Green's generic syzygy conjecture for curves of even genus lying on a $K3$ surface

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1 Introduction

If C is a smooth projective curve of genus g and K_C is its canonical bundle, the theorem of Noether asserts that the multiplication map

$$\mu_0 : H^0(C, K_C) \otimes H^0(C, K_C) \to H^0(C, K_C^{\otimes 2})$$

is surjective when C is non hyperelliptic.

The theorem of Petri concerns then the ideal I of C in its canonical embedding, assuming C is not hyperelliptic. It says that I is generated by its elements of degree 2 if C is neither trigonal nor a plane quintic.

In [7], M. Green introduced and studied the Koszul complexes

$$\bigwedge^{p+1} H^0(X,L)\otimes H^0(X,L^{q-1}) \xrightarrow{\delta} \bigwedge^{p} H^0(X,L)\otimes H^0(X,L^{q}) \xrightarrow{\delta} \bigwedge^{p-1} H^0(X,L)\otimes H^0(X,L^{q+1})$$

for X a variety and L a line bundle on X. Denoting by $K_{p,q}(X, L)$ the cohomology at the middle of the sequence above, one sees immediately that the surjectivity of the map μ_0 is equivalent to $K_{0,2}(C, K_C) = 0$, and that if this is the case, the ideal I is generated by quadrics if and only if $K_{1,2}(C, K_C) = 0$. On the other hand, C being non hyperelliptic is equivalent to the fact that the Clifford index $\mathrm{Cliff}\, C$ is strictly positive, where

$$\mathrm{Cliff} C := Min\{d - 2r,\, \exists L \in Pic\, C, d^0 L = d,\, h^0(L) = r + 1 \geq 2,\, h^1(L) \geq 2\}.$$

Similarly, C is neither hyperelliptic, nor trigonal nor a plane quintic if and only if $\mathrm{Cliff}\, C > 1$.

Green's conjecture on syzygies of canonical curves generalizes then the theorems of Noether and Petri as follows

Conjecture 1 *[7] For a smooth projective curve C in characteristic 0, the condition $\mathrm{Cliff}\, C > l$ is equivalent to the fact that $K_{l',2}(C, K_C) = 0, \forall l' \leq l$.*

The interest of this formulation of Noether and Petri's theorems is already illustrated in [9], where these theorems are given a modern proof, using geometric technics of computation of syzygies.

For our purpose, and as is done in [7], it is convenient to use the duality (cf [7])

$$K_{p,2}(C, K_C) \cong K_{g-p-2,1}(C, K_C)^*$$

to reformulate the conjecture as follows

Conjecture 2 *[7] For a smooth projective curve C of genus g in characteristic 0, the condition* $\mathrm{Cliff}\, C > l$ *is equivalent to the fact that* $K_{g-l'-2,1}(C, K_C) = 0$, $\forall l' \leq l$.

If C is now a generic curve, the theorem of Brill-Noether (cf [2], [11]) implies that

$$\mathrm{Cliff}\, C = \mathrm{gon(C)} - 2$$

where the gonality $\mathrm{gon(C)} := Min\,\{d,\, \exists L \in Pic\, C,\, d^0 L = d,\, h^0(L) \geq 2\}$, and that

$$\mathrm{gon(C)} = \frac{g+3}{2},\ if\ g\ is\ odd,$$

$$\mathrm{gon(C)} = \frac{g+2}{2},\ if\ g\ is\ even.$$

Hence we arrive at the following conjecture (the generic Green conjecture on syzygies of a canonical curve) :

Conjecture 3 *Let C be a generic curve of genus g. Then if $g = 2k+1$ or $g = 2k$, we have $K_{k,1}(C, K_C) = 0$.*

Remark 1 *The actual conjecture is $K_{l,1}(C, K_C) = 0$, $\forall l \geq k$; but it is easy to prove that*

$$K_{k,1}(C, K_C) = 0 \Rightarrow K_{l,1}(C, K_C) = 0,\ \forall l \geq k.$$

Notice that in the appendix to [7], Green and Lazarsfeld prove the conjecture 1 in the direction $\Leftarrow$ (i. e. they produce non zero syzygies from special linear systems.) Hence the conjecture above cannot be improved, namely, under the assumptions above, we have $K_{k-1,1}(C, K_C) \neq 0$.

Teixidor [16] has recently proposed an approach to the conjecture 3. Her method uses a degeneration to a tree of elliptic curves and the theory of limit linear series of Eisenbud and Harris [6], adapted to vector bundles of higher rank. It is very likely that her method will lead to a proof of the generic Green conjecture.

We propose here a completely different approach, which does not prove Conjecture 3 in odd genus, but proves Green's Conjecture 2 for generic curves C of genus $g(C)$ and gonality gon(C) in the range

$$\frac{g(C)}{3} + 1 \leq \mathrm{gon(C)} \leq \frac{g(C)}{2} + 1.$$

The inequality on the left says that the gonality has to be not too small, but the only case which is excluded by the inequality on the right is that of a generic curve of odd genus g, which has gonality $\frac{g+3}{2}$.

Recall from [11] that if S is a $K3$ surface endowed with a ample line bundle L such that L generates $Pic\, S$ and $L^2 = 2g-2$, the smooth members $C \in \mid L \mid$ are of genus g and generic in the sense of Brill-Noether, so that in particular they have the same Clifford index as a generic curve. Hence conjecture 1 predicts that their syzygies vanish as stated in conjecture 3. This is indeed what we prove here, in the case where the genus is even. Note first that the hyperplane restriction theorem [7] says that

$$K_{k,1}(C, K_C) = K_{k,1}(S, L) \tag{1.1}$$

whenever C is a hyperplane section of a $K3$ surface S (note that $K_C = L_{|C}$ in this case). Conjecture 3 for curves of even genus is therefore implied by

Theorem 1 *The pair (S, L) being as above, with $g = 2k$, we have*

$$K_{k,1}(S, L) = 0. \tag{1.2}$$

The body of the paper will be devoted to the proof of (1.2). It turns out that Theorem 1 in turn has much stronger consequences than the generic syzygy conjecture for curves of even genus, and we shall explain this now. In fact we have the following corollary :

Corollary 1 *For any $\delta \leq \frac{k}{2}$, the generic curve of genus $2k-\delta$ which is $k+1-\delta$-gonal satisfies*

$$K_{k,1}(C, K_C) = 0$$

or equivalently by the duality theorem

$$K_{k-\delta-2,2}(C, K_C) = 0.$$

Notice that this result is optimal and exactly predicted by Green's conjecture 1, since the Clifford index of such curve is less than or equal to $k-1-\delta$. An easy computation shows that the pairs

$$g = 2k-\delta,\ \mathrm{gon} = \mathrm{k}+1-\delta,\ \mathrm{k} \geq 0,\ \delta \leq \frac{\mathrm{k}}{2}$$

are exactly the pairs satisfying the inequalities

$$\frac{g}{3} + 1 \leq \mathrm{gon} \leq \frac{g}{2} + 1.$$

Hence Green's Conjecture is proved for generic curves of genus and gonality in this range. On the other hand, Teixidor [17] has proved the Green conjecture for generic curves of fixed gonality in the range $\mathrm{gon} \leq \frac{g}{3}$. Combining this and our corollary we see that Green's conjecture is proved for generic curves of any fixed gonality, with the exception of generic curves of odd genus (which satisfy $\mathrm{gon} = \frac{g}{2} + 1 + \frac{1}{2}$).

Proof of Corollary 1. Let (S, L) be as in theorem 1. A generic member X of $\mid L \mid$ is $k+1$-gonal. As in section 2, and following [11], it follows that there is a rank 2 vector bundle E on S with $det\, E = L$, $c_2(E) = k+1$, and $h^0(E) = k+2$. The zero set of a generic section of E is a generic member of a g^1_{k+1} of a generic curve $X \in \mid L \mid$.

Now let $x_1, \ldots, x_\delta$ be generic points of S. Beacause $\delta \leq \frac{k}{2}$, the space

$$H_{x.} = H^0(S, E \otimes \mathcal{I}_{x_1} \otimes \ldots \otimes \mathcal{I}_{x_\delta})$$

has rank at least 2. One checks that for α, β generic in this space, the curve X defined by the equation

$$det\,(\alpha \wedge \beta) \in H^0(S, det\, E) = H^0(S, L)$$

is nodal with nodes exactly as the x_i's. On the other hand, the two sections α, β generate a rank 1 subsheaf of the restriction $E_{|X}$. Let now

$$n : C \to X$$

be the normalization. The rank 1 subsheaf introduced above induces a line subbundle

$$D \subset n^* E$$

with two sections, and it is obvious that the moving part of this linear system on C is of degree $k+1-\delta$, since the sections $\lambda\alpha + \mu\beta$ of E vanish at the x_i's, so that the moving part of their zero sets is of degree $k+1-\delta$. Hence C is $k+1-\delta$-gonal. It remains to show that

$$K_{k,1}(C, K_C) = 0. \tag{1.3}$$

Now we have by the hyperplane restriction theorem and by theorem 1 the vanishing

$$K_{k,1}(X, K_X) = 0. \tag{1.4}$$

We prove now that this implies (1.3). Notice that there is an identification of $H^0(C, K_C)$ with a subspace of $H^0(X, K_X)$, namely the last space is a space of meromorphic forms on C with logarithmic singularities over the nodes satisfying the condition that the sum of the residues over each node vanishes. $H^0(C, K_C)$ is then the subspace of forms which are regular.

From this inclusion $j : H^0(C, K_C) \hookrightarrow H^0(X, K_X)$, we can deduce a commutative diagram of Koszul complexes

$$\begin{array}{ccccc}
\bigwedge^{k+1} H^0(C, K_C) & \to & \bigwedge^{k} H^0(C, K_C) \otimes H^0(C, K_C) & \to & \bigwedge^{k-1} H^0(C, K_C) \otimes H^0(C, K_C^{\otimes 2}) \\
j \downarrow & & j \downarrow & & j \downarrow \\
\bigwedge^{k+1} H^0(X, K_X) & \to & \bigwedge^{k} H^0(X, K_X) \otimes H^0(X, K_X) & \to & \bigwedge^{k-1} H^0(X, K_X) \otimes H^0(X, K_X^{\otimes 2}).
\end{array}$$

We claim that this induces an inclusion

$$j : K_{k,1}(C, K_C) \hookrightarrow K_{k,1}(X, K_X).$$

Indeed, consider in general the Koszul differential

$$\delta : \bigwedge^{l} H^0(Y,\mathcal{L}) \to H^0(Y,\mathcal{L}) \otimes \bigwedge^{l-1} H^0(Y,\mathcal{L}).$$

Then if

$$\wedge : H^0(Y,\mathcal{L}) \otimes \bigwedge^{l-1} H^0(Y,\mathcal{L}) \to \bigwedge^{l} H^0(Y,\mathcal{L})$$

is the wedge product map, one has

$$\wedge \circ \delta = \pm l Id. \tag{1.5}$$

Consider now the inclusion

$$j : H^0(C,K_C) \otimes \bigwedge^{k} H^0(C,K_C) \to H^0(X,K_X) \otimes \bigwedge^{k} H^0(X,K_X).$$

Let $\alpha \in H^0(C,K_C) \otimes \bigwedge^k H^0(C,K_C)$ such that $\delta\alpha = 0$ and $j(\alpha) = \delta\beta$. Then (1.5) gives

$$\begin{aligned} j(\alpha) = \delta\beta &= \pm\frac{1}{k+1}\delta(\wedge \circ \delta\beta) \\ &= \pm\frac{1}{k+1}\delta(\wedge(j(\alpha))). \end{aligned}$$

But $\wedge(j(\alpha)) = j(\wedge\alpha)$, so that this implies by injectivity of j that $\alpha = \pm\frac{1}{k+1}\delta(\wedge\alpha)$. Hence α is in fact exact. Hence our claim is proven. ∎

In the missed case of a generic curve of odd genus, we have the following corollary :

Corollary 2 *Let C be a generic curve of genus $g = 2k-1$; then*

$$K_{k,1}(C,K_C) = 0.$$

(Notice that the generic Green conjecture predicts in fact that $K_{k-1,1}(C,K_C) = 0$.)

Proof of Corollary 2. The $K3$ surface S being as above, let X be a member of $\mid L \mid$ with exactly one node as singularity. Let C be the normalization of X. Then the genus of C is equal to $2k-1$.

We have as before an inclusion

$$H^0(C,K_C) \hookrightarrow H^0(X,K_X) \tag{1.6}$$

which induces by the same argument as in the proof of corollary 1 an inclusion

$$K_{k,1}(C,K_C) \hookrightarrow K_{k,1}(X,K_X).$$

The hyperplane restriction theorem can be applied to $X \subset S$, and together with the vanishing (1.2), it gives

$$K_{k,1}(X,K_X) = 0.$$

Hence $K_{k,1}(C,K_C) = 0$. ∎

We conclude this introduction with a sketch of the main ideas in the proof of theorem 1. The very starting point is the following observation : denote by $S^{[l]}$ the Hilbert scheme parametrizing 0-dimensional length l subschemes of S. Let $I_l \subset S \times S^{[l]}$ be the incidence subscheme and

$$\begin{array}{ccc} I_l & \stackrel{\pi_l}{\rightarrow} & S^{[l]} \\ q\downarrow & & \\ S & & \end{array}$$

be the incidence correspondence. Let

$$\mathcal{E}_L := R^0 \pi_{l*} q^* L$$

and $L_l := det\, \mathcal{E}_L$. Then we have

Fact. $K_{l-1,1}(S, L) = 0$ *if and only if*

$$H^0(I_l, \pi_l^* L_l) = \pi_l^* H^0(S^{[l]}, L_l).$$

Our strategy will be then to construct a subvariety Z of $S^{[k+1]}$, such that

$$H^0(\tilde{Z}, \pi_l^* L_l) = \pi_l^* H^0(Z, L_l)$$

where $\tilde{Z} := \pi_l^{-1}(Z)$, and the restriction map

$$H^0(I_l, \pi_l^* L_l) \to H^0(\tilde{Z}, \pi_l^* L_l)$$

is injective.

As in the papers [11], [8], the key role in constructing our variety Z and verifying the conditions above will be played by the Lazarsfeld-Mukai vector bundle on S associated with minimal degree base-point free linear systems on smooth members of $\mid L \mid$.

Terminology. In this paper, we shall say that a Zariski open subset $U \subset X$ is large if the complementary closed subset $Z = X - U$ has codimension non smaller that 2 in X. In the considered cases, the variety X will be normal, and we will use freely the fact that for a line bundle $\mathcal{L}$ on X

$$H^0(X, \mathcal{L}) \cong H^0(U, \mathcal{L}_{|U})$$

for U a large open subset of X.

2 Strategy of the proof

We start with the following observation : Let X be a smooth projective variety. Denote by $X^{[k]}_{curv}$ the Hilbert scheme parametrizing curvilinear 0-dimensional subschemes of X of length k. $X^{[k]}_{curv}$ is smooth, and if X is a curve or a surface, it is a large open set in the Hilbert scheme $X^{[k]}$ which is smooth.

