AFFABLE EQUIVALENCE RELATIONS AND ORBIT STRUCTURE OF CANTOR DYNAMICAL SYSTEMS

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ABSTRACT. We prove several new results about AF-equivalence relations, and relate these to Cantor minimal systems (i.e. to minimal Z-actions). The results we obtain turn out to be crucial for the study of the topological orbit structure of more general countable group actions (as homeomorphisms) on Cantor sets, which will be the topic of a forthcoming paper. In all this, Bratteli diagrams and their dynamical interpretation, are indispensable tools.

1. INTRODUCTION

In the present paper we will prove some new results about AF-equivalence relations (cf. Definition 3.7) that, besides being of interest in their own right, turn out to be powerful new tools for the study of the topological orbit structure of countable group actions as homeomorphisms on Cantor sets. In a forthcoming paper we will apply these new techniques to prove that certain minimal and free $\mathbb{Z}^2$-actions on Cantor sets are (topologically) orbit equivalent to Cantor minimal systems (X, T), or, equivalently, to minimal $\mathbb{Z}$ actions. The strategy is to prove that the equivalence relation $R_{\mathbb{Z}^2}$ associated to the given $\mathbb{Z}^2$-action is *affable* ("AF-able"), i.e. may be given an AF-equivalence structure. To prove affability of $R_{\mathbb{Z}^2}$ we need a delicate "glueing" procedure, the technical part of which is stated in this paper as Lemma 4.15, the "key lemma". We demonstrate the power of Lemma 4.15 by establishing results that are highly non-trivial, concerning the intimate link that exists between minimal AF-equivalence relations and Cantor minimal systems (Thm. 4.16, Thm. 4.17; cf. also Thm. 4.8).

Our ultimate goal is to attack the following question (which is an analogue in the topological dynamical setting to the celebrated Connes-Feldman-Weiss result in the measure-theoretic setting [CFW]):

Let G be a countable, amenable group acting minimally (i.e. every G-orbit is dense) and freely (i.e. $gx = x$ for some $x \in X$, implies g =the identity element of G) as homeomorphisms on the Cantor set X. Then (X, G) is topologically orbit equivalent to a Cantor minimal system (Y, T), i.e. there exists a homeomorphism $F : X \to Y$ mapping G-orbits onto T-orbits.

[By Theorem 4.8 and Theorem 4.16 this is equivalent to show that the equivalence relation R_G associated to G (cf. Example 2.7(i)) is affable. Previously, this is known to be true for locally finite groups G (in fact, in this case we do not need to require free action, only that $fix\,(g) = \{x \in X | gx = x\}$ is a clopen set for each $g \in G$), and for $\mathbb{Z}^n$-actions that split as Cartesian products, and also for the case that $G = \mathbb{Z} \times H$, where H is a finite cyclic group. These facts can be deduced from results contained in [Kr], [GPS], [Jo]. It is also noteworthy that in the (standard) Borel setting the analogous question has an affirmative answer for $\mathbb{Z}^n$-actions [W].]

We shall need the key concept of a Bratteli diagram, both ordered and unordered, and we refer to [HPS] and [GPS] for details and proofs of basic results. (Cf. also Example 2.7(ii).) We will state two results that we shall need in the sequel, concerning the interplay between Bratteli diagrams and Cantor minimal systems.

Proposition 1.1 (HPS; Section 3)**.** *Let (V,E) be a simple (standard) Bratteli diagram, and let $X = X_{(V,E)}$ be the Cantor set consisting of the (infinite) paths associated to (V,E). Let $x, y \in X$ be two paths that are not cofinal. There exists a Cantor minimal system (X,T) such that T preserves cofinality, except that $Tx = y$.*

Proof. Without loss of generality we may assume (by appropriately telescoping the original diagram), that for every $n = 0, 1, 2, ...$, there is at least one edge between every vertex v at level n and every vertex w at level $n+1$ of (V,E). Furthermore, we may assume that the n'th edge of x is distinct from the n'th edge of y. It is an easy observation that (V,E) may be given a proper ordering (also called simple ordering) such that x becomes the unique max path, and y becomes the unique min path. The associated Bratteli-Vershik system (X,T) has the desired properties. □

Theorem 1.2 (GPS; Lemma 5.1)**.** *Let (X,T) be Cantor minimal system, and let Y be a closed (non-empty) subset of X that meets each T-orbit at most once. There exists an ordered Bratteli diagram $B_Y = (V,E,\geq)$, where (V,E) is a simple (standard) Bratteli diagram, such that the associated Bratteli-Vershik system $\left(X_{(V,E)}, T_{B_Y}\right)$ is conjugate to (X,T). The conjugating map $F : X \to X_{(V,E)}$ maps Y onto the set of maximal paths, and $T(Y)$ onto the set of minimal paths. Furthermore, if $y \in Y$, the backward orbit of y, $\{T^n y | n \leq 0\}$, is mapped onto the set of paths cofinal with $F(y)$, while the forward orbit $\{T^n y | n \geq 1\}$ is mapped onto the set of paths cofinal with $F(Ty)$. Any T-orbit, $\{T^n x | n \in \mathbb{Z}\}$, that does not meet Y is mapped onto the set of paths cofinal with $F(x)$.*

Corollary 1.3 (HPS; Theorem 4.7)**.** If $Y = \{y\}$, then $B_{\{y\}} = (X, V, \geq)$ is a properly ordered (also called simply ordered) Bratteli diagram, with $F(y)$ equal to the unique max path and $F(Ty)$ equal to the unique min path.

2. Étale equivalence relations

Let X be a Hausdorff locally compact, second countable (hence metrizable) space. For the most part we shall be considering the case when X is zero-dimensional, i.e., X has a countable basis of closed and open (clopen) sets. (This is equivalent to X being totally disconnected.) Of particular importance will be the case when X is a *Cantor set*, i.e., X is a totally disconnected compact metric space with no isolated points — it is a well known fact (going back to Cantor and Hausdorff) that all such spaces are homeomorphic.

We shall be considering countable equivalence relations R on X, i.e. $R \subset X \times X$ is an equivalence relation so that each equivalence class $[x]_R = \{y \in X \mid (x,y) \in R\}$ is countable (or finite) for each x in X. R has a natural (principal) *groupoid* structure, with *unit space* equal to the diagonal set $\Delta = \{(x,x) \mid x \in X\}$, which we may identify with X. Specifically, if (x,y), $(y,z) \in R$, then the product of this *composable* pair is defined as

$$(x,y) \cdot (y,z) = (x,z),$$

and the inverse of $(x,y) \in R$ is $(x,y)^{-1} = (y,x)$. The unit space of R is by definition the set consisting of products of elements of R with their inverses, and so equals Δ. Assume R is given a Hausdorff locally compact, second countable (hence metrizable) topology $\mathcal{T}$, so that the product of composable pairs (with the topology inherited from the product topology on $R \times R$) is continuous. Also, the inverse map on R shall be a homeomorphism. With this structure $(R, \mathcal{T})$ is a *locally compact (principal) groupoid*, cf. [Re1].

The range map $r : R \to X$ and the source map: $s : R \to X$ are defined by $r((x, y)) = x$ and $s((x, y)) = y$, respectively, where $(x, y) \in R$ — both maps being surjective.

Definition 2.1 (Étale equivalence relation)**.** The locally compact groupoid $(R, \mathcal{T})$, where R is a countable equivalence relation on the locally compact metric space X, is *étale* if $r : R \to X$ is a local homeomorphism, i.e. for every $(x, y) \in R$ there exists an open neighborhood $U^{(x,y)} \in \mathcal{T}$ of (x, y) so that $r(U^{(x,y)})$ is open in X and $r : U^{(x,y)} \to r(U^{(x,y)})$ is a homeomorphism. In particular, r is an open map. If X is zero-dimensional, we may clearly choose $U^{(x,y)}$ to be a clopen set.

We will call $(R, \mathcal{T})$ an *étale equivalence relation* on X, and we will occasionally refer to the local homeomorphism condition as the *étaleness condition.*

Remark 2.2. The definition we have given of étaleness — which is the most convenient to use for our objects of study — is equivalent to the various definitions of an étale (or r-discrete) locally compact groupoid (applied to our setting) that can be found in the literature. Confer for instance [Re1; Def. 2.6 and Prop. 2.8] and [Pat; Def. 2.2.1 and Def. 2.2.3] — the existence of an (essential) unique Haar system consisting of counting measures follows from our definition, cf. [Pat; Prop. 2.2.5]. Furthermore, one can prove that the diagonal $\Delta = \Delta_X = \{(x, x) \mid x \in X\}$ is a clopen subset of R [Re1; Prop. 2.8]. Also, Δ is homeomorphic to X, and so we are justified in identifying Δ with X.

We observe that s is a local homeomorphism, since $s((x, y)) = r((x, y)^{-1})$. It is easily deduced that $r^{-1}(x) = \{(x, y) \in R\}$, as well as $s^{-1}(x) = \{(y, x) \in R\}$, are (countable) discrete topological spaces in the relative topology for each $x \in X$. Clearly R can be written as a union of graphs of local homeomorphisms of the form $s \circ r^{-1}$.

Note that the topology $\mathcal{T}$ on R ($\subset X \times X$) is rarely the topology $\mathcal{T}_{\text{rel}}$ inherited from the product topology on $X \times X$. Necessarily $\mathcal{T}$ is finer than $\mathcal{T}_{\text{rel}}$. For details on topological groupoids, in particular locally compact and étale (r-discrete) groupoids, and the associated C^*-algebras, we refer to [Re1], [Pat], [Put].

The following proposition is the analogue in our setting of Theorem 1 of [FM], where countable (standard) Borel equivalence relations were studied. Our proof mimics the proof in [FM].

Proposition 2.3. *Let $(R, \mathcal{T})$ be an étale equivalence relation on the zero-dimensional space X. There exists a countable group G of homeomorphisms of X so that $R = R_G$, where $R_G = \{(x, gx) \mid x \in X, g \in G\}$.*

Proof. Let $\{C_k\}_{k=1}^{\infty}$ be a clopen partition of $R \setminus \Delta$, where Δ is the diagonal $\{(x, x) \mid x \in X\}$, so that for each k, the maps r and s are homeomorphisms from C_k onto the clopen sets $r(C_k) \subset X$ and $s(C_k) \subset X$, respectively. We refine the partition $\{C_k\}$ so that $r(C_k) \cap s(C_k) = \emptyset$ for each k. In fact, this may be achieved as follows. Let $\{I_i \times J_i\}_{i=1}^{\infty}$ be a clopen covering of $(X \times X) \setminus \Delta$ so that $I_i \cap J_i = \emptyset$ for every i, and each I_i and J_i are clopen. Define $D_k^i = C_k \cap (I_i \times J_i)$. Then $\{D_k^i\}_{i,k=1}^{\infty}$ is a clopen partition of $R \setminus \Delta$, so that r and s are homeomorphisms of D_k^i onto the clopen sets $r(D_k^i) \subset X$ and $s(D_k^i) \subset X$, respectively, and $r(D_k^i) \cap s(D_k^i) = \emptyset$ for all i, k. We relabel the non-empty sets in $\{D_k^i\}$, and so get a sequence $\{E_i\}_{i=1}^{\infty}$, which is a clopen refinement of $\{C_k\}_{k=1}^{\infty}$, with the property that $r(E_i) \cap s(E_i) = \emptyset$

for every i. For each i we define the continuous function

$$g(x) = \begin{cases} y(= s_i r_i^{-1}(x)) & \text{if } (x,y) \in E_i, \\ y(= r_i s_i^{-1}(x)) & \text{if } (y,x) \in E_i, \\ x & \text{otherwise,} \end{cases}$$

where r_i and s_i denote the restriction to E_i of r and s, respectively. Observe that $g_i^2 = \mathrm{id}$, and so g_i is a homeomorphism. The graph $\Gamma(g_i)$ of g_i is easily seen to be $E_i \cup \theta(E_i) \cup (\Delta \cap (F_i \times F_i))$, where $\theta : X \times X \to X \times X$ denotes the flip map $(x,y) \to (y,x)$, and $F_i = X \setminus (r(E_i) \cup s(E_i))$. Hence $\Gamma(g_i) \subset R$. Let G be the (countable) group generated by the g_i's. Clearly $R_G \subset R$. On the other hand, $\bigcup_{i=1}^{\infty} \Gamma(g_i) \supset R \setminus \Delta$, since $\{E_i\}$ is a covering of $R \setminus \Delta$. Since clearly $\Delta \subset R_G$, we conclude that $R = R_G$. □

Open Problem. Assume that for every $x \in X$, the equivalence class $[x]_R$ is dense in X, where $(R, \mathcal{T})$ is as in Proposition 2.3. Is it possible to choose G so that $R = R_G$ and G acts freely on X (i.e., $gy = y$ for some $y \in X$, $g \in G$, implies that $g =$ identity element)?

The analogous question has a negative answer in the setting of countable (standard) Borel equivalence relations [Ad]. Furthermore, in the ergodic, measure-preserving case there are examples of countable equivalence relations that cannot be generated by an essentially free action of a countable group [Fu; Theorem D].

Let $(R_1, \mathcal{T}_1)$ and $(R_2, \mathcal{T}_2)$ be two étale equivalence relations on X_1 and X_2, respectively. There is an obvious notion of isomorphism, namely a homeomorphism of $(R_1, \mathcal{T}_1)$ onto $(R_2, \mathcal{T}_2)$ respecting the groupoid operations. Since the unit spaces $\Delta_i = \{(x,x) \mid x \in X_i\}$ — which we identify with X_i — are equal to $\{aa^{-1} \mid a \in R_i\}$, $i = 1, 2$, the definition of isomorphism may be given as follows.

Definition 2.4 (Isomorphism and orbit equivalence)**.** Let $(R_1, \mathcal{T}_1)$ and $(R_2, \mathcal{T}_2)$ be two étale equivalence relations on X_1 and X_2 respectively. $(R_1, \mathcal{T}_1)$ is *isomorphic* to $(R_2, \mathcal{T}_2)$ — we use the notation $(R_1, \mathcal{T}_1) \cong (R_2, \mathcal{T}_2)$ — if there exists a homeomorphism $F : X_1 \to X_2$ so that

(i) $(x,y) \in R_1 \iff (F(x), F(y)) \in R_2$
(ii) $F \times F : (R_1, \mathcal{T}_1) \to (R_2, \mathcal{T}_2)$ is a homeomorphism, where $F \times F((x,y)) = (F(x), F(y))$, $(x,y) \in R_1$. We say F *implements* an isomorphism between $(R_1, \mathcal{T}_1)$ and $(R_2, \mathcal{T}_2)$.

