GENERATING FUNCTIONS FOR MOMENTS OF THE QUASI–NILPOTENT DT–OPERATOR

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Abstract. We prove a recursion formula for generating functions of certain renormalizations of $*$–moments of the $\mathrm{DT}(\delta_0,1)$–operator T, involving an operation $\odot$ on formal power series and a transformation $\mathcal{E}$ that converts $\odot$ to usual multiplication. This recursion formula is used to prove that all of these generating functions are rational functions, and to find a few of them explicitly.

Introduction

In combinatorics, one of the most useful methods for studying a sequence is to give its generating functions. The two most common types of generating functions are ordinary generating functions $\sum f(n)x^n$, and exponential generating functions $\sum f(n)x^n/n!$. In this paper we study the $*$–moment generating functions — a family of multivariable power series F_n— of a particular operator T that arose in the theory of free probability. We prove that F_n's are all rational by applying a linear transformation between these two types of generating functions.

The central object of this paper is the collection of $*$–moments of a particular bounded operator T on Hilbert space, which was constructed in [1] and which is a candidate for an operator without a nontrivial hyperinvariant subspace. (A hyperinvariant subspace of an operator T on a Hilbert space $\mathcal{H}$ is a closed subspace $\mathcal{H}_0 \subseteq \mathcal{H}$ that is invariant under every operator S that commutes with T, i.e. $S(\mathcal{H}_0) \subseteq \mathcal{H}_0$. It is an open problem whether every operator on Hilbert space that is not a multiple of the identity has a nontrivial hyperinvariant subspace.) The von Neumann algebra generated by T has a unique normalized trace τ, and by the $*$–moments of T we mean the values

$$M(k_1,\ell_1,\ldots,k_n,\ell_n) = \tau\big((T^*)^{k_1}T^{\ell_1}\ldots(T^*)^{k_n}T^{\ell_n}\big),$$

with $n \in \mathbf{N}$, $k_1,\ldots,k_n,\ell_1,\ldots,\ell_n \in \mathbf{N}\cup\{0\}$. These $*$–moments determine a representation of T on a Hilbert space, (which can be shown to be bounded, see [1]), via the construction of Gelfand, Naimark and Segal, (cf. [2]) and hence they encode all essential properties of the operator. Our effort to understand the $*$–moments is part of an attempt better to understand the operator T.

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To be precise, T is a $\mathrm{DT}(\delta_0, 1)$–operator, in the class of DT–operators constructed in [1]. T can be realized as the limit in $*$-moments of strictly upper triangular random matrices having i.i.d. complex gaussian entries. In [1], it was proved that T is quasi–nilpotent, i.e. has spectrum $\{0\}$. It was shown that $M(k_1,\ell_1,\ldots,k_n,\ell_n)=0$ if $k_1+\cdots+k_n\neq\ell_1+\cdots+\ell_n$, and also a recursion formula for M was proved (cf. [1, Theorem 8.5]). It was conjectured in [1] that

$$\tau\big(((T^*)^kT^k)^n\big)=\frac{n^{nk}}{(nk+1)!}\qquad(k,n\in\mathbf{N}).\tag{1}$$

In this paper, we will consider the quantities

$$N(k_1,\ell_1,\ldots,k_n,\ell_n)=\begin{cases}0 & \text{if } k_1+\cdots+k_n \\ & \quad\neq\ell_1+\cdots+\ell_n \\ (m+1)!\,M(k_1,\ell_1,\ldots,k_n,\ell_n) & \text{if } m:=k_1+\cdots+k_n \\ & \quad=\ell_1+\cdots+\ell_n\end{cases}$$

(which must be nonnegative integers by [1, Lemma 2.1]), and their generating functions

$$F_n(z_1,w_1,\ldots,z_n,w_n)=\sum_{\substack{k_1,\ldots,k_n\geq0\\ \ell_1,\ldots,\ell_n\geq0}}N(k_1,\ell_1,\ldots,k_n,\ell_n)z_1^{k_1}w_1^{\ell_1}\ldots z_n^{k_n}w_n^{\ell_n}.\tag{2}$$

We will prove a recursion formula for the functions F_n, involving an operation $\odot$ on power series, which can be decsribed as multiplication with reweighting of homogeneous parts. In order to compute these generating functions, we will define a linear transformation of formal power series, called the $\mathcal{E}$–transform, which maps these power series to exponential generating functions of a new variable q, and converts the operation $\odot$ to the usual multiplication. This will allow us to prove that F_n is a rational function for all n, and to give an algorithm for computing F_n.

In §1, the operation $\odot$ and the transform $\mathcal{E}$ on formal power series are introduced, and several properties are proved. In §2, recursion formulas for $N(k_1,\ell_1,\ldots,k_n,\ell_n)$ and for the generating functions F_n are proved. The latter are shown to be rational and several examples are computed. In §3, we use the method of contour integration to verify the conjecture (1) in the case $n=2$.

1. Formal power series

In this section, fix variables $x_1,x_2,\ldots$ and fix $N\in\mathbf{N}$. (In the next section we will use only the case $N=2$.) Let $\Theta=\Theta_N$ be the algebra of all formal power series in $x_1,x_2,\ldots$ of the form

$$f(x_1,\ldots,x_n)=\sum_{k_1,\ldots,k_n\geq0}c_{k_1,\ldots,k_n}x_1^{k_1}\ldots x_n^{k_n}\tag{3}$$

for some $n\in\mathbf{N}$, and such that $c_{k_1,\ldots,k_n}=0$ whenever $k_1+\cdots+k_n$ is not divisible by N. Given a formal power series as in (3) and given $k\in\mathbf{N}\cup\{0\}$, write $f^{(k)}$ for its

Nk–homogeneous part:

$$f^{(k)}(x_1,\dots,x_n) = \sum_{k_1+\dots+k_n=Nk} c_{k_1,\dots,k_n} x_1^{k_1}\dots x_n^{k_n}. \tag{4}$$

Definition 1.1. Let $\odot = \odot_N$ be the binary operation on Θ given by

$$f \odot g = \sum_{k,\ell\geq 0} \binom{k+\ell}{k} f^{(k)} g^{(\ell)}.$$

Note that $\odot$ is bilinear, commutative and associative. If $f_i \in \Theta$ $(i \in \{1,\dots,p\})$ then

$$(f_1 \odot f_2 \odot \dots \odot f_p) = \sum_{k_1,\dots,k_p\geq 0} \binom{k_1+\dots+k_p}{k_1,k_2,\dots,k_p} \prod_{i=1}^{p} f_i^{(k_i)}.$$

Let $\widetilde{\Theta}$ be the set of all formal power series in variables $x_1, x_2, \dots$ and the additional variable q.

Definition 1.2. The $\mathcal{E}$–transform $\mathcal{E} = \mathcal{E}_N : \Theta \to \widetilde{\Theta}$ is the map given by

$$\mathcal{E}f = \sum_{k\geq 0} \frac{q^k}{k!} f^{(k)}.$$

Note that $\mathcal{E}$ is linear.