Let

$$\begin{array}{ccc} I_k & \stackrel{\pi_k}{\to} & X^{[k]}_{curv} \\ q\downarrow & & \\ X & & \end{array}$$

be the incidence correspondence. For a line bundle L on X denote by $\mathcal{E}_L$ the vector bundle on $X^{[k]}_{curv}$ defined by $\mathcal{E}_L = R^0\pi_{k*}q^*L$, and let

$$L_k := det\,\mathcal{E}_L.$$

We have

Lemma 1 *There is a natural isomorphism*

$$K_{k,1}(X,L) \cong H^0(I_{k+1}, \pi^*_{k+1}L_{k+1})/\pi^*_{k+1}H^0(X^{[k+1]}_{curv}, L_{k+1}).$$

In particular, $K_{k,1}(X,L) = 0$ *is equivalent to*

$$H^0(I_{k+1}, \pi^*_{k+1}L_{k+1}) = \pi^*_{k+1}H^0(X^{[k+1]}_{curv}, L_{k+1}).$$

Proof. Recall that $K_{k,1}(X,L)$ is the cohomology at the middle of the sequence

$$\bigwedge^{k+1} H^0(X,L) \stackrel{\delta}{\to} H^0(X,L)\otimes \bigwedge^{k} H^0(X,L) \stackrel{\delta}{\to} H^0(X,L^{\otimes 2})\otimes \bigwedge^{k-1} H^0(X,L). \quad (2.7)$$

Now note that there is a natural morphism

$$\tau : I_{k+1} \to X \times X^{[k]}_{curv} \quad (2.8)$$

which to (x,z), $x \in Supp\, z$ associates (x,z'), where z' is the residual scheme of x in z. This morphism is well defined because we are working with curvilinear schemes.

One shows easily that τ identifies I_{k+1} to a large open subset of the blow-up of $X \times X^{[k]}_{curv}$ along the incidence subscheme I_k. (Indeed, away from I_k the inverse τ^{-1} of τ is given by

$$\tau^{-1}((x,z)) = x \cup z,\ x \in X,\ z \in X^{[k]}_{curv}.)$$

Furthermore, if $D \subset I_{k+1}$ is the exceptional divisor one has

$$\pi^*_{k+1}L_{k+1} = \tau^*(L \boxtimes L_k)(-D). \quad (2.9)$$

This isomorphism is obtained by studying the natural morphism of vector bundles over I_{k+1}

$$\pi^*_{k+1}\mathcal{E}_L \to \tau^*(pr_1^*E \oplus pr_2^*\mathcal{E}_L),$$

which at a point $\tilde{z}$ such that $\tau(\tilde{z}) = (x,z')$ identifies to the restriction map

$$H^0(L_{|\tilde{z}}) \to L_{|x} \oplus H^0(L_{|z'}).$$

It is immediate to see that the cokernel of this morphism is supported on the exceptional divisor D of τ and is of rank 1 on D.

It follows that

$$\begin{aligned} &H^0(I_{k+1}, \pi_{k+1}^* L_{k+1}) \\ &= \mathrm{Ker}\,(H^0(X, L) \otimes H^0(X_{curv}^{[k]}, L_k) \stackrel{rest}{\rightarrow} H^0(I_k, L \boxtimes L_{k|I_k})). \end{aligned} \tag{2.10}$$

We now apply the description above to I_k : we note that denoting by p_i, $i = 1, 2$, the compositions of the projections with the inclusion $I_k \hookrightarrow X \times X_{curv}^{[k]}$, we have

$$p_2 = \pi_k,\ p_1 = pr_1 \circ \tau,$$

where

$$\tau : I_k \to X \times X_{curv}^{[k-1]}$$

is defined as in (2.8). Hence applying formula (2.9), we get

$$L \boxtimes L_{k|I_k} = \tau^*(L^2 \boxtimes L_{k-1})(-D),$$

where D is now the exceptional divisor of the blowing-down morphism $\tau : I_k \to X \times X_{curv}^{[k-1]}$. So we conclude that there is a natural inclusion

$$i : H^0(I_k, L \boxtimes L_{k|I_k}) \subset H^0(X, L^{\otimes 2}) \otimes H^0(X_{curv}^{[k-1]}, L_{k-1}).$$

Hence we have constructed from formula (2.10) an exact sequence

$$0 \to H^0(I_{k+1}, \pi_{k+1}^* L_{k+1}) \stackrel{j}{\rightarrow} H^0(X, L) \otimes H^0(X_{curv}^{[k]}, L_k)$$

$$\stackrel{i \circ rest}{\rightarrow} H^0(X, L^{\otimes 2}) \otimes H^0(X_{curv}^{[k-1]}, L_{k-1}).$$

Next, it is a standard fact that the natural map

$$\bigwedge^l H^0(X, L) \to H^0(X_{curv}^{[l]}, L_l) \tag{2.11}$$

induced by the evaluation map

$$H^0(X, L) \otimes \mathcal{O}_{X_{curv}^{[k]}} \to \mathcal{E}_L$$

are isomorphisms for any l. To check this, one considers the large open subset $U^{[l]}$ of $X_{curv}^{[l]}$, made of subschemes z which have at most one point of multiplicity 2 as singularity. This set $U^{[l]}$ has the following description : one considers inside X^l the large open set X_0^l made of l-uples $(x_1, \ldots, x_l)$, where at most two of the x_i's coincide. Inside X_0^l, one blows-up the generalized diagonal $\cup_{i \neq j}\{x_i = x_j\}$, which is smooth there, and one takes the quotient of the resulting variety $\widetilde{X_0^l}$ by the action of the symmetric group S_l. By the same argument as above one finds that if

$$r : \widetilde{X_0^l} \to U^{[l]}$$

is the quotient map, one has

$$r^*L_l \cong L \boxtimes \ldots \boxtimes L(-E),$$

where E is the exceptional divisor of the blowing-up map $\tau' : \widetilde{X_0^l} \to X_0^l$. It follows that one has an identification

$$H^0(X^{[l]}_{curv}, L_l) = H^0(U^{[l]}, L_l) \cong H^0(\tilde{X_0^l}, \tau'^*(\boxtimes_{i=1}^{i=l} L)(-E))^{S_l}.$$

The last space is a subspace of

$$H^0(X_0^l, \boxtimes_{i=1}^{i=l} L)^{S_l} \cong H^0(X^l, \boxtimes_{i=1}^{i=l} L)^{S_l} = (\otimes_{i=1}^{i=l} H^0(X, L))^{S_l}.$$

But looking more closely at the action of S_l on the line bundle r^*L_k, one verifies that the induced action of S_l on $\otimes_{i=1}^{i=l} H^0(X, L)$ is the twisted action. Hence the invariant subspace

$$(\otimes_{i=1}^{i=l} H^0(X, L))^{S_l}$$

is isomorphic to $\bigwedge^l H^0(X, L)$. One verifies then that the injective map

$$H^0(X^{[l]}_{curv}, L_l) \to \bigwedge^l H^0(X, L)$$

so obtained is a left inverse for the map (2.11). ∎

Using the isomorphisms (2.11), the exact sequence above becomes

$$0 \to H^0(I_{k+1}, \pi_{k+1}^* L_{k+1}) \xrightarrow{j} H^0(X, L) \otimes \bigwedge^k H^0(X, L)$$

$$\xrightarrow{i \circ rest} H^0(X, L^{\otimes 2}) \otimes \bigwedge^{k-1} H^0(X, L),$$

hence it remains it remains only to show that the maps $j \circ \pi_{k+1}^*$ and $i \circ rest$ identify via the isomorphisms (2.11) to the differentials δ of the sequence (2.7), which is easy. ∎

We consider now a $K3$ surface S endowed with an ample line bundle L generating $Pic\, S$ and satisfying

$$L^2 = 2g - 2,\ g = 2k,\ k > 1.$$

We now explain our strategy to prove the vanishing

$$K_{k,1}(S, L) = 0.$$

Assume we have a subscheme $T \subset S^{[k+1]}$ such that, if $\tilde{T}$ denotes the subvariety $\pi_{k+1}^{-1}(T)$ of I_{k+1}, the following conditions are satisfied :

1. We have an isomorphism

$$H^0(\tilde{T}, \pi_{k+1}^* L_{k+1}) = \pi_{k+1}^* H^0(T, L_{k+1}).$$

2. The restriction map

$$H^0(I_{k+1}, \pi_{k+1}^* L_{k+1}) \to H^0(\tilde{T}, \pi_{k+1}^* L_{k+1})$$

is injective.

Then we claim that $K_{k,1}(S, L) = 0$.

Indeed we have the trace maps

$$tr : H^0(I_{k+1}, \pi_{k+1}^* L_{k+1}) \to H^0(S_{curv}^{[k+1]}, L_{k+1})$$

$$tr_T : H^0(\tilde{T}, \pi_{k+1}^* L_{k+1}) \to H^0(T, L_{k+1})$$

which commute with the restriction maps and which compose to $(k+1)\, Id$ with the pull-back maps. If $\sigma \in H^0(I_{k+1}, \pi^* L_{k+1})$, there exists by property 1 a $\beta \in H^0(T, L_{k+1})$ such that

$$\sigma_{|\tilde{T}} = \pi_{k+1}^* \beta.$$

Then

$$\beta = \frac{1}{k+1} tr_T(\sigma_{|\tilde{T}}) = (\frac{1}{k+1} tr\, \sigma)_{|T}.$$

Hence the section

$$\sigma' = \sigma - \pi^*(\frac{1}{k+1} Tr\, \sigma)$$

vanishes on $\tilde{T}$, hence it is zero by property 2. So we have

$$H^0(I_{k+1}, \pi^* L_{k+1}) = \pi^* H^0(S_{curv}^{[k+1]}, L_{k+1})$$

and this proves our claim, using lemma 1.

We will have to weaken the assumptions above as follows : Suppose we have a normal scheme Z together with a morphism

$$j : Z \to I_{k+1}$$

such that $\pi \circ j$ is generically one to one on its image, which is not contained in the branch locus of π_{k+1}. Suppose also that we have a normal scheme Z' together with a proper degree k morphism $\pi' : Z' \to Z$ and a morphism $j' : Z' \to I_{k+1}$ satisfying the conditions that

$$\pi_{k+1} \circ j' = j \circ \pi'$$

and the union $j(Z) \cup j'(Z')$ is equal set theoretically to $\pi_{k+1}^{-1}(\pi_{k+1} \circ j(Z))$. Finally assume there are subschemes $Z_1' \subset Z'$, $Z_1 \subset Z$ such that

$$\pi'_{|Z_1'} =: \phi : Z_1' \to Z_1$$

is a birational isomorphism and $j \circ \phi = j'_{|Z_1'}$.

(Hence roughly speaking, and up to birational maps, $\pi_{k+1}^{-1}(\pi_{k+1} \circ j(Z))$ is the scheme obtained by gluing Z' and Z along $Z_1' \cong Z_1$.)

Assume now that they satisfy the following set (H) of hypotheses

1. The map

$$\pi'^* : H^0(Z, (\pi_{k+1} \circ j)^* L_{k+1}) \to H^0(Z', (\pi_{k+1} \circ j')^* L_{k+1})$$

   is an isomorphism.

2. The restriction map

$$H^0(Z, (\pi_{k+1} \circ j)^* L_{k+1}) \to H^0(Z_1, (\pi_{k+1} \circ j)^* L_{k+1|Z_1})$$

   is injective.

3. The restriction map

$$j^* : H^0(I_{k+1}, \pi_{k+1}^* L_{k+1}) \to H^0(Z, (\pi_{k+1} \circ j)^* L_{k+1})$$

   is injective.

Then we claim that $K_{k,1}(S, L) = 0$.

Indeed by Lemma 1 we have to show that

$$H^0(I_{k+1}, \pi_{k+1}^* L_{k+1}) = \pi_{k+1}^* H^0(S_{curv}^{[k+1]}, L_{k+1}).$$

Now if $\sigma \in H^0(I_{k+1}, \pi_{k+1}^* L_{k+1})$, by hypothesis H1, $j'^* \sigma = \pi'^* \alpha$ for some $\alpha \in H^0(Z, (\pi_{k+1} \circ j)^* L_{k+1})$. We show now that $j^* \sigma = \alpha$. Indeed, by property H2, it suffices to show that this is true after restriction to Z_1, and since $\phi : Z'_1 \to Z_1$ is dominating, it suffices to show that

$$\phi^*(\alpha_{|Z_1}) = \phi^*(j^* \sigma_{|Z_1}).$$

But this follows from $j \circ \phi = j'_{|Z'_1}$ and from $j'^* \sigma = \pi'^* \alpha$, with $\phi = \pi'_{|Z'_1}$.

Finally it follows from the equalities $\alpha = j^* \sigma$ and $j'^* \sigma = \pi'^* \alpha$ that

$$\sigma' = \sigma - \pi^*(\frac{1}{k+1} Tr\, \sigma)$$

vanishes along $j(Z) \cup j'(Z')$. (Indeed, because we know that $j(Z)$ is not contained in the ramification locus of π_{k+1} and $\pi_{k+1} \circ j : Z \to X_{curv}^{[k+1]}$ is generically one to one on its image, the map π_{k+1} restricted to $j(Z) \cup j'(Z')$ is generically a degree $k+1$ unramified map onto its image. On the other hand, the two equalities above say that $\sigma_{|j(Z) \cup j'(Z')}$ is generically a pull-back of a section of L_{k+1} on this image. Hence the previously used trace argument applies to an open set of this image.)

Now this implies that σ' vanishes by hypothesis H3. This concludes the proof of our claim. ■

We conclude this section with the description of the schemes Z, Z' we will be considering.

Recall from [8], [11], [12], that there is a unique stable bundle E of rank 2 on S, (the Lazarsfeld-Mukai vector bundle,) which satisfies the following properties:

$$det\, E = L,\; c_2(E) = k+1,\; h^0(E) = k+2.$$

Such vector bundle is obtained by choosing a line bundle D on a generic member C of $\mid L \mid$, such that $h^0(D) = 2$ and $\deg D = k+1$. Such a line bundle exists by Brill-Noether theory, and it is generated by global sections since C does not carry a g^1_k by Lazarsfeld [11]. Then we have a vector bundle F on S defined by the exact sequence

$$0 \to F \to H^0(D) \otimes \mathcal{O}_S \to D \to 0 \tag{2.12}$$

and E is defined as the dual of F. The stability of E follows from the fact that $Pic\, S = \mathbb{Z}L$ and $H^0(S, E(-L)) = 0$. The uniqueness of such E follows then from the fact that $\chi(E, E') = 2$ for any other vector bundle E' with the same numerical properties, so that either $Hom\,(E, E') \neq 0$ or $Hom\,(E', E) \neq 0$. But then by stability, $E = E'$.

The property $h^0(S, E) = k+2$ follows from the sequence dual to (2.12)

$$0 \to H^0(D)^* \otimes \mathcal{O}_S \to E \to K_C - D \to 0, \tag{2.13}$$

and from Riemann-Roch which gives $h^0(K_C - D) = k$.

Another way to construct the bundle E is via Serre's construction. By Riemann-Roch the divisors D of degree $k+1$ on smooth members C of $\mid L \mid$ which satisfy $h^0(C, D) = 2$ are exactly the subschemes z of degree $k+1$ on S contained in a smooth member C of $\mid L \mid$ and satisfying the condition that the restriction map

$$H^0(S, L) \to H^0(L_{|z})$$

is not surjective. Note that since the curves C are general in the sense of Brill-Noether, the corank of this map is exactly 1 and furthermore for any $z' \subsetneq z$ the restriction map

$$H^0(S, L) \to H^0(L_{|z'})$$

is surjective. Hence, since K_S is trivial, to such z corresponds a vector bundle E together with a section σ_z vanishing on z. This E is an extension

$$0 \to \mathcal{O}_S \stackrel{\sigma_z}{\to} E \stackrel{\wedge\sigma_z}{\to} \mathcal{I}_z(L) \to 0. \tag{2.14}$$

Computing the numerical invariants of this bundle E, and arguing as before by stability, we see that this bundle is isomorphic to the one constructed above. Notice that each g^1_{k+1}, D on a smooth member $C \in\mid L \mid$ provides by (2.13) a rank 2 subspace of sections of E, and that the zero sets of these sections identify to the members of $\mid D \mid$, as subschemes of S.

