i-1)g \cup g\mathcal{R}(i-1)f \cup (g\mathcal{R}(i-1)g)^{\mathrm{o}}, \{v_{k(i)}, v_{k(i)}^*\}\big) \subseteq \ker E\ .$$

Because f and g are minimal projections in C and $fCg = \{0\}$,

$$\begin{aligned}(f\mathcal{R}(i-1)f)^{\mathrm{o}} \cup f\mathcal{R}(i-1)g \cup g\mathcal{R}(i-1)f \cup (g\mathcal{R}(i-1)g)^{\mathrm{o}} \subseteq \\ \subseteq \overline{\mathrm{span}}\left(\Lambda^{\mathrm{o}}(C^{\mathrm{o}}, (p_{[1,m+1]}\mathcal{Q})^{\mathrm{o}})\backslash C^{\mathrm{o}}\right)\ .\end{aligned}$$

But since $Cv_{k(i)}C \subseteq \ker E$, we have

$$\Lambda^{\mathrm{o}}\big(\Lambda^{\mathrm{o}}(C^{\mathrm{o}}, (p_{[1,m+1]}\mathcal{Q})^{\mathrm{o}})\backslash C^{\mathrm{o}}, \{v_{k(i)}, v_{k(i)}^*\}\big) \subseteq \ker E\ .$$

Thus $w_i^*v_{k(i)}$ is a Haar unitary and is $*$–free from $f\mathcal{R}(i-1)f$. This gives a $*$–isomorphism

$$p_{[1,m]}\mathcal{R}(i)p_{[1,m]} \xrightarrow{\sim} p_{[1,m]}\mathcal{R}(i-1)p_{[1,m]} * \left[\tfrac{\alpha(k(i))}{\beta[1,m]}\ ,\ L(\mathbf{Z})\right] \tag{12}$$

intertwining the inclusion $p_{[1,m]}\mathcal{R}(i-1)p_{[1,m]} \hookrightarrow p_{[1,m]}\mathcal{R}(i)p_{[1,m]}$ and the canonical embedding of $p_{[1,m]}\mathcal{R}(i-1)p_{[1,m]}$ into the RHS of (12).

Since $\mathcal{P}(m+1) = \overline{\bigcup_{i\in L}\mathcal{R}(i)}$, by composing the isomorphisms (11) and (12), we get an isomorphism

$$p_{[1,m]}\mathcal{P}(m+1)p_{[1,m]} \xrightarrow{\sim} \left(\mathcal{P}(m) * \left[\tfrac{\gamma(m+1)}{\beta[1,m]}, \mathcal{N}(m+1)_{\frac{\gamma(m+1)}{\beta(m+1)}}\right]\right) \mathop{*}_{i\in L} \left[\tfrac{\alpha(k(i))}{\beta[1,m]}, L(\mathbf{Z})\right] \tag{13}$$

intertwining the inclusion $\mathcal{P}(m) \hookrightarrow p_{[1,m]}\mathcal{P}(m+1)p_{[1,m]}$ and the canonical embedding of $\mathcal{P}(m)$ in the RHS of (13). Using that $\mathcal{N}(m+1) \cong Q(m+1) * L(\mathbf{F}_{s(m+1)})$ with

$$s(m+1) = 1 - \sum_{j\in K_m^{(1)}\cup J_m} \left(\tfrac{\alpha(j)}{\beta(m+1)}\right)^2,$$

by results from [4] there is an isomorphism

$$p_1\mathcal{P}(m+1)p_1 \xrightarrow{\sim} (p_1\mathcal{P}(m)p_1 * L(\mathbf{F}_{r(m+1)})) * \left[\tfrac{\gamma(m+1)}{\beta(1)}, Q(m+1)_{\frac{\gamma(m+1)}{\beta(m+1)}}\right], \tag{14}$$

where

$$r(m+1) = \frac{1}{\beta(1)^2}\left(\beta(m+1)^2 - \gamma(m+1)^2 - \sum_{j\in J_m}\alpha(j)^2\right),$$

intertwining the inclusion $p_1\mathcal{P}(m)p_1 \hookrightarrow p_1\mathcal{P}(m+1)p_1$ and the canonical embedding of $p_1\mathcal{P}(m)p_1$ in the RHS of (14). Taking the inductive limit as $m\to\infty$, we obtain the isomorphism (9). □

The next corollary follows directly from Proposition 3.1 and results of [6] and [4].