Proposition 1.3. (i) *If $f, g \in \Theta$ then $\mathcal{E}(f \odot g) = (\mathcal{E}f)(\mathcal{E}g)$.*

(ii) *If $f \in \Theta$ and if $a \in \Theta$ is a homogeneous polynomial of degree N then $\frac{d}{dq}\big(\mathcal{E}(af)\big) = a\,\mathcal{E}f$, i.e.*

$$\mathcal{E}(af)(x_1,x_2,\dots,q) = a\int_0^q \mathcal{E}f(x_1,x_2\dots,t)\,dt.$$

(iii) *If $r \in \mathbf{N}$ and if $u_1,\dots,u_r \in \Theta$ are distinct homogeneous polynomials of degree N, then*

$$\mathcal{E}\left(\frac{1}{(1-u_1)(1-u_2)\dots(1-u_r)}\right) = \sum_{i=1}^{r} \frac{u_i^{r-1} e^{qu_i}}{\prod_{j\neq i}(u_i-u_j)},$$

which in the case $r = 1$ means $\mathcal{E}(1/(1-u_1)) = e^{qu_1}$.

(iv) *Let $r \in \mathbf{N}$ and $k \in \{0,1,\dots,r-1\}$. If $u_1,\dots,u_r,a_1,\dots,a_k \in \Theta$ are homogeneous polynomials of degree N with $u_1,\dots,u_r$ distinct, then*

$$\mathcal{E}\left(\frac{a_1\dots a_k}{(1-u_1)(1-u_2)\dots(1-u_r)}\right) = a_1\dots a_k \sum_{i=1}^{r} \frac{u_i^{r-k-1} e^{qu_i}}{\prod_{j\neq i}(u_i-u_j).}$$

Proof. For (i),

$$\mathcal{E}(f\odot g) = \sum_{r\geq 0}\frac{q^r}{r!}\sum_{k+\ell=r}\binom{r}{k} f^{(k)} g^{(\ell)} = \sum_{k,\ell\geq 0}\left(\frac{q^k}{k!}f^{(k)}\right)\left(\frac{q^\ell}{\ell!}g^{(\ell)}\right) = (\mathcal{E}f)(\mathcal{E}g).$$

For (ii),

$$\frac{d}{dq}\mathcal{E}(af) = \frac{d}{dq}\mathcal{E}\left(\sum_{k=0}^{\infty} af^{(k)}\right) = a\frac{d}{dq}\sum_{k=0}^{\infty}\frac{q^{k+1}}{(k+1)!}f^{(k)} = a\sum_{k=0}^{\infty}\frac{q^k}{k!}f^{(k)} = a\,\mathcal{E}f.$$

For (iii),

$$\mathcal{E}\left(\frac{1}{(1-u_1)\dots(1-u_r)}\right) = \sum_{k=0}^{\infty}\frac{q^k}{k!}\sum_{k_1+\dots+k_p=k} u_1^{k_1}\dots u_r^{k_r}.$$

On the other hand,

$$\sum_{i=1}^{r}\frac{u_i^{r-1}e^{qu_i}}{\prod_{j\neq i}(u_i-u_j)} = \sum_{k=0}^{\infty}\frac{q^k}{k!}\sum_{i=1}^{r}\frac{u_i^{k+r-1}}{\prod_{j\neq i}(u_i-u_j).}$$

Hence it will suffice to show

$$\sum_{k_1+\dots+k_r=k} u_1^{k_1}\dots u_r^{k_r} = \sum_{i=1}^{r}\frac{u_i^{k+r-1}}{\prod_{j\neq i}(u_i-u_j)} \tag{5}$$

for all $k \geq 0$. We will proceed by induction on r, treating $u_1, \dots, u_r$ in (5) as any n distinct elements in a field. The case $r = 1$ is clear. If $r = 2$, then we have

$$\begin{aligned}\sum_{k_1+k_2=k} u_1^{k_1}u_2^{k_2} &= u_2^k\sum_{k_1=0}^{k}\left(\frac{u_1}{u_2}\right)^{k_1} = u_2^k\frac{(u_1/u_2)^{k+1}-1}{(u_1/u_2)-1}\\ &= \frac{u_1^{k+1}-u_2^{k+1}}{u_1-u_2} = \frac{u_1^{k+1}}{u_1-u_2}+\frac{u_2^{k+1}}{u_2-u_1},\end{aligned}$$

as required. Assume $r \geq 3$. Using the induction hypothesis, we have

$$\begin{aligned}\sum_{k_1+\dots+k_r=k} u_1^{k_1}\dots u_r^{k_r} &= u_1^k\sum_{k'=0}^{k}\sum_{k_2+\dots+k_r=k'}\left(\frac{u_2}{u_1}\right)^{k_2}\left(\frac{u_3}{u_1}\right)^{k_3}\dots\left(\frac{u_r}{u_1}\right)^{k_r}\\ &= u_1^k\sum_{k'=0}^{k}\sum_{i=2}^{r}\frac{(u_i/u_1)^{k'+r-2}}{\prod_{j\neq 1,i}\big((u_i/u_1)-(u_j/u_1)\big)}\\ &= u_1^k\sum_{i=2}^{r}\frac{\big((u_i/u_1)^{k+1}-1\big)(u_i/u_1)^{r-2}}{\big((u_i/u_1)-1\big)\prod_{j\neq 1,i}\big((u_i/u_1)-(u_j/u_1)\big)}\\ &= \sum_{i=2}^{r}\frac{u_i^{k+r-1}}{\prod_{j\neq i}(u_i-u_j)} - u_1^{k+1}\sum_{i=2}^{r}\frac{u_i^{r-2}}{\prod_{j\neq i}(u_i-u_j).}\end{aligned}$$

We must show

$$-\sum_{i=2}^{r}\frac{u_i^{r-2}}{\prod_{j\neq i}(u_i-u_j)} = \frac{u_1^{r-2}}{\prod_{j\neq 1}(u_1-u_j)}$$

or, equivalently,

$$\sum_{i=1}^{r} u_i^{r-2}\prod_{\substack{1\leq k<\ell\leq r\\ k,\ell\neq i}}(u_\ell-u_k) = 0. \tag{6}$$

Using the Vandermonde determinant

$$\det\begin{pmatrix} 1 & 1 & \dots & 1 \\ u_1 & u_2 & \dots & u_{r-1} \\ \vdots & \vdots & \vdots & \vdots \\ u_1^{r-2} & u_2^{r-2} & \dots & u_{r-1}^{r-2} \end{pmatrix} = \prod_{1\le k<\ell\le r-1} (u_\ell - u_k)$$

and expanding the determinant

$$\det\begin{pmatrix} u_1^{r-2} & u_2^{r-2} & \dots & u_r^{r-2} \\ 1 & 1 & \dots & 1 \\ u_1 & u_2 & \dots & u_r \\ \vdots & \vdots & \vdots & \vdots \\ u_1^{r-2} & u_2^{r-2} & \dots & u_r^{r-2} \end{pmatrix}$$

along the first row, we find the latter is equal to the LHS of (6), but is also clearly equal to zero.

