EVEN POWERS OF DIVISORS AND ELLIPTIC ZETA VALUES

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Abstract. We introduce and study *elliptic zeta values*, a two-parameter deformation of the values of Riemann's zeta function at positive integers. They are essentially Taylor coefficients of the logarithm of the elliptic gamma function, and share the $\mathrm{SL}(3,\mathbb{Z})$ modular properties of this function. Elliptic zeta values at even integers are related to Eisenstein series and thus to sums of odd powers of divisors. The elliptic zeta values at odd integers can be expressed in terms of generating series of sums of even powers of divisors.

Dedicated to Igor Frenkel on the occasion of his 50^{th} birthday

1. Introduction

The generating function of the sum of kth powers of divisors of positive integers

$$\sum_{n=1}^{\infty} \sigma_k(n) q^n, \qquad \sigma_k(n) = \sum_{d\,|\,n} d^k,$$

converges in $|q| < 1$. If k is odd, it has interesting transformation properties under the modular group $\mathrm{SL}(2,\mathbb{Z})$. Indeed, it is closely related to the Eisenstein series G_k (see e. g. [2]): let

$$D_k(q) = \frac{(-2\pi i)^{k+1}}{k!} \sum_{n=1}^{\infty} \sigma_k(n) q^n.$$

Then

$$G_k(\tau) = 2\zeta(2k) + 2D_{2k-1}(q), \qquad q = e^{2\pi i \tau}, \qquad k = 1, 2, 3, \ldots,$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function. For $k \geq 2$, these functions are modular forms of weight $2k$:

$$G_k\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{2k} G_k(\tau), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}),$$

as can be seen from the representation $G_k(\tau) = \sum_{(n,m)\neq(0,0)} (n + m\tau)^{-2k}$. It can easily be deduced from this that

$$\lim_{\tau \to 0} \tau^{2k} D_{2k-1}(q) = \zeta(2k).$$

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In other words, we may regard $\tau^{k+1}D_k(q)$ as a one-parameter deformation of the zeta value $\zeta(k+1)$, and this is true also if k is even. For even k, however, $D_k(q)$ does not have obvious modular properties. The purpose of this note is to show that $D_k(q)$ can be embedded into a two-parameter deformation of $\zeta(k+1)$, the *elliptic zeta value* at $k+1$, which obeys identities of modular type. These identities are essentially equivalent to modularity in the even case but are of a different nature in the odd case. They are related to (and a consequence of) the $\mathrm{SL}(3,\mathbb{Z})$ properties of the elliptic gamma function [1].

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2. Differences of modular forms

Let H be the upper half-plane $\operatorname{Im}\tau > 0$.

Proposition 2.1. *Let k be a positive integer. Suppose $Z(\tau,\sigma)$ is a holomorphic function on $H \times H$ admitting an expansion*

$$Z(\tau,\sigma) = \sum_{n,m=0}^{\infty} a_{n,m} q^n r^m, \qquad q = e^{2\pi i\tau}, \quad r = e^{2\pi i\sigma},$$

and obeying $Z(\tau,\sigma) = -Z(\sigma,\tau)$. Then the following statements are equivalent:

(i) *$Z(\tau,\sigma) = G(\tau) - G(\sigma)$ for some modular form G of weight $2k$.*
(ii) *Z obeys the three-term relations*

$$\begin{aligned} Z(\tau,\sigma) &= Z(\tau,\tau+\sigma) + Z(\tau+\sigma,\sigma), \\ Z(\tau,\sigma) &= \tau^{-2k} Z\left(-\frac{1}{\tau}, \frac{\sigma}{\tau}\right) + \sigma^{-2k} Z\left(-\frac{\tau}{\sigma}, -\frac{1}{\sigma}\right), \end{aligned}$$

for all $\sigma,\tau \in H$ such that $\sigma/\tau \in H$.