It follows from the exact sequence (2.14) twisted by E that $h^0(S, E\otimes\mathcal{I}_z) = 1$ for any z as above. Hence the morphism

$$\mathbb{P}(H^0(S,E)) \to S^{[k+1]}$$

which to σ associates its zero set, is in fact an embedding. One sees easily that the open set $\mathbb{P}(H^0(S,E))_{curv}$ corresponding to curvilinear subschemes is large in $\mathbb{P}(H^0(S,E))$.

Let now $W := \pi_{k+1}^{-1}(\mathbb{P}(H^0(S,E))_{curv}) \subset I_{k+1}$. W is easily shown to be smooth. There is a natural morphism

$$\psi : W \to S^{[k]}_{curv}$$

defined as the restriction of $pr_2 \circ \tau$ to W. This ψ can be shown to be generically of degree one on its image.

Consider the blow-up $\widetilde{S\times W}$ of $S\times W$ along $K := (Id,\psi)^{-1}(I_k)$. It admits a morphism $\widetilde{(Id,\psi)}$ to the blow-up of $S\times S^{[k]}_{curv}$ along I_k, and the later contains I_{k+1} as a large open set. One verifies that $\widetilde{(Id,\psi)}^{-1}(I_{k+1})$ is a large open set of $\widetilde{S\times W}$. This will be our scheme Z. The morphism $j : Z \to I_{k+1}$ will be simply the restriction to Z of $\widetilde{(Id,\psi)}$.

Again one can show (using now the assumption that $k > 1$) that the morphism $\pi_{k+1}\circ j : Z \to S^{[k+1]}$ is generically of degree one on its image.

Next let $\pi'' : \tilde{W} \to W$ be the degree k cover obtained by completing the Cartesian diagram

$$\begin{array}{ccc} \tilde{W} & \to & I_k \\ \pi'' \downarrow & & \pi_k \downarrow \\ W & \stackrel{\psi}{\to} & S^{[k]}_{curv} \end{array}.$$

Consider the rational map

$$j' : S\times\tilde{W} - - - - > I_{k+1}$$

which to (s, s_1, w), $s_1 \in Supp\,\psi(w)$ associates $(s_1, s\cup\psi(w))$. This morphism becomes well defined after blowing-up $K' := (Id,\pi'')^{-1}(K)$ and restricting to a large open subset. Our scheme Z' will be this large open set. The morphism $\pi' : Z' \to Z$ is the restriction to Z' of the morphism $Bl_{K'}(S\times\tilde{W}) \to Bl_K(S\times Z)$ induced by (Id,π''). The morphism $j' : Z' \to I_{k+1}$ is induced by the rational map j' above. We have

$$\pi_{k+1}\circ j' = \pi_{k+1}\circ j\circ\pi'.$$

Indeed, both maps send (s, s_1, w), $s_1 \in Supp\,\psi(w)$ to $s\cup\psi(w)$. It is obvious that $\pi_{k+1}^{-1}(\pi_{k+1}\circ j(Z))$ is equal to $j(Z)\cup j'(Z')$. Indeed, the fiber over $s\cup\psi(w) \in \pi_{k+1}\circ j(Z)$ consists in choosing one point in the scheme $s\cup\psi(w)$. This point may be s, in which case we are in $j(Z)$, or has to be a point s_1 contained in $Supp\,\psi(w)$ in which case it determines a point (s_1, w) of $\tilde{W}$ over w, and we are then in $j'(Z')$.

Remark 2 *The scheme Z is non necessarily smooth, but one can show that K is reduced, so that its singular locus is of codimension at least two in $S \times W$. The same thing is true for Z' and K'. If one wants to work with smooth schemes Z_0 and Z_0' (so as to be exactly in the conditions (H) described above), it suffices to restrict to the blowing-ups of $S \times W - K_{sing}$ along $K - K_{sing}$ and $S \times W - K'_{sing}$ along $K' - K'_{sing}$. All what follows will be true for these subschemes.*

To conclude, it remains now to construct Z_1 and Z_1'. Z_1 will be the exceptional divisor of Z (recall that Z is a large open set in $Bl_K(S \times W)$). Hence Z_1 is the inverse image under the blow-up map $Z \to S \times W$ of

$$K = \{(s, w) \in S \times W,\, s \in Supp\, \psi(w)\}.$$

We now construct a generic lifting of Z_1 in Z', the closure of the image of which will be Z_1'. By definition of Z' as a large open set of $Bl_{K'}(S \times \tilde{W})$, it suffices to construct a lifting of K to a component of K' in $S \times \tilde{W}$. But if $(s, w) \in K$, we have $s \in Supp\, \psi(w)$ so that (s, w) identifies to an element $\tilde{w}$ of $\tilde{W}$. Our lifting sends simply (s, w) to $(s, \tilde{w})$.

It remains finally to see that the morphisms j' and $j \circ \pi'$ agree on Z_1'. Since I_{k+1} is contained in $S \times S_{curv}^{[k+1]}$, it suffices to prove that $pr_1 \circ j'$ and $pr_1 \circ j \circ \pi'$ agree on Z_1' and that $pr_2 \circ j'$ and $pr_2 \circ j \circ \pi'$ agree on Z_1', with $pr_2 = \pi_{k+1}$ on I_{k+1}. For the first one this is obvious since both maps factor through the contraction $Z_1' \to K'$, and are equal on $K' \subset S \times \tilde{W}$ to the first projection on S, as follows from the definition of the lifting $K \to K'$.

As for the second one, it follows from the fact that, by construction, $\pi_{k+1} \circ j'$ and $\pi_{k+1} \circ j \circ \pi'$ agree on Z'. ∎

3 Proof of the assumptions H2 and H3

We start with the proof of hypothesis H2.

Proposition 1 *Let*

$$Z_1 \subset Z \overset{\pi_{k+1} \circ j}{\to} S^{[k+1]}$$

be as in the previous section. Then the restriction map

$$H^0(Z, (\pi_{k+1} \circ j)^* L_{k+1}) \to H^0(Z_1, (\pi_{k+1} \circ j)^* L_{k+1|Z_1})$$

is injective.

The proof will be obtained by restricting the construction to a generic smooth member $C \in\mid L \mid$. Indeed, recall that Z is a large open set in the blow-up of $S \times W$ along the incidence subscheme $K = (id, \psi)^{-1}(I_k)$, where

$$W = \{(x, \sigma) \in S \times \mathbb{P}(H^0(S, E))_{curv},\, \sigma(x) = 0\},$$

and $\psi : W \to S^{[k]}$ sends (x,σ) to the residual scheme of x in $V(\sigma)$. Now since $k \geq 1$, the generic element $z = V(\sigma)$ is supported in a pencil of elements of $\mid L \mid$, the generic member being smooth. It follows that a generic element of $S \times W$ is of the form (s_1, s_2, z), $z = V(\sigma)$, $\sigma(s_2) = 0$ and there exists a smooth member $C \in\mid L \mid$ such that s_1, s_2, z are supported on C. Hence it suffices to prove the analogue of proposition 1 with Z replaced by Z_C, the proper transform of $C \times W_C$ in $Z \subset Bl_K(S \times W)$, where

$$W_C := \{(c,\sigma) \in C \times \mathbb{P}(H^0(S,E)),\, \sigma(c) = 0,\, V(\sigma) \subset C\},$$

and Z_1 is replaced by $Z_{1,C} := Z_1 \cap Z_C$.

Proposition 2 *The restriction map*

$$H^0(Z_C, (\pi_{k+1} \circ j)^* L_{k+1|Z_C}) \to H^0(Z_1, (\pi_{k+1} \circ j)^* L_{k+1|Z_{1,C}})$$

is injective.

Proof. By the description of the bundle E given in the previous section, we note that the set

$$\{\sigma \in \mathbb{P}(H^0(S,E)),\ V(\sigma) \subset C\},$$

identifies by the map $\sigma \mapsto V(\sigma)$ to the disjoint union of the $\mathbb{P}^1 \subset C^{(k+1)}$ corresponding to g^1_{k+1}'s on C. If D is such a g^1_{k+1} on C, D gives a morphism of degree $k+1$

$$\phi_D : C \to \mathbb{P}^1$$

or a line bundle L_D on C of degree $k+1$ with two sections. By definition, W_C identifies (via ψ) to the disjoint union of copies C_D of C contained in $C^{(k)}$ and indexed by the g^1_{k+1}'s D of C, where the map

$$\psi_D : C \cong C_D \to C^{(k)}$$

is given by

$$c \mapsto \text{the unique effective divisor equivalent to } D - c.$$

Finally Z_C identifies to a disjoint union of surfaces $Z_{C,D}$ isomorphic to $C \times C$, since the pull-back Δ_D to $C \times C_D$ of the incidence scheme in $C \times C^{(k)}$ is of pure codimension 1, so that the blow-up of $C \times C_D$ along Δ_D is isomorphic to $C \times C_D$. Note that under this isomorphism, the intersection of Z_1 with $C \times C_D$ becomes identified to Δ_D.

Recall now that on the large open set $Z \subset \widetilde{S \times W}$, we have

$$(\pi_{k+1} \circ j)^* L_{k+1} = \tau^*(L \boxtimes \psi^* L_k)(-Z_1).$$

We have $L_{|C} = K_C$ and in the sequel we will use the notation H_D for the line bundle $L_{k|C_D}$. (It will be shown that $H_D \equiv kL_D$ but this will not be used now.) Restricting the equality above to Z_C, we have to show that for each D the restriction map

$$H^0(C \times C, K_C \boxtimes H_D(-\Delta_D)) \to H^0(\Delta_D, K_C \boxtimes H_D(-\Delta_D)_{|\Delta_D})$$

is injective. In other words we want to show that

$$H^0(C \times C, K_C \boxtimes H_D(-2\Delta_D)) = 0. \tag{3.15}$$

Now, since Δ_D is the restriction to $C \times C_D$ of the incidence scheme, and since C_D parametrizes the effective divisors of the form $L_D - x$, $x \in C$, it is clear that

$$\Delta_D = (\phi_D, \phi_D)^{-1}(diag\,(\mathbb{P}^1)) - diag\,(C).$$

Hence we have

$$\Delta_D \equiv L_D \boxtimes L_D - diag\,(C)$$

in $C \times C$. It follows that

$$K_C \boxtimes H_D(-2\Delta_D) \equiv (K_C - 2L_D) \boxtimes (H_D - 2L_D) + 2diag\,C.$$

Now we have the equality

$$H^0(C, K_C - 2L_D) = 0, \tag{3.16}$$

which is proven in [11], since C is generic in S. (Indeed for a base point free pencil, $\mid L_D \mid$, the condition that the μ_0-map

$$H^0(C, L_D) \otimes H^0(C, K_C - L_D) \to H^0(C, K_C)$$

is injective is equivalent by the base-point free pencil trick to the condition

$$H^0(C, K_C - 2L_D) = 0.$$

The equality (3.15) follows now from (3.16) and from the fact that the map $H^0(C, 2L_D) \to H^0(2L_{D|2x})$ is surjective for generic x in C. Hence by Riemann-Roch, $H^0(C, K_C - 2L_D) = 0$ implies $H^0(C, K_C - 2L_D + 2x) = 0$ for generic $x \in C$. It follows that

$$H^0(C \times C, (K_C - 2L_D) \boxtimes (H_D - 2L_D) + 2diag\,C) = 0,$$

which proves the proposition 2, and hence proposition 1 is proven. ∎

We turn now to the proof of hypothesis H3.

Proposition 3 *The morphism $Z \stackrel{j}{\to} I_{k+1}$ being defined as in the previous section, the pull-back map*

$$j^* : H^0(I_{k+1}, \pi_{k+1}^* L_{k+1}) \to H^0(Z, (\pi_{k+1} \circ j)^* L_{k+1})$$

is injective.

The proof proceeds in several steps, and occupies the remainder of this section. Recall that I_{k+1} is a large open set in the blow-up of $S\times S^{[k]}$ along the incidence subscheme I_k and that we have the following formula

$$\pi_{k+1}^*L_{k+1}=\tau^*(L\boxtimes L_k)(-D),$$

where D is the exceptional divisor and τ is the blowing-up map. Since Z is a large open set in the proper transform of this blowing-up under the morphism $(Id,\psi):S\times W\rightarrow S\times S^{[k]}$, it suffices to prove

Proposition 4 *The restriction map*

$$\psi^*:H^0(S^{[k]},L_k)\rightarrow H^0(W,\psi^*L_k)$$

is injective.

In order to prove this proposition, we first show

Lemma 2 *Denoting by $\pi:W\rightarrow\mathbb{P}(H^0(S,E))$ the restriction of the morphism $\pi_{k+1}:I_{k+1}\rightarrow S^{[k+1]}$, we have the formula*

$$\psi^*L_k=\pi^*\mathcal{O}_{\mathbb{P}(H^0(S,E))}(k).$$

Proof. By definition, $\psi^*L_k=det\,\psi^*\mathcal{E}_{L,k}$, where the bundle $\mathcal{E}_{L,k}$ has for fiber $H^0(L_{|z})$ at a point $z\in S^{[k]}$. Now, if $z\in W$, the scheme $z'=\psi(z)$ has length k, hence the restriction map

$$H^0(S,L)\rightarrow H^0(L_{|z'})$$

is surjective. On the other hand if $z''=\pi(z)$, we have $z'\subset z''$ and the restriction map

$$H^0(S,L)\rightarrow H^0(L_{|z''})$$

is not surjective. Hence we have

$$H^0(S,L\otimes\mathcal{I}_{z'})=H^0(S,L\otimes\mathcal{I}_{z''}),$$

and the fiber of $\psi^*\mathcal{E}_{L,k}$ at z is canonically isomorphic to $H^0(S,L)/H^0(S,L\otimes\mathcal{I}_{\pi(z)})$. Hence we have

$$\psi^*L_k=-\pi^*det\mathcal{F},$$

where the bundle $\mathcal{F}$ on $\mathbb{P}(H^0(S,E))$ is the bundle with fiber $H^0(S,L\otimes\mathcal{I}_{z_\sigma})$ at σ, $z_\sigma=V(\sigma)$. Now recall that for each σ we have the exact sequence

$$0\rightarrow\mathcal{O}_S\stackrel{\sigma}{\rightarrow}E\stackrel{\wedge\sigma}{\rightarrow}\mathcal{I}_{z_\sigma}(L)\rightarrow 0.$$

This induces the exact sequence

$$0\rightarrow<\sigma>\rightarrow H^0(S,E)\stackrel{\wedge\sigma}{\rightarrow}H^0(S,\mathcal{I}_{z_\sigma}(L))\rightarrow 0.$$

We conclude immediately from this that $\mathcal{F}$ fits into the exact sequence

$$0\rightarrow\mathcal{O}_{\mathbb{P}(H^0(S,E))}(-2)\rightarrow H^0(S,E)\otimes\mathcal{O}_{\mathbb{P}(H^0(S,E))}(-1)\rightarrow\mathcal{F}\rightarrow 0.$$

Since $rank\ H^0(S,E)=k+2$, it follows that $det\,\mathcal{F}=\mathcal{O}_{\mathbb{P}(H^0(S,E))}(-k)$. ∎

It follows from this lemma that we have a natural inclusion

$$S^k H^0(S,E)^* \hookrightarrow H^0(W, \psi^* L_k). \tag{3.17}$$

(It will be proven in the next section that this inclusion is in fact an isomorphism, but we shall not need this here.)