To prove (iv), use (iii) and iterate (ii). $\square$

Remark 1.4. If one knows that $h \in \widetilde{\Theta}$ is in the image of $\mathcal{E}$, then to find $\mathcal{E}^{-1}(h)$, simply replace every occurrence of q^k by $k!$. This can sometimes be used to compute $f \odot g$.

We now turn to some cases of $f \odot g$ which will be used in §2.

Proposition 1.5. *If $m \in \mathbf{N}$, $k \in \{0, 1, \dots, m-1\}$ and if $a_1, \dots, a_k, u_1, \dots, u_m, v \in \Theta$ are homogeneous polynomials of degree N such that $u_1, \dots, u_m$ are distinct, then*

$$\frac{a_1 \dots a_k}{(1-u_1)\dots(1-u_m)} \odot \frac{1}{(1-v)} = \frac{a_1 \dots a_k (1-v)^{m-k-1}}{(1-u_1-v)(1-u_2-v)\dots(1-u_m-v)}.$$

Proof. Using Proposition 1.3, we have

$$\begin{aligned}
&\frac{a_1 \dots a_k}{(1-u_1)\dots(1-u_m)} \odot \frac{1}{(1-v)} \\
&\quad \overset{\mathcal{E}}{\mapsto} \quad \left(a_1 \dots a_k \sum_{i=1}^m \frac{u_i^{m-k-1} e^{qu_i}}{\prod_{j\neq i}(u_i - u_j)} \right) e^{qv} \\
&\quad = \quad a_1 \dots a_k \sum_{i=1}^m \frac{\big((u_i+v)-v\big)^{m-k-1} e^{q(u_i+v)}}{\prod_{j\neq i}(u_i-u_j)} \\
&\quad = \quad a_1 \dots a_k \sum_{p=0}^{m-k-1} (-1)^p \binom{m-k-1}{p} v^p \sum_{i=1}^m \frac{(u_i+v)^{m-k-p-1} e^{q(u_i+v)}}{\prod_{j\neq i}\big((u_i+v)-(u_j+v)\big)} \\
&\quad \overset{\mathcal{E}^{-1}}{\mapsto} \quad \sum_{p=0}^{m-k-1} (-1)^p \binom{m-k-1}{p} v^p \frac{a_1 \dots a_k}{\prod_i (1-u_i-v)} \\
&\quad = \quad \frac{a_1 \dots a_k (1-v)^{m-k-1}}{(1-u_1-v)(1-u_2-v)\dots(1-u_m-v)}.
\end{aligned}$$

$\square$

Proposition 1.6. *Let $m, n \in \mathbf{N}$ and let $k \in \{0, 1, \dots, m-1\}$, $\ell \in \{0, 1, \dots, n-1\}$. Let $u_1, \dots, u_m, a_1, \dots, a_k, v_1 \dots, v_n, b_1, \dots, b_\ell \in \Theta$ be homogeneous polynomials of degree N such that $(u_i + v_j)_{1 \le i \le m, 1 \le j \le n}$ is a family of mn distinct polynomials. Then*

$$\frac{a_1 \dots a_k}{(1-u_1)\dots(1-u_m)} \odot \frac{b_1 \dots b_\ell}{(1-v_1)\dots(1-v_n)} = \frac{a_1 \dots a_k \, b_1 \dots b_\ell \, P_{m,n}^{k,\ell}}{\prod_{i,j}(1-u_i-v_j),}$$

where $P_{m,n}^{k,\ell} = P_{m,n}^{k,\ell}(u_1, \dots, u_m, v_1, \dots, v_n)$ is a polynomial in the variables $(u_i)_1^m$ and $(v_j)_1^n$, having degree bounded above by $mn - k - \ell - 1$.

Proof. Using Proposition 1.3 and Remark 1.4, we have

$$\begin{aligned}
&\frac{a_1 \dots a_k}{\prod_i (1-u_1)} \odot \frac{b_1 \dots b_\ell}{\prod_j (1-v_j)} \\
&\quad \overset{\mathcal{E}}{\mapsto} \quad a_1 \dots a_k \, b_1 \dots b_\ell \left(\sum_{i=1}^{m} \frac{u_i^{m-k-1} e^{qu_i}}{\prod_{i' \neq i}(u_i - u_{i'})} \right) \left(\sum_{j=1}^{n} \frac{v_j^{n-\ell-1} e^{qv_i}}{\prod_{j' \neq j}(v_j - v_{j'})} \right) \\
&\quad \overset{\mathcal{E}^{-1}}{\mapsto} \quad a_1 \dots a_k \, b_1 \dots b_\ell \sum_{i,j} \frac{u_i^{m-k-1} v_j^{n-\ell-1}}{(1-u_i-v_j)\big(\prod_{i' \neq i}(u_i - u_{i'})\big)\big(\prod_{j' \neq j}(v_j - v_{j'})\big)} \\
&\quad = \quad \left(\frac{a_1 \dots a_k \, b_1 \dots b_\ell}{\prod_{i,j}(1-u_i-v_j)} \right) \frac{Q}{\big(\prod_{p<q}(u_p-u_q)\big)\big(\prod_{r<s}(v_r-v_s)\big)},
\end{aligned}$$

where $Q = Q(u_1, \dots, u_m, v_1, \dots, v_n)$ is the polynomial

$$\begin{aligned}
Q = \sum_{i,j} (-1)^{i+j} u_i^{m-k-1} v_j^{n-\ell-1} &\left(\prod_{(i',j') \neq (i,j)} (1 - u_{i'} - v_{j'}) \right) \cdot \\
&\cdot \left(\prod_{p<q,\, p \neq i,\, q \neq i} (u_p - u_q) \right) \left(\prod_{r<s,\, r \neq j,\, s \neq j} (v_r - v_s) \right).
\end{aligned} \tag{7}$$

We will show that for every $1 \le a < b \le m$, $u_a - u_b$ divides Q. It will suffice to show that substituting $u_a = x$ and $u_b = x$ into the RHS of equation (7) yields zero. This substitution makes all terms of the sum zero except possibly when $i \in \{a, b\}$. These remaining terms are equal to the common factor

$$\begin{aligned}
x^{m-k-1} &\left(\prod_{\substack{i' \notin \{a,b\} \\ 1 \le j' \le n}} (1 - u_{i'} - v_{j'}) \right) \left(\prod_{\substack{p<q \\ p,q \notin \{a,b\}}} (u_p - u_q) \right) \cdot \\
&\qquad \cdot \sum_{j=1}^{n} (-1)^j v_j^{n-\ell-1} (1 - x - v_j) \prod_{j' \neq j} (1 - x - v_{j'})^2
\end{aligned}$$

times the quantity

$$(-1)^a \prod_{\substack{p<q \\ b \in \{p,q\} \\ a \notin \{p,q\}}} (u_p - u_q) + (-1)^b \prod_{\substack{p<q \\ a \in \{p,q\} \\ b \notin \{p,q\}}} (u_p - u_q). \tag{8}$$