Corollary 3.2. *Let A, B, $\mathcal{Q} = \bigoplus_{i\in I} Q(i)$ and*

$$(\mathcal{P}, E) = (\mathcal{Q}, E_B^{\mathcal{Q}}) *_B (A, E_B^A)$$

with accompanying notation be as in Proposition 3.1.

(A) *If $I = \{1,\dots,n\}$ is finite then*

$$\mathcal{P} \cong Q(1)_{\frac{1}{\beta(1)}} * Q(2)_{\frac{1}{\beta(2)}} * \cdots * Q(n)_{\frac{1}{\beta(n)}} * L(\mathbf{F}_t)$$

where

$$\begin{aligned} t &= -n+1+\sum_{i=1}^n \beta(i)^2 - \sum_{j\in J}\alpha(j)^2 \\ &= -n+1+\operatorname{fdim}(A) - \operatorname{fdim}(B). \end{aligned}$$

(B) *If $I = \mathbf{N}$ is infinite and if one or more of the following conditions hold:*

(i) $Q(i) \cong Q(i) * L(\mathbf{F}_\infty)$ *for some $i \in \mathbf{N}$,*

(ii) $\sum_{i=1}^{\infty} \gamma(i)^2 = \infty$,

(iii) $r = \infty$,

then

$$p_1 \mathcal{P} p_1 \cong \mathop{*}_{i=1}^{\infty} Q(i)_{\frac{\beta(1)}{\beta(i)}} .$$

4. Subfactors

In this section, we will recall constructions and results of Popa and Shlyakhtenko [8] and we will use them together with the results from §2 and §3 to prove the existence of subfactors in free subproducts and free products, including those mentioned in the introduction.

Let Q be any II_1–factor. Let $M_{-1} \subset M_0$ be a II_1–subfactor of finite index λ, let $\mathcal{G} = \mathcal{G}_{M_{-1},M_0}$ be the associated standard λ–lattice and let Γ be the standard graph and Γ' the second standard graph of $M_{-1} \subset M_0$. Popa and Shlyakhtenko [8] construct semifinite von Neumann algebras

$$\begin{array}{ccc} \mathcal{M}_{-1} & \subset & \mathcal{M}_0 \\ \cup & & \cup \\ \mathcal{A}^0_{-1} & \subset & \mathcal{A}^0_0 \\ \cup & & \cup \\ \mathcal{A}^{-1}_{-1} & \subset & \mathcal{A}^{-1}_0 \end{array} \tag{15}$$

with a specified faithful semifinite trace Tr on $\mathcal{M}_0$ and they prove the following:

I. The trace preserving conditional expectations make (15) a diagram of commuting squares.
II. $\mathcal{M}_i \cong M_i \otimes B(\mathcal{H})$, $(i = -1, 0)$.
III. Each algebra $\mathcal{A}^k_i$ is a type I von Neumann algebra with atomic center, and Tr is finite on all minimal projections in $\mathcal{A}^k_i$.
IV. The graphs of the inclusions $\mathcal{A}^{-1}_{-1} \subset \mathcal{A}^{-1}_0$ and $\mathcal{A}^0_{-1} \subset \mathcal{A}^0_0$ are Γ and Γ', respectively.
V. The commuting square

$$\begin{array}{ccc} \mathcal{A}^0_{-1} & \subset & \mathcal{A}^0_0 \\ \cup & & \cup \\ \mathcal{A}^{-1}_{-1} & \subset & \mathcal{A}^{-1}_0 \end{array} \tag{16}$$

depends only on the standard λ–lattice $\mathcal{G}$, and the commuting square (16) is functorial in $\mathcal{G}$, (see [8, Thm. 2.9] for details).