We say that $(R_1, \mathcal{T}_1)$, or R_1, is *orbit equivalent* to $(R_2, \mathcal{T}_2)$, or to R_2, if (i) is satisfied, and we call F an *orbit map* in this case.

Remark 2.5. Observe that $(R_1, \mathcal{T}_1)$ is orbit equivalent to $(R_2, \mathcal{T}_2)$ via the orbit map $F : X_1 \to X_2$ if and only if $F([x]_{R_1}) = [F(x)]_{R_2}$ for each $x \in X_1$. So F maps equivalence classes onto equivalence classes.

Note that if $R_i = R_{G_i}$ for some countable group G_i, $i = 1, 2$, then the equivalence classes coincide with G_i-orbits, and so the term orbit equivalence is appropriate, cf. Proposition 2.3.

There is a notion of invariant probability measure associated to an étale groupoid $(R, \mathcal{T})$. Suffice to say here that the probability measure μ on X is $(R, \mathcal{T})$-invariant iff μ is G-invariant, where G is as in Proposition 2.3. It is straightforward to show that if $(R_1, \mathcal{T}_1)$ and $(R_2, \mathcal{T}_2)$ on X_1 and X_2, respectively, are orbit equivalent via the orbit map $F : X_1 \to X_2$, then F maps the set of $(R_1, \mathcal{T}_1)$-invariant probability measures injectively onto the set of $(R_2, \mathcal{T}_2)$-invariant probability measures.

Remark 2.6. It is very important to be aware of the fact that a countable equivalence relation R on X may be given distinct non-isomorphic topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, so that $(R, \mathcal{T}_1)$

and $(R, \mathcal{T}_2)$ are étale equivalence relations. In fact, one may give examples of non-isomorphic $(R, \mathcal{T}_1)$ and $(R, \mathcal{T}_2)$ where every equivalence class is dense. Specifically, $(R, \mathcal{T}_1)$ may be chosen to be the étale equivalence relation associated to a Cantor minimal system (X, T), while $(R, \mathcal{T}_2)$ is the cofinal relation associated to a (simple) standard Bratteli diagram, appropriately topologized — see the description given in the two examples below. (Confer also Section 4 and [GPS, Thm. 2.3]).

The above fact contrasts with the situation in the countable (standard) Borel equivalence relation setting, where the Borel structure is uniquely determined by $R(\subset X \times X)$.

Examples 2.7 (Two étale equivalence relations)**.**

(i) Let G be a countable discrete group acting freely as homeomorphisms on the locally compact metric space X. Let

$$R_G = \{(x, gx) \mid x \in X, g \in G\} \subset X \times X,$$

i.e., the R_G-equivalence classes are simply the G-orbits. Topologize R_G by transferring the product topology on $X \times G$ to R_G via the bijection $(x, g) \to (x, gx)$. (This is a bijection since G acts freely on X.) Then it is easily verified that R_G becomes an étale equivalence relation. (If G do not act freely, we get a bijection between R_G and a closed subset of $X \times G \times X$ by the map $(x, gx) \to (x, g, gx)$, and we transfer the product topology on $X \times G \times X$ to R_G.)

We shall be especially interested in the situation when G acts *minimally* (and freely) on the Cantor set X, i.e., each orbit $Gx = \{gx \mid g \in G\}$ is dense in X for every x in X. In particular, when $G = \mathbb{Z}$, we let (X, T), where T is the (necessarily minimal) homeomorphism corresponding to $1 \in \mathbb{Z}$, denote the associated *Cantor* (dynamical) system. We will use the term "T-orbit" instead of "$\mathbb{Z}$-orbit"in this case.

Let (X, T) and (Y, S) be two Cantor minimal systems, and denote the associated étale equivalence relations by $R(X,T)$ and $R(Y,S)$, respectively. We claim that $R(X,T) \cong R(Y,S)$ if and only if (X,T) is flip conjugate to (Y,S) (i.e. (X,T) is conjugate to either (Y,S) or (Y,S^{-1})). In fact, one direction is obvious. Conversely, assume $R(X,T)$ is isomorphic to $R(Y,S)$ via the implementing map $F : X \to Y$. Let (x_i) be a sequence in X so that $x_i \to x$. Then $(x_i, Tx_i) \to (x, Tx)$ in $R(X,T)$. Now $F \times F((x,Tx)) = (Fx, S^m(Fx))$, $F \times F((x_i, Tx_i)) = (Fx_i, S^{m_i}(Fx_i))$, for some $m, m_i \in \mathbb{Z}$. Since $F \times F((x_i, Tx_i)) \to F \times F((x, Tx))$ in $R(Y,S)$, this implies that $m_i = m$ for all but finitely many i's. By a theorem of M.Boyle (cf. Thm. 1.4. of [GPS]), this implies that (X,T) and (Y,S) are flip conjugate.

(ii) We begin with a special infinite directed graph (V, E), called a *Bratteli diagram*, which consists of a vertex set V and an edge set E, where V and E can be written as a countable disjoint union of non-empty finite sets:

$$V = V_0 \cup V_1 \cup V_2 \cup \cdots \quad \text{and} \quad E = E_0 \cup E_1 \cup E_2 \cup \cdots$$

with the following property: An edge e in E_n goes from a vertex in V_{n-1} to one in V_n, which we denote by $i(e)$ and $f(e)$, respectively. We call i the *source map* and r the *range map*. We require that there are no *sinks*, i.e. $i^{-1}(v) \neq \emptyset$ for all $v \in V$.

It is convenient to give a diagrammatic presentation of a Bratteli diagram with V_n the vertices at level n and E_n the edges (downward directed) between V_{n-1} and V_n, see Figure 1.

We let $X = X_{(V,E)}$ denote the space of infinite paths in the diagram beginning at some *source* $v \in V$, i.e., $f^{-1}(v) = \emptyset$. Say $v \in V_n$ is a source, let

$$X_v = \{(e_{n+1}, e_{n+2}, \dots) \mid i(e_{n+1}) = v,\ i(e_{k+1}) = f(e_k),\ k > n\}$$

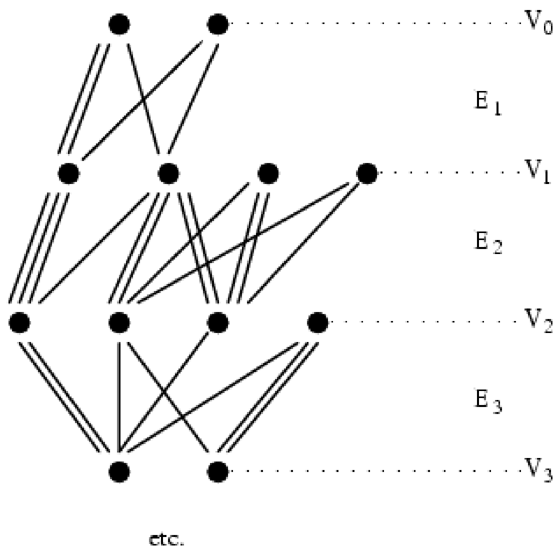

FIGURE 1

which is given the relative topology of the product space $\prod_{k>n} E_k$, and is therefore compact, metrizable and zero-dimensional. We let X be the disjoint union of the X_v's with the topological sum topology. Then X is locally compact, metrizable and zero-dimensional, which has a basis consisting of clopen *cylinder* sets, i.e. sets of the form $U_{(e_{n+1},\ldots,e_m)} = \{(f_{n+1}, f_{n+2}, \ldots) \in X \mid f_{n+1} = e_{n+1}, \ldots, f_m = e_m; i(e_{n+1}) \in V_n \text{ is a source}\}$. The equivalence R on X shall be *cofinal* or *tail equivalence*: two paths are equivalent if they agree from some level on. For $N = 0, 1, 2, \ldots$, let

$$R_N = \{\left((e_{m+1}, e_{m+2}, \ldots), (e'_{n+1}, e'_{n+2}, \ldots)\right) \in X \times X \mid m, n \le N \text{ and } e_k = e'_k \text{ for all } k > N\}.$$

Give R_N the relative topology $\mathcal{T}_N$ of $X \times X$. Then R_N is compact and is an open subset of R_{N+1} for all N. Let $R = \bigcup_{N=0}^{\infty} R_N$, and give R the inductive limit topology $\mathcal{T}$, so that a set U is in $\mathcal{T}$ if and only if $U \cap R_N$ is in $\mathcal{T}_N$ for each N. This means that a sequence $\{(x_n, y_n)\}$ in R converges to (x, y) in R if and only if $\{x_n\}$ converges to x, $\{y_n\}$ converges to y (in X) and, for some N, (x_n, y_n) is in R_N for all but finitely many n.

It is now a simple task to verify that $(R, \mathcal{T})$ is an étale equivalence relation. We shall prove in Section 3 (Theorem 3.9) that this Bratteli diagram example is the prototype of an AF-equivalence relation — the latter will be defined in Section 3 (Definition 3.7). Therefore we will denote $(R, \mathcal{T})$ by $AF(V, E)$.

There is an obvious (countable) locally finite group G of homeomorphisms of X so that $R = R_G = \{(x, gx) \mid x \in X, g \in G\}$, where the fixed point set, $fix(g) = \{x \in X \mid g(x) = x\}$, is a clopen subset of X for every $g \in G$ (cf. Proposition 2.3). In fact, $G = \bigcup_{n=0}^{\infty} G_n$, where $\{id\} = G_0 \subset G_1 \subset G_2 \subset \cdots$ is an increasing sequence of finite groups with $G_n = \bigoplus_{v \in V_n} G_v$. Here G_v is the group consisting of those homeomorphisms g of $X = X_{(V,E)}$, such that $g(x) = x$ for those paths $x \in X$ that do not pass through v, and otherwise $g(x)$ is obtained by permuting the initial segments (above level n) of the various x's passing through v, leaving the tails unchanged. We omit the details. Conversely, let G be a locally finite group acting as homeomorphisms on a zero-dimensional space X, such that the fixed point set of each $g \in G$ is clopen. Then one can construct a Bratteli diagram (V, E) so that X may be identified with $X_{(V,E)}$, and R_G will coincide with the cofinal equivalence relation, cf. [Kr] and [Jo].

If the Bratteli diagram (V, E) has only one source $v_0 \in V$ — which necessarily entails that $V_0 = \{v_0\}$ — we will call (V, E) a *standard* Bratteli diagram. The path space $X_{(V,E)}$ associated to a standard Bratteli diagram (V, E) is compact. We observe that if (V', E') is a *telescope* of (V, E), i.e. (V', E') is obtained from (V, E) by telescoping (V, E) to certain levels $0 < n_1 < n_2 < n_3 < \cdots$, then $AF(V, E) \cong AF(V', E')$. In fact, there is a natural homeomorphism $\alpha : X_{(V,E)} \to X_{(V',E')}$, and α clearly implements the isomorphism, according to the description we have given of convergence in $AF(V, E)$, respectively $AF(V', E')$.

The standard Bratteli diagram (V, E) is *simple* if for each n there is an $m > n$ so that by telescoping the diagram between levels n and m, every vertex v in V_n is connected to every vertex w in V_m. It is a simple observation that (V, E) is simple if and only if every $AF(V, E)$-equivalence class is dense in $X_{(V,E)}$.

3. AF-equivalence relations

We recall some terminology that we shall use. Let R be an equivalence relation on X and let $A \subset X$. We will denote by $R|_A$ the *restriction* of R to A, that is, $R|_A = R \cap (A \times A)$. We say that A is *R-invariant* if $(x, y) \in R$ and $x \in A$, implies $y \in A$. In other words, every R-equivalence class that meets A lies entirely inside A.

If R' is another equivalence relation on X, we say that R' is a *subequivalence relation* of R if $R' \subset R$.

Definition 3.1 (Compact étale equivalence relation (CEER))**.** Let $(R, \mathcal{T})$ be an étale equivalence relation on the locally compact space X, and let $\Delta = \Delta_X \subset R$ be the diagonal in $X \times X$ (i.e., the unit space of R). We say that $(R, \mathcal{T})$ is a *compact étale equivalence relation* (CEER for short) if $R \setminus \Delta$ is a compact subset of R. If X itself is compact this is equivalent to say that R is compact, since Δ then is compact.

Proposition 3.2. *Let $(R, \mathcal{T})$ be a CEER on X. Then:*

(i) $\mathcal{T}$ is the relative topology from $X \times X$.

(ii) R is a closed subset of $X \times X$ (with the product topology), and the quotient topology of the quotient space X/R is Hausdorff.

(iii) R is uniformly finite, that is, there is a natural number N such that the number $\#([x]_R)$ of elements in each equivalence class $[x]_R$ is at most N.