However, since $a < b$, we see that the quantity (8) is equal to

$$(-1)^a\Big(\prod_{p<b,\,p\neq a}(u_p - x)\Big)\Big(\prod_{q>b}(x-u_q)\Big) + (-1)^b\Big(\prod_{p<a}(u_p - x)\Big)\Big(\prod_{q>a,\,q\neq b}(x-u_q)\Big)$$
$$=\big((-1)^a(-1)^{b-a-1} + (-1)^b\big)\Big(\prod_{p<a}(u_p - x)\Big)\Big(\prod_{q>a,\,q\neq b}(x-u_q)\Big) = 0.$$

By symmetry, we have that $v_c - v_d$ divides the polynomial Q, for every $1 \le c < d \le n$. Thus

$$P^{k,\ell}_{m,n} = \frac{Q}{\big(\prod_{p<q}(u_p - u_q)\big)\big(\prod_{r<s}(v_r - v_s)\big)} \tag{9}$$

is a polynomial in $u_1, \dots, u_m, v_1, \dots, v_n$. The upper bound on its degree is easily computed from (9) and the expression (7). □

Proposition 1.5 shows $P^{k,0}_{m,1} = (1 - v_1)^{m-k-1}$ and $P^{0,\ell}_{1,n} = (1 - u_1)^{n-\ell-1}$. Here are some other examples.

Examples 1.7.

$$\begin{aligned}
P^{0,1}_{2,2} &= 1 - u_1u_2 - v_1 - v_2 + v_1v_2\\
P^{1,1}_{2,2} &= 2 - u_1 - u_2 - v_1 - v_2\\
P^{1,1}_{2,3} &= 2 - 3u_1 + u_1^2 - 3u_2 + 4u_1u_2 - u_1^2u_2 + u_2^2 - u_1u_2^2 - v_1 + u_1v_1 + u_2v_1 - u_1u_2v_1\\
&\quad - v_2 + u_1v_2 + u_2v_2 - u_1u_2v_2 - v_3 + u_1v_3 + u_2v_3 - u_1u_2v_3 + v_1v_2v_3\\
P^{1,2}_{2,3} &= 3 - 3u_1 + u_1^2 - 3u_2 + u_1u_2 + u_2^2 - 2v_1 + u_1v_1 + u_2v_1 - 2v_2 + u_1v_2 + u_2v_2\\
&\quad + v_1v_2 - 2v_3u_1v_3 + u_2v_3 + v_1v_3 + v_2v_3.
\end{aligned}$$

2. Generating functions

We will now define $N(k_1, \ell_1, \dots, k_n, \ell_n)$, but extending the definition found in the introduction to allow k_j, ℓ_j to be -1. The reason for this extension is to allow a recursion formula to be proved for $N(k_1, \ell_1, \dots, k_n, \ell_n)$ that holds for all $k_j, \ell_j \in \mathbf{N} \cup \{0\}$.

Definition 2.1. Let $n \in \mathbf{N}$ and $k_1, \dots, k_n, \ell_1, \dots, \ell_n \in \mathbf{N} \cup \{0, -1\}$. If $k_1 + \cdots + k_n \neq \ell_1 + \cdots + \ell_n$ then set $N(k_1, \ell_1, \dots, k_n, \ell_n) = 0$. Suppose now $k_1 + \cdots + k_n = \ell_1 + \dots, \ell_n$, and let $m = k_1 + \cdots + k_n$. If $k_j \ge 0$ and $\ell_j \ge 0$ for all j, then let

$$N(k_1, \ell_1, \dots, k_n, \ell_n) = (m+1)!\tau((T^*)^{k_1}T^{\ell_1} \dots (T^*)^{k_n}T^{\ell_n}).$$

In the case $n = 1, k_1 = \ell_1 = -1$ let $N(-1, -1) = 1$. In all other cases, (where $n \ge 2$ and $k_j = -1$ or $\ell_j = -1$ for some j), let $N(k_1, \ell_1, \dots, k_n, \ell_n) = 0$.

The following is immediate from [1, Proposition 8.4].

Proposition 2.2. (i) $N(k_1, \ell_1, \dots, k_n, \ell_n) = N(\ell_1, k_2, \ell_2, \dots, k_n, \ell_n, k_1)$.
(ii) $N(k_1, \ell_1, \dots, k_n, \ell_n) = N(\ell_n, k_n, \ell_{n-1}, k_{n-1}, \dots, \ell_1, k_1)$.

(iii) *If* $n \geq 2$, *if* $k_1, \ldots, k_n, \ell_1, \ldots, \ell_n \geq 0$ *and if* $k_1 = 0$, *then*

$$N(k_1, \ell_1, \ldots, k_n, \ell_n) = N(k_2, \ell_2, \ldots, k_{n-1}, \ell_{n-1}, k_n, \ell_n + \ell_1).$$

Theorem 2.3. *If* $n \in \mathbf{N}$ *and* $k_1, \ldots, k_n, \ell_1, \ldots, \ell_n \in \mathbf{N} \cup \{0\}$, *then*

$$\begin{aligned} N(k_1, \ell_1, \ldots, k_n, \ell_n) = \sum_{r=1}^{n} \sum_{1 \leq j(1) < \cdots < j(r) \leq n} &\mathrm{Nom}(\ell_1, \ldots, \ell_n; j(1), \ldots, j(r)) \\ &\Big(N(k_1, \ell_1, \ldots, k_{j(1)-1}, \ell_{j(1)-1}, k_{j(1)} - 1, \ell_{j(r)} - 1, k_{j(r)+1}\ell_{j(r)+1}, \ldots, k_n, \ell_n) \\ &\prod_{i=1}^{r-1} N(\ell_{j(i)} - 1, k_{j(i)+1}, \ell_{j(i)+1}, \ldots, k_{j(i+1)-1}, \ell_{j(i+1)-1}, k_{j(i+1)} - 1) \Big), \end{aligned} \tag{10}$$

where $\mathrm{Nom}(\ell_1, \ldots, \ell_n; j(1), \ldots, j(r))$ *is the multinomial coefficient*

$$\binom{\ell_1 + \cdots + \ell_n}{\ell_1 + \cdots + \ell_{j(1)-1} + \ell_{j(r)} + \cdots + \ell_n,\ \ell_{j(1)} + \cdots + \ell_{j(2)-1},\ \ldots,\ \ell_{j(r-1)} + \cdots + \ell_{j(r)-1}}.$$

Proof. If $k_1 + \cdots + k_n \neq \ell_1 + \cdots + \ell_n$ then both sides of the equality are zero. So suppose $k_1 + \cdots + k_n = \ell_1 + \cdots + \ell_n$. If $k_1, \ldots, k_n, \ell_1, \ldots, \ell_n \geq 1$ then the equality follows directly from the recursion formula [1, Theorem 8.5]. To verify the equality (10) in general, we will proceed by induction on n. Let $N'(k_1, \ell_1, \ldots, k_n, \ell_n)$ be the RHS of (10). In the case $n = 1$, we need only check that $N'(0, 0) = 1$, which is clear. Assume $n \geq 2$. Writing $k_1, \ell_1, \ldots, k_n, \ell_n$ on a circle, it is clear that both sides of equation (10) are invariant under the following permutation of order n:

$$k_1 \to k_2 \to \cdots \to k_n \to k_1, \qquad \ell_1 \to \ell_2 \to \cdots \to \ell_n \to \ell_1.$$

Therefore, it will be enough to consider three cases: (i) $k_1 = 0$, $\ell_n \neq 0$, (ii) $k_1 \neq 0$, $\ell_n = 0$, (iii) $k_1 = \ell_n = 0$.