Our strategy to prove proposition 4 will be first to construct an isomorphism

$$H^0(S^{[k]}, L_k) = \wedge^k H^0(S,L) \cong S^k H^0(S,E)^* \tag{3.18}$$

and then to show that composed with the inclusion (3.17), it is equal, up to a coefficient, to the pull-back map ψ^*.

Construction of the isomorphism (3.18). We note first that the determinant map

$$det : \bigwedge^2 H^0(S,E) \to H^0(S, det\, E) = H^0(S,L)$$

does not vanish on any element of rank 2. Indeed, such an element of rank 2 is given by a subspace W of rank 2 of $H^0(S,E)$, and if its determinant would vanish this would imply that W generates a rank 1 subsheaf of E with at least two sections. But since $Pic\, S$ is generated by L and $H^0(S,E(-L)) = 0$ this is impossible.Hence det provides a morphism

$$d : G_2 \to \mathbb{P}(H^0(S,L)),$$

where G_2 is the Grassmannian of rank two vector subspaces of $H^0(S,E)$, or dually a base-point free linear system

$$K := H^0(S,L)^* \stackrel{d^*}{\to} H^0(G_2, \mathcal{L}) = \wedge^2 H^0(S,E)^*,$$

where $\mathcal{L}$ is the Plücker polarization on G_2. Notice that since $rank\, K = 2k+1$, and $dim\, G_2 = 2k$, d^* has to be injective. Since K is base-point free, we have the exact Koszul complex on G_2

$$0 \to \bigwedge^{2k+1} K \otimes \mathcal{L}^{-(2k+1)} \to \ldots \to K \otimes \mathcal{L}^{-1} \to \mathcal{O}_{G_2} \to 0.$$

We can now tensor this sequence with $S^k\mathcal{E}$, where the rank 2 vector bundle $\mathcal{E}$ on G_2 is dual to the tautological rank two subbundle and satisfies $H^0(G_2, S^k\mathcal{E}) = S^k H^0(S,E)^*$.

This provides the exact complex

$$0 \to \bigwedge^{2k+1} K \otimes \mathcal{L}^{-(2k+1)} \otimes S^k\mathcal{E} \to \ldots \to K \otimes \mathcal{L}^{-1} \otimes S^k\mathcal{E} \to S^k\mathcal{E} \to 0. \tag{3.19}$$

In this complex $\mathcal{K}^\cdot$, the term $S^k\mathcal{E}$ is put in degree 0. The hypercohomology $\mathbb{H}^0(G_2, \mathcal{K}^\cdot)$ vanishes. Now we have a spectral sequence

$$E_1^{p,q} = H^q(G_2, \mathcal{K}^p) \Rightarrow \mathbb{H}^{p+q}(G_2, \mathcal{K}^\cdot).$$

It is obvious for degree reasons that all differentials d_r starting from the term $E_r^{0,0}$ vanish. On the other hand the terms $E_1^{p,q}$ with $p+q=-1$ are of the form

$$H^q(G_2, \bigwedge^{q+1} K \otimes \mathcal{L}^{-q-1} \otimes S^k\mathcal{E}).$$

Using the proposition 9 proven in the appendix, we see that these terms are all 0, except for

$$E_1^{-k-1,k} = H^k(G_2, \bigwedge^{k+1} K \otimes \mathcal{L}^{-k-1} \otimes S^k\mathcal{E}),$$

which is equal to $\bigwedge^{k+1} K$. It follows that there is only one non zero differential which arrives in some $E_r^{0,0}$, namely

$$d_{k+1} : E_{k+1}^{-k-1,k} \to E_{k+1}^{0,0}.$$

This implies that

$$E_{k+1}^{0,0} = E_1^{0,0} = H^0(G_2, S^k\mathcal{E}) = S^k H^0(S,E)^*$$

and that the differential d_{k+1} above is surjective, since the spectral sequence abuts to 0. Hence we have build a surjective map d_{k+1} from a subquotient of $E_1^{-k-1,k} = \bigwedge^{k+1} K$ to $S^k H^0(S,E)^*$. Since

$$dim \bigwedge^{k+1} K = dim\, S^k H^0(S,E)^* = C_{2k+1}^{k+1}$$

this subquotient must in fact be equal to $\bigwedge^{k+1} K$ and the map d_{k+1} has to be an isomorphism. Finally, since $rank\, K = 2k+1$,

$$\bigwedge^{k+1} K = (\bigwedge^{k} K)^* = \bigwedge^{k} H^0(S,L).$$

Hence we have constructed our isomorphism

$$d_{k+1} : \bigwedge^{k} H^0(S,L) \to S^k H^0(S,E)^*.$$

■

To conclude the proof of proposition 4, it remains only to show :

Proposition 5 *The map d_{k+1} constructed above identifies up to a coefficient to the map*

$$\psi^* : H^0(S^{[k]}, L_k) \cong \bigwedge^{k} H^0(S,L) \to H^0(W, \psi^* L_k),$$

which takes values in $S^k H^0(S,E)^* \subset H^0(W, \psi^* L_k)$.

Proof. First of all it is clear that ψ^* takes values in $\pi^* H^0(\mathbb{P}(H^0(S,E)), \mathcal{O}(k)) = S^k H^0(S,E)^*$. Indeed, this map is the pull-back map associated to the morphism

$$W \to Grass(k+1, H^0(S,L))$$
$$z \mapsto H^0(S, L \otimes \mathcal{I}_{z'}),\ z' = \psi(z).$$

But as mentioned in the proof of lemma 2, this morphism factors through $\pi : W \to \mathbb{P}(H^0(S,E))$. More precisely, we noticed in the proof of lemma 2 that the restriction map

$$\psi^* : \wedge^k H^0(S,L) \to S^k H^0(S,E)^*$$

corresponds to the morphism

$$\mathbb{P}(H^0(S,E)) \to Grass(k+1, H^0(S,L))$$
$$\sigma \mapsto det(\sigma \wedge H^0(S,E)).$$

But this morphism is the composition of the morphism

$$\beta : \mathbb{P}(H^0(S,E)) \to Grass(k+1, \bigwedge^2 H^0(S,E))$$
$$\sigma \mapsto \sigma \wedge H^0(S,E).$$

and of the rational map

$$det : Grass(k+1, \bigwedge^2 H^0(S,E)) \to Grass(k+1, H^0(S,L))$$

induced by the determinant map $det : \bigwedge^2 H^0(S,E) \to H^0(S,L)$.

Next, we note that, with the same spectral sequence argument, and replacing $K = H^0(S,L)^* \subset \bigwedge^2 H^0(S,E)^*$ by the base point free linear system $K' = \bigwedge^2 H^0(S,E)^*$ on G_2, we could have constructed more generally a surjective map

$$D_{k+1} : \bigwedge^{k+1}(\bigwedge^2 H^0(S,E)^*) \to S^k H^0(S,E)^*,$$

whose restriction to $\bigwedge^{k+1} K$ is equal to d_{k+1}.

Hence proposition 5 will follow from the following

Lemma 3 *The maps D_{k+1} and β^* from $\bigwedge^{k+1}(\bigwedge^2 H^0(S,E)^*)$ to $S^k H^0(S,E)^*$ coincide up to a coefficient.*

Proof. We could argue by $Sl(k+2)$-equivariance. A more direct way to prove this is to note the following : If $W \subset \bigwedge^2 H^0(S,E)^*$ is a rank $k+1$ vector subspace in general position, it defines a codimension $k+1$ complete intersection subvariety G_W of G_2. Consider the incidence correspondence

$$\begin{array}{ccc} P & \xrightarrow{f} & \mathbb{P}(H^0(S,E)) \\ g \downarrow & & \\ G_2 & & \end{array}$$

Then we have a hypersurface $X_W = f(g^{-1})(G_W)$ of $\mathbb{P}(H^0(S,E))$, which is easily proven to be of degree k. It is clear that

$$H^0(G_2, S^k\mathcal{E} \otimes \mathcal{I}_{G_W}) = H^0(\mathbb{P}(H^0(S,E)), \mathcal{O}_{\mathbb{P}(H^0(S,E))}(k)(-X_W)).$$

On the other hand, from the linear system W we can construct a Koszul complex which is a resolution of $\mathcal{I}_{G_W}$. Hence it is clear that

$$D_{k+1}(\bigwedge^{k+1} W) \subset H^0(G_2, S^k\mathcal{E} \otimes \mathcal{I}_{G_W}).$$

In other words, if η is a generator of $\bigwedge^{k+1} W$, $D_{k+1}(\eta)$ is a defining equation of X_W or 0. It remains then only to prove that $\beta^*\eta$ also vanishes on X_W. But by definition

$$X_W = \{x \in \mathbb{P}(H^0(S,E)),\, \exists 0 \neq \gamma \in \mathbb{P}(H^0(S,E)/ < x >),\, x \wedge \gamma \perp W\}.$$

This means that for $x \in X_W$, the composed map

$$W \hookrightarrow \bigwedge^2 H^0(S,E)^* \to (x \wedge H^0(S,E))^*$$

is not an isomorphism, hence its determinant vanishes. But this determinant is equal to $\beta^*\eta(x)$. ∎

4 Proof of the assumption H1

Recall that we have a Cartesian diagram

$$\begin{array}{ccc} Z' & \stackrel{\pi'}{\to} & Z \\ \tau' \downarrow & & \tau \downarrow \\ S \times \tilde{W} & \stackrel{(Id,\pi'')}{\to} & S \times W \end{array}$$

where the vertical maps τ, τ' are blow-ups and the degree k morphism π'' fits into the Cartesian diagram

$$\begin{array}{ccc} \tilde{W} & \to & I_k \\ \pi'' \downarrow & & \pi_k \downarrow \\ W & \stackrel{\psi}{\to} & S^{[k]}_{curv} \end{array}.$$

We have the morphisms

$$j' : Z' \to I_{k+1},\ j : Z \to I_{k+1}$$

such that $\pi_{k+1} \circ j' = \pi_{k+1} \circ j \circ \pi'$ and the formula

$$(\pi_{k+1} \circ j)^* L_{k+1} = \tau^*(L \boxtimes \psi^* L_k)(-D)$$

where D is the exceptional divisor of τ. Pulling-back this equality to Z', we have

$$(\pi_{k+1} \circ j')^* L_{k+1} = \tau'^* (L \boxtimes (\psi \circ \pi'')^* L_k)(-D'),$$

where D' is the exceptional divisor of τ'. Since $D' = {\pi'}^{-1}(D)$ and π' is surjective, we conclude that in order to prove H1, that is the fact that the pull-back map

$$\pi'^* : H^0(Z, (\pi_{k+1} \circ j)^* L_{k+1}) \to H^0(Z', (\pi_{k+1} \circ j')^* L_{k+1})$$

is surjective, it suffices to show that the pull-back map

$$\pi''^* : H^0(W, \psi^* L_k) \to H^0(\tilde{W}, (\psi \circ \pi'')^* L_k)$$

is surjective.

Now recall that we have a morphism

$$\pi : W \to \mathbb{P}(H^0(S, E))$$

such that (cf lemma 2)

$$\psi^* L_k = \pi^* \mathcal{O}_{\mathbb{P}(H^0(S,E))}(k).$$

Denoting by $r := \pi \circ \pi'' : \tilde{W} \to \mathbb{P}((H^0(S, E))$, we shall prove the following stronger statement

Theorem 2 *The pull-back map*

$$r^* : H^0(\mathbb{P}(H^0(E)), \mathcal{O}_{\mathbb{P}(H^0(E))}(k)) \to H^0(\tilde{W}, (\psi \circ \pi'')^* L_k) \tag{4.20}$$

is surjective.

The end of this section will be devoted to the proof of this theorem, which proceeds in several steps. In what follows, we shall use the notation $H^0(E)$ for $H^0(S, E)$.

Notice to begin with that $\tilde{W}$ is a large open set in the subscheme

$$W' \subset \widetilde{S \times S} \times \mathbb{P}(H^0(E)),$$

where $\widetilde{S \times S}$ is the blow-up of $S \times S$ along the diagonal, defined as

$$W' := \{(x, y, \eta, \sigma),\ \sigma_{|\eta=0},\ \{x\},\ \{y\} \subset \eta\}.$$

(Here η is a subscheme of length 2 of S, and we see elements of $\widetilde{S \times S}$ as elements (x, y) of $S \times S$ together with a schematic structure η of length 2 on $\{x\} \cup \{y\}$.)

The map r is just the restriction to W' of the second projection. Hence we have

$$H^0(\tilde{W}, (\psi \circ \pi'')^* L_k) = H^0(W', pr_2^* \mathcal{O}_{\mathbb{P}(H^0(E))}(k))$$

and the surjectivity of (4.20) is equivalent to the condition

$$H^1(\widetilde{S\times S}\times\mathbb{P}(H^0(E)),pr_2^*\mathcal{O}(k)\otimes\mathcal{I}_{W'})=0. \tag{4.21}$$

Now notice that there is a vector bundle $\tilde{E}_2$ on $\widetilde{S\times S}$ such that W' is the zero set of a section σ of $\tilde{E}_2\boxtimes\mathcal{O}_{\mathbb{P}(H^0(E))}(1)$. Indeed it suffices to take for $\tilde{E}_2$ the vector bundle with fiber $H^0(E_{|\eta})$ at the point (x,y,η) of $\widetilde{S\times S}$. Then the section σ takes the value $\tau_{|\eta}$ at the point (x,y,η,τ) of $\widetilde{S\times S}\times\mathbb{P}(H^0(E))$. One checks easily that W' is reduced of codimension 4. Hence we have a Koszul resolution of $\mathcal{I}_{W'}$

$$0\to\bigwedge^4\tilde{E}_2^*\boxtimes\mathcal{O}(-4)\to\ldots\to\tilde{E}_2^*\boxtimes\mathcal{O}(-1)\to\mathcal{I}_{W'}\to 0. \tag{4.22}$$

Our first goal will be to compute the cohomology groups of $\widetilde{S\times S}\times\mathbb{P}(H^0(E))$ with value in $\bigwedge^i\tilde{E}_2^*\boxtimes\mathcal{O}(k-i)$. Since $k\geq 2$, $i\leq 4$, $\mathcal{O}(k-i)$ has no higher cohomology on $\mathbb{P}(H^0(E))=\mathbb{P}^{k+1}$. Hence we have

$$H^l(\widetilde{S\times S}\times\mathbb{P}(H^0(E)),\bigwedge^i\tilde{E}_2^*\boxtimes\mathcal{O}(k-i))=H^l(\widetilde{S\times S},\bigwedge^i\tilde{E}_2^*)\otimes S^{k-i}H^0(S,E)^*.$$

We have now the following proposition

Proposition 6 *1.* $H^2(\widetilde{S\times S},\tilde{E}_2^*)=pr_1^*H^2(S,E^*)\oplus pr_2^*H^2(S,E^*)$ *and*

$$H^1(\widetilde{S\times S},\tilde{E}_2^*)=0.$$

2. $H^2(\widetilde{S\times S},\bigwedge^2\tilde{E}_2^*)=pr_1^*H^2(S,-L)\oplus pr_2^*H^2(S,-L)$.