Proof. (i) The (identity) map $r \times s : R \setminus \Delta \to X \times X$ is continuous and injective, and so is a homeomorphism onto its image, since $R \setminus \Delta$ is compact. Now Δ is a clopen subset of R, and its relative topology with respect to $\mathcal{T}$ and the product topology of $X \times X$ coincide (Remark 2.2). Hence the assertion follows. (ii) Clearly R is a closed subset of $X \times X$ by (i). Now the quotient map $q : X \to X/R$ is open since r and s are open maps. In fact, if U is an open subset of X, the set $q(U) = s(r^{-1}(U))$ is open in X/R. By [Ke; Thm. 11], X/R is Hausdorff. (iii) Let $X_1 = r(R \setminus \Delta)$. The subset $X_1 \subset X$ is compact, open (since Δ is clopen in R) and R-invariant, in the sense that $x \in X_1$ implies $[x]_R \subset X_1$. Since $r^{-1}(x)$ is a closed, discrete subset of R, it follows that $\#([x]_R) < \infty$ for every $x \in X_1$. Clearly $\#([x]_R) = 1$ for $x \in X \setminus X_1$. To prove (iii) it is sufficient to show that $\Phi : x \to \#([x]_R)$ is a continuous map from X into $\mathbb{N}$. Now Φ is always lower semicontinuous. In fact, let $\Phi(x) = k$ for some $x \in X$, and let $r^{-1}(x) = \{(x, y_1), \ldots, (x, y_k)\}$. By the étaleness condition there exist in $\mathcal{T}$ disjoint clopen neighbourhoods $U^{(x,y_1)}$,..., $U^{(x,y_k)}$ of (x,y_1),..., (x,y_k), respectively, such that the restriction of r to each $U^{(x,y_i)}$ is a homeomorphism onto its open image. Hence there is an open neighbourhood $U=\bigcap_{i=1}^{k} r(U^{(x,y_i)})$ of x such that $x'\in U$ implies $\#([x']_R)\geq k$. Consequently Φ is lower semicontinuous at x. To prove that Φ is upper semicontinuous at x, we need to use our assumption that $R\setminus\Delta$ is compact. It is enough to establish upper semicontinuity at a point x in the open, compact subset X_1 of X. It follows from our assumption that $R\cap(X_1\times X_1)$ is a compact subset of R. We retain the notation we used above, letting $\Phi(x)=k$, with $r^{-1}(x)=\{(x,y_1),\ \ldots,\ (x,y_k)\}$, etc. Let $x_n\to x$, where $\{x_n\}$ is a sequence in X_1, and assume by contradiction that $\Phi(x_n)>k$ for every n. We choose $k+1$ distinct points $y_{n,1},\ldots,y_{n,k+1}$ for each n so that $r^{-1}(x_n)=\{(x_n,y_{n,1}),\ \ldots,\ (x_n,y_{n,k+1})\}$, and we may assume that $(x_n,y_{n,i})\in U^{(x,y_i)}$ for $i\in 1,2,\ldots,k$ and every n. Passing to a subsequence we may assume by compactness that $(x_n,y_{n,k+1})\to(x',y')$, where $(x',y')\in R\setminus\bigcup_{i=1}^{k}U^{(x,y_i)}$. Clearly $x'=x$. But this yields a contradiction, since

$$(x,y')\notin\{(x,y_1),\ \ldots,\ (x,y_k)\}=r^{-1}(x).$$

This completes the proof. □

Remark 3.3. It is not true that an étale equivalence relation satisfying (ii) and (iii) of Proposition 3.2 is CEER, even when X is compact. In fact, let X be the unit interval $[0,1]$, and let the graph of R in $X\times X$ be the union of the diagonal Δ and the graph of the function $f(t)=1-t$. The equivalence classes have cardinality two, except the equivalence class of $\frac{1}{2}$, which has cardinality one. Clearly R is a closed (hence compact) subset of $X\times X$. It is easy to see that R may be given a topology $\mathcal{T}$ so that $(R,\mathcal{T})$ is an étale equivalence relation on X. However, $\mathcal{T}$ is not the relative topology of $X\times X$, and so is not CEER. [One can construct a similar example with X equal to the Cantor set.]

One can show that an étale equivalence relation $(R,\mathcal{T})$ on X satisfying (i) of Proposition 3.2 is characterized by the property that the map r is a local homeomorphism of R, when the latter is given the relative topology of $X\times X$.

The *disjoint union* of a finite set of étale equivalence relations is defined in the obvious way: Let $(R_i,\mathcal{T}_i)$ be an étale equivalence relation on X_i for $i=1,2,\ldots,k$, where the X_i's are disjoint. Let $X=\bigsqcup_{i=1}^{k}X_i$ be the disjoint union and let $R=\bigsqcup_{i=1}^{k}R_i$ be the equivalence relation on X defined in the obvious way. Let $\mathcal{T}=\sqcup_{i=1}^{k}\mathcal{T}_i$ be the disjoint union topology (also called the sum topology). Then $(R,\mathcal{T})$ — the *disjoint union* of $\{(R_i,\mathcal{T}_i)\}_{i=1}^{k}$ — is an étale equivalence relation on X.

The *product* of two étale equivalence relations is also defined in the obvious way: Let $(R_i,\mathcal{T}_i)$ be an étale equivalence relation on X_i, for $i=1,2$. The *product* $(R,\mathcal{T})$ of $(R_1,\mathcal{T}_1)$ and $(R_2,\mathcal{T}_2)$ is an étale equivalence relation on $X=X_1\times X_2$ in an obvious way, where $R=R_1\times R_2$ is given the product topology.

The following lemma, besides giving the structure of CEERs on zero-dimensional spaces, will be used below in connection with the construction of various Bratteli diagrams that we shall associate to AF-equivalence relations. We also point out the relevance of Remark 3.6 in this regard.

Lemma 3.4. *Let* $(R,\mathcal{T})$ *be a CEER* (Definition 3.1) *on* X*, where* X *is a zero-dimensional space. Then* $(R,\mathcal{T})$ *is isomorphic to a finite disjoint union of CEERs* $\big\{(R_i,\mathcal{T}_i)\big\}_{i=1}^{k}$ *of type* $m_1,\ldots,m_k$*, respectively, where* $X=\bigsqcup_{i=1}^{k}X_i$ *and* R_i *is an equivalence relation on* X_i *of type*

m_i for $i = 1, \dots, k$. Specifically, X_i is (homeomorphic to) $Y_i \times \{1, 2, \dots, m_i\}$ for some natural number m_i, where R_i is the product of the trivial equivalence relation on $\{1, 2, \dots, m_i\}$ (all points are equivalent) with the cotrivial equivalence relation (the identity relation) on Y_i. Furthermore, if X is non-compact, then Y_k is the only non-compact set of the Y_i's, and $m_k = 1$.

Conversely, an étale equivalence relation of the type described is CEER.

The family of sets

$$\mathcal{O} = \big\{Y_i \times \{j\} \mid 1 \le j \le m_i \text{ for } i = 1, 2, \dots, k\big\}$$

is a finite clopen partition of X. If $\mathcal{P}$ is an initially given finite clopen partition of X, we may choose the X_i's so that $\mathcal{O}$ is finer than $\mathcal{P}$. Furthermore, $\mathcal{O}$ gives rise to a clopen partition $\mathcal{O}'$ of R in a natural way, namely $\mathcal{O}'$ consists of the graphs of the local homeomorphisms $\gamma_{lm}^{(i)} : Y_i \times \{l\} \to Y_i \times \{m\}$ where

$$\gamma_{lm}^{(i)}\left((y, l)\right) = (y, m) \;\; ; \;\; 1 \le l, m \le m_i \text{ and } i = 1, ..., k.$$

(So the maps $\gamma_{lm}^{(i)}$ are of the form $s \circ r^{-1}$, appropriately restricted.) If $\mathcal{P}'$ is an initially given finite clopen partition of R, we may choose the X_i's so that $\mathcal{O}'$ is finer than $\mathcal{P}'$.

Proof. Let $X_1 = r(R \setminus \Delta)$. Then X_1 is a compact, clopen subset of X that is R-invariant. Now the restriction $R|_{X \setminus X_1}$ of R to the invariant set $X \setminus X_1$ coincides with $\Delta|_{X \setminus X_1}$. Hence $(R, \mathcal{T})$ is the disjoint union of $R|_{X_1}$ and $\Delta|_{X \setminus X_1}$ (with the relative topologies), and so we may assume at the outset that X is compact, and hence a fortiori $(R, \mathcal{T})$ is compact. For $x \in X$, let $r^{-1}(x) = \big\{(x, y_1), \dots, (x, y_m)\big\}$ — in other words, $[x]_R = \{y_1, \dots, y_m\}$. (We may restrictions of r and s, respectively, to $U^{(y_i, y_j)}$ are homeomorphisms onto their clopen images. Furthermore, $r\big(U^{(y_i, y_j)}\big) = U^{y_i}$ is independent of j and is a clopen neighbourhood of y_i, and the sets $U^{y_1}, \dots, U^{y_m}$ are disjoint. Likewise, $s\big(U^{(y_i, y_j)}\big)$ equals U^{y_j} and so is independent of i. In fact, by the étaleness condition it follows easily that there exist disjoint clopen neighborhoods $U^{(y_1, y_j)}$, for $j = 1, \dots, m$, such that both r and s, restricted to $U^{(y_1, y_j)}$, are homeomorphisms onto their respective clopen images, with $r\left(U^{(y_1, y_j)}\right) = U^{y_1}$ for every j. Set $U^{y_j} = s\big(U^{(y_1, y_j)}\big)$. By appropriately restricting the r and s maps we construct from this homeomorphisms $\gamma_{ij} : U^{y_i} \to U^{y_j}$ for each i, j. The graph of γ_{ij} is $U^{(y_i, y_j)}$. We omit the details. Let $\tilde{U}^x = \bigcup_{i=1}^m U^{y_i}$, and recall that $U^{y_1} = U^x$. It is tempting to restrict R to $\tilde{U}^x$, that is, $R \cap (\tilde{U}^x \times \tilde{U}^x)$, which is easily seen to be isomorphic to the product of the cotrivial and the trivial equivalence relations on $U^x \times \{1, \dots, m\}$. However, $\tilde{U}^x$ need not be R-invariant, so we have to proceed more carefully. Therefore, let $W^x = \bigcup_{i,j=1}^m U^{(y_i, y_j)}$. Then $\{W^x \mid x \in X\}$ is a clopen covering of R, and so by compactness of R there exists a finite set $\{x_1, \dots, x_n\}$ in X such that $R = \bigcup_{i=1}^n W^{x_i}$. Observe that $r(W^{x_i}) = \tilde{U}^{x_i}$ for $i = 1, \dots, n$, where we retain the notation introduced above. Assume we have ordered the x_i's so that

$$\#\big([x_1]_R\big) \ge \#\big([x_2]_R\big) \ge \dots \ge \#\big([x_n]_R\big).$$

Now

$$V^{x_1} = \tilde{U}^{x_1}, V^{x_2} = \tilde{U}^{x_2} \setminus \tilde{U}^{x_1}, \dots, V^{x_n} = \tilde{U}^{x_n} \setminus \bigcup_{i=1}^{n-1} \tilde{U}^{x_i}$$

is a clopen partition of X, and one verifies that these sets are R -invariant. Hence $(R, \mathcal{T})$ is isomorphic to the disjoint union of R restricted to those sets that are non-empty (with the relative topologies). Each of these restrictions is of the desired form. We omit the details.

Clearly an étale equivalence relation of the described type is CEER. To ensure that the associated partition $\mathcal{O}$ of X is finer than the given $\mathcal{P}$, simply choose for each $x \in X$ the U^x so small that each U^{y_i} that occurs in $\tilde{U}^x = \bigcup_{i=1}^m U^{y_i}$ is contained in some element of $\mathcal{P}$, where we again use the notation above. Similarly we may choose U^x so small that the graph $U^{(y_i,y_j)}$ of the local homeomorphism $\gamma_{ij} : U^{y_i} \to U^j$ is contained in some element of $\mathcal{P}'$ for each $1 \leq i,j \leq m$. This will ensure that $\mathcal{O}'$ is finer than $\mathcal{P}'$. □

Corollary 3.5. Let $(R, \mathcal{T})$ be a CEER on the zero-dimensional space X. If $\mathcal{P}$ and $\mathcal{P}'$ are finite clopen partitions of X and R, respectively, there exist clopen partitions $\mathcal{O}$ and $\mathcal{O}'$ of X and R, respectively, as described in Lemma 3.4, so that $\mathcal{O}$ is finer than $\mathcal{P}$ and $\mathcal{O}'$ is finer than $\mathcal{P}'$. In fact, $\mathcal{O} = \Delta \cap \mathcal{O}'$, where $\Delta = \Delta_X$ is the diagonal in $X \times X$. (We make the obvious identification between X and Δ.)

Remark 3.6. It is instructive to draw a picture to illustrate the content of the above lemma: X can be composed into n (disjoint) compact, clopen towers — the k-th tower being V^{x_k} of height $\#([x_k]_R)$ — with clopen floors, and possibly one non-compact tower of height 1. See Figure 2 (we assume that the various towers are non-empty). The equivalence classes of R are formed by the sets of points lying vertically above or below one another in each tower. The clopen partition $\mathcal{O}$ of X is the set consisting of the floors of the various towers. The partition $\mathcal{O}'$ of R can also be easily described in terms of the towers in Figure 2. In fact, for each tower there is between every pair of floors a local homeomorphism of the form $s \circ r^{-1}$(appropriately restricted). The graphs of these maps make up $\mathcal{O}'$. With this picture it is obvious how we may identify $\mathcal{O}$ with $\mathcal{O}' \cap \Delta_X$, where Δ_X is the diagonal in $X \times X$. In Figure 2 we have also shown the "contribution" of one of the towers, say V^{x_i} of height three (i.e. $\#([x_i]_R) = 3$), to the partition $\mathcal{O}'$ of R — we have drawn the graphs of the local homeomorphism associated to V^{x_i} in boldface.

Notice that $\mathcal{O}'$ is a very special partition of R. In fact, $\mathcal{O}'$ has a natural (abstract) principal groupoid structure, with unit space identifiable with $\mathcal{O}$, that we shall now describe. If we define $U \cdot V$ for two subsets U, V of R to be

$$U \cdot V = \{(x,z) \mid (x,y) \in U, (y,z) \in V \text{ for some } y \in X\}$$

then we can list the properties of $\mathcal{O}'$ as follows:

(i) $\mathcal{O}'$ is a finite clopen partition of R finer than $\{\Delta, R \setminus \Delta\}$.

(ii) For all $U \in \mathcal{O}'$, the maps $r, s : U \to X$ are local homeomorphisms, and if $U \in \mathcal{O}' \cap (R \setminus \Delta)$, then $r(U) \cap s(U) = \emptyset$.

(iii) For all $U, V \in \mathcal{O}'$, we have $U \cdot V = \emptyset$ or $U \cdot V \in \mathcal{O}'$. Also, U^{-1} $(= \{(y,x) \mid (x,y) \in U\})$ is in $\mathcal{O}'$ for every U in $\mathcal{O}'$.

(iv) With $\mathcal{O}'^{(2)} = \{(U,V) \mid U, V \in \mathcal{O}', U \cdot V \neq \emptyset\}$, define $(U,V) \in \mathcal{O}'^{(2)} \longrightarrow U \cdot V \in \mathcal{O}'$. Then $\mathcal{O}'$ becomes a principal groupoid with unit space equal to $\{U \in \mathcal{O}' | U \subset \Delta\}$, which clearly may be identified with $\mathcal{O}$.

We will call $\mathcal{O}'$ a *groupoid partition* of $(R, \mathcal{T})$, or simply R. Notice that if we define the equivalence relation $\sim_{\mathcal{O}'}$ on $\mathcal{O}$ by $A \sim_{\mathcal{O}'} B$ if there exists $U \in \mathcal{O}'$ so that $U^{-1} \cdot U = A$, $U \cdot U^{-1} = B$, then the equivalence classes, denoted $[\ ]_{\mathcal{O}'}$, are exactly the towers in Figure 2.

The compact étale equivalence relations (CEER) are the building blocks with which we will define an *AF-equivalence relations*. (AF stands for "approximately finite-dimensional", and we refer to [Re1] and [GPS] for further explanation of the terminology.)