In case (i), all terms in the sum (10) having $j(1) = 1$ are zero. If $j(1) \geq 2$, then using Proposition 2.2 we have

$$\begin{aligned} &N(k_1, \ell_1, \ldots, k_{j(1)-1}, \ell_{j(1)-1}, k_{j(1)} - 1, \ell_{j(r)} - 1, k_{j(r)+1}\ell_{j(r)+1}, \ldots, k_n, \ell_n) = \\ &\quad = \begin{cases} N(k_2, \ell_2, \ldots, k_{j(1)-1}, \ell_{j(1)-1}, k_{j(1)} - 1, \\ \qquad \ell_{j(r)} - 1, k_{j(r)+1}\ell_{j(r)+1}, \ldots, k_{n-1}, \ell_{n-1}, k_n, \ell_n + \ell_1) & \text{if } j(r) < n \\ N(k_2, \ell_2, \ldots, k_{j(1)-1}, \ell_{j(1)-1}, k_{j(1)} - 1, \ell_{j(r)} + \ell_1 - 1) & \text{if } j(r) = n. \end{cases} \end{aligned}$$

Therefore,

$$\begin{aligned} N'(k_1, \ell_1, \ldots, k_n, \ell_n) &= N'(k_2, \ell_2, \ldots, k_{n-1}, \ell_{n-1}, k_n, \ell_n + \ell_1) \\ &= N(k_2, \ell_2, \ldots, k_{n-1}, \ell_{n-1}, k_n, \ell_n + \ell_1) \\ &= N(0, \ell_1, k_2, \ell_2, \ldots, k_n, \ell_n), \end{aligned}$$

where the second equality is by the induction hypothesis and the third is from Proposition 2.2.

In case (ii), ($k_1 \neq 0$, $\ell_n = 0$), one similarly shows

$$\begin{aligned} N'(k_1, \ell_1, \ldots, k_n, \ell_n) &= N'(k_1 + k_n, \ell_1, k_2, \ell_2, \ldots, k_{n-1}, \ell_{n-1}) \\ &= N(k_1 + k_n, \ell_1, k_2, \ell_2, \ldots, k_{n-1}, \ell_{n-1}) \\ &= N(k_1, \ell_1, , \ldots k_{n-1}, \ldots, k_{n-1}, \ell_{n-1}, k_n, 0). \end{aligned}$$

In case (iii), ($k_1 = \ell_n = 0$), given $1 \leq r \leq n$ and $1 < j(1) < \cdots < j(r) \leq n$, let $t(j(1), \ldots, j(r))$ be the corresponding term in the sum (10) for $N'(k_1, \ell_1, \ldots, k_n, \ell_n)$, and, if $j(r) < n$, let $\tilde{t}(j(1), \ldots, j(r))$ be the term corresponding to $1 \leq j(1) < \cdots < j(r) \leq n-1$ in the analogous sum for $N'(k_2, \ell_2, \ldots, k_{n-1}, \ell_{n-1}, k_n, \ell_1)$. If $j(1) > 1$ and $j(r) < n$, then using Proposition 2.2 we have

$$N(k_1, \ell_1, \ldots, k_{j(1)-1}, \ell_{j(1)-1}, k_{j(1)} - 1, \ell_{j(r)} - 1, k_{j(r)+1}, \ell_{j(r)+1}, \ldots, k_n, \ell_n) =$$

$$N(k_2, \ell_2, \ldots, k_{j(1)-1}, \ell_{j(1)-1}, k_{j(r)} - 1, \ell_{j(r)} - 1, k_{j(r)+1}, \ell_{j(r)+1}, \ldots, k_{n-1}, \ell_{n-1}, k_n, \ell_1),$$

which, taken in the sum (10), shows

$$t(j(1), \ldots, j(r)) = \tilde{t}(j(1) - 1, \ldots, j(r) - 1).$$

If $j(1) = 1$ and $j(r) < n$ or if $j(1) > 1$ and $j(r) = n$, then $t(j(1), \ldots, j(r)) = 0$. If $j(1) = 1$ and $j(r) = n$ then since

$$\begin{aligned} N(-1, -1)N(\ell_1 - 1,& k_2, \ell_2, \ldots, k_{j(2)-1}, \ell_{j(2)-1}, k_{j(2)} - 1) \\ &= N(k_2, \ell_2, \ldots, k_{j(2)-1}, \ell_{j(2)-1}, k_{j(2)} - 1, \ell_1 - 1), \end{aligned}$$

we have

$$t(j(1), \ldots, j(r) = \tilde{t}(j(2) - 1, \ldots, j(r-1) - 1, n - 1).$$

Therefore,

$$\begin{aligned} N'(k_1, \ell_1, \ldots, k_n, \ell_n) &= N'(k_2, \ell_2, \ldots, k_{n-1}, \ell_{n-1}, k_n, \ell_1) \\ &= N(k_2, \ell_2, \ldots, k_{n-1}, \ell_{n-1}, k_n, \ell_1) \\ &= N(0, \ell_1, k_2, \ell_2, \ldots, k_{n-1}, \ell_{n-1}, k_n, 0). \end{aligned}$$

□

Recall the generating functions F_n, defined in the introduction. The following theorem is a recursion formula for these, using the operation $\odot = \odot_2$ introduced in §1. Here, as promised, we are using objects defined in §1 with $N = 2$.

Theorem 2.4. *If $n \in \mathbf{N}$ and $n \geq 2$, then*

$$(1 - z_1 w_1 - \cdots - z_n w_n) F_n(z_1, w_1, \ldots, z_n, w_n) = \sum_{r=2}^{n} \sum_{1 \leq j(1) < \cdots < j(r) \leq n} \tag{11}$$

$$\big(z_{j(1)} w_{j(r)} \widetilde{F}_{j(1)+n-j(r)}(z_1, w_1, \ldots, z_{j(1)-1}, w_{j(1)-1}, z_{j(1)}, w_{j(r)}, z_{j(r)+1}, w_{j(r)+1}, \ldots, z_n, w_n)\big)$$

$$\odot \left(\bigodot_{i=1}^{r-1} \big(w_{j(i)} z_{j(i+1)} \widetilde{F}_{j(i+1)-j(i)}(w_{j(i)}, z_{j(i)+1}, w_{j(i)+1}, \ldots, z_{j(i+1)-1}, w_{j(i+1)-1}, z_{j(i+1)})\big) \right),$$

where

$$\widetilde{F}_j(x_1,y_1,\dots,x_j,y_j) := \begin{cases} F_j(x_1,y_1,\dots,x_j,y_j) & \textit{if } j \geq 2 \\ F_1(x_1,y_1)/(x_1y_1) & \textit{if } j = 1. \end{cases}$$