3. $H^4(\widetilde{S\times S},\bigwedge^4\tilde{E}_2^*)$ *is dual to* $\mathrm{Ker}\,(H^0(S,L)\otimes H^0(S,L)\to H^0(S,2L))$.

4. $H^3(\widetilde{S\times S},\bigwedge^3\tilde{E}_2^*)=0$ *and* $H^4(\widetilde{S\times S},\bigwedge^3\tilde{E}_2^*)$ *admits as a quotient*

$$H^4(\widetilde{S\times S},\tau^*(E^*\boxtimes(-L))(2\Delta))\oplus H^4(\widetilde{S\times S},\tau^*((-L)\boxtimes E^*)(2\Delta)),$$

which is dual to the direct sum of two copies of

$$\mathrm{Ker}\,(H^0(S,E)\otimes H^0(S,L)\to H^0(S,E\otimes L)).$$

(Here $\Delta\subset\widetilde{S\times S}$ is the exceptional divisor.)

Proof.

1. The bundle $\tilde{E}_2$ fits into the exact sequence

$$0\to\tilde{E}_2\to\tau^*(pr_1^*E\oplus pr_2^*E)\to\tau'^*E\to 0 \tag{4.23}$$

where $\tau:\widetilde{S\times S}\to S\times S$ is the blowing-down map, and where $\tau':\Delta\to Diag\,S$ is its restriction to the exceptional divisor.

Dualizing, we get the following exact sequence

$$0 \to \tau^*(pr_1^*E^* \oplus pr_2^*E^*) \to \tilde{E}_2^* \to \tau'^*E^* \otimes \mathcal{O}_\Delta(\Delta) \to 0. \tag{4.24}$$

Now $R^0\tau'_*\mathcal{O}_\Delta(\Delta) = R^1\tau'_*\mathcal{O}_\Delta(\Delta) = 0$ hence the sheaf on the right has no cohomology. It follows that

$$\begin{aligned} H^i(\widetilde{S\times S}, \tilde{E}_2^*) &= H^i(\widetilde{S\times S}, \tau^*(pr_1^*E^* \oplus pr_2^*E)) \\ &= H^i(S\times S, pr_1^*E^* \oplus pr_2^*E). \end{aligned}$$

Since E^* has no odd dimensional cohomology, nor $\mathcal{O}_S$, it follows from Künneth formula that the same is true for $pr_1^*E^* \oplus pr_2^*E$ on $S \times S$. Finally we have

$$H^2(S\times S, pr_1^*E^*) = H^2(S, E^*)$$

since $H^0(S, E^*) = 0$. This proves 1.

2. From (4.24) we deduce that $\bigwedge^2 \tilde{E}_2^*$ has a filtration whose successive terms are

$$\bigwedge^2 \tau^*(pr_1^*E^* \oplus pr_2^*E^*),\ \tau^*(pr_1^*E^* \oplus pr_2^*E^*) \otimes \tau'^*E^* \otimes \mathcal{O}_\Delta(\Delta),\ \bigwedge^2 \tau'^*E^* \otimes \mathcal{O}_\Delta(2\Delta).$$

The sheaf $(pr_1^*E^* \oplus pr_2^*E^*) \otimes \tau'^*E^* \otimes \mathcal{O}_\Delta(\Delta)$ has no cohomology, since $\mathcal{O}_\Delta(\Delta)$ has no cohomology along the fibers of τ'. Hence we have an exact sequence

$$H^1(\Delta, \bigwedge^2 \tau'^*E^* \otimes \mathcal{O}_\Delta(2\Delta)) \to H^2(\widetilde{S\times S}, \tau^* \bigwedge^2(pr_1^*E^* \oplus pr_2^*E^*)) \to H^2(\widetilde{S\times S}, \bigwedge^2 \tilde{E}_2^*)$$

$$\to H^2(\Delta, \bigwedge^2 \tau'^*E^* \otimes \mathcal{O}_\Delta(2\Delta)) \ldots$$

But since

$$R^1\tau'_*(2\Delta_{|\Delta}) = \mathcal{O}_S,\ R^0\tau'_*(2\Delta_{|\Delta}) = 0,$$

the term on the left is equal to $H^0(S, \bigwedge^2 E^*) = 0$ and the term on the right is equal to $H^1(S, \bigwedge^2 E^*) = 0$. Hence we have

$$\begin{aligned} H^2(\widetilde{S\times S}, \bigwedge^2 \tilde{E}_2^*) &= H^2(\widetilde{S\times S}, \tau^* \bigwedge^2(pr_1^*E^* \oplus pr_2^*E^*)) \\ &= H^2(S\times S, \bigwedge^2(pr_1^*E^* \oplus pr_2^*E^*)) \end{aligned}$$

Finally

$$\bigwedge^2(pr_1^*E^* \oplus pr_2^*E^*) = pr_1^* \bigwedge^2 E^* \oplus E^* \boxtimes E^* \oplus pr_2^* \bigwedge^2 E^*.$$

The central term has no cohomology in degree 2 by Künneth formula, because $H^1(S, E^*) = H^0(S, E^*) = 0$, and we have

$$H^2(S\times S, pr_1^* \bigwedge^2 E^*) = H^2(S, \bigwedge^2 E^*) = H^2(S, -L),$$

where the first equality follows from Künneth formula and $H^0(S, \bigwedge^2 E^*) = 0$. This proves 2.

3. We have $det\, \tilde{E}_2^* = \tau^*((-L) \boxtimes (-L))(2\Delta)$ by the exact sequence (4.23). Hence

$$\bigwedge^3 \tilde{E}_2^* = \tilde{E}_2 \otimes det\, \tilde{E}_2^* = \tilde{E}_2 \otimes \tau^*((-L) \boxtimes (-L))(2\Delta). \tag{4.25}$$

The exact sequence (4.23) gives now the long exact sequence

$$H^2(\Delta, \tau'^*(E(-2L))(2\Delta_{|\Delta})) \to H^3(\widetilde{S \times S}, \bigwedge^3 \tilde{E}_2^*)$$

$$\to H^3(\widetilde{S \times S}, \tau^*((pr_1^* E \oplus pr_2^* E) \otimes ((-L) \boxtimes (-L)))(2\Delta)).$$

Since $R^0\tau'_*\mathcal{O}_\Delta(2\Delta) = 0$, $R^1\tau'_*\mathcal{O}_\Delta(2\Delta) = \mathcal{O}_S$, the left hand side is equal to $H^1(S, E(-2L))$, which is easily seen to be 0.

Next we have $K_{\widetilde{S\times S}} = \mathcal{O}_{\widetilde{S\times S}}(\Delta)$, hence

$$H^3(\widetilde{S \times S}, \tau^*(pr_1^* E \oplus pr_2^* E) \otimes ((-L) \boxtimes (-L)))(2\Delta))$$

is dual to

$$H^1(\widetilde{S \times S}, \tau^*((pr_1^* E^* \oplus pr_2^* E^*) \otimes (L \boxtimes L))(-\Delta)). \tag{4.26}$$

Writing the exact sequence

$$0 \to \tau^*((pr_1^* E^* \oplus pr_2^* E^*) \otimes (L \boxtimes L))(-\Delta) \to \tau^*((pr_1^* E^* \oplus pr_2^* E^*) \otimes (L \boxtimes L))$$

$$\to \tau^*((pr_1^* E^* \oplus pr_2^* E^*) \otimes (L \boxtimes L))_{|\Delta} \to 0$$

where

$$\tau^*((pr_1^* E^* \oplus pr_2^* E^*) \otimes (L \boxtimes L))_{|\Delta} \cong \tau'^*((E^* \oplus E^*)(2L)),$$

and using the isomorphism

$$E^* \otimes L \cong E,$$

we see that the vanishing of the cohomology group (4.26) follows from the fact that the multiplication map

$$H^0(S, E) \otimes H^0(S, L) \to H^0(S, E \otimes L)$$

is surjective, which is easily checked, and from the vanishing

$$H^1(S \times S, (pr_1^* E^* \oplus pr_2^* E^*) \otimes L \boxtimes L) = 0.$$

Finally the equality (4.25) and the exact sequence (4.23) also show that $H^4(\widetilde{S \times S}, \bigwedge^3 \tilde{E}_2^*)$ admits

$$H^4(\widetilde{S \times S}, \tau^*((pr_1^* E \oplus pr_2^* E) \otimes ((-L) \boxtimes (-L)))(2\Delta))$$

as a quotient. By Serre's duality this space is dual to

$$H^0(\widetilde{S\times S},\tau^*((pr_1^*E^*\oplus pr_2^*E^*)\otimes(L\boxtimes L))(-\Delta)). \tag{4.27}$$

But this is equal to

$$H^0(S\times S,(pr_1^*E^*\oplus pr_2^*E^*)\otimes(L\boxtimes L)\otimes\mathcal{I}_{Diag}).$$

We use then the fact that

$$pr_1^*E^*\otimes(L\boxtimes L)=E\boxtimes L$$

to conclude that (4.27) is equal to the sum of two copies of

$$\mathrm{Ker}\,(H^0(S,E)\otimes H^0(S,L)\to H^0(S,E\otimes L)).$$

4. We already noticed that

$$\bigwedge^4\tilde{E}_2^*=det\,\tilde{E}_2^*=\tau^*((-L)\boxtimes(-L))(2\Delta).$$

It follows then from Serre's duality and $K_{\widetilde{S\times S}}=\mathcal{O}_{\widetilde{S\times S}}(\Delta)$ that $H^4(\widetilde{S\times S},\bigwedge^4\tilde{E}_2^*)$ is dual to

$$H^0(\widetilde{S\times S},\tau^*(L\boxtimes L)(-\Delta))=\mathrm{Ker}\,(H^0(S,L)\otimes H^0(S,L)\to H^0(S,2L))).$$

Hence 4 is proven.

■

Coming back to the Koszul resolution of $\mathcal{I}_{W'}\otimes pr_2^*\mathcal{O}(k)$ induced by (4.22), we see that in order to prove the vanishing (4.21), it suffices to show :

a) $H^1(\widetilde{S\times S}\times\mathbb{P}(H^0(E)),\tilde{E}_2^*\boxtimes\mathcal{O}(k-1))=0.$

b) The interior product with σ

$$int(\sigma):H^2(\widetilde{S\times S}\times\mathbb{P}(H^0(E)),\bigwedge^2\tilde{E}_2^*\boxtimes\mathcal{O}(k-2))$$

$$\to H^2(\widetilde{S\times S}\times\mathbb{P}(H^0(E)),\tilde{E}_2^*\boxtimes\mathcal{O}(k-1))$$

is injective.

c) $H^3(\widetilde{S\times S}\times\mathbb{P}(H^0(E)),\bigwedge^3\tilde{E}_2^*\boxtimes\mathcal{O}(k-3))=0.$

d) The interior product with σ

$$int(\sigma):H^4(\widetilde{S\times S}\times\mathbb{P}(H^0(E)),\bigwedge^4\tilde{E}_2^*\boxtimes\mathcal{O}(k-4))$$

$$\to H^4(\widetilde{S\times S}\times\mathbb{P}(H^0(E)),\bigwedge^3\tilde{E}_2^*\boxtimes\mathcal{O}(k-3))$$

is injective.

The conditions a) and c) have been established in proposition 6. We now dualize property b) as follows : by proposition 6, 1 and 2, we have

$$H^2(\widetilde{S\times S}\times\mathbb{P}(H^0(E)),\bigwedge^2\tilde{E}_2^*\boxtimes\mathcal{O}(k-2))$$

$$=(pr_1^*H^2(S,-L)\oplus pr_2^*H^2(S,-L))\otimes S^{k-2}H^0(S,E)^*,$$

$$H^2(\widetilde{S\times S}\times\mathbb{P}(H^0(E)),\tilde{E}_2^*\boxtimes\mathcal{O}(k-1))$$
$$=(pr_1^*H^2(S,E^*)\oplus pr_2^*H^2(S,E^*))\otimes S^{k-1}H^0(S,E)^*.$$

Dualizing, we get

$$H^2(\widetilde{S\times S}\times\mathbb{P}(H^0(E)),\bigwedge^2\tilde{E}_2^*\boxtimes\mathcal{O}(k-2))^*$$

$$=(H^0(S,L)\oplus H^0(S,L))\otimes S^{k-2}H^0(S,E),$$
$$H^2(\widetilde{S\times S}\times\mathbb{P}(H^0(E)),\tilde{E}_2^*\boxtimes\mathcal{O}(k-1))^*$$
$$=(H^0(S,E)\oplus H^0(S,E))\otimes S^{k-1}H^0(S,E).$$

It is then immediate to check that the transpose of the map $int(\sigma)$ is the map $\wedge\sigma$, so that b) translates into the condition that

$$\wedge\sigma:(H^0(S,E)\oplus H^0(S,E))\otimes S^{k-1}H^0(S,E)\to(H^0(S,L)\oplus H^0(S,L))\otimes S^{k-2}H^0(S,E)$$

is surjective.

Now retracing through the isomorphisms given by proposition 6, one checks that the map $\wedge\sigma$ is up to sign equal to the direct sum of two copies of the composed map

$$\mu:H^0(S,E)\otimes S^{k-1}H^0(S,E)\to H^0(S,E)\otimes H^0(S,E)\otimes S^{k-2}H^0(S,E)$$

$$\stackrel{det\otimes id}{\to}H^0(S,L)\otimes S^{k-2}H^0(S,E).$$

Similarly statement d) dualizes as follows : by proposition 6, the space

$$H^4(\widetilde{S\times S},\bigwedge^4\tilde{E}_2^*\boxtimes\mathcal{O}(k-4))\cong H^4(\widetilde{S\times S},\bigwedge^4\tilde{E}_2^*)\otimes S^{k-4}H^0(S,E)^*$$

is dual to

$$\mathrm{Ker}\,(H^0(S,L)\otimes H^0(S,L)\to H^0(S,2L))\otimes S^{k-4}H^0(S,E).$$

Next, we know by proposition 6, 4, that

$$H^4(\widetilde{S\times S},\bigwedge^3\tilde{E}_2^*\boxtimes\mathcal{O}(k-3))\cong H^4(\widetilde{S\times S},\bigwedge^3\tilde{E}_2^*)\otimes S^{k-3}H^0(S,E)^*$$

admits a quotient which is dual to the direct sum of two copies of

$$\mathrm{Ker}\,(H^0(S,E)\otimes H^0(S,L)\to H^0(S,E\otimes L))\otimes S^{k-3}H^0(S,E)).$$

Denoting by $Q_{E,L} := \mathrm{Ker}\,(H^0(S,E)\otimes H^0(S,L)\to H^0(S,E\otimes L))$, $Q_{L,E} := \mathrm{Ker}\,(H^0(S,L)\otimes H^0(S,E)\to H^0(S,E\otimes L))$ and $Q_{L,L} = \mathrm{Ker}\,(H^0(S,L)\otimes H^0(S,L)\to H^0(S,2L))$, we have an inclusion

$$(Q_{L,E}\oplus Q_{E,L})\otimes S^{k-3}H^0(S,E)\subset H^4(\widetilde{S\times S},\bigwedge^3\tilde{E}_2^*\boxtimes\mathcal{O}(k-3))^*$$

and to prove d) it suffices to show that the map dual to $int(\sigma)$ restricts on this subspace to a surjection

$$\wedge\sigma:(Q_{L,E}\oplus Q_{E,L})\otimes S^{k-3}H^0(S,E)\to Q_{L,L}\otimes S^{k-4}H^0(S,E).$$

But retracing through the isomorphisms of proposition 6 and recalling the definition of σ, one checks easily that the first component

$$\wedge\sigma_1:Q_{L,E}\otimes S^{k-3}H^0(S,E)\to Q_{L,L}\otimes S^{k-4}H^0(S,E)$$

of the map above is the following composite

$$\begin{aligned}\mu':Q_{L,E}\otimes S^{k-3}H^0(S,E)\subset H^0(S,L)\otimes H^0(S,E)\otimes S^{k-3}H^0(S,E)\to\\ H^0(S,L)\otimes H^0(S,E)\otimes H^0(S,E)\otimes S^{k-4}H^0(S,E)\\ \stackrel{id\otimes det\otimes id}{\to} H^0(S,L)\otimes H^0(S,L)\otimes S^{k-4}H^0(S,E),\end{aligned}$$

which takes obviously value in $Q_{L,L}\otimes S^{k-4}H^0(S,E)$, while the second component is equal to the first composed with the permutation exchanging factors on both sides.