Definition 3.7 (AF-equivalence relation)**.** Let $\{(R_n, \mathcal{T}_n)\}_{n=1}^{\infty}$ be a sequence of CEERs on a zero-dimensional (second countable, locally compact Hausdorff) space X, so that R_n is an

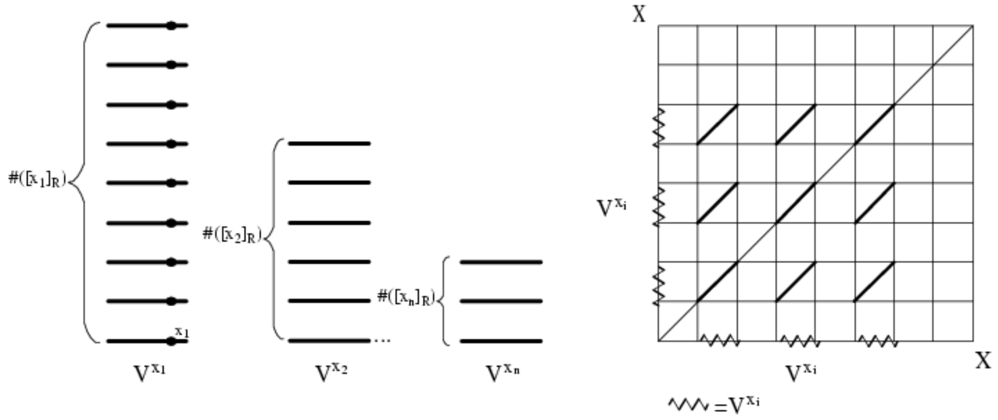

FIGURE 2

open subequivalence relation of R_{n+1}, i.e. $R_n \subset R_{n+1}$ and $R_n \in \mathcal{T}_{n+1}$ for every n. (Note that this implies that R_n is a clopen subset of R_{n+1}, since $R_n \setminus \Delta$ is compact.) Let $(R, \mathcal{T})$ be the *inductive limit* of $\{(R_n, \mathcal{T}_n)\}$ with the *inductive limit topology* $\mathcal{T}$, i.e., $R = \bigcup_{n=1}^{\infty} R_n$ and $U \in \mathcal{T}$ if and only if $U \cap R_n \in \mathcal{T}_n$ for every n. We say that $(R, \mathcal{T})$ is an *AF-equivalence relation* on X, and we use the notation $(R, \mathcal{T}) = \varinjlim (R_n, \mathcal{T}_n)$.

We say that $(R, \mathcal{T})$ is *minimal* if each equivalence class $[x]_R$, $x \in X$, is dense in X.

Theorem 3.8. *Let G be a countable group acting minimally and freely on the Cantor set X, and let $(R_G, \mathcal{T})$ be the associated étale equivalence relation (cf. Example 2.7(i)). Then $(R_G, \mathcal{T})$ is AF if and only if G is locally finite.*

Proof. Assume G is locally finite, and let $G_1 \subset G_2 \subset \cdots \subset G = \bigcup_n G_n$, where G_n is a finite group for every n. It is easy to see that $(R_G, \mathcal{T}) = \varinjlim (R_{G_n}, \mathcal{T}_n)$, where $(R_{G_n}, \mathcal{T}_n)$ is obviously CEER. In fact, R_{G_n} may be identified with $X \times G_n$, and $\mathcal{T}_n$ is the product topology. Hence $(R_G, \mathcal{T})$ is an AF-equivalence relation. Conversely, if $(R_G, \mathcal{T}) = \varinjlim (R_n, \mathcal{T}_n)$ is an AF-equivalence relation, then — identifying R_G with $X \times G$ — let F be a finite subset of G. To show that G is locally finite it suffices to show that the subgroup generated by F is finite. Since $X \times F$ is compact, it is contained in some R_n. Define $H = \{g \in G \mid X \times \{g\} \subset R_n\}$. Clearly $F \subset H$, and since R_n is an equivalence relation it follows that H is a subgroup of G. As R_n is compact, H is finite, and so the subgroup generated by F is finite. □

It is straightforward to verify that an AF-equivalence relation is an étale equivalence relation. Furthermore, one verifies that the étale equivalence relation of Example 2.7(ii) is an AF-equivalence relation. The following theorem is the converse result, as alluded to in Example 2.7(ii).

Theorem 3.9. *Let $(R, \mathcal{T}) = \varinjlim (R_n, \mathcal{T}_n)$ be an AF-equivalence relation on X. There exists a Bratteli diagram (V, E) such that $(R, \mathcal{T})$ is isomorphic to the AF-equivalence relation $AF(V, E)$ associated to (V, E).*

If X is compact, the Bratteli diagram (V, E) may be chosen to be standard. Furthermore, (V, E) is simple if and only if $(R, \mathcal{T})$ is minimal.

Proof. Choose an increasing sequence $\{K_m\}_{m=1}^{\infty}$ of compact, clopen subsets of X, such that $X = \bigcup_{m=1}^{\infty} K_m$ (if X is itself compact, we let $K_1 = K_2 = \cdots = X$), and such that $r(R_m \setminus \Delta) \subset K_m$ for each m. We will choose and increasing sequence $\mathcal{P}_1 \prec \mathcal{P}_2 \prec \cdots$ of finite clopen partitions of X whose union generates the topology of X, and which is related to the sequence $\{K_m\}$ in a way we shall describe. In fact, if $\mathcal{P}_m = \{A_1, \ldots, A_{n_m}\}$, we require that $K_m = \bigcup_{i=1}^{n_m-1} A_i$, and hence $A_{n_m} = X \setminus K_m$. (In particular, if X is compact, $A_{n_m} = \emptyset$. Henceforth we will assume that X is not compact — and so $A_{n_m} \neq \emptyset$ for every m — the compact case being of course a simplified version. We will also assume that $K_m \subset K_{m+1}$ for every m, so that $A_{n_m} \setminus A_{n_{m+1}} \neq \emptyset$ for every m.) It is easily seen that K_m and $X \setminus K_m$ are R_m-invariant sets, so $(R_m, \mathcal{T}_m)$ is the disjoint union of the restrictions of R_m to K_m and $X \setminus K_m$, respectively. (It is to be understood that when we restrict we take the relative topology.) Now $R_m|_{X\setminus K_m}$ is equal to $\Delta|_{X\setminus K_m}$. By Lemma 3.4 we can decompose K_m into a finite number of disjoint towers, with $R_m|_{K_m}$ being as described in Lemma 3.4 and in Remark 3.6. We will use the terminology suggested by Remark 3.6, including the terms "tower", "floor", "height" and "groupoid partition". Keeping the notation above, as well as then one used in Lemma 3.4 and Remark 3.6, we now describe how to construct the Bratteli diagram (V, E). First, let $V_0 = \{v_0\}$ be a one-point set. We will do the next two steps in the construction, which should make it clear how one proceeds.

Step 1. Applying Lemma 3.4 to $(R_1, \mathcal{T}_1)$, we get a groupoid partition $\mathcal{O}_1'$ of R_1, so that $R_1|_{K_1}$ is represented (as explained in Remark 3.6) by l_1 (compact) towers of heights $h_1, ..., h_{l_1}$ and $R_1|_{X\setminus K_1} = \Delta|_{X\setminus K_1}$ is a single non-compact tower of height one, so that the associated clopen partition $\mathcal{O}_1$ of X is finer than $\mathcal{P}_1$ and contains $A_{n_1} = X \setminus K_1$. We let $V_1 = \{v_1, ..., v_{l_1}, v_{l_1+1}\}$, where v_i corresponds to the i'th tower of height h_i when $1 \leq i \leq l_1$, and v_{l_1+1} corresponds to the non-compact tower A_{n_1} of height one. The number of edges between v_0 and v_i is the height h_i of the tower v_i for $1 \leq i \leq l_1$. There are no edges between v_0 and v_{l_1+1} (so v_{l_1+1} will be a source in V). This defines E_1.

Step 2. Let $\tilde{\mathcal{P}}_2 = \mathcal{P}_2 \vee \mathcal{O}_1$ be the join of $\mathcal{P}_2$ and $\mathcal{O}_1$, and so $\tilde{\mathcal{P}}_2$ is a finite clopen partition of X. Applying Corollary 3.5 to $(R_2, \mathcal{T}_2)$ we get a groupoid partition $\mathcal{O}_2'$ of R_2 that is finer than $\{\mathcal{O}_1', R_2 \setminus R_1\}$, and so that the associated clopen partition $\mathcal{O}_2$ of X is finer than $\tilde{\mathcal{P}}_2$ and contains $A_{n_2} = X \setminus K_2$. Hence $R|_{K_2}$ will be represented by l_2 (compact) towers of heights $\tilde{h}_1, ..., \tilde{h}_{l_2}$, and $R|_{X\setminus K_2} = \Delta|_{X\setminus K_2}$ a single non-compact tower of height one. We let $V_2 = \{\tilde{v}_1, ..., \tilde{v}_{l_2}, \tilde{v}_{l_2+1}\}$, where $\tilde{v}_j$ corresponds to the j'th tower of height $\tilde{h}_j$ when $1 \leq j \leq l_2$, and $\tilde{v}_{l_2+1}$ corresponds to the non-compact tower A_{n_2} of height one. Let $\mathcal{O}_2'' = \{U \in \mathcal{O}_2' | U \subset R_1\}$. It is a simple observation that $\mathcal{O}_2''$ is a groupoid partition of R_1 that is finer than $\mathcal{O}_1'$, and with unit space equal to $\mathcal{O}_2$. The set E_2 of edges between V_1 and V_2 is labelled by $\mathcal{O}_2 \setminus \{A_{n_2}\}$ modulo $\mathcal{O}_2''$, i.e. E_2 consists of $\sim_{\mathcal{O}_2''}$ equivalence classes (denoted by $[\ ]_{\mathcal{O}_2''}$) of $\mathcal{O}_2 \setminus \{A_{n_2}\}$. Specifically, if $A, B \in \mathcal{O}_2 \setminus \{A_{n_2}\}$, we have $A \sim_{\mathcal{O}_2''} B$ if there exists $U \in \mathcal{O}_2''$ so that $U^{-1} \cdot U = A$, $U \cdot U^{-1} = B$. As we explained in Remark 3.6, the vertex set V_j ($j = 1, 2$) may be identified with the $\sim_{\mathcal{O}_j'}$ equivalence classes $[\ ]_{\mathcal{O}_j'}$ of $\mathcal{O}_j$. Doing this we may write down the source and range maps $i : E_2 \to V_1$, $f : E_2 \to V_2$, associated to the Bratteli diagram. In fact, if $[A]_{\mathcal{O}_2''} \in E_2$, where $A \in \mathcal{O}_2 \setminus \{A_{n_2}\}$, then $f\left([A]_{\mathcal{O}_2''}\right) = [A]_{\mathcal{O}_2'}$, $i\left([A]_{\mathcal{O}_2''}\right) = [B]_{\mathcal{O}_1'}$, where B is the unique element of $\mathcal{O}_1$ such that $A \subset B$. The vertex $v_{l_2+1} \in V_2$ (corresponding to the tower A_{n_2}) will be a source in V. [Using the pictorial presentation of Figure 2, we can give a more intuitive explanation of our construction of the edge set E_2 between V_1 and V_2. In fact, for $1 \leq j \leq l_2$, let x be any point in the tower $\tilde{v}_j \in V_2$. The R_2-equivalence class $[x]_{R_2}$ of x consists of the $\tilde{h}_j$ vertically lying points of $\tilde{v}_j$ including x. Now $[x]_{R_2}$ is a disjoint

union of distinct R_1-equivalence classes. If $[x]_{R_2}$ contains h_{ij} distinct R_1-equivalence classes "belonging" to the tower $v_i \in V_1$, we connect v_i to $\tilde{v}_j$ by h_{ij} edges. There are no edges between $\tilde{v}_{l_2+1}$ and any vertex in V_1.] Continuing in the same manner we construct the Bratteli diagram (V, E). There is an obvious map F from X to the path space associated to (V, E). In fact, if $x \in X$ let $v_i \in V_{n-1}$ be the tower at level $n-1$ that contains x. Likewise let $\tilde{v}_j \in V_n$ be the tower at level n that contains x, and assume that x lies in floor $A (\in \mathcal{O}_n)$ of $\tilde{v}_j$. The n'th edge $e_n \in E_n$ of $F(x) = (e_1, e_2, ...)$ is then $[A]_{\mathcal{O}_n''}$. (We use similar notation at level n as we used above at level two.) One verifies that F is an homeomorphism between X and the path space associated to (V, E). Furthermore, it is straightforward to show that the map F establishes an isomorphism between $(R, \mathcal{T})$ and the AF-equivalence relation associated to (V, E). In fact, $(x, y) \in R$ and $(x, y) \notin R_{n-1}$, $(x, y) \in R_n$ if and only if $F(x)$ and $F(y)$ become cofinal from level n on. We omit the details.
The two last assertions of the theorem are immediate consequences of our construction and the comments we made in Example 2.7 (ii). ☐

Remark 3.10. Even though the Bratteli diagram model for $(R, \mathcal{T})$ is not unique, it is true that two such models give rise to isomorphic dimension groups, and so the diagrams themselves are related by a telescoping procedure, cf. Example 2.7(ii) and Lemma 4.13. (A relevant reference for Theorem 3.9 is [Re2; Theorem 3.1].)

We now consider the situation that (V, E) is a standard Bratteli diagram and (W, F) is a (standard) subdiagram, i.e. $W \subset V$, $F \subset E$ and $W_0 = V_0 = \{v_0\}$. It is easy to see that $X_{(W,F)}$ is a closed subset of $X_{(V,E)}$. It is also clear that $AF(W, F)$ is the intersection of $AF(V, E)$ with $X_{(W,F)} \times X_{(W,F)}$. Moreover, it is easy to check that the relative topology on $AF(W, F)$ coming from $AF(V, E)$ agrees with the usual topology on $AF(W, F)$. We have the following realization theorem for this situation.