Proof. We will use the recursion formula (10) for the coefficients in the definition (2) of F_n. Since $n \geq 2$, if $j \in \{1,\dots,n\}$, then

$$\begin{aligned}\sum_{\substack{k_1,\dots,k_n\geq 0\\ \ell_1,\dots,\ell_n\geq 0}} & N(k_1,\ell_1,\dots,k_{j-1},\ell_{j-1},k_j-1,\ell_j-1,k_{j+1},\ell_{j+1},\dots,k_n,\ell_n) z_1^{k_1}w_1^{\ell_1}\dots z_n^{k_n}w_n^{\ell_n} \\ &= \quad z_jw_jF_n(z_1,w_1,\dots,z_n,w_n).\end{aligned}$$

This takes care of all terms in the sum (10) with $r=1$, and these together with the LHS of equation (10) give the LHS of formula (11). Let $2 \leq r \leq n$ and $1 \leq j(1) < \dots < j(r) \leq n$. Then

$$\begin{aligned}\sum_{\substack{k_1,\dots,k_n\geq 0\\ \ell_1,\dots,\ell_n\geq 0}} & \mathrm{Nom}(\ell_1,\dots,\ell_n;j(1),\dots,j(r))\cdot \\ & \cdot\Big(N(k_1,\ell_1,\dots,k_{j(1)-1},\ell_{j(1)-1},k_{j(1)}-1,\ell_{j(r)}-1,k_{j(r)+1}\ell_{j(r)+1},\dots,k_n,\ell_n) \\ & \quad \prod_{i=1}^{r-1} N(\ell_{j(i)}-1,k_{j(i)+1},\ell_{j(i)+1},\dots,k_{j(i+1)-1},\ell_{j(i+1)-1},k_{j(i+1)}-1)\Big)\cdot \\ & \quad \cdot z_1^{k_1}w_1^{\ell_1}\dots z_n^{k_n}w_n^{\ell_n}\end{aligned}$$

$$= \sum_{p_1,\dots,p_r\geq 0}\binom{p_1+\dots+p_r}{p_1,\ \dots,\ p_r}$$

$$\Bigg(\Bigg(\sum_{\substack{k_1+\ell_1+\dots+k_{j(1)-1}+\ell_{j(1)-1}+k_{j(1)}+\\ +\ell_{j(r)}+k_{j(r)+1}+\ell_{j(r)+1}+\dots+k_n+\ell_n=p_1}} \tag{12}$$

$$N(k_1,\ell_1,\dots,k_{j(1)-1},\ell_{j(1)-1},k_{j(1)}-1,\ell_{j(r)}-1,k_{j(r)+1},\ell_{j(r)+1},\dots,k_n,\ell_n)\cdot$$

$$\cdot z_1^{k_1}w_1^{\ell_1}\dots z_{j(1)-1}^{k_{j(1)-1}}w_{j(1)-1}^{\ell_{j(1)-1}}z_{j(1)}^{k_{j(1)}}w_{j(r)}^{\ell_{j(r)}}z_{j(r)+1}^{k_{j(r)+1}}w_{j(r)+1}^{\ell_{j(r)+1}}\dots z_n^{k_n}w_n^{\ell_n}\Bigg)\cdot$$

$$\cdot\Bigg(\prod_{i=1}^{r-1}\sum_{\substack{\ell_{j(i)}+k_{j(i)+1}+\ell_{k(i)+1}+\dots+\\ +k_{j(i+1)-1}+\ell_{j(i+1)-1}+k_{j(i+1)}=p_{i+1}}} \tag{13}$$

$$N(\ell_{j(i)}-1,k_{j(i)+1},\ell_{j(i)+1},\dots,k_{j(i+1)-1},\ell_{j(i+1)-1},k_{j(i+1)}-1)\cdot$$

$$\cdot w_{j(i)}^{\ell_{j(i)}}z_{j(i)+1}^{k_{j(i)+1}}w_{j(i)+1}^{\ell_{j(i)+1}}\dots z_{j(i+1)-1}^{k_{j(i+1)-1}}w_{j(i+1)-1}^{\ell_{j(i+1)-1}}z_{j(i+1)}^{k_{j(i+1)}}\Bigg)\quad\Bigg)$$

Consider a summation beginning at (13) for some fixed i. If $j(i+1)-j(i)\geq 2$ then all terms with either $\ell_{j(i)}=0$ or $k_{j(i+1)}=0$ vanish and the summation is equal to the $2p_{i+1}$–homogeneous part of

$$w_{j(i)}z_{j(i+1)}F_{j(i+1)-j(i)}(w_{j(i)},z_{j(i)+1},w_{j(i)+1},\ldots,z_{j(i+1)-1},w_{j(i+1)-1},z_{j(i+1)}).$$

On the other hand, if $j(i+1)-j(i)=1$, then, since $N(k-1,\ell-1)=N(k,\ell)$ for all $k,\ell\in\mathbf{N}\cup\{0\}$, this summation is equal to the $2p_{i+1}$–homogenous part of $F_1(w_{j(i)},z_{j(i+1)})$. In either case, the summation is equal to the $2p_{i+1}$–homogeneous part of

$$w_{j(i)}z_{j(i+1)}\widetilde{F}_{j(i+1)-j(i)}(w_{j(i)},z_{j(i)+1},w_{j(i)+1},\ldots,z_{j(i+1)-1},w_{j(i+1)-1},z_{j(i+1)}).$$

Similarly, we see that the summation beginning at (12) is equal to the $2p_1$–homogeneous part of

$$z_{j(1)}w_{j(r)}\widetilde{F}_{j(1)+n-j(r)}(z_1,w_1,\ldots,z_{j(1)-1},w_{j(1)-1},z_{j(1)},w_{j(r)},z_{j(r)+1},w_{j(r)+1},\ldots,z_n,w_n).$$

Now using the definition of the operation $\odot$ and summing over $2\leq r\leq n$ and all $1\leq j(1)<\cdots<j(r)\leq n$, one proves the recursion formula (10). $\square$

It is straightforward from the recursion formula that

$$F_1(z_1,w_1)=\frac{1}{1-z_1w_1}.$$

Starting from this, one can (in principle) compute F_n for arbitrary given n, using the recursion formula (10). Below are the results for $n=2,3,4$. We were motivated to find and write down F_4, although it is rather long, because, while we were able to find F_2 and F_3 using ad hoc methods to work out, in the notation of Proposition 1.5,

$$\frac{1}{1-u_1}\odot\frac{1}{1-v}\qquad\text{and}\qquad\frac{a}{(1-u_1)(1-u_2)}\odot\frac{1}{1-v},$$

we were unable to find, in the notation of Proposition 1.6,

$$\frac{a}{(1-u_1)(1-u_2)}\odot\frac{b}{(1-v_1)(1-v_2)},$$

which is needed in computing F_4, until we discovered the $\mathcal{E}$–transform.