VI. Consider the amalgamated free products of von Neumann algebras

$$\mathcal{P}_0 = \mathcal{A}^0_0 *_{\mathcal{A}^{-1}_0} (Q \otimes \mathcal{A}^{-1}_0) \tag{17}$$

$$\mathcal{P}_{-1} = \mathcal{A}^0_{-1} *_{\mathcal{A}^{-1}_{-1}} (Q \otimes \mathcal{A}^{-1}_{-1}) , \tag{18}$$

where the amalgamation is with respect to the trace preserving conditional expectations. Then $\mathcal{P}_{-1} \subset \mathcal{P}_0$ is an inclusion of type II_∞ factors whose standard invariant (i.e. the standard invariant of $p\mathcal{P}_{-1}p \subset p\mathcal{P}_0 p$ where $p \in \mathcal{P}_{-1}$ is any finite projection) is $\mathcal{G}$.

VII. Consider the amalgamated free products of von Neumann algebras

$$\hat{\mathcal{M}}_0 = \mathcal{M}_0 *_{\mathcal{A}_0^{-1}} (Q \otimes \mathcal{A}_0^{-1}) \tag{19}$$

$$\hat{\mathcal{M}}_{-1} = \mathcal{M}_{-1} *_{\mathcal{A}_{-1}^{-1}} (Q \otimes \mathcal{A}_{-1}^{-1}) \,, \tag{20}$$

where the amalgamation is with respect to the trace preserving conditional expectations. Then $\hat{\mathcal{M}}_{-1} \subset \hat{\mathcal{M}}_0$ is an inclusion of type II_∞ factors whose standard invariant (i.e. the standard invariant of $p\hat{\mathcal{M}}_{-1}p \subset p\hat{\mathcal{M}}_0 p$ where $p \in \hat{\mathcal{M}}_{-1}$ is any finite projection) is $\mathcal{G}$.

4.1. Consider the commuting square (16) determined by $\mathcal{G}$. Let $\mathcal{P}_0$ be as in (17) and let $q(0) \in \mathcal{A}_0^{-1}$ be the sum of a maximal family of mutually orthogonal and nonequivalent minimal projections of $\mathcal{A}_0^{-1}$. Letting $B = q(0)\mathcal{A}_0^{-1}q(0)$, we have that B is commutative and

$$q(0)\mathcal{P}_0 q(0) = q(0)\mathcal{A}_0^0 q(0) *_B (Q \otimes B) \,,$$

where the amalgamation is with respect to the trace preserving conditional expectations. Fix any minimal projection $q_1(0)$ in B. Let $m(0) \in \mathbf{N} \cup \{\infty\}$ be the number of minimal projections in B and let $(\beta(0,i))_{i=1}^{m(0)}$ be the values of Tr on the minimal projections in B, with $\beta(0,1) = \mathrm{Tr}(q_1(0))$. Proposition 3.1 shows that for cretain $r(0) \in [0,\infty]$ and $\gamma(0,i) \le \beta(0,i)$, we have

$$q_1(0)\mathcal{P}_0 q_1(0) \cong \begin{cases} Q * L(\mathbf{F}_{r(0)}) & \text{if } m(0) = 1 \\ \big(Q * L(\mathbf{F}_{r(0)})\big) \mathop{*}\limits_{i=2}^{m(0)} \left[\dfrac{\gamma(0,i)}{\beta(0,1)}, Q_{\frac{\gamma(0,i)}{\beta(0,i)}}\right] & \text{if } m(0) > 1 \,. \end{cases} \tag{21}$$