Theorem 3.11. *Let $(R, \mathcal{T})$ be an AF-equivalence relation on the compact (zero-dimensional) space X. Suppose that Z is a closed subset of X such that $R \cap (Z \times Z)$, with the relative topology from $(R, \mathcal{T})$, is an étale equivalence relation on Z. Then there exists a Bratteli diagram (V, E), a subdiagram (W, F) and a homeomorphism $h : X_{(V,E)} \to X$ such that*

(i) *h implements an isomorphism between $AF(V, E)$ and $(R, \mathcal{T})$.*
(ii) *$h(X_{(W,F)}) = Z$ and the restriction of h to $X_{(W,F)}$ implements on isomorphism between $AF(W, F)$ and $R \cap (Z \times Z)$.*

Proof. Let us take $(R', \mathcal{T}')$, a compact open subequivalence relation of $(R, \mathcal{T})$, where $\mathcal{T}'$ is the relative topology. As noted in Proposition 3.2, the topology $\mathcal{T}'$ on R' coincides with the relative topology from $X \times X$. We note that $R' \cap (Z \times Z)$ is a clopen, compact subequivalence relation of $R \cap (Z \times Z)$ and therefore, in particular, is étale. For each pair (x, y) in R', we choose a clopen neighbourhood U in $\mathcal{T}'$ (hence in $\mathcal{T}$) as follows. First, if (x, y) is not in $Z \times Z$, we use the fact that Z, and hence $Z \times Z$, is closed to find a clopen neighbourhood U which is disjoint from $Z \times Z$. If (x, y) is in $Z \times Z$, then we use the fact that $R' \cap (Z \times Z)$ is étale to find a clopen subset $U \subset R'$ such that $r : U \cap (Z \times Z) \to r(U) \cap Z$ and $s : U \cap (Z \times Z) \to s(U) \cap Z$ are both homeomorphisms. [To achieve this, let $V \in \mathcal{T}'$ be a clopen neighbourhood of (x, y) such that $r : V \cap (Z \times Z) \to A \cap Z$ and $s : V \cap (Z \times Z) \to B \cap Z$ are homeomorphisms, where A and B are clopen subsets of X. Choose U to be $V \cap r^{-1}(A) \cap s^{-1}(B)$.] Moreover, we can choose U sufficiently small so that r, s are also local homeomorphisms from U to $r(U)$ and $s(U)$, respectively. In this way, we cover R' with clopen sets. We then extract a finite subcover and then choose a groupoid partition of R' which is finer than this subcover, such that each member of the partition satisfies the properties just listed. (Cf. the proof of Lemma 3.4 and Remark 3.6.) In the end, we obtain a groupoid partition $\mathcal{O}'$ of R' such that

$$\mathcal{O}' \mid Z = \{U \cap (Z \times Z) \mid U \in \mathcal{O}', U \cap (Z \times Z) \neq \emptyset\}$$

is a groupoid partition of $R' \cap (Z \times Z)$. Moreover, it is clear from the construction that these may both be made finer than any pair of pre-assigned clopen partitions of R' and $R' \cap (Z \times Z)$. We also note that it is easy to verify from the construction, any element U of $\mathcal{O}'$ meets $Z \times Z$ if and only if $r(U)$ and $s(U)$ both meet Z. (Referring to Figure 2 as representing $\mathcal{O}'$, this means that if Z intersects any two floors in the same tower, these intersections project onto each other.) Now we consider a sequence

$$R_1 \subset R_2 \subset R_3 \subset \cdots$$

of CEERs in R, with union R, each open in the next, so that $(R, \mathcal{T})$ is the inductive limit. For each n, we construct a groupoid partition $\mathcal{O}'_n$ of R_n with the properties as above. We do this so that the restriction of $\mathcal{O}'_{n+1}$ to R_n is finer than $\mathcal{O}'_n$, and so that the restriction of $\mathcal{O}'_{n+1} \mid Z$ to $R_n \cap (Z \times Z)$ is finer than $\mathcal{O}'_n \mid Z$. This sequence is also chosen so as to generate the respective topologies of R and $R \cap (Z \times Z)$. In the proof of Theorem 3.9, it was shown how $\mathcal{O}'_n$ defines a Bratteli diagram, (V, E). We now describe the subdiagram (W, F). Let Δ_X and Δ_Z denote the diagonals in $X \times X$ and $Z \times Z$, respectively (which we identify with X and Z, respectively). The vertices of V_n correspond to towers in $\mathcal{O}'_n$; that is, equivalence classes of sets in $\Delta_X \cap \mathcal{O}'_n$, modulo $\mathcal{O}'_n$. The vertices of W_n are those classes having a representative which meets Δ_Z. (Again referring to Figure 2, the vertices of V_n correspond to the towers, while the vertices of W_n correspond to those towers that meet Z.) The edges of E_n are the equivalence classes of sets in $\Delta_X \cap \mathcal{O}'_n$, modulo $\mathcal{O}''_n = \{U \in \mathcal{O}'_n | U \subset R_{n-1}\}$. (For an interpretation of this in terms of Figure 2, we refer to the remarks made in the proof of Theorem 3.9.) We let F_n be those equivalence classes having a representative which meets Δ_Z. It follows from the properties of the partitions $\mathcal{O}'_n$ described above that (V, E) and (W, F) satisfy the desired conclusion. We leave the details to the reader. □

The inductive limit $(R, \mathcal{T})$ of a sequence of étale equivalence relations $\{(R_n, \mathcal{T}_n)\}_{n=1}^{\infty}$ on X (notation: $(R, \mathcal{T}) = \varinjlim (R_n, \mathcal{T}_n)$) is defined as in Definition 3.7, with the obvious modifications. It is an easy exercise to show that $(R, \mathcal{T})$ is an étale equivalence relation. The following proposition is a stabilization result with respect to AF-equivalence relations.

Proposition 3.12. *(i) Let $(R, \mathcal{T}) = \varinjlim (R_n, \mathcal{T}_n)$ be an inductive limit of a sequence $\{(R_n, \mathcal{T}_n)\}$ of AF-equivalence relations on X. Then $(R, \mathcal{T})$ is an AF-equivalence relation on X.*

(ii) Let $(R, \mathcal{T})$ be an AF-equivalence relation on X, and let $R' \subset R$ be a subequivalence relation which is open, i.e., $R' \in \mathcal{T}$. Then $(R', \mathcal{T}')$ is an AF-equivalence relation, where $\mathcal{T}'$ is the relative topology of R.

Proof. (i). For each n, let $(R_n, \mathcal{T}_n) = \varinjlim (R_{n,k}, \mathcal{T}_{n,k})$, where $(R_{n,k}, \mathcal{T}_{n,k})$ is a CEER (Definition 3.1) on X for every k. So we have

$$R_1 \subset R_2 \subset R_3 \subset \ldots \subset R = \bigcup_{n=1}^{\infty} R_n,$$

$$R_{n,1} \subset R_{n,2} \subset R_{n,3} \subset \ldots \subset R_n = \bigcup_{k=1}^{\infty} R_{n,k} \ ; n = 1, 2, \ldots,$$

where each set is open in the next one containing it with respect to the relevant topology. Define $R'_1 = R_{1,1}$. Now $R_{1,2} \subset R_1 \subset R_2 = \bigcup_{k=1}^{\infty} R_{2,k}$, and so we may choose $k_2 \geq 2$ so large that $R_{1,2} \subset R_{2,k_2}$. Define $R'_2 = R_{2,k_2}$. Continuing in this manner we get an ascending sequence $\{R'_n\}_{n=1}^{\infty}$ of equivalence relations on X so that R'_n contains all $R_{l,m}$'s, provided l and m are at most n. Clearly $R = \bigcup_{n=1}^{\infty} R'_n$, and we claim that $(R, \mathcal{T}) = \varinjlim(R'_n, \mathcal{T}'_n)$, where $(R'_n, \mathcal{T}'_n) = (R_{n,k_n}, \mathcal{T}_{n,k_n})$ is a CEER for each n. This will finish the proof of (i). (Note that R'_n is open in R'_{n+1}, i.e. $R'_n \in \mathcal{T}'_{n+1}$ for every n.) Now, if $U \in \mathcal{T}$ then $U \cap R_n \in \mathcal{T}_n$ for every n. Hence $U \cap R'_n = U \cap R_{n,k_n} = (U \cap R_n) \cap R_{n,k_n} \in \mathcal{T}_{n,k_n} = \mathcal{T}'_n$ for every n, and so $U \in \tilde{\mathcal{T}}$, where by definition $(R, \tilde{\mathcal{T}})$ equals $\varinjlim R'_n, \mathcal{T}'_n)$. Hence $\mathcal{T} \subset \tilde{\mathcal{T}}$. Conversely, assume $U \in \tilde{\mathcal{T}}$. Then

$$U \cap R'_n = U \cap R_{n,k_n} \in \mathcal{T}'_n = \mathcal{T}_{n,k_n} = \mathcal{T}_n \cap R_{n,k_n} \subset \mathcal{T}_n$$

for every n. To show that $U \in \mathcal{T}$ we must show that $U \cap R_m \in \mathcal{T}_m$ for any given m. This is again equivalent to show that

$$U \cap R_{m,l} = (U \cap R_m) \cap R_{m,l} \in \mathcal{T}_{m,l}$$

for any l. Now choose $n \geq \max\{m, l\}$. Then $R_{m,l} \subset R'_n$ and, since clearly $R_{m,l} \subset R_m$, we get $U \cap R_{m,l} = ((U \cap R'_n) \cap R_m) \cap R_{m,l} \in \mathcal{T}_{m,l}$ from the fact that $U \cap R'_n \in \mathcal{T}_n$ and $R_{m,l}$ is open in R_m, which again is open in R_n. This proves that $U \in \mathcal{T}$, and so $\tilde{\mathcal{T}} \subset \mathcal{T}$, which finishes the proof of (i). (ii). R' being an open subequivalence relation of R implies that $(R', \mathcal{T}')$ is an étale equivalence relation, where $\mathcal{T}'$ is the relative topology of R—a fact that is easily shown. Let $(R, \mathcal{T}) = \varinjlim(R_n, \mathcal{T}_n)$, where $(R_n, \mathcal{T}_n)$ is a CEER on X for every n. It is easily verified that $(R', \mathcal{T}') = \varinjlim(R'_n, \mathcal{T}'_n)$, where $R'_n = R' \cap R_n$ is an open subequivalence relation of R_n, and $\mathcal{T}'_n$ is the relative topology of R_n. By (i) it is sufficient to show that $(R'_n, \mathcal{T}'_n)$ is an AF-equivalence relation for every n. So the proof boils down to showing that an open subequivalence relation of a CEER is an AF-equivalence relation. This again may be reduced further by Lemma 3.4 to showing that an open subequivalence relation of a CEER of type m, with $m \geq 2$, on a compact space is an AF-equivalence relation. (Indeed, it is straightforward to show that a finite disjoint union of AF-equivalence relations is again an AF-equivalence relation). So we may assume that $(R, \mathcal{T})$ is equal to the product of the cotrivial and trivial equivalence relations on $X = Y \times \{1, \ldots, m\}$, where Y is compact, and $R' \subset R$ is an open subequivalence relation. The proof will be completed by showing that $(R', \mathcal{T}')$ is an AF-equivalence relation, where $\mathcal{T}'$ is the relative topology of R. By Proposition 3.2(i) we know that both $\mathcal{T}$ and $\mathcal{T}'$ are the relative topology from $X \times X$. For $y \in Y$, let

$$R'(y) = \big\{(i, j) \mid ((y, i), (y, j)) \in R'\big\}$$

and observe that $R'(y)$ is an equivalence relation on $\{1, \ldots, m\}$. For $(i, j) \in R'(y)$, there exists a clopen neighbourhood $U_{i,j}$ of y such that

$$\big\{((y', i), (y', j)) \mid y' \in U_{i,j}\big\} \subset R',$$

since R' is open in R. Let $U_y = \bigcup_{(i,j) \in R'(y)} U_{i,j}$. Then U_y is a clopen neighbourhood of y such that $R'(y) \subset R'(y')$ for all $y' \in U_y$. Fix $\epsilon > 0$. For each $y \in Y$ select U_y as above so that $U_y \subset B(y, \epsilon)$, where $B(y, \epsilon)$ is the open ball around y of radius ϵ. Select a finite subcover $U_{y_1}, \ldots, U_{y_k}$ of the clopen cover $\{U_y \mid y \in Y\}$ of Y. For $y \in Y$, let $R''(y) = \bigcap_{\{i \mid y \in U_i\}} R'(y_i)$. Let $\mathcal{P}$ be the clopen partition of Y generated by $\{U_{y_1}, \ldots, U_{y_{k_n}}\}$. Then $R''|_E$ is constant for every $E \in \mathcal{P}$. In an obvious way R'' defines an equivalence relation on $X = Y \times \{1, \ldots, m\}$.

Furthermore, we have shown that R'' is a subequivalence relation of R' which is a CEER in the relative topology. Clearly R'' is an open subset of R'. Now we let R'_1 be some R'' corresponding to $\epsilon = 1$. Assume R'_n has been defined to be some R'' corresponding to $\epsilon = 1/n$, and let $\{U_{y_1}, \dots, U_{y_{k_n}}\}$ be the associated clopen cover of Y as explained above. For $\epsilon = 1/(n+1)$, choose a finite clopen cover $\{U_{z_1}, \dots, U_{z_{k_{n+1}}}\}$ of Y as before so that every U_{z_i} is contained in some U_{y_j}, and define R'_{n+1} to be the associated R''. One observes that $R'_n \subset R'_{n+1}$, and that R'_n is open in R'_{n+1}, where each equivalence relation is given the relative topology of $X \times X$. Now $\bigcup_{n=1}^{\infty} R'_n \subset R'$, since each R'_n is contained in R'. On the other hand, let $y \in Y$ and let U_y be, as before, a clopen neighbourhood of y such that $y' \in U_y \Rightarrow R(y) \subset R(y')$. Choose n so large that $B(y, 1/n) \subset U_y$. With $\epsilon = 1/n$ let $\{U_{y_1}, \dots, U_{y_{k_n}}\}$ be the clopen cover of Y associated to R'_n. In particular, $U_{y_i} \subset B(y_i, 1/n)$ for each i. If $y \in U_{y_i}$, then $y \in B(y_i, 1/n)$ and so $y_i \in B(y, 1/n) \subset U_y$. Hence $R'(y) \subset R'(y_i)$, and so we get $R'(y) \subset R'_n(y)$. Consequently, $R' = \bigcup_{n=1}^{\infty} R'_n$ and one verifies easily that $(R', \mathcal{T}') = \varinjlim (R'_n, \mathcal{T}'_n)$, where $\mathcal{T}'_n$ is the relative topology of $X \times X$. This finishes the proof of (ii). □

4. Affable equivalence relations

Definition 4.1 (Affable equivalence relation)**.** Let R be a countable equivalence relation on the zero-dimensional (Hausdorff locally compact, second countable) space X. We say that R is *affable* if R may be given a topology $\mathcal{T}$ so that $(R, \mathcal{T})$ is an AF-equivalence relation.