To conclude then that

$$\wedge\sigma:(Q_{L,E}\oplus Q_{E,L})\otimes S^{k-3}H^0(S,E)\to Q_{L,L}\otimes S^{k-4}H^0(S,E)$$

is surjective, it suffices to show that

$$\mu'_-:Q_{L,E}\otimes S^{k-3}H^0(S,E)\to Q^-_{L,L}\otimes S^{k-4}H^0(S,E)$$

and

$$\mu'_+:Q_{L,E}\otimes S^{k-3}H^0(S,E)\to Q^+_{L,L}\otimes S^{k-4}H^0(S,E)$$

are surjective, where $Q^+_{L,L}$, (resp. $Q^-_{L,L}$) are the symmetric, resp. antisymmetric part of $Q_{L,L}$ and μ'_+ (resp. μ'_-) are the composition of μ' with the projections on the symmetric (resp. antisymmetric) part of $Q_{L,L}$.

In conclusion, theorem 2 will be a consequence of the following propositions

Proposition 7 *The composed map*

$$\mu : H^0(S,E)\otimes S^{k-1}H^0(S,E)\to H^0(S,E)\otimes H^0(S,E)\otimes S^{k-2}H^0(S,E)$$

$$\stackrel{det}{\to} H^0(S,L)\otimes S^{k-2}H^0(S,E)$$

is surjective.

Proposition 8 *a) The map*

$$\mu'_- : Q_{L,E}\otimes S^{k-3}H^0(S,E)\to Q^-_{L,L}\otimes S^{k-4}H^0(S,E)$$

defined above is surjective.

b) The map

$$\mu'_+ : Q_{L,E}\otimes S^{k-3}H^0(S,E)\to Q^+_{L,L}\otimes S^{k-4}H^0(S,E)$$

defined above is surjective.

Proof of proposition 7. Let $\alpha,\ \beta\in H^0(S,E)$ and $\gamma\in H^0(S,L)$ such that

$$\gamma = det(\alpha\wedge\beta).$$

Then we observe first that if $D\subset H^0(S,E)$ is the rank 2 vector subspace generated by α and β, we have

$$\gamma\otimes S^{k-2}D\subset Im\,\mu$$

since the composite

$$D\otimes S^{k-1}D\to D\otimes D\otimes S^{k-2}D\to \bigwedge^2 D\otimes S^{k-2}D$$

is surjective.

Recall now that the map det determines a morphism

$$d : G_2\to \mathbb{P}H^0(S,L)$$

which is surjective and finite since both spaces are of the same dimension $2k$. The fiber $d^{-1}(\gamma)$ is then a finite subscheme $Z_\gamma\subset G_2$ which is the complete intersection of a space W of hyperplane sections of the Grassmannian G_2.

Now by the above observation, and since d is surjective, it suffices to show that the subspaces $S^{k-2}D$ for $D\in Z_\gamma$ generate $S^{k-2}H^0(S,E)$. If we dualize, this is equivalent to say that the dual map

$$S^{k-2}H^0(S,E)^*\to \oplus_{D\in Z_\gamma}S^{k-2}D^*$$

is injective. But this map identifies to the restriction

$$H^0(G_2,S^{k-2}\mathcal{E})\to H^0(Z_\gamma,S^{k-2}\mathcal{E}_{|Z_\gamma}),$$

at least for a reduced Z_γ, which will be the case for a generic γ.

Hence it suffices to show that

$$H^0(G_2, S^{k-2}\mathcal{E} \otimes \mathcal{I}_{Z_\gamma}) = 0. \tag{4.28}$$

Now we use the Koszul resolution

$$0 \to \bigwedge^{2k} W \otimes \mathcal{L}^{-2k} \to \ldots \to W \otimes \mathcal{L}^{-1} \to \mathcal{I}_{Z_\gamma} \to 0$$

The vanishing (4.28) will then follow from the vanishing

$$H^i(G_2, S^{k-2}\mathcal{E} \otimes \mathcal{L}^{-i-1}),\ i = 0,\ 2k-1$$

which is proved in proposition 9 of the appendix. ∎

Proof of proposition 8, a). Notice first that the natural composed map

$$\bigwedge^3 H^0(S,E) \to \bigwedge^2 H^0(S,E) \otimes H^0(S,E)$$

$$\stackrel{det\otimes id}{\to} H^0(S,L) \otimes H^0(S,E)$$

has its image contained in $Q_{L,E}$. Hence it suffices to show that the following composite

$$\mu'' : \bigwedge^3 H^0(S,E) \otimes S^{k-3}H^0(S,E) \to \bigwedge^2 H^0(S,E) \otimes H^0(S,E) \otimes S^{k-3}H^0(S,E)$$

$$\stackrel{det\otimes\mu}{\to} H^0(S,L) \otimes H^0(S,L) \otimes S^{k-4}H^0(S,E) \to \bigwedge^2 H^0(S,L) \otimes S^{k-4}H^0(S,E)$$

is surjective.

Now note that for $\alpha_1,\ \alpha_2,\ \alpha_3 \in H^0(S,E)$

$$\mu''(\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \otimes \alpha_3^{k-3}) = 2(k-3)det(\alpha_2 \wedge \alpha_3) \wedge det(\alpha_1 \wedge \alpha_3) \otimes \alpha_3^{k-4}. \tag{4.29}$$

Fix now $\gamma \in H^0(S,L)$ and consider the set of couples (α_1, α_3) such that

$$det(\alpha_1 \wedge \alpha_3) = \gamma.$$

For any α_2 and any such (α_1, α_3), we have

$$\mu''(\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \otimes \alpha_3^{k-3}) = 2(k-3)det(\alpha_2 \wedge \alpha_3) \wedge \gamma \otimes \alpha_3^{k-4}.$$

Note that the vector α_3 for such pairs takes arbitrary value in some of the lines $D \in Z_\gamma$, where the notations are as in the previous proposition.

Now we have the map

$$\mu''' : H^0(S,E) \otimes S^{k-3}H^0(S,E) \to H^0(S,L) \otimes S^{k-4}H^0(S,E)$$

analogous to μ and the formula above shows that

$$\mu''(\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \otimes \alpha_3^{k-3}) = 2\gamma \wedge \mu'''(\alpha_2 \otimes \alpha_3^{k-3}).$$

With the same proof as in the previous proposition, one shows now that the $S^{k-3}D$, $D \in Z_\gamma$ generate $S^{k-3}H^0(S,E)$ and that μ''' is surjective. Hence the $\alpha_2 \otimes \alpha_3^{k-3}$, $\alpha_3 \in D$, $D \in Z_\gamma$ generate $H^0(S,E) \otimes S^{k-3}H^0(S,E)$ and the $\mu'''(\alpha_2 \otimes \alpha_3^{k-3})$, $\alpha_3 \in D$, $D \in Z_\gamma$ generate by the surjectivity of μ''' the space $H^0(S,L) \otimes S^{k-4}H^0(S,E)$. Hence $Im\, \mu''$ contains $\gamma \wedge H^0(S,L) \otimes S^{k-4}H^0(S,E)$, and since γ was generic, we conclude that μ'' is surjective.

∎

Proof of proposition 8, b). We want to prove that

$$\mu'_+ : Q_{L,E} \otimes S^{k-3}H^0(S,E) \to Q^+_{L,L} \otimes S^{k-4}H^0(S,E)$$

is surjective. Denote similarly, for C a generic member of $\mid L \mid$,

$$Q_{K_C,E} := \mathrm{Ker}\,(H^0(C,K_C) \otimes H^0(C,E_{|C}) \to H^0(C, E \otimes K_C)),$$

$$Q^+_{K_C,K_C} := \mathrm{Ker}\,(S^2H^0(C,K_C) \to H^0(C, K_C^{\otimes 2})).$$

Then we can define

$$\mu'_{+,C} : Q_{K_C,E} \otimes S^{k-3}H^0(C,E_{|C}) \to Q^+_{K_C,K_C} \otimes S^{k-4}H^0(C,E_{|C})$$

as the composite

$$Q_{K_C,E} \otimes S^{k-3}H^0(C,E_{|C}) \subset H^0(C,K_C) \otimes H^0(C,E_{|C}) \otimes S^{k-3}H^0(C,E_{|C})$$

$$\to H^0(C,K_C) \otimes H^0(C,E_{|C}) \otimes H^0(C,E_{|C}) \otimes S^{k-4}H^0(C,E_{|C}) \stackrel{id\otimes det\otimes id}{\to}$$

$$H^0(C,K_C) \otimes H^0(C,K_C) \otimes S^{k-4}H^0(C,E_{|C}) \to S^2H^0(C,K_C) \otimes S^{k-4}H^0(C,E_{|C}).$$

Now the restriction map $H^0(S,E) \to H^0(C,E_{|C})$ is an isomorphism, and the restriction map $H^0(S,L) \to H^0(C,K_C)$ is surjective with kernel σ, the defining equation of C. Hence the restrictions induce a surjection

$$Q_{L,E} \to Q_{K_C,E}$$

and an isomorphism

$$Q^+_{L,L} \cong Q^+_{K_C,K_C},$$

and it suffices to show that $\mu'_{+,C}$ is surjective. A fortiori it suffices to show that the composite

$$\mu'_C : Q_{K_C,E} \otimes S^{k-3}H^0(C,E_{|C}) \subset H^0(C,K_C) \otimes H^0(C,E_{|C}) \otimes S^{k-3}H^0(C,E_{|C})$$

$$\to H^0(C,K_C) \otimes H^0(C,E_{|C}) \otimes H^0(C,E_{|C}) \otimes S^{k-4}H^0(C,E_{|C})$$

$$\stackrel{id\otimes det\otimes id}{\to} H^0(C,K_C) \otimes H^0(C,K_C) \otimes S^{k-4}H^0(C,E_{|C})$$

which takes value in $Q_{K_C,K_C}:=\mathrm{Ker}\,(H^0(C,K_C)^{\otimes 2}\to H^0(C,K_C^{\otimes 2}))$, is surjective on this last space.

Let us now consider the following diagram of exact sequences

$$\begin{array}{ccccc} 0 & \to & Q_{K_C,E}\otimes S^{k-3}H^0(C,E_{|C}) & \to & H^0(C,K_C)\otimes H^0(C,E_{|C})\otimes S^{k-3}H^0(C,E_{|C}) \\ & & \mu'_C\downarrow & & id\otimes\mu_C\downarrow \\ 0 & \to & Q_{K_C,K_C}\otimes S^{k-4}H^0(C,E_{|C}) & \to & H^0(C,K_C)\otimes H^0(C,K_C)\otimes S^{k-4}H^0(C,E_{|C}) \end{array}$$

$$\begin{array}{cccc} \to & H^0(C,E\otimes K_C)\otimes S^{k-3}H^0(C,E_{|C}) & \to & 0 \\ & \mu_{C,K_C}\downarrow & & , \\ \to & H^0(C,K_C^{\otimes 2})\otimes S^{k-4}H^0(C,E_{|C}) & \to & 0 \end{array}$$

where the vertical maps μ_C and μ_{C,K_C} are defined in a way similar to μ e.g μ_C is the composite

$$H^0(C,E_{|C})\otimes S^{k-3}H^0(C,E_{|C})\subset H^0(C,E_{|C})\otimes H^0(C,E_{|C})\otimes S^{k-4}H^0(C,E_{|C})$$

$$\stackrel{det\otimes id}{\to} H^0(C,K_C)\otimes S^{k-4}H^0(C,E_{|C}),$$

and μ_{C,K_C} is defined similarly with a twist by K_C.

One checks easily the surjectivity of the multiplication maps on the left.

The proof of proposition 7 shows as well that μ_C is surjective, as is μ_{C,K_C} by the commutativity of the diagram above. Hence the surjectivity of μ'_C will follow by diagram chasing from the surjectivity of the induced multiplication map

$$H^0(C,K_C)\otimes\mathrm{Ker}\,\mu_C\to\mathrm{Ker}\,\mu_{C,K_C}. \tag{4.30}$$

In what follows we will use again the notation $H^0(E)$ for $H^0(S,E)=H^0(C,E_{|C})$. Define the vector bundle $\mathcal{Q}$ on C as the kernel of the surjective composite morphism of vector bundles

$$S^{k-3}H^0(E)\otimes E\subset S^{k-4}H^0(E)\otimes H^0(E)\otimes E\stackrel{id\otimes det}{\to}S^{k-4}H^0(E)\otimes K_C.$$

Then we clearly have

$$\mathrm{Ker}\,\mu_C=H^0(C,\mathcal{Q}),\ \mathrm{Ker}\,\mu_{C,K_C}=H^0(C,\mathcal{Q}\otimes K_C)$$

so that the surjectivity of the map (4.30) is equivalent to the surjectivity of the multiplication map

$$H^0(C,\mathcal{Q})\otimes H^0(C,K_C)\to H^0(C,\mathcal{Q}\otimes K_C). \tag{4.31}$$

Now we proceed as follows : let $\sigma\in H^0(S,L)$ be the defining equation for C. Recall the finite reduced subscheme $Z_\sigma=d^{-1}(\sigma)\subset G_2$ made of the rank 2 vector subspaces D of $H^0(S,E)$ such that $det\,D=\sigma$. For each such D, there is a line subbundle L_D of E on C, of degree $k+1$ with two sections without common zeroes (see section 2). The space D identifies naturally to $H^0(C,L_D)$.

Clearly the image of the inclusion

$$S^{k-3}H^0(C,L_D)\otimes L_D \subset S^{k-3}H^0(E)\otimes E$$

is contained in $\mathcal{Q}$.