Proof: It is easy to check that if G is a modular form then $G(\tau) - G(\sigma)$ obeys the three-term relations. Conversely, let us extend $a_{n,m}$ to all integers n, m by setting $a_{n,m} = 0$ if n or m is negative. The first three-term relation implies that $a_{n-m,m} + a_{n,m-n} = a_{n,m}$. It follows that $a_{n,m} = a_{n-m,m}$ if $n > m$ and $a_{n,m} = a_{n,m-n}$ if $m > n$. By the Euclidean algorithm we see that for $n, m > 0$, $a_{n,m} = a_{N,N}$, where $N = (n,m)$ is the greatest common divisor. But $a_{N,N} = 0$ since Z is odd under interchange of q and r. Thus only $a_{n,0}, a_{0,m}$ may be non-zero and $Z(\tau,\sigma) = G(\tau) - G(\sigma)$ for some function G admitting an expansion in powers of q. In particular, $G(\tau+1) = G(\tau)$. The second three-term relation then implies that $G(-1/\tau) = \tau^{2k} G(\tau)$ possibly after redefining G by adding a constant. Thus G is a modular form of weight $2k$. □

3. Elliptic zeta values

We now define two-parameter deformations of the values of the zeta function at positive integers, which we call *elliptic zeta values*:

$$(1)\quad Z_n(\tau,\sigma) = -\frac{(2\pi i)^n}{(n-1)!}\sum_{j=1}^{\infty} j^{n-1}\frac{q^j-(-1)^n r^j}{(1-q^j)(1-r^j)}\,, \quad q=e^{2\pi i\tau}, \quad r=e^{2\pi i\sigma}, \quad n\in\mathbb{Z}_{\geq 1}.$$

Clearly, $Z_n(\tau,\sigma) = -(-1)^n Z_n(\sigma,\tau)$. The relation to the functions D_k is obtained by expanding the elliptic zeta values in a power series in q and r. The result is the following.

If $n=2k$ is even, then

$$Z_{2k}(\tau,\sigma) = D_{2k-1}(r) - D_{2k-1}(q).$$

If $n=2k+1$ is odd, then

$$Z_{2k+1}(\tau,\sigma) = D_{2k}(q) + D_{2k}(r) + 2\sum_{(a,b)=1} D_{2k}(q^a r^b).$$

The sum is over pairs of relatively prime pairs of positive integers a,b.

Theorem 3.1.

(i) *Let $n\geq 4$. Then Z_n obeys the three-term relations*

$$\begin{aligned} Z_n(\tau,\sigma) &= Z_n(\tau,\tau+\sigma) + Z_n(\tau+\sigma,\sigma),\\ Z_n(\tau,\sigma) &= \tau^{-n} Z_n\left(-\frac{1}{\tau},\frac{\sigma}{\tau}\right) + (-\sigma)^{-n} Z_n\left(-\frac{\tau}{\sigma},-\frac{1}{\sigma}\right), \end{aligned}$$

for all $\sigma,\tau\in H$ such that $\sigma/\tau\in H$.

(ii) $\lim_{\sigma\to 0}\lim_{\tau\to i\infty}\sigma^n Z_n(\tau,\sigma) = \zeta(n)$ *if $n\geq 2$.*

(iii) *For $n=1$, $\lim_{\sigma\to 0}\lim_{\tau\to i\infty}(\sigma Z_1(\tau,\sigma)+\ln(-2\pi i\sigma))$ is the Euler constant $\gamma = 0.577\ldots$ (the logarithm is real if σ is imaginary).*

In particular, by taking the limit of Z_n as $\tau\to i\infty$ we obtain

$$\begin{aligned} \lim_{\sigma\to 0}(\sigma D_0(e^{2\pi i\sigma}) + \ln(-2\pi i\sigma)) &= \gamma,\\ \lim_{\sigma\to 0}\sigma^{n+1}D_n(e^{2\pi i\sigma}) &= \zeta(n+1), \qquad n=1,2,\ldots \end{aligned}$$

We prove Theorem 3.1 in the next section.

4. The elliptic gamma function

Theorem 3.1 follows from the results of [1] where the modular properties of Ruijsenaars' elliptic gamma function were discovered. In the normalization of [1], the elliptic gamma function is defined by the double infinite product

$$\Gamma(z,\tau,\sigma) = \prod_{j,k=0}^{\infty}\frac{1-q^{j+1}r^{k+1}e^{-2\pi i z}}{1-q^j r^k e^{2\pi i z}} \qquad z\in\mathbb{C}, \tau,\sigma\in H.$$

It obeys the functional relation $\Gamma(z+\sigma,\tau,\sigma) = \theta_0(z,\tau)\Gamma(z,\tau,\sigma)$, which is an elliptic version of Euler's functional equation for the gamma function. Here θ_0 denotes the theta function

$$\theta_0(z,\tau) = \prod_{j=0}^{\infty}(1-q^{j+1}e^{-2\pi iz})(1-q^je^{2\pi iz}) \qquad z\in\mathbb{C}, \tau\in H.$$