Remark 4.2. To say that the countable equivalence relation R on X is affable is the same as to say that R is orbit equivalent (cf. Definition 2.4) to some $(R', \mathcal{T}')$, where $(R', \mathcal{T}')$ is an AF-equivalence relation on X', i.e. there exists a homeomorphism $F : X \to X'$ such that $(x, y) \in R \iff \big(F(x), F(y)\big) \in R'$. In fact, using F to pull back the topology $\mathcal{T}'$ on R' to get the topology $\mathcal{T}$ on R, we get that $(R, \mathcal{T})$ is an AF-equivalence relation on X.

The next result is an immediate consequence of Corollary 1.3. Because of its importance we state it is as a theorem.

Theorem 4.3. *Let (X, T) be a Cantor minimal system, and let $(R, \mathcal{T})$ be the associated étale equivalence relation on X* (cf. Example 2.7(i))*. Let x be an arbitrary point of X. The subequivalence relation $R_{\{x\}}$ of R whose equivalence classes are the full T-orbits, except that the T-orbit of x is split into two at x — the forward orbit $\{T^n x \mid n \geq 1\}$ and the backward orbit $\{T^n x \mid n \leq 0\}$ — is open in R. Furthermore, $(R_{\{x\}}, \mathcal{T}_{\{x\}})$ is an AF-equivalence relation on X, where $\mathcal{T}_{\{x\}}$ is the relative topology. In particular, $R_{\{x\}}$ is affable.*

Proof. By Corollary 1.3 we may assume that (X, T) is the Bratteli-Vershik system associated to the properly ordered Bratteli diagram $(V, E, \geq)$, and where x, respectively Tx, is the unique max path, respectively the unique min path. The set of paths that are cofinal with the unique max path equals the backward orbit of x, and the set of paths that are cofinal with the unique min path equals the forward orbit of x. As for the other T-orbits, they agree with the cofinal equivalence relation of (V, E). Thus $R_{\{x\}}$ coincides with the cofinal equivalence relation associated to (V, E). Let $(y, T^k y) \in R_{\{x\}}$, where $y = (e_1, e_2, \dots) \in X$ and k is an integer. So y and $T^k y$ are paths that agree from a certain level on, say N. Let U be the open (and closed) neighbourhood of y defined by

$$U = U_{(e_1, e_2, \dots, e_N)} = \big\{(f_1, f_2, \dots) \in X_B \mid (f_1, \dots, f_N) = (e_1, \dots, e_N)\big\}.$$

Then $W = \{(z, T^k z) \mid z \in U\}$ is an open neighbourhood of $(y, T^k y)$ in $(R, \mathcal{T})$. Furthermore, $W \subset R_{\{x\}}$ since z and $T^k z$ agree from level N on for every $z \in U$. Hence $R_{\{x\}}$ is open in R. The argument we have just given also shows that $\mathcal{T}_{\{x\}}$ coincides with the topology associated to (V, E) as described in Example 2.7(ii), and so $(R_{\{x\}}, \mathcal{T}_{\{x\}}) = AF(V, E)$. Hence $(R_{\{x\}}, \mathcal{T}_{\{x\}})$ is an AF-equivalence relation on X. □

Definition 4.4. Let (X, T) be a Cantor minimal system, and let Y be a non-empty closed subset of X. We say that Y is *regular* (with respect to (X, T)) if the positive and negative return time maps λ^+ and λ^- for T on Y are continuous, where $\lambda^+, \lambda^- : Y \to \mathbb{N} \cup \{+\infty\}$ are given by

$$\lambda^+(y) = \inf\{k \geq 1, +\infty | T^k y \in Y\},$$
$$\lambda^-(y) = \inf\{k \geq 1, +\infty | T^{-k} y \in Y\},$$

and $\mathbb{N} \cup \{+\infty\}$ is given the "one-point compactification topology".

Remark 4.5. Let (X, T) be a Cantor minimal system. If Y is a closed subset of X that meets each T-orbit at most once, then Y is regular. In fact, in this case $\lambda^+(y) = \lambda^-(y) = +\infty$ for each $y \in Y$.

The following theorem, besides being a considerable generalization of Theorem 4.3, turns out to be a useful tool in the study of affability of equivalence relations associated to certain group actions on the Cantor set. (See also the remarks prior to Theorem 4.8.)

Theorem 4.6. *Let (X, T) be a Cantor minimal system, and let $(R, \mathcal{T})$ be the associated étale equivalence relation on X. Let $Y \subset X$ be a regular set that contains a point $y \in Y$ so that $\lambda^+(y) = \lambda^-(y) = +\infty$, i.e. Y meets the T-orbit of y at y only. Let R_Y be the subequivalence relation of R defined by*

$$R_Y = \{(x, T^k x), (T^k x, x) \mid x \in X, k \geq 0,\ \#(\{0 \leq i < k \mid T^i x \in Y\}) \text{ is an even number}\}.$$

In particular, if Y meets each T-orbit at most once, R_Y is obtained from R by splitting the T-orbits meeting Y in the forward and backward orbits at Y.

Then $(R_Y, \mathcal{T}_Y)$ is an AF-equivalence relation on X, where $\mathcal{T}_Y$ is the relative topology from R. In particular, R_Y is affable.

Proof. Clearly R_Y is a subequivalence relation of $R_{\{y\}}$. By Theorem 4.3 we have that $R_{\{y\}}$ is an open subrelation of R and $(R_{\{y\}}, \mathcal{T}_{\{y\}})$ is an AF-equivalence relation, $\mathcal{T}_{\{y\}}$ being the relative topology from R. So by Proposition 3.12 (ii) the proof will be completed if we can show that R_Y is an open subset of R, i.e. $R_Y \in \mathcal{T}$. Let $x \in X$ and $k \geq 1$. It is obviously sufficient to show that $(x, T^k x) \in R_Y$ has an open neighbourhood $U \in \mathcal{T}$ so that $U \subset R_Y$. Let $T^{i_1}x, \ldots, T^{i_l}x$, where $0 \leq i_1 < \cdots < i_l \leq k-1$, be the points on the T-orbit of x lying between x and $T^{k-1}x$ that meet Y. By assumption l is an even number. Obviously

$$\begin{array}{ll} \lambda^+(T^{i_1}x) = i_2 - i_1, & \lambda^-(T^{i_1}x) > i_1, \\ \lambda^+(T^{i_j}x) = i_{j+1} - i_j, & \lambda^-(T^{i_j}x) = i_j - i_{j-1} \text{ if } 1 < j < l; \\ \lambda^+(T^{i_l}x) > k - 1 - i_l, & \lambda^-(T^{i_l}x) = i_l - i_{l-1}. \end{array}$$

By continuity of λ^+ and λ^- on Y, we may find an open neighbourhood V of x so that if $x' \in V$, the number l' of points in Y lying on the T-orbit of x' between x' and $T^{k-1}x'$ is even. In fact, we choose V so small that for $1 \leq i \leq k-1$, $T^i V$ does not meet Y if $i \notin \{i_1, \ldots, i_l\}$, and if $T^{i_j}x' \in Y, 1 \leq j \leq l$, then $\lambda^+(T^{i_j}x') = \lambda^+(T^{i_j}x)$ and $\lambda^-(T^{i_j}x') = \lambda^-(T^{i_j}x)$ if these

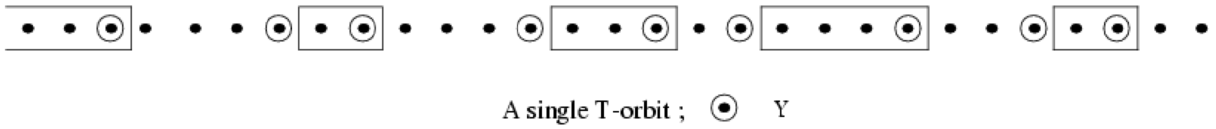

FIGURE 3

values are finite (with obvious modification if some of these values are $+\infty$). By a simple argument it follows that if $T^{i_j}x' \in Y$ for some $i_j \in \{i_1, ..., i_l\}$, then $l' = l$. Thus either $l' = l$ or $l' = 0$. In either case l' is even. This shows that $U = \{(x', T^k x') \mid x' \in V\}$ is contained in R_Y. By definition of the topology $\mathcal{T}$ we have $U \in \mathcal{T}$. □

Remark 4.7. We illustrate by a figure the equivalence classes of R_Y. Let us draw a T-orbit (ordered from left to right) as dots, and circle those dots that are in Y, see Figure 3. The T-orbit splits into two R_Y-equivalence classes (assuming the orbit in question meets Y). One class is everything inside the boxes, while the other class is everything else. Note that if Y meets the T-orbit at exactly one point, say y, we get the splitting at y into the forward and backward T-orbits.

Henceforth all Bratteli diagrams (V, E) that we shall consider will be standard. This entails in particular that the associated path space $X_{(V,E)}$ is compact.

One of our aims is to prove that the étale equivalence relation $(R, \mathcal{T})$ associated to a Cantor minimal system (X, T) is affable. But more than that, we want to prove that (X, T)—or $(R, \mathcal{T})$— is orbit equivalent (cf. Remark 4.2) to the subequivalence relation R_Y of R, obtained by splitting the T-orbit in the forward and backward orbits at Y, where Y is any (non-empty) closed subset Y of X that meets each T-orbit at most once. By Theorem 4.6 we know that R_Y with the relative topology from $(R, \mathcal{T})$ is an AF-equivalence relation. We shall obtain our result without involving the full power of the main theorem, Theorem 2.2 in [GPS], and hence we will avoid using homological algebra as well as C^*-algebra ingredients in our proof. More importantly, the key lemma, Lemma 4.15, that will give us our desired result as a corollary, is of independent interest. It turns out to be a powerful tool in handling more general group actions, something that will be treated in a forthcoming paper.

First, however, we want to prove a converse of the above — a result which is more easily accessible. This converse result is actually an immediate corollary of Lemma 6.1 in [GPS], but we give here a simplified and more direct proof avoiding any mentioning of K-Theory and C^*-algebras.

Theorem 4.8. *Let $(R, \mathcal{T})$ be a minimal AF-equivalence relation on the Cantor set X. Then $(R, \mathcal{T})$ is orbit equivalent to a Cantor minimal system (Y, S), i.e. there exists a homeomorphism $F : X \to Y$ so that $F([x]_R) = orbit_S(F(x))$ for every $x \in X$.*

Proof. By Theorem 3.9 we may assume at the outset that $(R, \mathcal{T})$ is the AF-equivalence relation $AF(V, E)$ associated to a simple Bratteli diagram (V, E), where X is the path space $X_{(V,E)}$ associated to (V, E). Choose a proper ordering on (V, E), and denote the associated Bratteli-Vershik system by (X, T). Let $x_{\max}$, respectively $x_{\min}$, be the unique maximal, respectively minimal, path in X. Choose a point $x_0 \in X$ which is not cofinal with $x_{\max}$ or $x_{\min}$—in other words, $x_0 \notin orbit_T(x_{\max})$. Let $Z = \{x_{\max}, x_0\}$. By Theorem 1.2 there exists an ordered Bratteli diagram $B_Z = (\tilde{V}, \tilde{E}, \geq)$, where $(\tilde{V}, \tilde{E})$ is simple, such that (X, T) is conjugate to the Bratteli-Vershik system, denoted $(\tilde{X}, \tilde{T})$, associated to B_Z. Here $\tilde{X}$ is the path space $X_{(\tilde{V},\tilde{E})}$ associated to $(\tilde{V}, \tilde{E})$. Let $F : X \to \tilde{X}$ be the conjugating map, and let $(\tilde{R}, \tilde{\mathcal{T}}) = AF(\tilde{V}, \tilde{E})$ be the AF-equivalence relation associated to $(\tilde{V}, \tilde{E})$. By Theorem 1.2 we get that F implements an orbit equivalence between $AF(V, E)$ and the equivalence relation $\tilde{R} \subset \tilde{X} \times \tilde{X}$ generated by $AF(\tilde{V}, \tilde{E})(\subset \tilde{X} \times \tilde{X})$ and $(F(x_0), F(Tx_0)) \in \tilde{X} \times \tilde{X}$. Since $F(x_0)$, respectively $F(Tx_0)$, is a max path, respectively a min path, in $\tilde{X}$ with respect to the ordering $\geq$ on $(\tilde{V}, \tilde{E})$, we may give $(\tilde{V}, \tilde{E})$ a new ordering $\geq'$, which is proper, such that $F(x_0)$, respectively $F(Tx_0)$, is the unique max path, respectively unique min path. Let (Y, S) denote the associated Bratteli-Vershik system, where $Y = \tilde{X}$. Then the map $F : X \to Y$ above implements an orbit equivalence between $(R, \mathcal{T}) = AF(V, E)$ and (Y, S). □

Before we prove Lemma 4.15, alluded to above, we need some definitions and preliminary results.

Definition 4.9. Let $(R.\mathcal{T})$ be an étale relation on the space X. Let μ be a probability measure on X. Define the two σ-finite (regular) measures ν_r and ν_s on $(R, \mathcal{T})$ by

$$\nu_r(C) = \int_X \#\left(r^{-1}(x) \cap C\right) d\mu(x), \ \nu_s(C) = \int_X \#\left(s^{-1}(x) \cap C\right) d\mu(x)$$

where C is a Borel subset of R, and $\#(S)$ denotes the cardinality of a set S. We say that μ is *R-invariant* if $\nu_r = \nu_s$.

By [FM;Section 2] this is equivalent to say that μ is G-invariant, where G is any countable group such that $R = R_G$.

Remark 4.10. Let $(R, \mathcal{T})$ be a minimal AF-equivalence relation on the Cantor set X. By Theorem 3.9 we may assume that $(R, \mathcal{T}) = AF(V, E)$, where (V, E) is a simple Bratteli diagram with $X = X_{(V,E)}$. Then μ is a R-invariant probability measure iff μ is T-invariant, where (X, T) is the Bratteli-Vershik system associated to any properly ordered diagram $(V, E, \geq)$, cf. [HPS; Theorem 5.5].

Definition 4.11. Let $(R, \mathcal{T})$ be an étale equivalence relation on the space X. We say that a (non-empty) closed subset Z of X is *thin* (with respect to $(R, \mathcal{T})$) if $\mu(Z) = 0$, for all R-invariant probability measures μ on X.