Let now

$$\mathcal{N} := \oplus_{D\in Z_\sigma} S^{k-3}D\otimes L_D.$$

Then by the observation above we have a morphism

$$\alpha : \mathcal{N} \to \mathcal{Q}.$$

The surjectivity of 4.31 will follow from the following three lemmas :

Lemma 4 *The morphism α is surjective.*

Denoting $\mathcal{M} := \operatorname{Ker}\alpha$ we also prove

Lemma 5 *The vector bundle $\mathcal{M}$ is generated by its sections.*

Lemma 6 *The space $H^0(C,\mathcal{M})$ is generated by the subspaces $H^0(C,\mathcal{M}(-x))$, $x\in C$.*

We explain first how these three lemmas imply our result. Using the exact sequence

$$0\to\mathcal{M}\to\mathcal{N}\to\mathcal{Q}\to 0$$

given by lemma 4, we see that the map (4.31) will be surjective if the multiplication map

$$H^0(C,\mathcal{N})\otimes H^0(C,K_C)\to H^0(C,\mathcal{N}\otimes K_C)$$

is surjective, and $H^1(C,\mathcal{M}\otimes K_C)=0$.

The first condition is easy to check. Indeed $\mathcal{N}$ is a direct sum of line bundles L_D corresponding to g^1_{k+1}'s on C, and the result is easy to prove for each of them. As for the second condition, it is equivalent to $H^0(C,\mathcal{M}^*)=0$ by Serre's duality. But since $\mathcal{M}$ is generated by sections by lemma 5, we have an inclusion

$$H^0(C,\mathcal{M}^*)\subset H^0(C,\mathcal{M})^*.$$

The image of this inclusion obviously vanishes on each subspace $H^0(C,\mathcal{M}(-x))$, hence it must be 0 since we know by lemma 6 that these subspaces generate $H^0(C,\mathcal{M})$. ∎

To conclude the proof of 8,b) it remains only to prove these three lemmas.

Proof of lemma 4. First of all we note that the bundle $\mathcal{Q}$ is generated by its sections, since there is a natural surjection

$$S^{k-2}H^0(E)\otimes\mathcal{O}_C\to\mathcal{Q}\to 0.$$

Hence it suffices to show that the map

$$H^0(C, \mathcal{N}) \to H^0(C, \mathcal{Q})$$

is surjective.

But by definition

$$H^0(C, \mathcal{Q}) = \operatorname{Ker}\left(H^0(E) \otimes S^{k-3} H^0(E) \stackrel{\mu_C}{\to} H^0(C, K_C) \otimes S^{k-4} H^0(E)\right)$$

and

$$H^0(C, \mathcal{N}) = \oplus_{D \in Z_\sigma} D \otimes S^{k-3} D.$$

Hence we need to show that the sequence

$$\oplus_{D \in Z_\sigma} D \otimes S^{k-3} D \to H^0(E) \otimes S^{k-3} H^0(E) \stackrel{\mu_C}{\to} H^0(C, K_C) \otimes S^{k-4} H^0(E)$$

is exact at the middle. Again this will follow from a cohomological computation on the Grassmannian G_2. Indeed, the notations being as in the proof of Propositions 4 and 7, the sequence above dualizes as

$$\begin{aligned} I_{Z_\sigma}(\mathcal{L}) \otimes S^{k-4} H^0(G_2, \mathcal{E}) \to H^0(G_2, \mathcal{E}) \otimes S^{k-3} H^0(G_2, \mathcal{E}) \\ \to H^0(\mathcal{E} \otimes S^{k-3} \mathcal{E}_{|Z_\sigma}), \end{aligned} \tag{4.32}$$

where the map

$$I_{Z_\sigma}(\mathcal{L}) \otimes S^{k-4} H^0(G_2, \mathcal{E}) \to H^0(G_2, \mathcal{E}) \otimes S^{k-3} H^0(G_2, \mathcal{E})$$

is composed of the inclusion

$$I_{Z_\sigma}(\mathcal{L}) \otimes S^{k-4} H^0(G_2, \mathcal{E}) \subset H^0(G_2, \mathcal{L}) \otimes S^{k-4} H^0(G_2, \mathcal{E})$$

$$\cong \wedge^2 H^0(G_2, \mathcal{E}) \otimes S^{k-4} H^0(G_2, \mathcal{E})$$

and of the (Koszul) map

$$\wedge^2 H^0(G_2, \mathcal{E}) \otimes S^{k-4} H^0(G_2, \mathcal{E}) \to H^0(G_2, \mathcal{E}) \otimes S^{k-3} H^0(G_2, \mathcal{E}).$$

One checks easily that $H^0(G_2, \mathcal{E}) \otimes S^{k-3} H^0(G_2, \mathcal{E}) \cong H^0(G_2, \mathcal{E} \otimes S^{k-3} \mathcal{E})$. Hence the kernel in the middle identifies to $H^0(G_2, \mathcal{E} \otimes S^{k-3} \mathcal{E} \otimes \mathcal{I}_{Z_\sigma})$. Furthermore $S^{k-4} H^0(G_2, \mathcal{E}) \cong H^0(G_2, S^{k-4} \mathcal{E})$ identifies to $H^0(G_2, \mathcal{E} \otimes S^{k-3} \mathcal{E} \otimes \mathcal{L}^{-1})$ via the (Koszul) inclusion

$$S^{k-4} \mathcal{E} \otimes \mathcal{L} = S^{k-4} \mathcal{E} \otimes \bigwedge^2 \mathcal{E} \subset \mathcal{E} \otimes S^{k-3} \mathcal{E}.$$

Hence the exactness at the middle of the sequence 4.32 will follow from the surjectivity of the multiplication map

$$H^0(G_2, \mathcal{E} \otimes S^{k-3} \mathcal{E} \otimes \mathcal{L}^{-1}) \otimes I_{Z_\sigma}(\mathcal{L}) \to H^0(G_2, \mathcal{E} \otimes S^{k-3} \mathcal{E} \otimes \mathcal{I}_{Z_\sigma}). \tag{4.33}$$

Now let $W := I_{Z_\sigma}(\mathcal{L})$. The Koszul resolution of $\mathcal{I}_{Z_\sigma}$

$$0 \to \bigwedge^{2k} W \otimes \mathcal{L}^{-2k} \to \ldots \to W \otimes \mathcal{L}^{-1} \to \mathcal{I}_{Z_\sigma} \to 0$$

twisted by $\mathcal{E} \otimes S^{k-3}\mathcal{E}$ shows that the surjectivity of (4.33) will hold if we know that

$$H^i(G_2, \mathcal{E} \otimes S^{k-3}\mathcal{E} \otimes \mathcal{L}^{-i-1}) = 0,\ 1 \le i < 2k.$$

Since we have the exact sequence

$$0 \to S^{k-4}\mathcal{E} \otimes \mathcal{L} \to \mathcal{E} \otimes S^{k-3}\mathcal{E} \to S^{k-2}\mathcal{E} \to 0,$$

it suffices to know that

$$H^i(G_2, S^{k-4}\mathcal{E} \otimes \mathcal{L}^{-i}) = 0,\ 1 \le i < 2k,$$

and

$$H^i(G_2, S^{k-2}\mathcal{E} \otimes \mathcal{L}^{-i-1}) = 0,\ 1 \le i < 2k.$$

This is proved in Proposition 9. ■

Proof of lemma 5. The bundles $\mathcal{N}$ and $\mathcal{Q}$ are generated by global sections. To prove that $\mathcal{M}$ is generated by global sections, it suffices to prove that for any $x \in C$, the restriction map $H^0(C, \mathcal{N}(-x)) \to H^0(C, \mathcal{Q}(-x))$ is surjective. For each g^1_{k+1} L_D on C, denote by $\sigma_{D,x} \in H^0(C, L_D) \cong D$ a generator for $H^0(C, L_D(-x))$. We need to show the exactness of the sequence

$$\begin{aligned}\oplus_{D \in Z_\sigma} \sigma_{D,x} \otimes S^{k-3}D \to H^0(C, E(-x)) \otimes S^{k-3}H^0(E) \\ \stackrel{\mu_C}{\to} H^0(C, K_C(-x)) \otimes S^{k-4}H^0(E).\end{aligned} \tag{4.34}$$

(Indeed, by definition, $\oplus_{D \in Z_\sigma} \sigma_{D,x} \otimes S^{k-3}D$ identifies to $H^0(C, \mathcal{N}(-x))$ and

$$\mathrm{Ker}\,(H^0(C, E(-x)) \otimes S^{k-3}H^0(E) \stackrel{\mu_C}{\to} H^0(C, K_C(-x)) \otimes S^{k-4}H^0(E))$$

identifies to $H^0(C, \mathcal{Q}(-x))$.)

Denote by $K_x \subset H^0(E)$ the subspace $H^0(C, E(-x))$. Note that via the identification $H^0(C, L_D) = D \subset H^0(E)$, $\sigma_{D,x}$ becomes a generator of the 1-dimensional vector space $D \cap K_x$. Furthermore, K_x being of codimension 2 in $H^0(E)$ determines a section $\tau_x \in \bigwedge^2 H^0(E)^*$ up to a coefficient. Clearly τ_x belongs to $H^0(C, K_C)^* \subset \bigwedge^2 H^0(E)^*$ and identifies also to the linear form on $H^0(C, K_C)$ defining $H^0(C, K_C(-x))$. Let $G_x \subset G_2$ be the hyperplane section defined by τ_x. The scheme Z_σ is a complete intersection of hyperplane sections of G_x. The variety G_x admits a desingularization $P_x \stackrel{g}{\to} G_x$ defined as

$$P_x = \{(u, \Delta) \in \mathbb{P}(K_x) \times G_2,\ u \in \Delta \cap K_x\}.$$

Note that if

$$\begin{array}{ccc} P & \stackrel{g}{\to} & G_2 \\ f \downarrow & & \\ \mathbb{P}(H^0(E)) & & \end{array}$$

is the incidence variety, P_x can also be defined as $f^{-1}(\mathbb{P}(K_x)) \subset P$.

Each line D parametrized by Z_σ meets K_x along a one dimensional vector space, because L_D has no base-point, so that $H^0(L_D(-x)) \cong D \cap K_x$ is 1-dimensional. It follows that the scheme Z_σ can also be seen as the complete intersection in P_x of hypersurfaces in $\mid g^*\mathcal{L} \mid$.

We now dualize the sequence (4.34). The space $H^0(C, K_C(-x))$ admits for dual the space $W \subset H^0(P_x, g^*\mathcal{L})$ defining $Z_\sigma \subset P_x$. The vector space $< \sigma_{D,x} >^*$ identifies clearly to the fiber of the line bundle $H_x := f^*\mathcal{O}_{\mathbb{P}(K_x)}(1)$ at the point $D \in Z_\sigma$. Hence our sequence dualizes as

$$\begin{aligned} W \otimes S^{k-4}H^0(E)^* \to f^*H^0(\mathbb{P}(K_x), \mathcal{O}(1)) \otimes H^0(P_x, g^*S^{k-3}\mathcal{E}) \\ \to H^0(S^{k-3}\mathcal{E} \otimes H_x \mid Z_\sigma). \end{aligned} \qquad (4.35)$$

The second space in this sequence is easily shown to identify to $H^0(P_x, g^*S^{k-3}\mathcal{E} \otimes H_x)$, so that the kernel at the middle is equal to

$$H^0(P_x, g^*S^{k-3}\mathcal{E} \otimes H_x \otimes \mathcal{I}_{Z_\sigma}).$$

The first map in (4.35) is induced by the isomorphism

$$S^{k-4}H^0(E)^* \cong H^0(P_x, g^*S^{k-4}\mathcal{E}),$$

the multiplication

$$W \otimes H^0(P_x, g^*S^{k-4}\mathcal{E}) \to H^0(P_x, g^*(S^{k-4}\mathcal{E} \otimes \mathcal{L}) \otimes \mathcal{I}_{Z_\sigma})$$

and the composed bundle map

$$g^*S^{k-4}\mathcal{E} \otimes \mathcal{L} \to g^*S^{k-3}\mathcal{E} \otimes \mathcal{E} \to g^*S^{k-3}\mathcal{E} \otimes H_x,$$

where the last map is induced by the natural surjective map $g^*\mathcal{E} \to H_x$.

The exactness of (4.35) will then follow from the surjectivity of

$$W \otimes H^0(P_x, g^*(S^{k-3}\mathcal{E} \otimes \mathcal{L}^{-1}) \otimes H_x) \to H^0(P_x, g^*S^{k-3}\mathcal{E} \otimes H_x \otimes \mathcal{I}_{Z_\sigma}) \quad (4.36)$$

and from the equality

$$H^0(P_x, g^*(S^{k-3}\mathcal{E} \otimes \mathcal{L}^{-1}) \otimes H_x) = H^0(P_x, g^*S^{k-4}\mathcal{E}). \qquad (4.37)$$

This last equality is proved as follows : on P_x we have the exact sequence

$$0 \to g^*\mathcal{L} \otimes H_x^{-1} \to g^*\mathcal{E} \to H_x \to 0,$$

which gives

$$0 \to g^*S^{k-4}\mathcal{E} \otimes g^*\mathcal{L} \otimes H_x^{-1} \to g^*S^{k-3}\mathcal{E} \to H_x^{k-3} \to 0.$$

Tensoring this with $H_x \otimes \mathcal{L}^{-1}$ we get

$$0 \to g^*S^{k-4}\mathcal{E} \to g^*(S^{k-3}\mathcal{E} \otimes \mathcal{L}^{-1}) \otimes H_x \to g^*\mathcal{L}^{-1} \otimes H_x^{k-2} \to 0.$$

But the right hand side has no non-zero sections since it is of negative degree on the fibers of f. Hence the equality (4.37).

Since $Z_\sigma \subset P_x$ is the complete intersection of the space W of sections of $g^*\mathcal{L}$, we have a Koszul resolution of $\mathcal{I}_{Z_\sigma}$, which takes the form

$$0 \to \bigwedge^{2k-1} W \otimes g^*\mathcal{L}^{-2k-1} \to \ldots \to W \otimes g^*\mathcal{L}^{-1} \to \mathcal{I}_{Z_\sigma} \to 0.$$

We can tensor it with $g^*S^{k-3}\mathcal{E} \otimes H_x$, and the surjectivity of the map (4.36) will follow from the following vanishing

$$H^i(P_x, g^*(S^{k-3}\mathcal{E} \otimes \mathcal{L}^{-i-1}) \otimes H_x) = 0,\ 1 \leq i < 2k-1 = dim\, P_x. \tag{4.38}$$

Recall now that $P_x \subset P$ is the complete intersection of two sections of $H = f^*\mathcal{O}_{\mathbb{P}(H^0(E))}(1)$, with $H_x = H_{|P_x}$. The vanishing (4.38) will then follow from

$$H^i(P, g^*(S^{k-3}\mathcal{E} \otimes \mathcal{L}^{-i-1}) \otimes H) = 0,\ 1 \leq i < 2k-1$$

$$H^{i+1}(P, g^*(S^{k-3}\mathcal{E} \otimes \mathcal{L}^{-i-1})) = 0,\ 1 \leq i < 2k-1$$

$$H^{i+2}(P, g^*(S^{k-3}\mathcal{E} \otimes \mathcal{L}^{-i-1}) \otimes H^{-1}) = 0,\ 1 \leq i < 2k-1.$$

The second equality follows immediately from the proposition 9, and the third is obvious since H^{-1} has no cohomology on the fibers of g. The first equality is proven as follows : we have

$$H^i(P, g^*S^{k-3}\mathcal{E} \otimes \mathcal{L}^{-i-1}) \otimes H) = H^i(G_2, S^{k-3}\mathcal{E} \otimes \mathcal{E} \otimes \mathcal{L}^{-i-1}),$$

since $R^0g_*H = \mathcal{E}$. Now we have the exact sequence on G_2

$$0 \to S^{k-4}\mathcal{E} \otimes \mathcal{L} \to S^{k-3}\mathcal{E} \otimes \mathcal{E} \to S^{k-2}\mathcal{E} \to 0.$$

Hence the needed equality will follow from the vanishings

$$H^i(G_2, S^{k-4}\mathcal{E} \otimes \mathcal{L}^{-i}) = 0,$$

$$H^i(G_2, S^{k-2}\mathcal{E} \otimes \mathcal{L}^{-i-1}) = 0,$$

for $1 \leq i < 2k-1$, which are proved in proposition (9). Hence lemma 5 is proven.