The Euler gamma function $\Gamma(z)$ is recovered in the limit

$$\Gamma(z) = \lim_{\sigma\to 0}\lim_{\tau\to i\infty}\theta_0(\sigma,\tau)^{1-z}\frac{\Gamma(\sigma z,\tau,\sigma)}{\Gamma(\sigma,\tau,\sigma)}. \tag{2}$$

Theorem 4.1. ([1], *p. 54*) *Suppose* $\tau,\sigma,\sigma/\tau\in H$. *Then*

$$\begin{aligned}\Gamma(z,\tau,\sigma) &= \Gamma(z+\tau,\tau,\sigma+\tau)\Gamma(z,\tau+\sigma,\sigma),\\ \Gamma(z/\tau,-1/\tau,\sigma/\tau) &= e^{i\pi Q(z;\tau,\sigma)}\Gamma((z-\tau)/\sigma,-\tau/\sigma,-1/\sigma)\Gamma(z,\tau,\sigma),\end{aligned}$$

for some polynomial $Q(z,\tau,\sigma)$ *of degree three in* z *with coefficients in* $\mathbb{Q}(\tau,\sigma)$.

It follows from the Weierstrass product representation $\Gamma(z+1) = e^{-\gamma z}\prod_{j=1}^{\infty}(1+z/j)^{-1}e^{jz}$ that the values of ζ at positive integers are essentially Taylor coefficients at 1 of the logarithm of the Euler gamma function:

$$\ln\Gamma(z+1) = -\gamma z + \sum_{j=2}^{\infty}\frac{\zeta(j)}{j}(-z)^j. \tag{3}$$

Here $\gamma = \lim_{n\to\infty}(\sum_{j=1}^{n-1}1/j - \ln n)$ is the Euler constant. The elliptic analog of this formula involves the elliptic zeta values:

$$\ln\frac{\Gamma(z+\sigma,\tau,\sigma)}{\Gamma(\sigma,\tau,\sigma)} = \sum_{j=1}^{\infty}\frac{Z_j(\tau,\sigma)}{j}(-z)^j. \tag{4}$$

This formula, with Z_j defined by eq. (1), is an easy consequence of the summation formula for $\ln\Gamma(z,\tau,\sigma)$ ([1] p. 51):

$$\ln\Gamma(z,\tau,\sigma) = -\frac{i}{2}\sum_{j=1}^{\infty}\frac{\sin(\pi j(2z-\tau-\sigma))}{j\sin(\pi j\tau)\sin(\pi j\sigma)}.$$

Proof of Theorem 3.1: The first claim of the theorem is proven by taking the logarithm of the identities of Theorem 4.1 and expanding them in powers of $z-\sigma$.

Taking the limit of (4), by using (2) to compare it with (3) implies (ii).

In the same way we can deduce (iii). However, if $n=1$, we have to take into account the factor θ_0^{1-z} in (2): we obtain $\gamma = \lim_{\sigma\to 0}\lim_{\tau\to\infty}(\sigma Z_1(\tau,\sigma)+\ln\theta_0(\sigma,\tau))$. Since $\ln\theta_0(\sigma,\tau) = \ln(-2\pi i\sigma)$ plus terms that vanish in the limit, the proof is complete. □

By using the explicit formula for the polynomial Q, we also obtain the exceptional relations for $n=1,2,3$:

Theorem 4.2. *Let* $1 \leq n \leq 3$ *and suppose* $\tau, \sigma, \sigma/\tau \in H$. *Then*

$$\begin{aligned} Z_n(\tau,\sigma) &= Z_n(\tau,\tau+\sigma) + Z_n(\tau+\sigma,\sigma), \\ Z_n(\tau,\sigma) &= \tau^{-n} Z_n\left(-\frac{1}{\tau}, \frac{\sigma}{\tau}\right) + (-\sigma)^{-n} Z_n\left(-\frac{\tau}{\sigma}, -\frac{1}{\sigma}\right) + i\pi a_n, \end{aligned}$$

where

$$\begin{aligned} a_1 &= -\frac{1}{2} + \frac{1}{2\tau} - \frac{1}{2\sigma} + \frac{\sigma}{6\tau} + \frac{\tau}{6\sigma} + \frac{1}{6\tau\sigma}, \\ a_2 &= -\frac{1}{\tau} + \frac{1}{\sigma} - \frac{1}{\tau\sigma}, \\ a_3 &= \frac{1}{\tau\sigma}. \end{aligned}$$

References

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