Analogously, if $\mathcal{P}_{-1}$ is as in (18) and if $q_1(-1)$ is any minimal projection in $\mathcal{A}_{-1}^{-1}$, if $m(-1)$ is the number of minimal projections in the center of $\mathcal{A}_{-1}^{-1}$, letting $(\beta(-1,i))_{i=1}^{m(-1)}$ be Tr applied to a maximal family of mutually inequivalent minimal projections in $\mathcal{A}_{-1}^{-1}$, with $\beta(-1,i) = \mathrm{Tr}(q_1(-1))$, we have

$$q_1(-1)\mathcal{P}_{-1} q_1(-1) \cong \begin{cases} Q * L(\mathbf{F}_{r(-1)}) & \text{if } m(-1) = 1 \\ \big(Q * L(\mathbf{F}_{r(-1)})\big) \mathop{*}\limits_{i=2}^{m(-1)} \left[\dfrac{\gamma(-1,i)}{\beta(-1,1)}, Q_{\frac{\gamma(-1,i)}{\beta(-1,i)}}\right] & \text{if } m(-1) > 1 \end{cases} \tag{22}$$

for certain $r(-1) \in [0,\infty]$ and $\gamma(-1,i) \le \beta(-1,i)$.

Theorem 4.2. *Let $\mathcal{G}$ be a standard lattice of finite depth. Then there are $m, n \in \mathbf{N}$, positive real numbers $s_1, \ldots, s_m$ and $t_1, \ldots, t_n$ and there are $a > 1 - m$ and $b > 1 - n$ such that for any II_1–factor Q, there is an inclusion $P_{-1} \subset P_0$ of II_1–factors whose standard invariant is $\mathcal{G}$ and such that*

$$P_0 = Q_{s_1} * \cdots * Q_{s_m} * L(\mathbf{F}_a)$$

$$P_{-1} = Q_{t_1} * \cdots * Q_{t_n} * L(\mathbf{F}_b) \,.$$

Proof. Since $\mathcal{G}$ is of finite depth, it follows from [8] (see point IV above) that the centers of $\mathcal{A}_0^{-1}$ and $\mathcal{A}_{-1}^{-1}$ are finite dimensional. Thus $m(0)$ and $m(1)$ in (21) and (22) are finite. By results in [4], from (21) and (22) we get

$$\begin{aligned} q_1(0)\mathcal{P}_0q_1(0) &\cong Q_{s'_1} * \cdots * Q_{s'_{m(0)}} * L(\mathbf{F}_{a'}) \\ q_1(-1)\mathcal{P}_{-1}q_1(-1) &\cong Q_{t'_1} * \cdots * Q_{t'_{m(1)}} * L(\mathbf{F}_{b'}) \end{aligned}$$

for some $a' > 1 - m(0)$ and $b' > 1 - m(-1)$, where $s'_i = \beta(0,i)/\beta(0,1)$ and $t'_i = \beta(-1,i)/\beta(-1,1)$. By results of [8] (see point VI above), the II_1–subfactor $q\mathcal{P}_{-1}q \subset q\mathcal{P}_0q$ has standard invariant $\mathcal{G}$, where q is any finite projection in $\mathcal{P}_{-1}$. But $q\mathcal{P}_0q$ is a rescaling of $q_1(0)\mathcal{P}_0q_1(0)$ by $\lambda := \mathrm{Tr}(q)/\mathrm{Tr}(q_1(0))$, and thus by results of [6] we have

$$q\mathcal{P}_0q \cong Q_{\lambda s'_1} * \cdots * Q_{\lambda s'_{m(0)}} * L(\mathbf{F}_a)$$

where $a = \lambda^{-2}a' + (m(0)-1)(\lambda^{-2}-1)$. Rescaling $q_1(-1)\mathcal{P}_{-1}q_1(-1)$ yields

$$q\mathcal{P}_{-1}q \cong Q_{t_1} * \cdots * Q_{t_{m(1)}} * L(\mathbf{F}_b)$$

for appropriate values of t_i and b. $\square$

Remark 4.3. In the proof of Theorem 4.2, λ can be chosen to arrange either $a = 0$, provided $m(0) > 1$, or $b = 0$, provided $m(1) > 1$.