■

Proof of lemma 6. Let $x_1, \ldots, x_{2k-1}$ be points of C in general position. We will show that the natural map

$$\oplus_i H^0(C, \mathcal{M}(-x_i)) \to H^0(C, \mathcal{M}) \tag{4.39}$$

is surjective.

Recall that

$$H^0(C, \mathcal{M}) = \text{Ker } \oplus_{D \in Z_\sigma} S^{k-3}D \otimes D \to S^{k-3}H^0(E) \otimes H^0(E).$$

It follows from this, using the identifications

$$H^0(E)^* = H^0(G_2, \mathcal{E}),\ D^* \cong \mathcal{E}_D$$

that

$$H^0(C, \mathcal{M})^* = Coker\, H^0(G_2, S^{k-3}\mathcal{E} \otimes \mathcal{E}) \to H^0(S^{k-3}\mathcal{E} \otimes \mathcal{E}_{|Z_\sigma})$$

$$= H^1(G_2, S^{k-3}\mathcal{E} \otimes \mathcal{E} \otimes \mathcal{I}_{Z_\sigma}).$$

Similarly

$$H^0(C, \mathcal{M}(-x_i)) = \mathrm{Ker}\ \oplus_{D \in Z_\sigma} S^{k-3}D \otimes \sigma_{D,x_i} \to S^{k-3}H^0(E) \otimes K_{x_i}$$

which, with the notations of the previous proof, dualizes to

$$H^0(C, \mathcal{M}(-x_i))^* = \mathrm{Coker}\,(H^0(P_{x_i}, g^*S^{k-3}\mathcal{E} \otimes H_{x_i}) \to H^0(Z_\sigma, S^{k-3}\mathcal{E} \otimes H_{x_i}))$$

$$= H^1(P_{x_i}, g^*S^{k-3}\mathcal{E} \otimes H_{x_i} \otimes \mathcal{I}_{Z_\sigma}),$$

where we view Z_σ as a subscheme of P_{x_i} as well. Hence we have to show that the natural map (induced by the morphism $g^*\mathcal{E} \to H_{x_i}$ on P_{x_i})

$$H^1(G_2, S^{k-3}\mathcal{E} \otimes \mathcal{E} \otimes \mathcal{I}_{Z_\sigma}) \to \oplus_i H^1(P_{x_i}, g^*S^{k-3}\mathcal{E} \otimes H_{x_i} \otimes \mathcal{I}_{Z_\sigma}) \qquad (4.40)$$

is injective.

Let $R \subset G_2$ be the curve which is the complete intersection of the sections $\sigma_{x_i} \in H^0(G_2, \mathcal{L})$. We have first

Fact. *The restriction map*

$$H^0(G_2, S^{k-3}\mathcal{E} \otimes \mathcal{E}) \to H^0(R, S^{k-3}\mathcal{E} \otimes \mathcal{E}_{|R})$$

is surjective.

Using the Koszul resolution of $\mathcal{I}_D$ this is obtained by application of the proposition 9. ■

¿From this we conclude that the restriction map

$$H^1(G_2, S^{k-3}\mathcal{E} \otimes \mathcal{E} \otimes \mathcal{I}_{Z_\sigma}) \to H^1(R, S^{k-3}\mathcal{E} \otimes \mathcal{E}_{|R} \otimes \mathcal{I}_{Z_\sigma})$$

is injective.

Consider now the fibered product

$$\tilde{R} := P_{x_1} \times_{G_2} \times \ldots \times_{G_2} P_{x_{2k-1}}.$$

Denote by $\tilde{g} : \tilde{R} \to R \subset G_2$ the natural morphism. One shows easily that the curve $\tilde{R}$ is isomorphic to R excepted over the intersection of R with the Grassmannian of lines in $\mathbb{P}(K_{x_i})$ for some i. Here R has nodes, which are replaced in $\tilde{R}$ by lines.

(This fact is obviously true set theoretically, and is proved scheme theoretically by the computation of the canonical bundles, which gives :

$$K_{\tilde{R}} = \tilde{g}^* K_R.)$$

The zero set Z_σ is supported away of this singular locus. For each i we have a natural restriction map

$$H^1(P_{x_i}, g^* S^{k-3}\mathcal{E} \otimes H_{x_i} \otimes \mathcal{I}_{Z_\sigma}) \to H^1(\tilde{R}, \tilde{g}^* S^{k-3}\mathcal{E} \otimes H_{x_i} \otimes \mathcal{I}_{Z_\sigma}),$$

since $\tilde{R} = P_{x_1} \times_{G_2} \ldots \times_{G_2} P_{x_{2k-1}}$ admits a natural morphism to P_{x_i}. Next we have by the above description of $\tilde{R}$ an isomorphism

$$H^1(R, S^{k-3}\mathcal{E} \otimes \mathcal{E}_{|R} \otimes \mathcal{I}_{Z_\sigma}) \cong H^1(\tilde{R}, \tilde{g}^* S^{k-3}\mathcal{E} \otimes \mathcal{E} \otimes \mathcal{I}_{Z_\sigma})$$

and it follows that the injectivity of the map (4.40) will be a consequence of the injectivity of the map

$$H^1(\tilde{R}, \tilde{g}^* S^{k-3}\mathcal{E} \otimes \mathcal{E} \otimes \mathcal{I}_{Z_\sigma}) \to \oplus_i H^1(\tilde{R}, \tilde{g}^* S^{k-3}\mathcal{E} \otimes H_{x_i} \otimes \mathcal{I}_{Z_\sigma}) \qquad (4.41)$$

induced by the morphisms $\tilde{g}^*\mathcal{E} \to H_{x_i}$ on $\tilde{R}$. Recall now that $Z_\sigma \subset R$ is defined by a section of $\mathcal{L}$ so that similarly $Z_\sigma \subset \tilde{R}$ is defined by a section of $\tilde{g}^*\mathcal{L}$. Hence we have

$$\mathcal{I}_{Z_\sigma} \cong \tilde{g}^*\mathcal{L}^{-1}.$$

Furthermore

$$K_{\tilde{R}} = \tilde{g}^* K_R = \tilde{g}^*(K_{G_2|R} \otimes \mathcal{L}^{2k-1}) = \tilde{g}^*\mathcal{L}^{k-3}.$$

Hence the map (4.41) dualizes by Serre's duality as the map

$$\oplus_i H^0(\tilde{R}, \tilde{g}^* S^{k-3}\mathcal{E}^* \otimes H^*_{x_i} \otimes \tilde{g}^*\mathcal{L} \otimes \tilde{g}^*\mathcal{L}^{k-3}) \qquad (4.42)$$
$$\to H^0(\tilde{R}, \tilde{g}^*(S^{k-3}\mathcal{E}^* \otimes \mathcal{E}^*) \otimes \tilde{g}^*\mathcal{L} \otimes \tilde{g}^*\mathcal{L}^{k-3}) \qquad (4.43)$$

given by the inclusions $H^*_{x_i} \subset \tilde{g}^*\mathcal{E}^*$ on $\tilde{R}$. Since $det\,\mathcal{E} = \mathcal{L}$, we have

$$\mathcal{E}^* \otimes \mathcal{L} \cong \mathcal{E},$$

Hence (4.42) rewrites as

$$\oplus_i H^0(\tilde{R}, \tilde{g}^*(S^{k-3}\mathcal{E} \otimes \mathcal{L}) \otimes H^*_{x_i}) \to H^0(\tilde{R}, \tilde{g}^*(S^{k-3}\mathcal{E} \otimes \mathcal{E})) \qquad (4.44)$$

given by the inclusions

$$H^*_{x_i} \otimes \tilde{g}^*\mathcal{L} \subset \tilde{g}^*\mathcal{E}.$$

We want to show that (4.41) is injective, or that (4.44) is surjective. We already noticed that the restriction map

$$H^0(G_2, S^{k-3}\mathcal{E} \otimes \mathcal{E}) = S^{k-3}H^0(E)^* \otimes H^0(E)^*$$

$$\to H^0(R, S^{k-3}\mathcal{E}\otimes\mathcal{E}) = H^0(\tilde{R}, \tilde{g}^*(S^{k-3}\mathcal{E}\otimes\mathcal{E}))$$

is surjective. On the other hand, consider the 2-dimensional subspace $H_{x_i} = K_{x_i}^{\perp} \subset H^0(E)^*$. It is obvious that it parametrizes sections of

$$\mathrm{Ker}\,(H^0(P_{x_i}, g^*\mathcal{E}) \to H^0(P_{x_i}, H_{x_i})) = H^0(P_{x_i}, g^*\mathcal{L}\otimes H_{x_i}^*).$$

Hence the surjective map

$$S^{k-3}H^0(E)^*\otimes H^0(E)^* \to H^0(\tilde{R}, \tilde{g}^*(S^{k-3}\mathcal{E}\otimes\mathcal{E}))$$

sends $S^{k-3}H^0(E)^*\otimes H_{x_i}$ in the subspace $H^0(\tilde{R}, \tilde{g}^*(S^{k-3}\mathcal{E}\otimes\mathcal{L})\otimes H_{x_i}^*)$.

Now since the x_i's are generic, the spaces H_{x_i} generate $H^0(E)^*$, hence the $S^{k-3}H^0(E)^*\otimes H_{x_i}$'s generate $S^{k-3}H^0(E)^*\otimes H^0(E)^*$. Hence we have shown that (4.44) is surjective.

■

5 Appendix

We consider the Grassmannian G_2 of rank 2 vector subspaces of a $k+2$-dimensional vector space V. Let $\mathcal{L}$ be the line bundle on G_2 whose sections give the Plücker embedding. If $\mathcal{E}$ is the dual of the tautological subbundle $\mathcal{S}\subset V\otimes\mathcal{O}_{G_2}$, we have $\mathcal{L} = det\,\mathcal{E}$. The cohomology groups $H^p(G_2, \mathcal{L}^{-q}\otimes S^{q'}\mathcal{E})$ are described in the following proposition.

Proposition 9 *For $q>0$, $q'>0$, we have*

$$H^p(G_2, \mathcal{L}^{-q}\otimes S^{q'}\mathcal{E}) = 0 \text{ if } p\neq k,\, 2k.$$

Furthermore, for $p=k$, we have

$$H^p(G_2, \mathcal{L}^{-q}\otimes S^{q'}\mathcal{E}) = 0 \text{ if } -q+q'+1<0,$$

and for $p=2k$, we have

$$H^p(G_2, \mathcal{L}^{-q}\otimes S^{q'}\mathcal{E}) = 0 \text{ if } -q+q'\geq -k-1.$$

Proof. Let

$$\begin{array}{ccc} P & \stackrel{g}{\to} & G_2 \\ f\downarrow & & \\ \mathbb{P}(V) & & \end{array}$$

be the incidence variety. P is a $\mathbb{P}^1$-bundle over G_2 and a $\mathbb{P}^k$-bundle over $\mathbb{P}(V)$. Let $H := f^*\mathcal{O}_{\mathbb{P}(V)}(1)$ and let $L' = g^*\mathcal{L}$. Then $\mathcal{E} = R^0g_*H$ and $S^{q'}\mathcal{E} = R^0g_*(q'H)$. It follows that we have

$$H^p(G_2, \mathcal{L}^{-q}\otimes S^{q'}\mathcal{E}) = H^p(P, -qL'+q'H).$$

Next the line bundle L' restricts to $\mathcal{O}(1)$ on the fibers of π. It follows from this that
$$K_P = -(k+1)L' - 2H,$$
and $K_{P/\mathbb{P}(V)} = -(k+1)L' + kH$.

Now since $q > 0$ we have $R^l f_*(-qL' + q'H) = 0$ for $l < k$ and hence
$$H^p(P, -qL' + q'H) = H^{p-k}(\mathbb{P}(V), R^k f_*(L' + q'H)). \tag{5.45}$$
By Serre's duality, we have
$$\begin{aligned} R^k f_*(-qL' + q'H) &= (R^0 f_*(qL' - q'H - (k+1)L' + kH))^* \\ &= (R^0 f_*((q-(k+1))L' + (k-q')H))^*. \end{aligned}$$
Now we have
$$R^0 f_*((q-(k+1))L') = 0 \tag{5.46}$$
if $q < k+1$, and
$$R^0 f_*((q-(k+1))L') \cong S^{q-k-1}(\Omega_{\mathbb{P}(V)}(2)) \tag{5.47}$$
for $q \geq k+1$. (The isomorphism (5.47) for $q - k + 1 = 1$ follows from the isomorphism
$$H^0(P, L') = H^0(G_2, \mathcal{L}) = \bigwedge^2 V^* = H^0(\mathbb{P}(V), \Omega_{\mathbb{P}(V)}(2))$$
and from the comparison of the kernels of the surjective evaluation maps
$$H^0(P, L') \to H^0(f^{-1}(x), L')$$
and
$$H^0(\mathbb{P}(V), \Omega_{\mathbb{P}(V)}(2)) \to \Omega_{\mathbb{P}(V)}(2)_x.)$$

Finally we conclude from (5.45), (5.46) and (5.47) that

1. $H^p(P, -qL' + q'H) = 0$ for $p < k$.
2. $H^p(P, -qL' + q'H) = 0$ for $q < k+1$.
3. For $p \geq k$, $q \geq k+1$,
$$H^p(P, -qL' + q'H) = H^{p-k}(\mathbb{P}(V), S^{q-k-1}(T_{\mathbb{P}(V)}(-2))(q'-k)).$$

To conclude, consider the Euler exact sequence
$$0 \to \mathcal{O}_{\mathbb{P}(V)}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}(V)} \to T_{\mathbb{P}(V)}(-1) \to 0.$$
It induces the exact sequences
$$0 \to S^{q-k-2}V \otimes \mathcal{O}_{\mathbb{P}(V)}(-q+q') \to S^{q-k-1}V \otimes \mathcal{O}_{\mathbb{P}(V)}(-q+q'+1)$$

$$\to S^{q-k-1}(T_{\mathbb{P}(V)}(-1))(-q+q'+1) \to 0.$$

Hence we conclude that the space $H^p(G_2, \mathcal{L}^{-q} \otimes S^{q'}\mathcal{E})$ which by the above is also isomorphic to

$$H^{p-k}(\mathbb{P}(V), S^{q-k-1}(T_{\mathbb{P}(V)}(-2))(q'-k))$$

$$= H^{p-k}(\mathbb{P}(V), S^{q-k-1}(T_{\mathbb{P}(V)}(-1))(-q+q'+1))$$

is equal to 0 for $p-k \neq 0$, k (since $p \leq 2k$), and that :

- for $p-k=0$ it is 0 if $-q+q'+1<0$;
- for $p-k=k$ it is 0 if $-q+q' \geq -k-1$. ∎

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