Examples 2.5. ($n=2$):

$$\begin{aligned}F_2(z_1,w_1,z_2,w_2)&=\frac{1}{1-u_1^{(2)}}\big(F_1(z_1,w_2)\odot F_1(w_1,z_2)\big)\\&=\frac{1}{1-u_1^{(2)}}\left(\frac{1}{1-z_1w_2}\odot\frac{1}{1-w_1z_2}\right)\\&=\frac{1}{(1-u_1^{(2)})(1-u_2^{(2)})}\end{aligned}$$

where $u_1^{(2)}=z_1w_1+z_2w_2$ and $u_2^{(2)}=z_1w_2+z_2w_1$.

$(n=3)$:

$$\begin{aligned}
&F_3(z_1,w_1,z_2,w_2,z_3,w_3) = \\
&= \frac{1}{1-u_1^{(3)}}\Big(\big(z_1w_2F_2(z_1,w_2,z_3,w_3)\big)\odot F_1(w_1,z_2) \\
&\qquad\qquad + \ F_1(z_1,w_3)\odot\big(w_1z_3F_2(w_1,z_2,w_2,z_3)\big) \\
&\qquad\qquad\qquad + \ \big(z_2w_3F_2(z_1,w_1,z_2,w_3)\big)\odot F_1(w_2,z_3) \\
&\qquad\qquad\qquad\qquad + \ F_1(z_1,w_3)\odot F_1(w_1,z_2)\odot F_1(w_2,z_3)\Big) \\
&= \frac{1}{(1-u_1^{(3)})(1-u_2^{(3)})}\left(1+\frac{z_1w_2}{1-u_3^{(3)}}+\frac{z_2w_3}{1-u_4^{(3)}}+\frac{z_3w_1}{1-u_5^{(3)}}\right)
\end{aligned}$$

where

$$\begin{aligned}
u_1^{(3)} &= z_1w_1+z_2w_2+z_3w_3 & u_2^{(3)} &= z_1w_3+z_2w_1+z_3w_2 \\
u_3^{(3)} &= z_1w_2+z_2w_1+z_3w_3 & u_4^{(3)} &= z_1w_1+z_2w_3+z_3w_2 \\
u_5^{(3)} &= z_1w_3+z_2w_2+z_3w_1.
\end{aligned}$$

$(n=4)$:

$$\begin{aligned}
&F_4(z_1,w_1,z_2,w_2,z_3,w_3,z_4,w_4) = \\
&\quad = \frac{1}{1-u_1^{(4)}}\Big((z_1w_2F_3(z_1,w_2,z_3,w_3,z_4,w_4))\odot F_1(w_1,z_2) \\
&\qquad\qquad + \quad (z_1w_3F_2(z_1,w_3,z_4,w_4))\odot(w_1z_3F_2(w_1,z_2,w_2,z_3)) \\
&\qquad\qquad + \quad F_1(z_1,w_4)\odot(w_1z_4F_3(w_1,z_2,w_2,z_3,w_3,z_4)) \\
&\qquad\qquad + \quad (z_2w_3F_3(z_1,w_1,z_2,w_3,z_4,w_4))\odot F_1(w_2,z_3) \\
&\qquad\qquad + \quad (z_2w_4F_2(z_1,w_1,z_2,w_4))\odot(w_2z_4F_2(w_2,z_3,w_3,z_4)) \\
&\qquad\qquad + \quad (z_3w_4F_3(z_1,w_1,z_2,w_2,z_3,w_4))\odot F_1(w_3,z_4) \\
&\qquad\qquad + \quad (z_1w_3F_2(z_1,w_3,z_4,w_4))\odot F_1(w_1,z_2)\odot F_1(w_2,z_3) \\
&\qquad\qquad + \quad F_1(z_1,w_4)\odot F_1(w_1,z_2)\odot(w_2z_4F_2(w_2,z_3,w_3,z_4)) \\
&\qquad\qquad + \quad F_1(z_1,w_4)\odot(w_1z_3F_2(w_1,z_2,w_2,z_3))\odot F_1(w_3,z_4) \\
&\qquad\qquad + \quad (z_2w_4F_2(z_1,w_1,z_2,w_4))\odot F_1(w_2,z_3)\odot F_1(w_3,z_4) \\
&\qquad\qquad + \quad F_1(z_1,w_4))\odot F_1(w_1,z_2)\odot F_1(w_2,z_3)\odot F_1(w_3,z_4)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(1-u_1^{(4)})(1-u_2^{(4)})}\Bigg(1 + \frac{z_3w_4}{1-u_3^{(4)}} + \frac{z_2w_3}{1-u_5^{(4)}} + \frac{z_4w_1}{1-u_6^{(4)}} + \frac{z_1w_2}{1-u_8^{(4)}} + \\
&\quad + \frac{z_2w_4}{1-u_9^{(4)}} + \frac{z_3w_1}{1-u_{10}^{(4)}} + \frac{z_4w_2}{1-u_{11}^{(4)}} + \frac{z_1w_3}{1-u_{12}^{(4)}} + \frac{z_1z_3w_2w_4}{(1-u_3^{(4)})(1-u_{14}^{(4)})} + \\
&\quad + \frac{z_2z_3w_4^2}{(1-u_3^{(4)})(1-u_9^{(4)})} + \frac{z_3^2w_1w_4}{(1-u_3^{(4)})(1-u_{10}^{(4)})} + \frac{z_2z_4w_2w_4}{(1-u_4^{(4)})(1-u_9^{(4)})} + \\
&\quad + \frac{z_2z_4w_2w_4}{(1-u_4^{(4)})(1-u_{11}^{(4)})} + \frac{z_2^2w_3w_4}{(1-u_5^{(4)})(1-u_9^{(4)})} + \frac{z_1z_2w_3^2}{(1-u_5^{(4)})(1-u_{12}^{(4)})} + \\
&\quad + \frac{z_2z_4w_1w_3}{(1-u_5^{(4)})(1-u_{13}^{(4)})} + \frac{z_3z_4w_1^2}{(1-u_6^{(4)})(1-u_{10}^{(4)})} + \frac{z_4^2w_1w_2}{(1-u_6^{(4)})(1-u_{11}^{(4)})} + \\
&\quad + \frac{z_2z_4w_1w_3}{(1-u_6^{(4)})(1-u_{13}^{(4)})} + \frac{z_1z_3w_1w_3}{(1-u_7^{(4)})(1-u_{10}^{(4)})} + \frac{z_1z_3w_1w_3}{(1-u_7^{(4)})(1-u_{12}^{(4)})} + \\
&\quad + \frac{z_1z_4w_2^2}{(1-u_8^{(4)})(1-u_{11}^{(4)})} + \frac{z_1^2w_2w_3}{(1-u_8^{(4)})(1-u_{12}^{(4)})} + \frac{z_1z_3w_2w_4}{(1-u_8^{(4)})(1-u_{14}^{(4)})}\Bigg),
\end{aligned}
$$