Let (V, E) be a Bratteli diagram and (W, F) be a subdiagram. If $X_{(W,F)}$ is a thin subset of $X_{(V,E)}$ (with respect to $AF(V, E)$), then we say that (W, F) is a *thin* subdiagram of (W, F).

Lemma 4.12. *Let (W, F) be a thin subdiagram of (V, E), where (V, E) is a simple Bratteli diagram. Given $m \geq 1, K \geq 1$, there exists $n \geq m$ so that for $w \in W_m$, $w' \in W_n$,*

$$K \cdot \#(\{\text{paths in } (W, F) \text{ from } w \text{ to } w'\}) \leq \#(\{\text{paths in } (V, E) \text{ from } w \text{ to } w'\})$$

Proof. For $n \geq m$, let $U_n = \bigcup\{U_{(f_1,,\ldots,f_n)} \mid (f_1, \ldots, f_n) \in P_n(W, F)\}$, where $P_n(W, F)$ denotes the paths in (W, F) from the top vertex $v_0 \in V_0 = W_0$ to a vertex $w \in W_n$. Here $U_{(f_1,\ldots,f_n)}$ is the clopen cylinder set in $X_{(V,E)}$ defined by $U_{(f_1,\ldots,f_n)} = \{(e_1, e_2, \ldots) \in X_{(V,E)} \mid (e_1, e_2, \ldots, e_n) = (f_1, f_2, \ldots, f_n)\}$. Clearly $\bigcap_{n=1}^{\infty} U_n = X_{(W,F)}$. Let

$$(*) \qquad 0 < \delta < \frac{1}{K} \cdot \inf\{\mu\left(U_{(f_1,\ldots,f_m)}\right) \mid (f_1, \ldots, f_m) \in P_m(W, F); \mu \in M(V, E)\}$$

where $M(V,E)$ denotes the (V,E)-invariant probability measures. (Note that $\mu(A) > 0$ for any non-empty clopen set $A \subset X$ and any $\mu \in M(V,E)$, since (V,E) is simple. In fact, $M(V,E)$ may be identified with the set of states on the simple dimension group $K_0(V,E)$, and A (being a finite union of cylinder sets) may naturally be identified with a non-zero element $[A]$ in $K_0(V,E)^+$.) Let $M_n = \{\mu \in M(V,E) \mid \mu(U_n) \geq \delta\}$. Then M_n is w^*-compact, since the characteristic function χ_{U_n} is continuous. Since obviously $M_m \supseteq M_{m+1} \supseteq \cdots$, we get by thinness of $X_{(W,F)}$ that $\bigcap_{m=n}^{\infty} M_n = \emptyset$. By compactness there exists an $n_1 \geq m$ so that $M_{n_1} = \emptyset$, and so $\mu(U_{n_1}) < \delta$ for all $\mu \in M(V,E)$. By $(*)$, we then get $\mu\left(U_{(f_1,\ldots,f_m)}\right) > K\mu(U_{n_1})$ for all $\mu \in M(V,E)$ and all $(f_1,\ldots,f_m) \in P_m(W,F)$. Hence $[U_{(f_1,\ldots,f_m)}] > K[U_{n_1}]$ in $K_0(V,E)$ for all $(f_1,\ldots,f_m) \in P_m(W,F)$, cf. [Ef; Ch.4]. This means that there exists $n \geq n_1$ so that $\chi_{U_{(f_1,\ldots,f_m)}}(v) > K\chi_{U_{n_1}}(v)$ for all $v \in V_n$ and all $(f_1,\ldots,f_m) \in P_m(W,F)$, where we have made the obvious identification of the characteristic function of a clopen set (being a finite union of cylinder sets) with a group element in $K_0(V,E)$. Now let $w \in W_m, w' \in W_n$, and choose $(f_1,\ldots,f_m) \in P_m(W,F)$ so that $f(f_m) = w$. Then

$$
\begin{aligned}
&\#(\{\text{paths from } w \text{ to } w' \text{ in } (V,E)\}) = \chi_{U_{(f_1,\ldots,f_m)}}(w') > K \cdot \chi_{U_{n_1}}(w') \\
&= K \cdot \#(\{\text{paths } (e_1,\ldots,e_n) \text{ from } v_0 \text{ to } w' \text{ in } (V,E) \text{ such that } (e_1,\ldots,e_{n_1}) \in P_{n_1}(W,F)\}) \\
&\geq K \cdot \#(\{\text{paths from } v_0 \text{ to } w' \text{ in } (W,F)\}) \geq K \cdot \#(\{\text{paths from } w \text{ to } w' \text{ in } (W,F)\})
\end{aligned}
$$

This completes the proof. □

Lemma 4.13. *Let (V,E) and (V',E') be two Bratteli diagrams. The following are equivalent:*

(i) $AF(V,E) \cong AF(V',E')$

(ii) $K_0(V,E) \cong K_0(V',E')$, *i.e. $K_0(V,E)$ is order-isomorphic to $K_0(V',E')$ by a map preserving the canonical order units.*

(iii) *There exists a so-called "aggregate" Bratteli diagram $(\tilde{V},\tilde{E})$, so that telescoping $(\tilde{V},\tilde{E})$ to odd levels $0 < 1 < 3 < 5 < \cdots$ yields a telescope of (V,E), while telescoping $(\tilde{V},\tilde{E})$ to even levels $0 < 2 < 4 < 6 < \cdots$ yields a telescope of (V',E').*

(iv) $(V,E) \sim (V',E')$, *where $\sim$ denotes the equivalence relation on Bratteli diagrams generated by telescoping.*

Proof. The equivalence of (ii), (iii) and (iv) is well known, cf. [GPS; Section 3]. The implication (iii)⇒(i) is immediate from the observation we made in Example 2.7 (ii) concerning telescoping of Bratteli diagrams. In fact, $AF(\tilde{V},\tilde{E})$ is isomorphic to both $AF(\tilde{V}_o,\tilde{E}_o)$ and $AF(\tilde{V}_e,\tilde{E}_e)$, where $(\tilde{V}_o,\tilde{E}_o)$, respectively $(\tilde{V}_e,\tilde{E}_e)$, is the telescope of $(\tilde{V},\tilde{E})$ to odd levels, respectively even levels.

We prove (i)⇒(iii). Let $AF(V,E) = \varinjlim(R_N,\mathcal{T}_N)$, $AF(V',E') = \varinjlim(R'_N,\mathcal{T}'_N)$, where $R_N,\mathcal{T}_N$, respectively $R'_N,\mathcal{T}'_N$, have the same meaning as in Example 2.7(ii). There is an obvious groupoid partition associated to R_N, respectively R'_N, which corresponds to the vertex set $V_N \in V$, respectively $V'_N \in V'$. Let $\alpha : X_{(V,E)} \to X_{(V',E')}$ implement the isomorphism between $AF(V,E)$ and $AF(V',E')$. Because of compactness of R_1, $\alpha \times \alpha(R_1)$ is contained in R_{n_1} for some $n_1 \geq 1$. We may choose n_1 so large that the groupoid partition associated to R'_{n_1} is finer than the one associated to $\alpha \times \alpha(R_1)$. By the same procedure as in the proof of Theorem 3.9 we associate edges between the vertices in V_1 and V'_{n_1}, keeping in mind that the groupoid partitions associated to R_1 and $\alpha\times\alpha(R_1)$, respectively, are isomorphic in an obvious way. Next we consider $\alpha^{-1}\times\alpha^{-1}\left(R'_{n_1}\right)$, which by compactness of R'_{n_1} is contained in some R_{n_2}. We choose $n_2>1$ so large that the groupoid partition associated to R_{n_2} is finer than the one associated to $\alpha^{-1}\times\alpha^{-1}\left(R'_{n_1}\right)$. Similarly as above we associate edges between V'_{n_1} and V_{n_2}. Continuing in this way we construct a Bratteli diagram $(\tilde V,\tilde E)$. It is now a simple matter to show that $(\tilde V,\tilde E)$ is an aggregate diagram with respect to (V,E) and (V',E'). We omit the details. □

Remark 4.14. Let (X,T) and (Y,S) be two Cantor minimal systems with associated étale equivalence relations R and R', respectively, cf. Example 2.7(i). Let $x\in X$, $y\in Y$, and let R_x, respectively R'_y, denote the AF-subequivalence relation of R, respectively R' (cf. Theorem 4.3). Then $R_x\cong R'_y$ iff (X,T) and (Y,S) are strong orbit equivalent. This is an immediate consequence of the lemma and Corollary 1.3, in combination with Theorem 2.1 of [GPS].

Lemma 4.15 (Key lemma)**.** *Let $(R,\mathcal{T})$ be isomorphic to $(R',\mathcal{T}')$, where $(R,\mathcal{T})$ and $(R',\mathcal{T}')$ are minimal AF-equivalence relations on the Cantor sets X and X', respectively. Let Z and Z' be closed, thin subsets of X and X' (with respect to $(R,\mathcal{T})$ and $(R',\mathcal{T}')$), respectively. Assume that*

(i) *the (not necessarily minimal) restrictions $R|_Z=R\cap(Z\times Z)$ (respectively $R'|_{Z'}=R'\cap(Z'\times Z')$) with the relative topologies are étale equivalence relations on Z (respectively Z').*
(ii) *there exists a homeomorphism $\alpha: Z\to Z'$ which implements an isomorphism between $R|_Z$ and $R'|_{Z'}$.*

There exists an extension $\tilde\alpha: X\to X'$ of α such that $\tilde\alpha$ implements an isomorphism between $(R,\mathcal{T})$ and $(R',\mathcal{T}')$.

Proof. By Theorem 3.11 we may assume that $(R,\mathcal{T})=AF(V,E)$, $(R',\mathcal{T}')=AF(V',E')$, and $R|_Z=AF(W,F)$, $R'|_{Z'}=AF(W',F')$, where (W,F) and (W',F') are thin subdiagrams of the (simple) Bratteli diagrams (V,E) and (V',E'), respectively. So $X=X_{(V,E)}$, $X'=X_{(V',E')}$, $Z=X_{(W,F)}$, $Z'=X_{(W',F')}$. The idea of the proof is to construct a Bratteli diagram $(\tilde V,\tilde E)$, together with a subdiagram $(\tilde W,\tilde F)$, so that $(\tilde V,\tilde E)$ is an aggregate diagram with respect to (V,E) and (V',E'), while $(\tilde W,\tilde F)$ is an aggregate diagram with respect to (W,F) and (W',F'). Furthermore, we will do this in such way that we can "read off" the map $\alpha: X_{(W,F)}\to X_{(W',F')}$ from $(\tilde W,\tilde F)$ as an "intertwining map" (to be explained below). We will then use $(\tilde V,\tilde E)$ to extend α to $\tilde\alpha: X_{(V,E)}\to X_{(V',E')}$, α being again an intertwining map. We begin the proof by first noticing that by telescoping (V,E), say, we automatically get a corresponding telescoping of (W,F), which again will be a thin subdiagram. Now there is a natural homeomorphism between the path spaces associated to a Bratteli diagram and a telescope of it (cf. Example 2.7(ii)). Thus we may by condition (i) — invoking Lemma 4.13 — assume at the start that there is a Bratteli diagram $(\overline{V},\overline{E})$, so that telescoping $(\overline{V},\overline{E})$ to odd levels $0<1<3<\cdots$ we get (V,E), while telescoping $(\overline{V},\overline{E})$ to even levels $0<2<4<\cdots$ we get (V',E'). Using the notation introduced in Example 2.7(ii), let $V=V_0\cup V_1\cup V_3\cup\cdots$, $E=E_1\cup E_2\cup\cdots$, $W=W_0\cup W_1\cup W_2\cup\cdots$, $F=F_1\cup F_2\cup\cdots$, and similarly for V', E', W', F'. Also, let $AF(W,E)=\varinjlim(R'_N,\mathcal{T}'_N)$, $AF(W',F')=\varinjlim(R'_N,\mathcal{T}'_N)$. Since R_1 is compact, we get by assumption (iii) that there exists $n'_1\geq 1$ so that $\alpha\times\alpha(R_1)\subset R'_{n'_1}$. Furthermore, we may assume n'_1 is chosen so large that the groupoid partition associated to $R'_{n'_1}$ is finer than the one associated to $\alpha\times\alpha\left(R_1\right)$. As in the proof of (i)⇒(iii) in Lemma 4.13 we associate edges between vertices in W_1 and $W'_{n'_1}$. Denote these edges by L. Now V_1 and $V'_{n'_1}$ correspond to level $n_1 = 1$ and level $2n'_1$, respectively, of $(\overline{V},\overline{E})$, and so by telescoping between these levels we get edges connecting vertices in $V_1 = V_{n_1}$ with vertices in $V'_{n'_1}$. Denote these edges by M. Since (W',F') is thin in (V',E') by condition (ii), we may apply Lemma 4.12 and choose n'_1 so large that for any $v\in W_1\subset V_1$ and $v'\in W'_{n'_1}\subset V'_{n'_1}$, the number of edges in M between v and v' is larger than the number of edges in L between v and v'.