Theorem 4.4. *Given a standard lattice $\mathcal{G}$, there are some positive real numbers $s_1, s_2, \ldots$ and $t_1, t_2, \ldots$ such that for any II_1–factor Q, there is an inclusion $P_{-1} \subset P_0$ of II_1–factors whose standard invariant is $\mathcal{G}$ and such that*

$$\begin{aligned} P_0 &\cong \mathop{*}_{i=1}^{\infty} Q_{s_i} \\ P_{-1} &\cong \mathop{*}_{i=1}^{\infty} Q_{t_i} \end{aligned} \tag{23}$$

Proof. If $\mathcal{G}$ is of finite depth, let $\widetilde{Q} = Q * Q * \cdots$ be a the free products of infinitely many copies of Q. By [5], a free product of infinitely many II_1–factors absorbs a free product with $L(\mathbf{F}_\infty)$, and the rescaling by λ of a free product of infinitely many factors is isomorphic to the free product of the same factors each rescaled by λ. Applying these facts and Theorem 4.2 to $\widetilde{Q}$, we find a subfactor $P_{-1} \subset P_0$ whose standard invariant is $\mathcal{G}$ and where

$$\begin{aligned} P_0 &\cong (Q_{s_1} * Q_{s_1} * \cdots) * \cdots * (Q_{s_n} * Q_{s_n} * \cdots) \\ P_{-1} &\cong (Q_{t_1} * Q_{t_1} * \cdots) * \cdots * (Q_{t_m} * Q_{t_m} * \cdots)\,. \end{aligned}$$

If $\mathcal{G}$ has infinite depth, let $\widetilde{Q} = Q * L(\mathbf{F}_\infty)$. Let $\mathcal{P}_0$ and $\mathcal{P}_{-1}$ be as in (17) and (18), but with Q replaced by $\widetilde{Q}$. Then we have the isomorphisms as in (21) and (22) but with Q replaced by $\widetilde{Q}$. Since $\mathcal{G}$ has infinite depth, $m(0)$ and $m(-1)$ are infinite. Since $\widetilde{Q}$ absorbs a free product with $L(\mathbf{F}_\infty)$, using [4, Theorem 3.11] and the above mentioned results about rescalings and absorbtion of $L(\mathbf{F}_\infty)$, we have the following isomorphisms:

$$\begin{aligned} q_1(0)\mathcal{P}_0 q_1(0) &\cong \widetilde{Q} * \left(\mathop{*}_{i=2}^{\infty} \widetilde{Q}_{\frac{\beta(0,i)}{\beta(0,1)}} \right) \cong \mathop{*}_{i=1}^{\infty} Q_{\frac{\beta(0,i)}{\beta(0,1)}} \\ q_1(-1)\mathcal{P}_{-1} q_1(-1) &\cong \widetilde{Q} * \left(\mathop{*}_{i=2}^{\infty} \widetilde{Q}_{\frac{\beta(-1,i)}{\beta(-1,1)}} \right) \cong \mathop{*}_{i=1}^{\infty} Q_{\frac{\beta(-1,i)}{\beta(-1,1)}} \end{aligned}$$

As in the proof of Theorem 4.2, we get a subfactor $P_{-1} = q\mathcal{P}_{-1}q \subset P_0 = q\mathcal{P}_0 q$ with isomorphisms (23) as desired. □

The following theorem is a special case of the previous one, and gives that II_1–factors other than $L(\mathbf{F}_\infty)$ have the universal property for subfactors.