where

$$
\begin{aligned}
u_1^{(4)} &= z_1w_1 + z_2w_2 + z_3w_3 + z_4w_4 & u_2^{(4)} &= z_1w_4 + z_2w_1 + z_3w_2 + z_4w_3 \\
u_3^{(4)} &= z_1w_1 + z_2w_2 + z_3w_4 + z_4w_3 & u_4^{(4)} &= z_1w_1 + z_2w_4 + z_3w_3 + z_4w_2 \\
u_5^{(4)} &= z_1w_1 + z_2w_3 + z_3w_2 + z_4w_4 & u_6^{(4)} &= z_1w_4 + z_2w_2 + z_3w_3 + z_4w_1 \\
u_7^{(4)} &= z_1w_3 + z_2w_2 + z_3w_1 + z_4w_4 & u_8^{(4)} &= z_1w_2 + z_2w_1 + z_3w_3 + z_4w_4 \\
u_9^{(4)} &= z_1w_1 + z_2w_4 + z_3w_2 + z_4w_3 & u_{10}^{(4)} &= z_1w_4 + z_2w_2 + z_3w_1 + z_4w_3 \\
u_{11}^{(4)} &= z_1w_4 + z_2w_1 + z_3w_3 + z_4w_2 & u_{12}^{(4)} &= z_1w_3 + z_2w_1 + z_3w_2 + z_4w_4 \\
u_{13}^{(4)} &= z_1w_4 + z_2w_3 + z_3w_2 + z_4w_1 & u_{14}^{(4)} &= z_1w_2 + z_2w_1 + z_3w_4 + z_4w_3.
\end{aligned}
$$

Theorem 2.6. *For every $n \in \mathbf{N}$ with $n \geq 2$, $F_n(z_1, w_1, \ldots, z_n, w_n)$ is a rational function in variables $z_1, \ldots, z_n, w_1, \ldots, w_n$. It is, moreover, a sum of terms of the form*

$$\frac{z_{i_1} \ldots z_{i_k} w_{j_1} \ldots w_{j_k}}{(1-u_1) \ldots (1-u_\ell)}$$

for some $\ell \in \{2, \ldots, n\}$, $k \in \{0, 1, \ldots, \ell-1\}$, $i_1, \ldots, i_k, j_1, \ldots, j_k \in \{1, \ldots, n\}$ and for $u_1, \ldots, u_\ell$ of the form

$$u_p = \sum_{i=1}^{n} z_i w_{\sigma_p(i)}$$

for distinct permutations $\sigma_1, \ldots, \sigma_\ell$ of $\{1, \ldots, n\}$.

Proof. This follows by induction on n from the recursion formula (10) and Proposition 1.6. $\square$

Examining the power series of F_n, for each n the conjecture (1) yields a conjectured identity involving multinomial coefficients. For example, when $n = 3$ we get the conjecture

$$3^{3p} = \sum_{\substack{j,k\geq 0\\ j+k=p}} \binom{3j}{j,j,j}\binom{3k}{k,k,k} +$$

$$+\, 3 \sum_{\substack{j,k,\ell\geq 0\\ j+k+\ell=p-1}} \quad \sum_{\substack{k',\ell'\geq 0\\ k'+\ell'=k+\ell+1}} \quad \sum_{\substack{j',\ell''\geq 0\\ j'+\ell''=j+\ell+1}} \binom{2j+j'}{j,\, j,\, j'}\binom{2k+k'}{k,\, k,\, k'}\binom{\ell+\ell'+\ell''}{\ell,\, \ell',\, \ell''}$$

for all $p \in \mathbf{N}$.

3. Contour integration

It follows from Theorem 2.6 that $F_n(z_1, w_1, \ldots, z_n, w_n)$ is a holomorphic function of complex variables $z_1, w_1, \ldots, z_n, w_n$ in a suitably small ball around the origin. It is possible, at least in principle, to verify the conjecture mentioned at equation (1) in the introduction for a given value of n from the generating function F_n by performing $2n-1$ contour integrations. For example, letting

$$G_n(x_1, \ldots, x_n) = \sum_{k_1,\ldots,k_n\geq 0} N(k_1, k_1, k_2, k_2, \ldots, k_n, k_n) x_1^{k_1} \ldots x_n^{k_n},$$

for suitably small $\epsilon > 0$ we have

$$G_n(y_1^2, \ldots, y_n^2) =$$
$$= \frac{1}{(2\pi i)^n} \int_{|\zeta_1|=\epsilon} \cdots \int_{|\zeta_n|=\epsilon} \zeta_1^{-1} \ldots \zeta_n^{-1} F_n(\zeta_1 y_1, \zeta_1^{-1} y_1, \ldots, \zeta_n y_n, \zeta_n^{-1} y_n)\, d\zeta_n \ldots d\zeta_1.$$

Letting

$$H_n(a) = \sum_{k=0}^{\infty} N(\underbrace{k, k, \ldots, k, k}_{2n \text{ times}}) a^k,$$

we have

$$H_n(b^n) =$$
$$= \frac{1}{(2\pi i)^{n-1}} \int_{|\eta_2|=\epsilon} \cdots \int_{|\eta_n|=\epsilon} \eta_2^{-1} \ldots \eta_n^{-1} G_n(\eta_2 \ldots \eta_n b, \eta_2^{-1} b, \eta_3^{-1} b, \ldots, \eta_n^{-1} b)\, d\eta_n \ldots d\eta_2.$$

The conjecture (1) is that

$$H_n(a) = \frac{1}{1 - n^n a}$$

for every $n \in \mathbf{N}$.

Although the counjecture can be verified without difficulty in the case $n = 2$ using combinatorial methods, we will illustrate the method described above in checking it. However, even for $n = 3$, we have been unable to prove the conjecture using the method of contour integration.

We have

$$G_2^{(1)}(y_1,z_2,w_2) := \frac{1}{2\pi i}\int_{|\zeta|=\epsilon} F_2(\zeta y_1,\zeta^{-1}y_1,z_2,w_2)\zeta^{-1}d\zeta$$
$$= \frac{1}{2\pi i(1-y_1^2-z_2w_2)}\int_{|\zeta|=\epsilon}\frac{1}{\zeta-\zeta^2y_1w_2-y_yz_2}d\zeta$$

and $\zeta-\zeta^2y_1w_2-y_1z_2=y_1w_1(r_1-\zeta)(\zeta-r_2)$ with

$$r_1,r_2=\frac{1\pm\sqrt{1-4y_1^2z_2w_2}}{2y_1w_2}.$$

If $|y_1|$, $|z_2|$, $|w_2|<\delta$ for $\delta>0$ small enough, r_1 can be forced arbitrarily close to zero and r_2 can be forced close to ∞. By the residue theorem,

$$G_2^{(1)}(y_1,z_2,w_2)=\frac{1}{(1-y_1^2-z_2w_2)y_1w_2(r_1-r_2)}$$
$$=\frac{1}{(1-y_1^2-z_2w_2)\sqrt{1-4y_1^2z_2w_2}}.$$

Therefore,

$$G_2(x_1,x_2)=\frac{1}{(1-x_1-x_2)\sqrt{1-4x_1x_2}}.$$

Finally,

$$H_2(b^2)=\frac{1}{2\pi i}\int_{|\eta|=1}\eta^{-1}G_2(\eta b,\eta^{-1}b)\,d\eta$$
$$=\frac{1}{2\pi i\sqrt{1-4b^2}}\int_{|\eta|=1}\frac{1}{\eta-b-\eta^2b}d\eta$$
$$=\frac{1}{1-4b^2}.$$

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