Next we consider $\alpha^{-1}\times\alpha^{-1}\left(R'_{n'_1}\right)$. Arguing the same way as above we may find $n_2 > n'_1$ with edge set L, respectively M, between vertices in $W'_{n'_1}$ and W_{n_2}, respectively $V'_{n'_1}$ and V_{n_2}, with the same properties as above. Continuing in this way we construct a Bratteli diagram $(\tilde{V},\tilde{E})$, which we will show has the desired properties. In fact, by its very construction there is a Bratteli subdiagram $(\tilde{W},\tilde{F})$ of $(\tilde{V},\tilde{E})$, so that $(\tilde{W},\tilde{F})$ is an aggregate diagram with respect to (W,F) and (W',F'), while $(\tilde{V},\tilde{E})$ itself is an aggregate diagram with respect to (V,E) and (V',E'). In fact, by our previous deliberations we may assume that telescoping $(\tilde{V},\tilde{E})$, respectively $(\tilde{W},\tilde{F})$, to odd levels $0<1<3<\cdots$, we get (V,E), respectively (W,F) — not just a telescope of these. Likewise, telescoping $(\tilde{V},\tilde{E})$, respectively $(\tilde{W},\tilde{F})$, to even levels $0<2<4<\cdots$, we get (V',E'), respectively (W',F'). Let $\tilde{E}=\tilde{E}_1\cup\tilde{E}_2\cup\cdots$ and $\tilde{F}=\tilde{F}_1\cup\tilde{F}_2\cup\cdots$. We define the composition of edges between levels $k-1$ and $k+1$, where $k\geq 1$, by

$$\begin{aligned}\tilde{E}_k\circ\tilde{E}_{k+1} &= \{(\tilde{e}_k,\tilde{e}_{k+1})\,|\tilde{e}_k\in\tilde{E}_k,\tilde{e}_{k+1}\in\tilde{E}_{k+1}, f\left(\tilde{e}_k\right)=i\left(\tilde{e}_{k+1}\right)\}\\ \tilde{F}_k\circ\tilde{F}_{k+1} &= \{(\tilde{f}_k,\tilde{f}_{k+1})|\tilde{f}_k\in\tilde{F}_k,\tilde{f}_{k+1}\in\tilde{F}_{k+1}, f(\tilde{f}_k)=i(\tilde{f}_{k+1})\}.\end{aligned}$$

We will establish bijections

$$(a)\begin{cases}\tilde{E}_{k-1}\circ\tilde{E}_k\longleftrightarrow E'_{\frac{k}{2}}\\ \tilde{E}_k\circ\tilde{E}_{k+1}\longleftrightarrow E_{\frac{k}{2}+1}\end{cases}\qquad (b)\begin{cases}\tilde{F}_{k-1}\circ\tilde{F}_k\longleftrightarrow F'_{\frac{k}{2}}\\ \tilde{F}_k\circ\tilde{F}_{k+1}\longleftrightarrow F_{\frac{k}{2}+1}\end{cases}$$

for every even number $k=2,4,\ldots$, which will respect the range and source maps — recalling that $\tilde{V}_m=V_{\frac{m+1}{2}}$, $\tilde{W}_m=W_{\frac{m+1}{2}}$ for m odd, and $\tilde{V}_m=V'_{\frac{m}{2}}$, $\tilde{W}_m=W'_{\frac{m}{2}}$ for m even. (In addition we have $\tilde{E}_1=E_1$, $\tilde{F}_1=F$, and $\tilde{V}_0=V_0=V'_0=\tilde{W}_0=W_0=W'_0$, all being equal to the top vertex of $(\tilde{V},\tilde{E})$.) The bijections will be chosen successively, and in such a way that we will be able to read off the given map $\alpha: X_{(W,F)}\to X_{(W',F')}$ by this intertwining process. Furthermore, the bijections in (a) shall extend the ones in (b), keeping in mind that the various edge sets occurring in (b) are contained in the corresponding ones occurring in (a).

We first consider the bijections (b), starting with $\tilde{F}_1=F_1$. By the embedding scheme that we outlined above, the inclusion $\alpha\times\alpha\left(R_1\right)\subset R'_1$ determines uniquely the edges $\tilde{F}_2$ between $\tilde{W}_1=W_1$ and $\tilde{W}_2=W'_1$. This in turn sets up a bijection $\tilde{F}_1\circ\tilde{F}_2\longleftrightarrow F'_1$ in an obvious way, respecting the range and source maps. Similarly, the inclusion $\alpha^{-1}\times\alpha^{-1}\left(R'_1\right)\subset R_2$ determines uniquely the edges $\tilde{F}_3$ between $\tilde{W}_2=W'_1$ and $\tilde{W}_3=W_2$, and this in turn sets up a bijection $\tilde{F}_2\circ\tilde{F}_3\longleftrightarrow F_2$ in an obvious way, respecting the range and source maps. Continuing in this way we get all the bijections in (b). Now these bijections induce a map between $X_{(W,F)}$ and $X_{(W',F')}$, which will be equal to α. In fact, if $x=(f_1,f_2,\ldots)\in X_{(W,F)}$, where $f_i\in F_i$, then the successive bijections $\tilde{F}_k\circ\tilde{F}_{k+1}\longleftrightarrow F_{\frac{k}{2}+1}$ of (b), starting with $\tilde{F}_1=F_1$,

will determine a unique path in $X_{(\tilde{W},\tilde{F})}$. Then, using the bijections $\tilde{F}_{k-1} \circ \tilde{F}_k \longleftrightarrow F'_{\frac{k}{2}}$ of (b), we conclude that x determines a unique path $y = (\tilde{f}_1, \tilde{f}_2, \ldots) \in X_{(W',F')}$, where $f'_i \in F'_i$. One shows easily that $y = \alpha(x)$ — we omit the details. We say that $\alpha : X_{(W,F)} \to X_{(W',F')}$ is defined as an *intertwining map* (via the aggregate diagram $(\tilde{W}, \tilde{F})$). (Cf. also [GPS;Theorem 2.1, proof of (ii)⇒(i)].)
To conclude the proof of the lemma we extend the bijection in (b) to (a). We do this successively, starting with $\tilde{E}_1 = E_1$. We may choose the various bijections in (a), which extend the ones in (b), in an arbitrary way, the only proviso being that the source and range maps are respected. The associated intertwining map $\tilde{\alpha} : X_{(V,E)} \to X_{(V',E')}$ will clearly be a homeomorphism that extends α. By its very construction, $\tilde{\alpha}$ clearly preserves cofinality. Furthermore, the n first edges $\{f'_1, f'_2, \ldots, f'_n\}$ of $y = \tilde{\alpha}(x) = (f'_1, f'_2, \ldots) \in X_{(V',E')}$, $f'_i \in E'_i$, is determined by the n first edges of $x = (e_1, e_2, \ldots) \in X_{(V,E)}$, $e_i \in E_i$. (A similar statement is true for $\tilde{\alpha}^{-1}$.) This implies that $\tilde{\alpha}$ implements an isomorphism between $AF(V,E)$ and $AF(V',E')$ (cf. the description we gave of convergence in Example 2.7(ii)). □

We are now in a position to prove the following non-trivial result, which is a converse to Theorem 4.8.

Theorem 4.16. *Let (X,T) be a Cantor minimal system, and let R denote the equivalence relation associated to (X,T), i.e. the R-equivalence classes are the T-orbits. Then R is affable.*

In fact, more is true: R is orbit equivalent to R_Y, where Y is any non-empty closed subset of X that meets each T-orbit at most once. (Cf. Theorem 4.6 for the definition of R_Y, and the proof that R_Y is affable.)

Proof. The main ingredient of the proof will be Lemma 4.15, in combination with Theorem 1.2. However, we need a preliminary result in order to set the stage, so to say, before we can apply Lemma 4.15. So let Y be any non-empty closed subset of X that meets each T-orbit at most once. We shall need the following technical result, which we state as a sublemma, and whose proof we postpone to the end in order not to interrupt the flow of the main argument.

Sublemma. *There exists a point $y \in X$ and a sequence of pairwise disjoint, non-empty clopen sets $\{U_n\}_{n=1}^{\infty}$, each of which are disjoint from Y, together with a sequence of non-empty closed sets $\{Y_n\}_{n=1}^{\infty}$, where $Y_n \subset U_n$ for each n, so that*

(i) *$U_n \to y$ (i.e. for each neighbourhood U of y, there exists N so that $n \geq N$ implies $U_n \subset U$).*

(ii) *The set $Z = Y \cup \{y\} \cup \bigcup_{n=1}^{\infty} Y_n$ is a closed set such that Z meets each T-orbit at most once.*

(iii) *There exists a sequence $\{h_n\}_{n=0}^{\infty}$ of homeomorphisms $h_0 : Y \to Y_1, h_1 : Y_1 \to Y_2, h_2 : Y_2 \to Y_3, \ldots$*

Applying the sublemma we may define a homeomorphism $h : Z \to Z'$, where $Z' = \{y\} \cup \bigcup_{n=1}^{\infty} Y_n$. In fact, define h by $h|_{Y_n} = h_n$, $n = 0, 1, 2, \ldots$, and set $h(y) = y$. Using Theorem 1.2 we construct a Bratteli-Vershik model for (X,T) — which we still will denote by (X,T) — based on the closed set Z, and we let $B = (V, E, \geq)$ denote the associated ordered Bratteli diagram. Hence $X = X_{(V,E)}$, and T is the Vershik map. Furthermore, since Z are the maximal paths and $T(Z)$ are the minimal paths in X, $Z \cup T(Z)$ will be the path space associated to a thin subdiagram (W, F) of (V, E), i.e. $Z \cup T(Z) = X_{(W,F)}$. In fact, (W, F) is a *tree* and it is obviously thin because of condition (ii) of the sublemma. By Theorem 1.2, R_Z is equal to the cofinal relation associated to (V, E). Furthermore, it is clear that

$$(*) \qquad R = R_Z \vee \{(z, Tz) \,|z \in Z\} \quad , \quad R_Y = R_Z \vee \{(z, Tz) \,|z \in Z'\},$$

where we here let $A \vee B$ denote the equivalence relation on X generated by the two subsets A, B of $X \times X$.
Set $(V', E') = (V, E)$, and let (W', E') be the thin subdiagram of (V', E') associated to $Z' \cup T(Z')$, i.e. $Z' \cup T(Z') = X_{(W',F')}$. (Note that (W', F') is a subdiagram of (W, F) .) There is a homeomorphism $\alpha : X_{(W,F)} \to X_{(W',F')}$, namely $\alpha|_Z = h$ and $\alpha(Tz) = T(h(z)), z \in Z$. Because of condition (ii) of the sublemma, α is well-defined and is a homeomorphism. Also, α clearly implements an isomorphism between $AF(W, F)$ and $AF(W', F')$, which is an immediate consequence of the fact that both the Bratteli diagrams (W, F) and (W', F') are trees. Hence all the conditions of Lemma 4.15 are satisfied. Let $\tilde{\alpha} : X_{(V,E)} \to X_{(V',E')}$ be the extension of α, with the properties stated in Lemma 4.15. Clearly $\tilde{\alpha} \times \tilde{\alpha}(R_Z) = R_Z$, since $\tilde{\alpha}$ preserves the cofinal relation associated to $(V, E) = (V', E')$. Also, $\tilde{\alpha} \times \tilde{\alpha}(\{(z, Tz)\,|z \in Z\}) = \alpha \times \alpha(\{(z, Tz)\,|z \in Z\}) = \{(z, Tz)\,|z \in Z'\}$. Hence, by $(*)$, $\tilde{\alpha} \times \tilde{\alpha}(R) = R_Y$. *Proof of sublemma.* We first show that if V is a non-empty clopen subset of X that is disjoint from Y, then we may find a closed subset Y' of V, such that $Y \cup Y'$ meets each T-orbit at most once, and, furthermore, there exists a homeomorphism $h : Y \to Y'$. To obtain this we pick a finite partition $\{A_1, \ldots, A_{n_1}\}$ of Y consisting of non-empty clopen sets (Y is given the relative topology from X) such that the diameters of each A_i is less than $\frac{1}{2}$. Also, pick $B_1, \ldots B_{n_1}$ to be non-empty clopen and pairwise disjoint subsets of V, each of diameter less than $\frac{1}{2}$, such that

$$(i) \quad T^i(B_k) \cap T^j(B_l) = \emptyset \quad \text{for } -2 \leq i, j \leq 2, \quad 1 \leq k, l \leq n_1.$$

$$(ii) \quad B_k \cap T^i(Y) = \emptyset \qquad \text{for } -2 \leq i \leq 2, \quad 1 \leq k \leq n_1.$$

This can clearly be achieved — as for (ii) we note that $\bigcup_{i=-2}^{2} T^i(Y)$ is a closed subset of X with empty interior. Next we partition each of the A_i's into non-empty clopen sets of diameter less than $\frac{1}{3}$, with corresponding picking of clopen subsets of the B_i's. For example, say the partition of A_1 is $\{A_1^1, \ldots, A_1^{m_1}\}$. We pick m_1 non-empty and pairwise disjoint clopen subsets $B_1^1, \ldots, B_1^{m_1}$ of B_1, each with diameter less than $\frac{1}{3}$, such that they satisfy properties (i) and (ii), where i and j now range between -3 and 3, and k and l range from 1 to m_1. Continuing like this we get nested sequences of A's converging to every point in Y, together with nested sequences of B's converging to points in V, and we denote this set of points by Y'. There is a map $h : Y \to Y'$ induced by the obvious $1-1$ correspondence between nested A- and B-sequences, and it is a routine matter to verify that Y' and h satisfy the desired properties. To finish the proof of the sublemma, choose $y \in X \setminus Y$ so that $Y_0' = Y \cup \{y\}$ meets each T-orbit at most once, and let $\{U_n\}_{n=1}^{\infty}$ be a sequence of pairwise disjoint, non-empty clopen sets (each disjoint from Y) converging to y. Applying what we just have proved, choose successively non-empty closed sets $Y_1', Y_2', Y_3', \ldots$ such that $Y_1' \subset U_1, Y_2' \subset U_2, Y_3' \subset U_3, \ldots$ and homeomorphisms $h_0' : Y_0' \to Y_1', h_1' : Y_0' \cup Y_1' \to Y_2', h_2' : Y_0' \cup Y_1' \cup Y_2' \to Y_3', \ldots$ so that for each k, $Y_0' \cup Y_1' \cup Y_2' \cup \ldots \cup Y_k'$ meets each T-orbit at most once. Define successively $Y_0 = Y, Y_1 = h_1'(Y_0), Y_2 = h_1'(Y_1), Y_3 = h_2'(Y_2), \ldots$ and $h_k = h_k'|_{Y_k}$ for $k \geq 0$. Then $Z = Y \cup \{y\} \cup \bigcup_{n=1}^{\infty} Y_n$ is a closed set that meets each T-orbit at most once. This finishes the proof of the sublemma, and consequently the proof of the theorem. □

The following theorem, which essentially is corollary of Theorem 4.16, can succinctly be stated to say that a finite extension of a minimal AF-equivalence relation is affable.

Theorem 4.17. *Let $(R, \mathcal{T})$ be a minimal AF-equivalence relation on the Cantor set X. Let $(x_1, y_1), \ldots, (x_n, y_n)$ be n pairs of points in $X \times X$. Let R' be the equivalence relation on X generated by R and $(x_1, y_1), \ldots, (x_n, y_n)$. The R' is affable.*

Proof. Clearly it is enough to prove the statement for the special case $n = 1$, since the general case follows by induction. By Theorem 3.9 we may assume that $(R, \mathcal{T}) = AF(V, E)$, where (V, E) is a simple, standard Bratteli diagram, and $X = X_{(V,E)}$. If x_1 and y_1 are cofinal paths, then $R' = R$, and there is nothing to prove. So assume x_1 and y_1 are not cofinal. According to Proposition 1.1, there exists a Cantor minimal system (X, T) such that T respects cofinality, except that $Tx_1 = y_1$. By Theorem 4.16 the equivalence relation R' associated to (X, T) is affable. Now R' is the equivalence relation generated by R and (x_1, y_1). □

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