Theorem 4.5. *Let Q be any II_1–factor whose fundamental group is equal to the positive real numbers, and let $P = Q * Q * \cdots$ be the free product of infinitely many copies of Q. Then for any standard lattice $\mathcal{G}$ there is a subfactor $N = N_{\mathcal{G}}$ of P whose standard invariant is $\mathcal{G}$; moreover N is isomorphic to P.*

Consider the commuting square (15) determined by the subfactor $M_{-1} \subset M_0$ and let $\hat{\mathcal{M}}_0$ be as in (19). With notation as in 4.1, we have

$$\begin{aligned} q(0)\hat{\mathcal{M}}_0 q(0) &= q(0)\mathcal{M}_0 q(0) *_B (Q \otimes B), \\ q(-1)\hat{\mathcal{M}}_{-1} q(-1) &= q(-1)\mathcal{M}_{-1} q(-1) *_B (Q \otimes B) \ , \end{aligned}$$

where the amalgamated free products are with respect to Tr–preserving conditional expectations. From Proposition 2.1, we have

$$q_1(0)\hat{\mathcal{M}}_0 q_1(0) \cong \begin{cases} (M_0)_{\beta(0,1)} & \text{if } m(0) = 1 \\ (M_0)_{\beta(0,1)} \mathop{*}_{i=2}^{m(0)} \left[\frac{\beta(0,i)}{\beta(0,1)}, Q\right] & \text{if } m(0) > 1 \end{cases}$$

and similarly

$$q_1(-1)\hat{\mathcal{M}}_0 q_1(-1) \cong \begin{cases} (M_{-1})_{\beta(-1,1)} & \text{if } m(-1) = 1 \\ (M_{-1})_{\beta(-1,1)} \mathop{*}_{i=2}^{m(-1)} \left[\frac{\beta(-1,i)}{\beta(-1,1)}, Q\right] & \text{if } m(-1) > 1 \ . \end{cases}$$

From these, and in a manner analogous to the proofs of Theorems 4.2, 4.4 and 4.5, one proves the following results.

Theorem 4.6. *Let Q be any II_1–factor. Then for any finite depth subfactor $M_{-1} \subset M_0$ of II_1–factors, there are $m, n \in \mathbf{N}$, positive real numbers $s_1, \ldots, s_m$ and $t_1, \ldots, t_n$ and there are $a > 1 - m$ and $b > 1 - n$, all depending only on the standard invariant of $M_{-1} \subset M_0$, and there is a II_1–subfactor $\hat{M}_{-1} \subset \hat{M}_0$ having the same standard invariant as $M_{-1} \subset M_0$ and where*

$$\begin{aligned} \hat{M}_0 &\cong M_0 * Q_{s_1} * \cdots * Q_{s_m} * L(\mathbf{F}_a) \\ \hat{M}_{-1} &= M_{-1} * Q_{t_1} * \cdots * Q_{t_n} * L(\mathbf{F}_b) \ . \end{aligned}$$

Theorem 4.7. *Let Q be any II_1–factor. Then for any finite depth subfactor $M_{-1} \subset M_0$ of II_1–factors, there are positive real numbers $s_1, \ldots, s_m$ and $t_1, \ldots, t_n$ depending only on the standard invariant of $M_{-1} \subset M_0$ and there is a II_1–subfactor $\hat{M}_{-1} \subset \hat{M}_0$ having the same standard invariant as $M_{-1} \subset M_0$ and where*

$$\hat{M}_0 \cong M_0 * \left(\mathop{*}_{i=1}^{\infty} Q_{s_i} \right)$$

$$\hat{M}_{-1} = M_{-1} * \left(\mathop{*}_{i=1}^{\infty} Q_{t_i} \right) .$$

Theorem 4.8. *Let Q be any II_1–factor having fundamental group equal to $\mathbf{R}_+^*$. Then for any subfactor $N \subset M$ of a II_1–factor, there is a subfactor $\hat{N} \subset \hat{M}$ whose standard lattice is equal to that of $N \subset M$ and such that*

$$\hat{N} \cong N * \left(\mathop{*}_{i=1}^{\infty} Q \right), \qquad \hat{M} \cong M * \left(\mathop{*}_{i=1}^{\infty} Q \right) .$$

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